SYMMETRIC SPACES

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ABSTRACT. These notes give an introduction to the theory of symmetric spaces and study symmetric spaces of non-compact type in greater detail.

They grew out of an outline of the theory of symmetric spaces by G. Link. The author relied on the books by S. Helgason and P. Eberlein to provide details. In their present form, these notes were taken, modified and typeset by S. Tornier when, with slight modification, part of the above mentioned material was taught by M. Burger in the course "Lie Groups II" at ETH Zurich in 2014.

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INTRODUCTION

We remark that a prerequisite for this course is a basic understanding of Riemannian geometry, see for instance [Boo86] or [Hel79], which at the same time is a standard reference for the theory of symmetric spaces. See also [Bor98] for a much more condensed version of the latter.

In 1926, É. Cartan began to study those Riemannian manifolds, for which *central symmetries* are distance preserving. The term *central symmetry* is going to be made precise later but you can think of it as a reflection about a point, as in $\mathbb{R}^n$ for instance. These spaces are now known as *symmetric spaces*; he noticed that the classification of these is essentially equivalent to the classification of real semisimple Lie algebras. The local definition of symmetric spaces is a generalisation of constant sectional curvature. Just as a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ is constant if its derivative vanishes identically, one may ask for the Riemann curvature tensor to have covariant derivative zero. As incomprehensible as this statement may be at the moment, it yields an important equivalent characterization of symmetric spaces.

Example. The following list of examples of symmetric spaces contains the constant curvature cases.

(i) Euclidean n-space $\mathbb{E}^n = (\mathbb{R}^n, g_{\text{eucl}})$ has constant sectional curvature zero and isometry group $\text{Iso}(\mathbb{E}^n) \cong \text{O}(n) \ltimes \mathbb{R}^n$.

(ii) The n-sphere $\mathbb{S}^n = (S^n, g_{\text{eucl}})$ has constant sectional curvature one. Its Riemannian metric arises through restriction of the Riemannian metric of the ambient space $\mathbb{R}^{n+1} \supset S^n$ to the tangent bundle of S^n. The isometry group of $\mathbb{S}^n$ is $\text{O}(n)$.

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(iii) Hyperbolic n-space $\mathbb{H}^n$ has constant sectional curvature minus one. To define hyperbolic n-space, consider the quadratic form

$$q(x_1,\dots,x_n,x_{n+1}) := x_1^2+\cdots+x_n^2-x_{n+1}^2$$

on $\mathbb{R}^n$ and define $\mathbb{H}^n := \{x\in\mathbb{R}^{n+1} \mid q(x)=-1,\ x_{n+1}>0\}$. For $n=2$, this set is the upper sheet of the following two-sheeted hyperboloid. The group $\mathrm{O}(n,1)=\{g\in\mathrm{GL}(n,\mathbb{R}) \mid q(g(x))=q(x)\ \forall x\in\mathbb{R}^{n+1}\}$ acts transitively on the two-sheeted hyperboloid and the group

$$\mathrm{Iso}(\mathbb{H}^n) := \mathrm{O}(n,1)_+ := \{g\in\mathrm{O}(n,1) \mid g(\mathbb{H}^n)=\mathbb{H}^n\}$$

does so on the upper sheet. It is the isometry group of $\mathbb{H}^n$ for the Riemannian metric which arises through restriction of the euclidean metric of the ambient space $\mathbb{R}^{n+1}$ to $\mathrm{T}_{e_{n+1}}\mathbb{H}^n$ and propagation to $\mathrm{T}_p\mathbb{H}^n$ for all $p\in\mathbb{H}^n$ using elements of $\mathrm{O}(n,1)_+$. This is well-defined since the scalar product on $\mathrm{T}_{e_{n+1}}\mathbb{H}^n$ is invariant under the induced action of the stabilizer $\mathrm{stab}_{\mathrm{O}(n,1)_+}(e_{n+1})$ on $\mathrm{T}_{e_{n+1}}\mathbb{H}^n$.

1. Overview

Before making precise all the terms used above, we provide an overview of the two characterizations of symmetric spaces and their relation.

1.1. **Riemannian Characterization.**

Let M be a Riemannian manifold. A *geodesic symmetry* about $p\in M$ is a map $s_p: U\to M$, defined on a neighbourhood U of p, which fixes p and reverses every geodesic through p. This definition still is not precise but suffices to give an idea about the geometric aspects we want to highlight in this section.

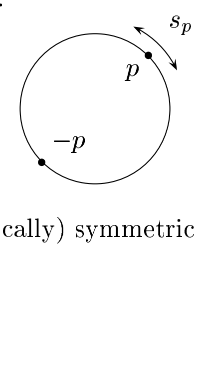
Definition 1.1. Let M be a connected Riemannian manifold. Then M is *locally symmetric* if for every $p\in M$ there is a geodesic symmetry about p which is an isometry. It is *(globally) symmetric* if it is locally symmetric and in addition every geodesic symmetry s_p $(p\in M)$ is defined on the whole of M.

The connectedness assumption in the definition above will become clearer later. For instance, we want symmetric spaces to be homogeneous under the action of a connected group. Globally symmetric spaces will often simply be called *symmetric.*

Example. Following the example in the Introduction, we have the following.

(i) Let $M=\mathbb{E}^n$ and let $p\in M$. Then $s_p: M\to M$ is given by $v\mapsto 2p-v$. Note that each s_p has a unique fixed point p and that by a composition of two such geodesic symmetries one obtains all translations. These observations will generalize later on.

(ii) Let $M=\mathbb{S}^n$ and let $p=e_{n+1}\in S^n\subseteq\mathbb{R}^{n+1}$. Then $\mathbb{R}^{n+1}$ decomposes as $\mathbb{R}^{n+1}=\mathbb{R}\,p\oplus(\mathbb{R}\,p)^\perp$. In this decomposition, $s_p: S^n\to S^n$ is given by $tp+w\mapsto tp-w$. Note that in this case, each s_p has exactly two fixed points, namely p and $-p$.



Some of the useful features of (locally) symmetric spaces are the following.

(i) The universal covering of a locally symmetric space is a globally symmetric space. Hence every locally symmetric space M is of the form $M = \Gamma\backslash\widetilde{M}$ where Γ is a subgroup of $\operatorname{Iso}(\widetilde{M})$ which acts properly discontinuously and without fixed points on $\widetilde{M}$.

(ii) A globally symmetric space X is homogeneous under the action of $\operatorname{Iso}(X)^\circ$ with compact stabilizers. In fact $\operatorname{Iso}(X)^\circ$ is going to be a Lie group of a specific kind for which there is a well-established theory. For locally symmetric spaces one then has to look at subgroups Γ of these as above.

Example 1.2. Here are some more examples of (locally) symmetric spaces that illustrate the above features.

(i) Compact, orientable connected surfaces are completely determined up to homeomorphism by their genus g. For $g = 0$, we have the globally symmetric space $\mathbb{S}^2$. For $g = 1$, there is the torus. It admits many locally symmetric metrics but they all come from bases of $\mathbb{R}^2$ and hence organize themselves in a parameter space. For instance, the torus can be realized as $\mathbb{R}^2 / \mathbb{Z}^2$. Higher genus surfaces all have universal cover $\mathbb{H}^2$.

(ii) Flat, compact, three-dimensional manifolds correspond to cristallographic groups acting on $\mathbb{E}^3$.

(iii) Consider $\mathbb{H}^n$ and the subgroup $\Gamma := \mathrm{O}(n,1,\mathbb{Z})_+$ of $\operatorname{Iso}(\mathbb{H}^n) = \mathrm{O}(n,1)_+$. Then Γ is discrete in $\operatorname{Iso}(\mathbb{H}^n)$ and the space $\Gamma\backslash\mathbb{H}^n$ is non-compact, but has finite volume, i.e. finite quotient measure.

(iv) In a sense, the mother example of all symmetric spaces is the space

$$\operatorname{Sym}_1^+(n) := \{X \in M_{n,n}(\mathbb{R}) \mid X^T = X,\ X \gg 0,\ \det X = 1\}.$$

If you think about it geometrically as a subset of $M_{n,n}(\mathbb{R})$ it seems like a rather complicated space. Anyway, it comes with the action

$$\mathrm{SL}(n,\mathbb{R}) \times \operatorname{Sym}_1^+(n) \to \operatorname{Sym}_1^+(n),\ (g,X) \mapsto g^T X g$$

for which $\operatorname{stab}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}_n) = \mathrm{SO}(n)$ and hence $\operatorname{Sym}_1^+(n) \cong \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

1.2. **Lie Characterization.** Lie-theoretically, we will start out with a connected Lie group G and an automorphism $\sigma : G \to G$ such that $\sigma^2 = \mathrm{id}$. Assume that $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is compact, hence closed, and therefore a Lie subgroup. Then G/G^σ is a manifold which by compactness of G^σ can be equipped with a Riemannian metric. Choose $(G^\sigma)^\circ \le K \le G^\sigma$ and consider $M := G/K$. Then for any G-invariant Riemannian metric, M is a symmetric space.

One may look at $D_e\sigma : \mathfrak{g} \to \mathfrak{g}$ which is an automorphism of the Lie algebra $\mathfrak{g}$ of G satisfying $(D_e\sigma)^2 = \mathrm{Id}$. Therefore $\mathfrak{g}$ decomposes as a vector space as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{k} = E_1(D_e\sigma)$ and $\mathfrak{m} = \mathrm{E}_{-1}(D_e\sigma)$ are the eigenspaces of $D_e\sigma$ for the eigenvalues one and minus one respectively. The relations $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{m}] \subseteq \mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{m}$ are then immediate. In particular, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. This kind of data classifies, as we shall see, globally symmetric spaces.

1.3. **Connection Between the Riemannian and the Lie Characterization.** If M is a symmetric space, then $G = \operatorname{Iso}(M)^\circ$ is a connected Lie group such that M is homogeneous under the action of G. Fix $p \in M$, let $K = \operatorname{stab}_G(p)$ and let s_p be the geodesic symmetry about p. Then $\sigma : G \to G,\ g \mapsto s_p g s_p$ is an involutory automorphism of G and $(G^\sigma)^\circ \le K \le G^\sigma$.

1.4. **Classification.** Notice that products of symmetric spaces are again symmetric. It holds true that any symmetric space M admits a decomposition

$$M \cong \mathbb{E}^n \times M_+ \times M_-.$$

The symmetric space M_+ is said to be of *compact type*. It has non-negative sectional curvature and $\mathrm{Iso}(M_+)$ is compact semisimple. Hence also M_+ is compact and in a sense generalizes $\mathbb{S}^n$. The symmetric space M_- is said to be of *non-compact type*. It has non-positive sectional curvature and $\mathrm{Iso}(M_-)$ is non-compact semisimple. Also, M_- is non-compact and in a sense generalizes $\mathbb{H}^n$.

There is a duality theory between symmetric spaces of compact type and those of non-compact type. In this theory, $\mathbb{S}^n$ and $\mathbb{H}^n$ are dual spaces.

An important invariant of a symmetric space is its *rank* which is the maximal dimension of a totally geodesic flat subspace. Apart from the $\mathbb{E}^n$ part, these may be contained in the non-compact type part. For instance, $\mathrm{Sym}_1^+(n)$ has rank $n-1$.

2. Generalities on Riemannian Symmetric Spaces

In this section, we thoroughly define symmetric spaces. We will always *assume smooth manifolds to be second-countable.* In particular, a manifold has countably many connected components and admits a countable dense subset.

2.1. **Isometries and the Isometry Group.** A *Riemannian metric* on a smooth manifold M is a map which to every $x \in M$ associates a scalar product g_x on the tangent space T_xM of M at x, such that in local coordinates the map $x \mapsto g_x$ is smooth: Let (U, φ) be any chart on M. Then we require the map $U \to \mathrm{Sym}_1^+(\dim M),\ x \mapsto (g_x((D_e\varphi)^{-1}(e_i), (D_e\varphi^{-1}(e_j))_{i,j}$ is smooth. Equivalently, a Riemannian metric is a smooth section of the appropriate bundle.

Let (M, g) be a Riemannian manifold. The *length* of a smooth curve $c : [0, 1] \to M$ is defined by $\int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\, dt$. The *distance* of $x, y \in M$ is defined by $d(x, y) := \inf\{l(c) \mid c : [0, 1] \to M \text{ smooth},\ c(0) = x,\ c(1) = y\}$.

Definition 2.1. Let (M, g) and (N, h) be Riemannian manifolds. A *Riemannian isometry* between (M, g) and (N, h) is a diffeomorphism $f : M \to N$ such that $f^*h = g$, i.e. $h_{f(p)}(D_pf(u), D_pf(v)) = g_p(u, v)\ \forall p \in M\ \forall u, v \in \mathrm{T}_pM$.

The fact that a Riemannian isometry is the same as a distance-preserving bijection reduces to the euclidean case for which it can be proven directly, see [Hel79, Thm. I.11.1]

Theorem 2.2. Let (M, g) be a Riemannian manifold with associated distance d. Further, let $f : M \to M$ be a map. Then the following statements are equivalent.

(i) f is a Riemannian isometry.
(ii) f is a distance-preserving bijection.

We shall call an *isometry* any self-map of a Riemannian manifold which satisfies the equivalent conditions of Theorem 2.2. Next, we prove a rigidity-type result for such isometries of Riemannian manifolds: They are determined globally by local data, in contrast to isometries of arbitrary metric spaces: An isometry of the simplicial realization of the 3-valent tree for instance is not determined by its restriction to any finite radius ball about a point. In the Riemannian setting, however, we have the following.

Lemma 2.3. Let M and N be Riemannian manifolds and let M be connected. Further, let $f_1, f_2 : M \to N$ be Riemannian isometries. If there is $p \in M$ such that $f_1(p) = f_2(p)$ and $D_pf_1 = D_pf_2$, then $f_1 \equiv f_2$.

For the proof, we recall the notion of a *normal* neighbourhood of a point $p \in M$ and of $0 \in T_p(M)$ respectively. Let $\mathrm{Exp}_p : \mathrm{T}_pM \to M$ be the Riemannian exponential map at p. A *normal* neighbourhood of $0 \in \mathrm{T}_pM$ is an open, star-shaped neighbourhood N_0 of $0 \in \mathrm{T}_pM$ such that $U := \mathrm{Exp}_p$ is open and $\mathrm{Exp}_p|_{N_0} \to U$ is a diffeomorphism. In this case U is a *normal* neighbourhood of $p \in M$.

Proof. (Lemma 2.3). The isometry $f := f_2^{-1} \circ f_1 : M \to M$ satisfies $f(p) = p$ and $D_p f = \mathrm{Id}$. We aim to show that $f \equiv \mathrm{id}$. If M was complete, we could argue as follows: There is a geodesic arc connecting p to q with tangent vector $v \in \mathrm{T}_p M$ at p. Since an isometry preserves geodesics, $D_p f = \mathrm{Id}$ implies that this geodesic arc is fixed, in particular $f(q) = q$. However, M need not be complete. Consider the set

$$S := \{q \in M \mid f(q) = q,\ D_q f = \mathrm{Id}\}.$$

Then S contains p and is closed by smoothness of f. It now suffices to show that S is open in which case $S = M$ by connectedness of M. Let $q \in S$ and let N_0 be a normal neighbourhood of $0 \in \mathrm{T}_q M$. Since f is an isometry, we have for all $v \in N_0$ and $t \in \mathbb{R}$ such that $tv \in N_0$:

$$f(\mathrm{Exp}_q(tv)) = \mathrm{Exp}_q(D_q f(tv)) = \mathrm{Exp}_q(tv)$$

where the first equality follows from the fact that f is an isometry, as well as local existence and uniqueness of geodesics. Hence f is the identity on $U := \mathrm{Exp}_q(N_0)$ which implies $U \subseteq S$ by openness of U. Therefore, S is open. □

Now, let $\mathrm{Iso}(M)$ denote the group of isometries of a Riemannian manifold (M, g). Recall that $\mathrm{Iso}(M)$ is a topological group when endowed with the compact-open topology a subbasis of which is given by

$$\mathcal{S} = \{W(C, U) \mid C \subseteq X \text{ compact},\ U \subseteq Y \text{ open}\}.$$

Since the topology of M is given by the distance d associated to g, this topology on $\mathrm{Iso}(M)$ coincides with the topology of uniform convergence on compact subsets. In fact, one can exploit the Riemannian setting further to show coincidence with the topology of pointwise convergence.

Lemma 2.4. Let M be a Riemannian manifold, $S \subseteq M$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of isometries. If $(f_n)_n$ converges pointwise on S then it does so on $\overline{S}$.

Idea of Proof. If M is complete, this is merely an argument using Cauchy sequences: If $p \in \overline{S}$, there is a Cauchy sequence $(p_k \in M)_{k\in\mathbb{N}}$ converging to p. Then $(f_n(p))_n$ is a Cauchy sequence as well since

$$d(f_n(p), f_m(p)) \leq d(f_n(p), f_n(p_k)) + d(f_n(p_k), f_m(p_k)) + d(f_m(p_k), f_m(p))$$

and hence convergent. If M is not complete one has to use local compactness to avoid issues of non-completeness.

Lemma 2.5. Let M be a Riemannian manifold. On $\mathrm{Iso}(M)$, the topology of uniform convergence on compact sets coincides with the topology of pointwise convergence.

Sketch of Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Iso}(M)$ such that $\lim_n f_n(p) =: f(p)$ exists for all $p \in M$. Then $d(f(p), f(q)) = \lim_n d(f_n(p), f_n(q)) = d(p, q)$ and hence f is distance-preserving. Showing that f is bijective requires more work. We now show that $(f_n)_n$ converges uniformly on compact sets. If not, there exists a compact set $C \subseteq M$, $\delta > 0$ and $(x_m)_{m\in\mathbb{N}}$ in C such that $d(f(x_m), f_m(x_m)) > \delta$ for all $m \in \mathbb{N}$. By compactness of C, we may assume by passing to a subsequence that $(x_m)_m$ converges to some $y \in C$. Pick $m_0 \in \mathbb{N}$ such that $d(x_m, y) \leq \delta/3$ for all $m \geq m_0$. Then $d(f_m(x_m), f_m(y)) = d(x_m, y) \leq \delta/3$ and $d(f(x_m), f(y)) = d(x_m, y) \leq \delta/3$ for all $m \geq m_0$ and hence

$$d(f_m(y), f(y)) \geq d(f(x_m), f_m(x_m)) - d(f(y), f(x_m)) - d(f_m(y), f_m(x_m)) \geq \delta/3$$

for all $m \geq m_0$ which contradicts pointwise convergence.

Using the preceeding lemmas, we obtain the following rigidity-type statement for sequences of isometries of Riemannian manifolds.

Theorem 2.6. Let M be a Riemannian manifold. Further, let $(f_n)_{n\geq 1}$ be a sequence in $\mathrm{Iso}(M)$ such that $\lim_n f_n(p)$ exists for some $p \in M$. Then there is a subsequence $(f_{n_k})_{k\geq 1}$ which converges uniformly on compact subsets to an isometry $f \in \mathrm{Iso}(M)$.

Idea of Proof. First, using local compactness of M, one shows

$$\{q \in M \mid (f_n(q))_{n\in\mathbb{N}} \text{ has compact closure}\} = M.$$

Again, this is easy in the case where M is complete using the Heine-Borel theorem.

We now turn to the desired subsequence of $(f_n)_n$. Using second-countability of M, pick a dense subset D of M. By the first step, $(f_n(d))_n$ has compact closure for all $d \in D$. Since D is countable we may thus use Cantor's diagonal procedure to produce a subsequence $(f_{n_k})_k$ of $(f_n)_n$ which converges pointwise for every $d \in D$. By Lemma 2.4 we conclude that $(f_{n_k})_k$ converges pointwise on $\overline{D} = M$. Lemma 2.5 then implies uniform convergence on compact sets.

In summary, we have exploited the nature of the involved maps being isometries to conclude uniform convergence from pointwise convergence on a single point. The following theorem collects what we know about the isometry group of a Riemannian manifold.

Theorem 2.7. Let M be a Riemannian manifold. Then $\mathrm{Iso}(M)$ is a locally compact, second countable topological group which acts continuously with compact stabilizers on M.

Idea of Proof. We show that $\mathrm{Iso}(M)$ is locally compact. For every $p \in M$ and every open, relatively compact neighbourhood $U \in \mathcal{U}(p)$ of p, the set

$$W(p,U) := \{f \in \mathrm{Iso}(M) \mid f(p) \in U\}$$

is a neighbourhood of $\mathrm{id} \in \mathrm{Iso}(M)$. By Theorem 2.6, every sequence in $W(p,U)$ has a convergent subsequence. Given that the compact-open topology on $\mathrm{Iso}(M)$ is metrisable and that $\mathrm{Iso}(M)$ is second-countable $W(p,U)$ is thus compact.

2.2. **Geodesic Symmetries.** We now turn to the precise definition of a (locally) symmetric space, which we keep as synthetic as possible, that is, avoiding explicit mention of the smooth structure.

Definition 2.8. Let M be a connected Riemannian manifold. Then M is *locally symmetric* if for all $p \in M$ there is a normal neighbourhood U of $p \in M$ and an isometry $s_p : U \to U$ such that $s_p^2 = \mathrm{id}$ and such that p is the unique fixed point of s_p in U. The space M is *globally symmetric* if it is locally symmetric and each s_p can be extended to an isometry of M.

Note that for a globally symmetric space M and $p \in M$, the geodesic symmetry s_p need not arise from choosing $U = M$ in the definition of a locally symmmetric space, see the example $\mathbb{S}^n$ in the Introduction.

The following lemma, combined with Lemma 2.3 implies that the extension of a locally defined geodesic symmetry to the whole space is unique if it exists.

Lemma 2.9. Let M be a locally symmetric space, $p \in M$ and $s_p : U \to U$ a geodesic symmetry as in Definition 2.8. Let N_0 be the normal neighbourhood of $0 \in \mathrm{T}_pM$ corresponding to U, in particular $\mathrm{Exp}_p(N_0) = U$. Then $D_ps_p = -\mathrm{Id}$ and $s_p(\mathrm{Exp}_p v) = \mathrm{Exp}_p(-v)$ $\forall v \in N_0$.

Proof. Since $s_p^2 = \mathrm{id}$, we have $(D_ps_p)^2 = \mathrm{Id}$, i.e. $D_ps_p \in \mathrm{GL}(\mathrm{T}_pM)$ is involution of $\mathrm{T}_p(M)$; it is hence diagonalizable with possible eigenvalues $\lambda \in \{-1, 1\}$. We aim to show that $\lambda := 1$ cannot be an eigenvalue. In fact, assume that $v \in \mathrm{T}_pM - \{0\}$ satisfies $D_ps_p(v) = v$. Then by rescaling, we may assume that $v \in N_0$. But then $s_p(\mathrm{Exp}_p tv) = \mathrm{Exp}_p(tD_ps_p(v)) = \mathrm{Exp}_p tv$ for all $t \in \mathbb{R}$ such that $tv \in N_0$. Hence there is a geodesic segment of fixed points which contradicts the assumption that p is the unique fixed point in U. $\square$

We now aim to prove the following criterion for a locally symmetric space to be globally symmetric.

Theorem 2.10. Let M be a complete, simply connected locally symmetric space. Then M is globally symmetric.

The converse of Theorem 2.10 does not hold. For instance, here is an example of a globally symmetric space which is not simply connected.

Example 2.11. Consider $\mathbb{S}^n$ and $\Gamma := \{\pm \mathrm{id}\} < \mathrm{O}(n+1,\mathbb{R}) = \mathrm{Iso}(\mathbb{S}^n)$. Then Γ acts by isometric covering transformations on $\mathbb{S}^n$, i.e. without fixed points and properly discontinuously. We claim that $M := \Gamma \backslash \mathbb{S}^n$ is a globally symmetric space and for $n \geq 2$ satisfies $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z}$, in particular it is not simply connected in this case. Since Γ acts by covering transformations, M is again a Riemannian manifold. We argue that it is symmetric: Let s_p denote the geodesic symmetry about $p \in S^n$. Since s_p commutes with Γ, it induces a well-defined diffeomorphism $\sigma_{\pi(p)}$ of M onto itself, making the following diagram commute:

$$\begin{array}{ccc} S^n & \xrightarrow{s_p} & S^n \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\sigma_{\pi(p)}]{} & M. \end{array}$$

Given $p \in M$, the map $\sigma_{\pi(p)}$ is indeed the geodesic symmetry about $\pi(p)$: By definition of the Riemannian metric on S^n, the map $D_q\pi : \mathrm{T}_q S^n \to \mathrm{T}_{\pi(q)} M$ is an isomorphism of inner product spaces for all $q \in S^n$, and the diagram

$$\begin{array}{ccc} \mathrm{T}_q S^n & \xrightarrow{D_q s_p} & \mathrm{T}_{s_p(q)} S^n \\ {\scriptstyle D_q\pi}\big\downarrow & & \big\downarrow{\scriptstyle D_{s_p(q)}\pi} \\ \mathrm{T}_{\pi(q)} M & \xrightarrow[D_{\sigma_{\pi(p)}(\pi(q))}\sigma_{\pi(p)}]{} & \mathrm{T}_{\sigma_{\pi(p)}(\pi(q))} M \end{array}$$

commutes. Thus, in particular, $\sigma_{\pi(p)}$ is an isometry. Specializing the diagram to $q = p$ shows that $D_{\pi(p)}\sigma_{\pi(p)} = -\mathrm{Id}$, hence $\pi(p)$ is an isolated fixed point of $\sigma_{\pi(p)}$.

The space M is typically denoted by $\mathbb{P}^n(\mathbb{R})$ and called *real projective n-space.*

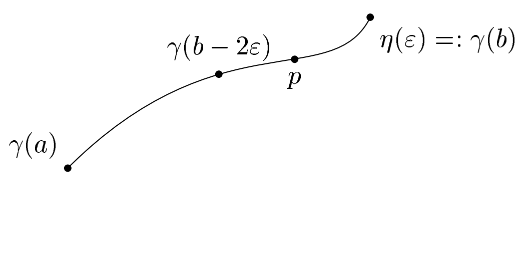
We now state several lemmas, of independent interest, to show that a globally symmetric space is a homogeneous space of the connected component of its isometry group. This will be useful to show that the latter is a Lie group.

Lemma 2.12. Let M be a globally symmetric space. Then M is complete.

Proof. By the Hopf-Rinow theorem, it suffices to show that any geodesic segment $\gamma : [a,b) \to M$ can be extended to $b \in \mathbb{R}$. Let $\varepsilon := (b-a)/4$, let $p := \gamma(b-\varepsilon)$. Define

$$\eta : [0, b-\varepsilon) \to M,\ t \mapsto s_p(\gamma(b-\varepsilon-t)).$$

Then η is a geodesic segment starting at $\eta(0) = s_p(p) = p$ with $\dot\eta(0) = \dot\gamma(b-\varepsilon)$. Thus, setting $t = \varepsilon$ we obtain the sought-for extension.



□

Lemma 2.13. Let M be a globally symmetric space. Then the action of $\mathrm{Iso}(M)$ on M is transitive.

Proof. Let $p, q \in M$. By Lemma 2.12, M is complete. Hence there exists a geodesic segment $\gamma : [0, d] \to M$ such that $\gamma(0) = p$ and $\gamma(d) = q$ with $d = d(p, q)$. Let $m = \gamma(d/2)$. Then $s_m(\gamma(t)) = \gamma(d - t)$: Indeed, let $\eta : [0, d] \to M, t \mapsto s_m(\gamma(t))$. Then $\eta(d/2) = m$ and $\dot\eta(d/2) = D_m s_m(\dot\gamma(d/2)) = -\dot\gamma(d/2)$. By uniqueness of geodesics, this implies the assertion. In particular we have $\eta(d) = p = s_m(\gamma(d)) = s_m(q)$, that is, $\mathrm{Iso}(M)$ acts transitively on M. □

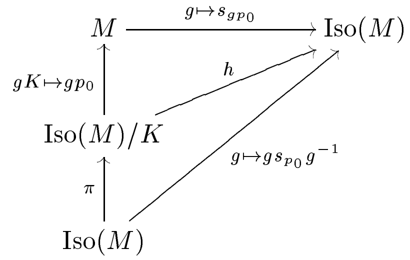
Given a symmetric space M, we would not only like $\mathrm{Iso}(M)$, but also $\mathrm{Iso}(M)^\circ$ to act transitively on M, thus allowing only orientation-preserving transformations. Connectedness of M is clearly a necessary condition for this. We shall prove that it is also sufficient using the following two lemmas.

Lemma 2.14. Let M be globally symmetric space. Further, given $p \in M$, define $K := \mathrm{stab}_{\mathrm{Iso}(M)}(p)$. Then the map $\mathrm{Iso}(M)/K \to M,\ gK \mapsto g(p)$ is a homeomorphism.

Prove this as an exercise using a Baire argument.

Lemma 2.15. Let M be a globally symmetric space. The map $M \to \mathrm{Iso}(M),\ p \mapsto s_p$ is continuous.

Proof. Let $p_0 \in M$. Given $g \in \mathrm{Iso}(M)$, it is immediate that $g s_{p_0} g^{-1}$ fixes gp_0 and has derivative $-\mathrm{Id}$ at gp_0. Hence $g s_{p_0} g^{-1} = s_{gp_0}$. Now, consider the following commutative diagram.



The map $\mathrm{Iso}(M) \to \mathrm{Iso}(M),\ g \mapsto g s_{p_0} g^{-1}$ is continuous since $\mathrm{Iso}(M)$ is a topological group. It factors through π via h which hence is a continuous map as well by the definition of the quotient topology. Since further by Lemma 2.14 the map orbital map $\mathrm{Iso}(M)/K \to M,\ gK \mapsto g(p_0)$ is a homeomorphism, we conclude the assertion. □

Proposition 2.16. Let M be a globally symmetric space. Then $\mathrm{Iso}(M)^\circ$ acts transitively on M.

Proof. Consider the continuous map $\varphi : M \times M \to \mathrm{Iso}(M),\ (m, m') \mapsto s_m \circ s_{m'}$. Its image contains $\mathrm{id} = s_m \circ s_m$ $(m \in M)$ and is thus contained in $\mathrm{Iso}(M)^\circ$. We now show that $\mathrm{im}\,\varphi$ acts transitively on M: Given $p, q \in M$, let $\gamma : [0, d] \to M$ be a geodesic connecting p to q as in the proof of Lemma 2.13. Then $s_{\gamma(d/2)} \circ s_p(p) = q$. Hence $\mathrm{Iso}(M)^\circ$ acts transitively on M. □

We are now in a position to sketch a proof of the theorem that $\mathrm{Iso}(M)^\circ$ is a Lie group. In fact, an argument by Myers-Steenrod shows that the isometry group of any Riemannian manifold is a Lie group. However, it is rather complicated. Our proof will utilize the fact that a symmetric space M is homogeneous under the action of $\mathrm{Iso}(M)^\circ$.

Theorem 2.17. Let M be a globally symmetric space. Then $\mathrm{Iso}(M)^\circ$ admits a smooth structure which turns it into a Lie group. The action of $\mathrm{Iso}(M)^\circ$ is then smooth and the map $\mathrm{Iso}(M)^\circ/K$ is a diffeomorphism.

Sketch of Proof. We are going to show that $\mathrm{Iso}(M)^\circ$ is locally homeomorphic to a smooth manifold. Let $p_0 \in M$. First, note that $K := \mathrm{stab}_{\mathrm{Iso}(M)^\circ}$ is compact and comes with the representation $\varphi : K \to \mathrm{O}(\mathrm{T}_{p_0}M),\ k \mapsto D_{p_0}k$ which is continuous and faithful by Lemma 2.3. Hence K is homeomorphic to its image $\mathrm{im}\,\varphi$ under φ which is a compact hence closed subgroup of the Lie group $\mathrm{O}(\mathrm{T}_{p_0})$. Therefore, by Cartan's theorem, K is a Lie group in its own right.

So we already know that in the expression $M \cong \mathrm{Iso}(M)^\circ/K$ both M and K are manifolds; hence there should be a compatible manifold structure on $\mathrm{Iso}(M)^\circ$. This comes about as follows: We show that the map $\pi : \mathrm{Iso}(M)^\circ \to M$ admits local cross-sections. Namely, let U be a normal neighbourhood of $p_0 \in M$ and let $N_0 \subseteq \mathrm{T}_{p_0}(M)$ be the corresponding normal neighbourhood of $0 \in \mathrm{T}_{p_0}(M)$. Then consider the map

$$\Phi_U : U \to \mathrm{Iso}(M)^\circ,\ \mathrm{Exp}_{p_0} v \mapsto s_{\mathrm{Exp}_{p_0}(v/2)} \circ s_{p_0}.$$

This is a continuous local cross-section of the map $\pi : \mathrm{Iso}(M)^\circ \to M,\ g \mapsto gp_0$; in other words, $\Phi_U : U \to \pi^{-1}(U)$ is a continuous right-inverse of π. In particular, Φ_U is a homeomorphism onto its image and we deduce that

$$U \times K \to \pi^{-1}(U),\ (p,k) \mapsto \Phi_U(p)k$$

is a homeomorphism between the manifold $U \times K$ and the open subset $\pi^{-1}(U)$ of $\mathrm{Iso}(M)^\circ$. This was the main step. It produces charts on $\mathrm{Iso}(M)^\circ$ and it remains to be checked that transition maps are smooth.

In the sequel, we shall make use of the principle of analytic continuation. The following remark establishes several preliminaries in this direction.

Remark 2.18. (Real analyticity). Let $U \subseteq \mathbb{R}^n$ $(n \in \mathbb{N})$ be open. A function $f : U \to \mathbb{R}$ is called *real analytic* on U if for all $x_0 \in U$ there is a power series $\sum_\alpha c_\alpha (x-x_0)^\alpha$ which converges in some ball $B(x_0,\varepsilon)$ to f. Here, $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index and $(x-x_0)^\alpha := (x_1-x_{01})^{\alpha_1}\cdots(x_n-x_{0n})^{\alpha_n}$. The multi-index notation allows one to handle multi-dimensional expressions as if they were one-dimensional. If $f : U \to \mathbb{R}$ is real analytic, then the power series at $x_0 \in U$ converges absolutely and uniformly in $\overline{B(x_0,r)}$ for all $r < \varepsilon$, the function f is smooth and $c_\alpha = (D^\alpha f)(x_0)/\alpha!$.

Let U be as above. A function $f : U \to \mathbb{R}^m$ is called *real analytic* if all its coordinate functions are real analytic. An atlas of a manifold is called *real analytic* if the associated transition maps are real analytic. The implicit function theorem holds also for real analytic maps.

One of the reasons to consider analytic objects is the principle of analytic continuation, to be described later on, and the following rigidity-type statement: Let $U \subseteq \mathbb{R}^n$ be open and connected. Further, let $f_1, f_2 : U \to \mathbb{R}^m$ be analytic maps such that $f_1|_V \equiv f_2|_V$ where $V \subseteq U$ is open. Then $f_1 \equiv f_2$ on U. To prove this, show that the set $\{x \in U \mid D^\alpha f_1(x) = D^\alpha f_2(x)\ \forall |\alpha| \geq 0\}$ is open and closed in U.

Real analyticity is relevant in our context because of the following: The exponential map $\mathrm{Exp} : M_n(\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is a real analytic map. More generally, given any Lie group G with Lie algebra $\mathfrak{g}$, let $\exp : \mathfrak{g} \to G$ be the corresponding exponential map. Fix an open neighbourhood $V \subseteq \mathfrak{g}$ of $0 \in \mathfrak{g}$ such that $\exp|_V : V \to \exp(V)$ is a diffeomorphism. Then $\{(g\exp(V), \exp^{-1} \circ L_{g^{-1}}) \mid g \in G\}$ is an atlas on G giving the existent smooth structure on G. Using a version of the Baker-Campbell-Hausdorff theorem, one can show that the transition maps with respect to this atlas are in fact real analytic. It also follows, that multiplication and inversion are real analytic maps with respect to this atlas.

In what follows, we will often assume manifolds to be real analytic. This is not really a restriction: It is a theorem that every smooth manifold admits a (unique) compatible real analytic structure. In the case of a Riemannian manifold, the Riemannian metric and the exponential maps are real analytic with respect to this structure.

To prove Theorem 2.10, we shall use the principle of real analytic continuation the statement and proof of which we prepare with the following Lemma.

Lemma 2.19. Let (M,g) and (N,h) be Riemannian, real analytic manifolds. Let $p \in M$. Choose $\varrho > 0$ such that $B(p,\varrho)$ is a normal neighbourhood of $p \in M$. Assume that for some $0 < r < \varrho$ we have an isometry $f : B(p,r) \to B(f(p),r)$. Then f is real analytic and extends to an isometry $F : B(p,\varrho) \to B(f(p),\varrho)$.

Proof. Let $N_r := \mathrm{Exp}_p^{-1}(B(p,r))$. Then since f is an isometry, the diagram

$$\begin{array}{ccc} N_r & \xrightarrow{D_p f} & T_{f(p)}N \\ {\scriptstyle \mathrm{Exp}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{Exp}_{f(p)}} \\ B(p,r) & \xrightarrow[f]{} & N \end{array}$$

commutes. Therefore, also the diagram

$$\begin{array}{ccc} \mathrm{T}_p M_r & \xrightarrow{D_p f} & T_{f(p)}N \\ {\scriptstyle \mathrm{Exp}_p^{-1}}\uparrow & & \downarrow{\scriptstyle \mathrm{Exp}_{f(p)}} \\ B(p,r) & \xrightarrow[f]{} & N \end{array}$$

commutes, from which we deduce that f is a composition of real analytic maps and hence real analytic. A candidate for the extension F of f to $B(p,\varrho)$ clearly is $F := \mathrm{Exp}_{f(p)} \circ D_p f \circ \mathrm{Exp}_p^{-1}$. We need to show that F is an isometry. Since f is an isometry we have for all $q \in B(p,r)$ and $X_q, Y_q \in \mathrm{T}_q M$:

$$h_{f(q)}(D_q f(X_q), D_q f(Y_q)) = g_q(X_q, Y_q).$$

As F coincides with f on $B(p,r)$, the two real analytic maps from $B(p,\varrho)$ to $\mathbb{R}$ given by $q \mapsto h_{F(q)}(D_q F(X_q), D_q F(Y_q))$ and $q \mapsto g_q(X_q, Y_q)$ respectively, also coincide on $B(p,r)$ and hence coincide on $B(p,\varrho)$ by Remark 2.18. □

The next lemma is at the heart of the analytic continuation principle. Recall the following fact from Riemannian geometry: Let (M,g) be a Riemannian manifold. For every $p \in M$, there is $\varepsilon > 0$ such that for all $r \leq \varepsilon$, the ball $B(p,r)$ is geodesically convex, i.e. there is a unique distance-minimising geodesic between x and y, entirely contained in $B(p,r)$. If for $p,q \in M$ the balls $B(p,r)$ and $B(q,l)$ are geodesically convex, then so is $B(p,r) \cap B(q,l)$ which hence in particular is connected. Given a compact subset C of M, there is $\varepsilon > 0$ such that for all $p \in C$, the ball $B(p,\varepsilon)$ is a normal, geodesically convex neighbourhood of $p \in C$, and for all $p,q \in C$ the subset $B(p,\varepsilon) \cap B(q,\varepsilon)$ of M is connected (the empty set being connected).

Lemma 2.20. Let (M,g) and (N,h) be complete, real analytic Riemannian manifolds, $p \in M$ and $U \in \mathcal{U}(p)$ open and normal. Further, let $f : U \to N$ be an isometry onto an open subset of N. Given a continuous path $\eta : [0,1] \to M$ starting at p, there is for all $t \in [0,1]$ a neighbourhood U_t of $\eta(t)$ and an isometry $f_t : U_t \to N$ such that

(i) $U_0 = U$, $f_0 \equiv f$, and
(ii) $\exists \varepsilon > 0 : \forall t,s \in [0,1]$ with $|t-s| < \varepsilon$: $U_t \cap U_s \neq \emptyset$ and $f_t|_{U_t \cap U_s} \equiv f_s|_{U_t \cap U_s}$.

Proof. Apply the above mentioned fact to $C := \eta([0,1])$: Let $r > 0$ such that $B(\eta(t), r)$ is a normal neighbourhood of $\eta(t)$ (uniformly) for every $t \in [0,1]$, and $B(\eta(t), r) \cap B(\eta(s), r)$ is connected for all $s, t \in [0,1]$.

Now, if $t \in [0,1]$ is such that $\eta(t) \in U$, choose $\varepsilon(t) > 0$ with $B(\eta(t), \varepsilon(t)) \subseteq U$. Then by Lemma 2.19, $f|_{B(\eta(t),\varepsilon(t))}$ extends to an isometry $f_t : B(\eta(t), r) \to N$. We may then continue in this fashion: Formally, let

$$I' := \left\{ s \in [0,1] \;\middle|\; \begin{array}{l} f \text{ satisfies the conclusions of Lemma 2.20} \\ \text{for } \eta|_{[0,1]} \text{ with } U_t := B(\eta(t), r) \end{array} \right\}.$$

Then the above argument shows that I' is open: Note first, that we have the following monotonicity property: If $s_1, s_2 \in [0,1]$ such that $s_1 \leq s_2$ and $s_2 \in I'$, then $s_1 \in I'$. Now, if $s_2 \geq s_1$ but with $\eta(s_2) \in B(\eta(s_1), r)$, then we may construct an isometry $f_{s_2} : B(\eta(s_2), r) \to N$ as before.

A similar argument shows that I' is closed: Let $s_0 := \sup I'$. Then, as before, $[0, s_0) \subseteq I'$, and we may choose $s \in [0, s_0)$ such that $s_0 \in B(\eta(s), r)$ to continue as before. Overall, I' is non-empty, open and closed in $[0,1]$, hence equal to $[0,1]$. □

Now, we can strengthen the shape of the analytic continuation obtained in Lemma 2.20 in the following way: Let (M, g) and $\eta : [0,1] \to M$ be as in Lemma 2.20. A pair $(r, \varepsilon) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ is *η-admissible* if

(i) $B(\eta(t), r)$ is a normal neighbourhood of $\eta(t)$ for all $t \in [0,1]$.
(ii) $B(\eta(t), r) \cap B(\eta(t'), r)$ is connected.
(iii) $d(\eta(t), \eta(t')) < r/2$ if $|t - t'| < \varepsilon$. (By uniform continuity of η).

Then if further U, f and (N, h) are as in Lemma 2.20 and (r, ε) is η-admissible, there is a unique family $(f_t, B(\eta(t), r))_{t\in[0,1]}$ such that

(i) $f_t : B(\eta(t), r) \to N$ is an isometry onto its image.
(ii) f_0 and f coincide on $U \cap B(p, r)$.
(iii) Whenever $|t - t'| < \varepsilon$, then f_t and $f_{t'}$ coincide on $B(\eta(t), r) \cap B(\eta(t'), r)$.

We call (f_t^η, U_t^η) an analytic continuation of (f, U) along η. The map $t \mapsto f_t(\eta(t))$ is independent of (r, ε) and if (f_t, U_t) and (f_t', U_t') are two analytic continuations of (f, U) along η, then f_1 and f_1' coincide on $B(\eta(1), r) \cap B(\eta(1), r')$. This goes into the phrasing of the following Lemma.

Lemma 2.21. Let (M, g) and (N, h) be complete, real analytic Riemannian manifolds, $p \in M$ and $U \in \mathcal{U}(p)$ open and normal. Further, let $f : U \to N$ be an isometry onto an open subset of N. If $\eta, \delta : [0,1] \to M$ are homotopic paths in M starting at p, having the same end point $\eta(1) = \delta(1)$, then f_1^η and f_1^δ coincide on $B(\eta(1), r) \cap B(\delta(1), r')$.

Proof. Let $H : [0,1] \times [0,1] \to M$, $(t, s) \mapsto H_s(t)$ be a homotopy from $H_0 \equiv \eta$ to $H_1 \equiv \delta$ with fixed end points. Since the image of H is compact, we may pick $r' \leq r$ such that for all $p \in \mathrm{im}(H)$, the ball $B(p, r')$ is geodesically convex. Further, by equicontinuity of H_s, let $\varepsilon > 0$ such that for all $t, t' \in [0,1]$ with $|t - t'| < \varepsilon$ we have $d(H_s(t), H_s(t')) \leq r/2$ for all $s \in [0,1]$.

Then the pair (r, ε) is H_s-admissible for all $s \in [0,1]$. Let $(f_t^{H_s}, U_t^{H_s})$ be the corresponding analytic continuation along H_s and consider

$$I' := \{s \in [0,1] \mid \forall s' \in [0, s] : \; f_1^{H_{s'}} \equiv f_1^{H_0} \equiv f_1^\eta\}.$$

We aim to show that I' is non-empty, open and closed in $[0,1]$. Clearly, I' contains $0 \in [0,1]$. Then observe that if $s_1, s_2 \in [0,1]$ satisfy $s_1 \leq s_2$ and $s_2 \in I'$, then $s_1 \in I'$. To show that I' is closed it therefore suffices to show that $l := \sup I'$ is contained in I': By uniform continuity, choose $s \in I'$ near enough to l such that for all $t \in [0,1]$

we have $d(H_l(t), H_s(t)) < r/2$. Further, choose $\varepsilon' > 0$ such that for $t, t' \in [0,1]$ with $|t - t'| < \varepsilon'$ we have $d(H_s(t'), H_s(t)) < r/4$. Then $B(H_l(t), r) \supseteq B(H_s(t), r/2)$ and

$$B(H_l(t), r) \cap B(H_l(t'), r) \supseteq B(H_s(t), r/2) \cap B(H_s(t'), r/2)$$

which is non-empty and connected. Then

$$\left(f_t^{H_l}|_{B(H_s(t),r/2)}, B(H_s(t), r/2)\right)$$

is an analytic continuation of (f, U) along H_s. In particular, $f^{H_l}|_{B(\eta(1),r/2)} \equiv f^{H_s}|_{B(\eta(1),r/2)}$; hence they coincide on $B(\eta(1), r)$. This shows that $l \in I'$. An analogous argument shows that I' is open. □

The proof of Theorem 2.10 now basically follows from the following result of our so far discussion.

Theorem 2.22. Let (M, g) and (N, h) complete, real-analytic Riemannian manifolds, $p \in M$ and $U \in \mathcal{U}(p)$ open and normal. Assume that M is simply connected. Further, let $f : U \to N$ be an isometry onto on open subset of N. Then f extends uniquely to a local isometry $F : M \to N$, i.e. for all $q \in M$ $\exists r > 0$ such that $F|_{B(q,r)} \to B(F(q), r)$ is an isometry.

The map F of Theorem 2.22 is not necessarily going to be globally injective, and therefore not a global isometry. Think of a covering map, e.g. $\pi : \mathbb{E}^2 \to \mathbb{Z}^2 \setminus \mathbb{E}^2$. However, if the target manifold N is simply connected as well, then F is a global isometry: Extend a local inverse along paths in N. Then the composition of F with the so obtained map is an extension of the identity and hence equal to the identity.

Proof. (Theorem 2.22). For every $q \in M$, pick a continuous path η from p to q. Let (f_t^η, U_t^η) be an analytic continuation of (f, U) along η and define $F(q) := f_1^\eta(\eta(1))$. Since M is simply connected, Lemma 2.21 implies that F does not depend on the chosen paths and hence is well-defined. To see that F is a local isometry at $q \in M$, let η be a path from p to q. Then for all $x \in B(q, r)$, the value $F(x)$ is given the value of the analytic continuation of (f, U) along the concatenation of η and a path η_x from q to x. Hence $F|_{U_1^\eta} \equiv f_1^\eta$ which is an isometry onto its image. □

Proof. (Theorem 2.10). Let $s_p : U \to U$ be a geodesic symmetry in a normal neighbourhood U of p. Let $F_p : M \to M$ be the extension of s_p given by Theorem 2.22. To see that F_p is a diffeomorphism, note that $F_p \circ F_p$ extends $s_p \circ s_p \equiv \mathrm{id}$; hence F_p is its own inverse. □

Overall we obtain the following algebraization of the theory of symmetric spaces: If M is a complete, real analytic locally symmetric space, then its universal cover $\widetilde{M}$ is globally symmetric and $M \cong \pi_1(M)\backslash\widetilde{M}$ where $\pi_1(M)$ is viewed as a subgroup of the isometry group of $\widetilde{M}$. Given a point $p \in \widetilde{M}$, we therefore have

$$M \cong \pi_1(M)\backslash \operatorname{Iso}(\widetilde{M})/K,$$

where $K := \operatorname{stab}_{\operatorname{Iso}(\widetilde{M})}(p)$, which describes M as a double coset space. The study of locally symmetric spaces therefore reduces to the study of certain pairs (G, K) where G is a Lie group and $K \leq G$ is compact, and certain discrete subgroups of G. In the above discussion, we may also write $M = \operatorname{Iso}(\widetilde{M})^\circ/K$ for $K = \operatorname{stab}_{\operatorname{Iso}(\widetilde{M})^\circ}(p)$ in which case we still have $M \cong \pi_1(M)\backslash \operatorname{Iso}(\widetilde{M})^\circ/K$ which however is not a double coset space as $\pi_1(M)$ may not be contained in $\operatorname{Iso}(\widetilde{M})^\circ$. This happens e.g. in the case of the Klein bottle.

Remark 2.23. Let X be a globally symmetric, simply connected space. Let $G := \operatorname{Iso}(X)^\circ$. Then for every $p \in X$ the stabilizer $\operatorname{stab}_G(p)$ is connected (and compact): Let K° be the connected component of K. We aim to show that $K^\circ = K$. Since

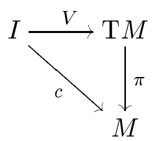
K is a Lie group, K° is an open (normal) subgroup of K and hence $K/K^\circ =: F$ is a finite group. The map $\pi : G/K^\circ \to G/K$ is G-equivariant. It is actually a Galois covering map with Galois group F: The map $K \times G/K^\circ \to G/K^\circ$ defines a right action of K on G/K°, because $K^\circ \trianglelefteq K$, for which K° acts trivially. Then π is F-invariant and one verifies that it is a covering map. If now G/K is simply connected and G/K° is connected, we must have $K = K^\circ$.

2.3. **Transvections and Parallel Transport.** Let M be a smooth manifold. A *connection* on M is a map $\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M),\ (X, Y) \mapsto \nabla_X Y$ which is designed to make sense of "the derivative of Y in the direction of X". It has to satisfy the following properties:

(i) ∇ is $C^\infty(M)$-linear in the first variable.
(ii) ∇ is $\mathbb{R}$-linear in the second variable.
(iii) (Leibniz) $\forall X, Y \in \mathrm{Vect}(M),\ \forall f \in C^\infty(M) :\ \nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

Properties (i) and (ii) imply that $(\nabla_X Y)(p)$ depends only on $X(p)$ and $Y|_U$ where U is open neighbourhood of $p \in M$. When we turn to the Lie-theoretic point of view on symmetric spaces, many connections will come into play. For a Riemannian manifold there is always the Levi-Civita connection associated to the metric but nonetheless the general concept is very important.

Property (i) above allows us to make sense of the "derivative of a vector field along a curve". Let $c : I \to M$ be a smooth curve where $I \subseteq \mathbb{R}$ is interval (if I is not open then smoothness means that c is the restriction of a smooth map defined on an open neighbourhood of I). Recall that a smooth vector field along c is a smooth map $V : I \to \mathrm{TM}$ which makes the diagram



commute. The *derivative of V along c* is a vector field DV/dt along c, the operator D/dt on vector fields along curves having the (characterizing) property that $DV/dt = \nabla_{\dot c(t)} Y$ whenever $V(t) = Y(c(t))$ where $Y \in \mathrm{Vect}(M)$.

Definition 2.24. Let $c : I \to M$ be a smooth curve. Further, let $V : I \to \mathrm{T}M$ be a vector field along c. Then V is *parallel* if $(DV/dt)(t) = 0$ for all $t \in I$.

Proposition 2.25. Let $c : I \to M$ be a smooth curve. Further, let $v \in \mathrm{T}_{c(0)}M$. Then there exists a unique smooth vector field V along c with $V(0) = v$ and $DV/dt \equiv 0$.

Thus, given a smooth curve $c : I \to M$ and $t_0, t_1 \in M$, Proposition 2.25 provides a well-defined linear map $P_{c;t_0,t_1} : \mathrm{T}_{c(t_0)} \to \mathrm{T}_{c(t_1)}$ which to $v_0 \in \mathrm{T}_{c(t_0)}$ associates the value at t_1 of the parallel vector field along c with initial value v_0. The uniqueness statement of Proposition 2.25 implies

$$\forall\ t_0, t_1, t_2 \in I :\ P_{c;t_1,t_2} \circ P_{c;t_0,t_1} \equiv P_{c;t_0,t_2}.$$

In particular, since $P_{c;t_0,t_0} = \mathrm{Id}$ for all $t_0 \in I$, all the $P_{c;t_0,t_1}$ $(t_0, t_1 \in I)$ are isomorphisms of vector spaces.

Now, given a Riemannian manifold $(M, \langle -, - \rangle)$, there exists a unique connection ∇ on M which satisfies

(i) $\nabla_X Y - \nabla_Y X = [X, Y]$
(ii) $X\langle Y, Z\rangle \equiv \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

Condition (ii) can be utilized to prove existence through symmetrizing the expression and (i) establishes uniqueness. This connection is called the *Levi-Civita* *connection.* Property (ii) has the following geometric content: Let $c : I \to M$ be a smooth curve and let V_1, V_2 be vector fields along c, each of which is parallel. Then

$$\dot{c}(t)\langle V_1(t), V_2(t)\rangle_{c(t)} = \left\langle \frac{DV_1}{dt}(t), V_2(t)\right\rangle_{c(t)} + \left\langle V_1(t), \frac{DV_2}{dt}(t)\right\rangle_{c(t)} = 0.$$

Hence $\langle V_1(t), V_2(t)\rangle_{c(t)}$ $(t \in I)$ is constant and therefore $P_{c;t_0,t_1} : \mathrm{T}_{c(t_0)} \to \mathrm{T}_{c(t_1)}M$ preserves the Riemannian metric.

A consequence of the uniqueness of the Levi-Civita connection is the following behaviour with respect to isometries which can be proven using the formula for the Levi-Civita connection obtained from the axioms.

Lemma 2.26. Let M be a Riemannian manifold with Levi-Civita connection ∇. Further, let $f \in \mathrm{Iso}(M)$. Then

$$\nabla_{f_*X}(f_*Y) = f_*(\nabla_X Y) \quad \text{where} \quad f_*Z(f(p)) = D_pf(Z(p)) \ (Z \in \mathrm{Vect}(M)).$$

Let us assume now, that M is a Riemannian manifold. Further, let $\gamma : \mathbb{R} \to M$ be a geodesic in M which is preserved by $f \in \mathrm{Iso}(M)$. Since $\mathrm{Iso}(\mathbb{R}) = \{\pm 1\} \ltimes \mathbb{R}$, we have $f \circ \gamma(t) = \gamma(at + b)$ for $a \in \{\pm 1\}$ and $b \in \mathbb{R}$. Thus, if $V : \mathbb{R} \to \mathrm{T}M$ is a vector field along γ then so is f_*V; and V is parallel along γ if and only if f_*V is parallel along γ. Let now M be a globally symmmetric space and let $\gamma : \mathbb{R} \to M$ be a geodesic. Then for every $t \in \mathbb{R}$, we may define

$$\mathcal{T}_t : M \to M, \ p \mapsto s_{\gamma(t/2)} \circ s_{\gamma(0)}(p).$$

For every geodesic $\gamma : \mathbb{R} \to M$ and $t \in \mathbb{R}$, the map $\mathcal{T}_t$ is an isometry of M, termed a *transvection.* If $M = \mathbb{R}^n$ $(n \in \mathbb{N})$, then $\mathcal{T}_t$ is a translation. Transvections are an important class of isometries because of the following.

Proposition 2.27. Let M be a Riemannian manifold and $\gamma : \mathbb{R} \to M$ a geodesic. Let $\mathcal{T}_t$ $(t \in \mathbb{R})$ denote the transvections along γ. Then

(i) $\forall s, t \in \mathbb{R} : \ \mathcal{T}_t(\gamma(s)) = \gamma(s + t)$,
(ii) $\forall s, t \in \mathbb{R} : \ D_{\gamma(s)}\mathcal{T}_t \equiv P_{\gamma;s,s+t}$,
(iii) $\forall t_1, t_2 \in \mathbb{R} : \ \mathcal{T}_{t_1} \circ \mathcal{T}_{t_2} \equiv \mathcal{T}_{t_1+t_2}$, and
(iv) $\mathcal{T}_t$ is independent of any orientation-preserving reparameterisation of γ.

Proposition 2.27 provides many one-parameter subgroups $\mathbb{R} \to \mathrm{Iso}(M)^\circ$, $t \mapsto \mathcal{T}_t$, i.e. elements of the Lie algebra of $\mathrm{Iso}(M)^\circ$. This point of view will be important later on. For now, we prove Proposition 2.27.

Proof. For (i), compute

$$\mathcal{T}_t(\gamma(s)) = s_{\gamma(t/2)} \circ s_{\gamma(0)}(\gamma(s)) = s_{\gamma(t/2)}(\gamma(-s)) = \gamma(t + s).$$

As for part (ii), observe that $s_{\gamma(l)}(\gamma(t)) = \gamma(2l - t)$, hence the isometry s_{γ_l} preserves the geodesic γ for all $l \in \mathbb{R}$ and acts as an orientation-preserving reparameterisation. Therefore, if V is a parallel vector field along γ, then so is $(s_{\gamma(l)})_*V$. Observe further, that $(s_{\gamma(l)})_*V(l) = -V(l)$ and hence $(s_{\gamma(l)})_*V \equiv -V$. This implies $(\mathcal{T}_t)_*V = V$ for all parallel vector fields V along γ, i.e.

$$(\mathcal{T}_t)_*V(l) = (D_{\gamma(l)}\mathcal{T}_t)(V(l)) = V(t + l)$$

and hence $D_{\gamma(l)}\mathcal{T}_t \equiv P_{\gamma;t,t+l}$.

For part (iii), we appeal to the rigidity-type Lemma 2.3 for isometries using

$$\begin{aligned} D_{\gamma(s)}(\mathcal{T}_t \circ \mathcal{T}_{t'}) &\equiv D_{\gamma(s+t')}\mathcal{T}_t \, D_{\gamma(s)}\mathcal{T}_{t'} \\ &\equiv P_{\gamma;s+t',s+t+t'}P_{\gamma;s,s+t'} \equiv P_{\gamma;s,s+t'+t} \equiv D_{\gamma(s)}\mathcal{T}_{t+t'} \,. \end{aligned}$$

Hence the isometries $\mathcal{T}_t \circ \mathcal{T}_{t'}$ and $\mathcal{T}_{t+t'}$ both map $\gamma(s)$ to $\gamma(s + t + t')$ and have the same derivative at $\gamma(s)$.

We check part (iv) on translations:

$$\begin{aligned} s_{\gamma(t/2+b)} \circ s_{\gamma(b)} &\equiv s_{\gamma(t/2+b)} \circ s_{\gamma(0)} \circ s_{\gamma(0)} \circ s_{\gamma(b)} \\ &\equiv \mathcal{T}_{t+2b} \circ (s_{\gamma(b)} \circ s_{\gamma(0)})^{-1} \equiv \mathcal{T}_{t+2b} \circ \mathcal{T}_{2b}^{-1} \equiv \mathcal{T}_t\,. \end{aligned}$$

□

2.4. **Lie Group Viewpoint.** We have seen that a globally symmetric space M is homogeneous under the action of $G := \mathrm{Iso}(M)^\circ$ with compact point stabilizer K. We are now going to characterize the pairs (G, K), consisting of a connected Lie group G and a compact subgroup $K \leq G$, that come from symmetric spaces as above. This will unveil deeper structures. A central notion will be the following.

Definition 2.28. Let G be a Lie group and let $\sigma : G \to G$ be an automorphism. Then σ is *involutive* if $\sigma^2 = \mathrm{id}$.

Proposition 2.29. Let M be a symmetric space, $o \in M$, s_o the geodesic symmetry about $o \in M$, $G := \mathrm{Iso}(M)^\circ$ and $K := \mathrm{stab}_G(o)$. Then the map $\sigma : G \to G$, $g \mapsto s_o g s_o = s_o g s_o^{-1}$ is an involution of G such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

Proof. First of all, σ is an involution of G since it is given by conjugation with s_o. To see that $K \subseteq G^\sigma$, note that $s_o k s_o(o) = o$. Taking the derivative of $s_o k s_o : M \to M$ at $o \in M$ yields

$$D_o(s_o k s_o) = (D_o s_o)(D_o k)(D_o s_o) = (-\mathrm{Id})(D_o k)(-\mathrm{Id}) = D_o k.$$

Hence the isometries $s_o k s_o$ and k coincide up to first order at $o \in M$. By Lemma 2.3 they are thus equal.

To prove $(G^\sigma)^\circ \subseteq K$, it is enough to show the existence of a neighbourhood U of e in G^σ contained in K. Then $(G^\sigma)^\circ \subseteq \langle U \rangle \subseteq K$. Let V be a neighbourhood of $o \in M$ such that o is the only fixed point of s_o in V and set $U := \{g \in G^\sigma \mid g(o) \in V\}$. Then U is an open neighbourhood of $e \in G^\sigma$. It is also contained in K: For every $g \in U$, we have $s_o g(o) = s_o g s_o(o) = \sigma(g)(o) = g(o)$. That is, $g(o) \in V$ is a fixed point of s_o. By definition of V, this implies $g(o) = o$ and hence $g \in K$. □

Example 2.30. The condition $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$ of Proposition 2.29 cannot be made more precise as the subsequent examples will show. However, recall that if M is simply connected then K is connected and hence $K = (G^\sigma)^\circ$ in this case.

(i) $M = \mathbb{S}^2$, $o = e_3$, $G := \mathrm{Iso}(M)^\circ = \mathrm{SO}(3)$. Here,

$$s_o = \begin{pmatrix} -\mathrm{Id}_2 & \\ & 1 \end{pmatrix} \in G \quad \text{and for} \quad g = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}$$

we have

$$\sigma(g) = \begin{pmatrix} -\mathrm{Id}_2 & \\ & 1 \end{pmatrix} \begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} -\mathrm{Id}_2 & \\ & 1 \end{pmatrix} = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}.$$

Therefore,

$$G^\sigma = \left\{ \begin{pmatrix} A & \\ & d \end{pmatrix} \middle|\, A \in \mathrm{O}(2),\ d \in \{\pm 1\},\ (\det A) d = 1 \right\}$$

which is disconnected because of the continuous map $d : G^\sigma \to \{\pm 1\}$; in fact G^σ has two connected components. Furtermore, by the opening remark,

$$K = (G^\sigma)^\circ = \left\{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} \middle|\, A \in \mathrm{SO}(2) \right\}.$$

(ii) $M = \mathbb{P}^2(\mathbb{R}) = \{\pm \mathrm{Id}\} \backslash \mathbb{S}^2$, $o = [e_3]$, $G := \mathrm{Iso}(M)^\circ = \mathrm{Iso}(M) = \mathrm{O}(3)/\{\pm \mathrm{Id}\}$. Here, $\mathrm{stab}_G(o)$ has two connected components and equals G^σ.

(iii) $M = \mathbb{S}^3$, $o = e_4$, $G := \operatorname{Iso}(M)^\circ = \operatorname{SO}(4)$. Here,
$$s_o = \begin{pmatrix} -\operatorname{Id}_3 & \\ & 1 \end{pmatrix} \notin G$$
and as in part (i), G^σ has two connected components whereas $K = (G^\sigma)^\circ$ is connected.

(iv) $M = \mathbb{P}^3(\mathbb{R}) = \{\pm \operatorname{Id}\}\backslash \mathbb{S}^3$, $o = [e_4]$. In this case we have
$$G := \operatorname{Iso}(M)^\circ = \operatorname{SO}(4)/\{\pm \operatorname{Id}\} \neq \operatorname{O}(4)/\{\pm \operatorname{Id}\} = \operatorname{Iso}(M).$$
and G^σ is connected.

As announced, we now characterize pairs (G, K) of a connected Lie group G and a compact subgroup $K \leq G$ coming from symmetric spaces.

Definition 2.31. Let G be a connected Lie group and let $K \leq G$ be a closed subgroup of G. Then (G, K) is a *Riemannian symmetric pair* if

(i) $\operatorname{Ad}_G(K) \leq \operatorname{GL}(\mathfrak{g})$ is compact, and
(ii) there exists an involution $\sigma : G \to G$ with $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

Recall, that the adjoint representation $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$ of a Lie group G is given by $\operatorname{Ad}(g) = D_e \operatorname{int}(g)$ where $\operatorname{int}(g) : G \to G,\ h \mapsto ghg^{-1}$, as in the following commutative diagram:
$$\begin{array}{ccc} G & \xrightarrow{\operatorname{int}(g)} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\operatorname{Ad}(g)]{} & \mathfrak{g}\,. \end{array}$$

The first condition of Definition 2.31 means that K is compact up to the center of G since $\ker \operatorname{Ad}_G = C_G(G^\circ)$ (thus for connected G we have: $\ker \operatorname{Ad}_G = \operatorname{Z}(G)$). In virtue of Proposition 2.29, every symmetric space M gives rise to a Riemannian symmetric pair. Conversely, we shall prove the following.

Theorem 2.32. Let (G, K, σ) be a Riemannian symmetric pair. Then there exists a G-invariant Riemmanian metric on the homogeneous space $M := G/K$ and M is a symmetric space with respect to any such metric. If $\pi : G \to G/K$ denotes the natural projection, $o := eK \in G/K$ and s_o is the geodesic symmetry about $o \in G/K$, then $s_o\pi = \pi\sigma$.

As a corollary, we note that the geodesic symmetry s_o of Theorem 2.32 is independent of the choice of the G-invariant Riemannian metric on G/K.

Example 2.33. Theorem 2.32 provides a powerful way to construct Riemannian symmetric spaces. Rather than thinking of its geometric shape, we may just look at Lie groups with certain subgroups. However, note that Theorem 2.32 is useless so far in that it does not say what the Riemannian metric is; however, this will be remedied by the proof.

(i) Let $G = \operatorname{SL}(n, \mathbb{R})$, $K = \operatorname{SO}(n)$ and $\sigma : G \to G,\ g \mapsto (g^{-1})^T$. Then σ is an involution of G and $G^\sigma = \{g \in \operatorname{SL}(n, \mathbb{R}) \mid g^T g = \operatorname{Id}\} = \operatorname{SO}(n) = K$. Now, for instance it is well known that $\operatorname{SL}(2, \mathbb{R})/\operatorname{SO}(2) \cong \mathbb{H}^2$. In this sense, $\operatorname{SL}(n, \mathbb{R})/\operatorname{SO}(n)$ is a good generalization of the hyperbolic plane.

(ii) Let L be a compact connected Lie group. Further, set $G = L \times L$ and $\sigma : G \to G,\ (x, y) \mapsto (y, x)$. Then σ is an involution of G and $G^\sigma = \Delta_{L\times L} =: K$ is compact. Hence, by Theorem 2.32, G/K is a symmetric space. Note that the map $L \times L \to L,\ (g, h) \mapsto gh^{-1}$ induces a diffeomorphism $L \times L/\Delta_{L\times L} \to L$. Hence L may be viewed as a symmetric space. In this case, the $G = L \times L$-action on $L = G/K$ is given by $(g, h)_* l = glh^{-1}$.

Proof. (Theorem 2.32). We first give an appropriate model for T_oM and its K-action. By assumption, we have $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$. Denoting by $\mathfrak{k}$ the Lie algebra of K, we conclude $\mathfrak{k} = \mathrm{Lie}((G^\sigma)^\circ) = \mathrm{Lie}(G^\sigma)$. Hence

$$\begin{aligned} \mathfrak{k} = \mathrm{Lie}(G^\sigma) &= \{X \in \mathfrak{g} \mid \exp tX \in G^\sigma \ \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{g} \mid \sigma(\exp tX) = \exp tX \ \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{g} \mid D_e\sigma X = X\} \end{aligned}$$

where the last equality follows from the fact σ is a Lie group homomorphism and therefore satisfies $\sigma(\exp tY) = \exp(tD_e\sigma(Y))$ for all $t \in \mathbb{R}$ and $Y \in \mathfrak{g}$. Note further, that $D_e\sigma \in \mathrm{GL}(\mathfrak{g})$, being the derivative of an automorphism of G. It satisfies $(D_e\sigma)^2 = D_e(\sigma^2) = \mathrm{Id}$. Overall, $D_e\sigma$ is an involutive automorphism of $\mathfrak{g}$ and as such is diagonalizable with eigenvalues in $\{\pm 1\}$. By the above, $\mathrm{E}_1(D_e\sigma) = \mathfrak{k}$. Thus, if $\mathfrak{p} := \mathrm{E}_{-1}(D_e\sigma)$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,. \tag{CD}$$

Now, since $\sigma(k) = k \ \forall k \in K$, we have $\sigma \circ \mathrm{int}(k) \equiv \mathrm{int}(k) \circ \sigma$ for all $k \in K$. Passing to the derivative at $e \in G$, we obtain $D_e\sigma \, \mathrm{Ad}_G(k) \equiv \mathrm{Ad}_G(k) D_e\sigma$ for all $k \in K$ which implies that $\mathrm{Ad}(K) \leq \mathrm{GL}(\mathfrak{g})$ preserves $\mathfrak{p}$. Since $\mathrm{Ad}(K) \leq \mathrm{GL}(\mathfrak{g})$ also preserves $\mathfrak{k}$, it preserves the decomposition (CD).

In the following, for the sake of clarity, we will denote by L_g the diffeomorphism $M = G/K \to G/K, \ xK \mapsto gxK$. For $k \in K$, consider the following commutative diagrams:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{int}(k)} & G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G/K & \xrightarrow[L_k]{} & G/K \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(k)} & \mathfrak{g} \\ {\scriptstyle D_e\pi}\downarrow & & \downarrow{\scriptstyle D_e\pi} \\ \mathrm{T}_oG/K & \xrightarrow[D_oL_k]{} & \mathrm{T}_oG/K; \end{array} \tag{T}$$

the right one being obtained by taking derivatives in the first. By construction of the differentiable structure on G/K, the map $\pi : G \to G/K$ is a submersion, in particular $D_e\pi : \mathfrak{g} \to \mathrm{T}_oG/K$ is surjective; and it has kernel $\ker D_e\pi = \mathfrak{k}$. Therefore, $D_e\pi|_\mathfrak{p} : \mathfrak{p} \to \mathrm{T}_oG/K$ is an isomorphism of vector spaces and by the above diagram an isomorphism of K-modules, i.e. $D_e\pi \, \mathrm{Ad}(k)|_\mathfrak{p} = D_oL_kD_e\pi|_\mathfrak{p}$ for all $k \in K$.

This description of T_oG/K and its K-action allows us to prove the existence of a G-invariant Riemannian metric on $M = G/K$. First of all, a G-invariant Riemannian metric $\{B_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R} \mid p \in M\}$ on M is determined by B_o since for $g \in G$, $p := L_go$ and $v, w \in \mathrm{T}_oM$ the G-invariance forces

$$B_p(D_oL_gv, D_oL_gw) = B_o(v, w).$$

The above equation also shows that B_o is a K*-invariant* scalar product on $\mathrm{T}_o(M)$. Conversely, any K-invariant scalar product B_o on T_oM gives rise to a well-defined, G-invariant Riemannian metric $\{B_p \mid p \in M\}$ on M via

$$B_{L_go}(v, w) = B_o(D_{L_go}L_{g^{-1}}v, D_{L_go}L_{g^{-1}}w).$$

Now, using the right diagram of (T), we see that K-invariant scalar products on T_oM correspond to $\mathrm{Ad}(K)$-invariant scalar products on $\mathfrak{p}$. By assumption, $\mathrm{Ad}(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact and hence so is $\mathrm{Ad}(K)|_\mathfrak{p} \leq \mathrm{GL}(\mathfrak{p})$. This implies the existence of an $\mathrm{Ad}(K)|_\mathfrak{p}$-invariant scalar product on $\mathfrak{p}$: If μ denotes the left Haar measure on $\mathfrak{p}$ and $\langle -, - \rangle : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ is any scalar product on $\mathfrak{p}$, then an $\mathrm{Ad}(K)|_\mathfrak{p}$-invariant scalar product on $\mathfrak{p}$ is given by

$$(u, v) := \int_{\mathrm{Ad}(K)|_\mathfrak{p}} \langle Au, Av \rangle \ \mu(A)$$

(If the action of $\mathrm{Ad}(K)|_{\mathfrak{p}}$ on $\mathfrak{p}$ is irreducible, this is essentially the only such product. Otherwise, the irreducible pieces of $\mathfrak{p}$ give rise to a decomposition of M into geometric pieces that are again symmetric.)

We have found a model for $\mathrm{T}_o G/K$ and its K-action, and used it to equip $M = G/K$ with a G-invariant Riemannian metric. It remains to provide the geodesic symmetries. Define $s_o(p) := L_{\sigma(g)}o$ whenever $L_g o = p$. This is well-defined since $L_{\sigma(gk)}(o) = L_{\sigma(g)}L_{\sigma(k)}o = L_{\sigma(g)}o$. We now aim to show that $D_o s_o = -\mathrm{Id}$. To this end, note that s_o commutes with π in the sense that

$$\begin{array}{ccc} G & \xrightarrow{\ \sigma\ } & G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G/K & \xrightarrow[s_o]{} & G/K \end{array}$$

commutes. Taking derivatives implies $D_o s_o D_e\pi = D_e\pi D_e\sigma : \mathfrak{g} \to \mathrm{T}_o G/K$. Restricting to $\mathfrak{p}$ now yields for all $X \in \mathfrak{p}$:

$$D_o s_o D_e\pi(X) = D_e\pi D_e\sigma(X) = -D_e\pi X.$$

Since $D_e\pi|_{\mathfrak{p}} : \mathfrak{p} \to T_o M$ is an isomorphism, this implies that $D_o s_o = -\mathrm{Id}$. It remains to show that s_o is an isometry with respect to any G-invariant metric $\{B_p \mid p \in M\}$ on M. We need to verify that for all $p \in M$ and for all $v, w \in T_p M$ we have

$$B_{s_o(p)}(D_p s_o v, D_p s_o w) = B_p(v, w).$$

To this end, the following prelimary computation will by useful:

$$s_o L_g(xK) = s_o(gxK) = \sigma(gx)K = \sigma(g)\sigma(x)K = \sigma(g)s_o(xK) = L_{\sigma(g)}s_o(xK).$$

Now we verify the isometry condition mentioned above. Write $p = L_g o$ and let $v = D_o L_g v_o$ as well as $w = D_o L_g w_o$ for $v_o, w_o \in \mathrm{T}_o M$. Then $B_p(v, w) = B_o(v_o, w_o)$ by G-invariance and

$$\begin{aligned} B_{s_o(p)}(D_p s_o v, D_p s_o w) &= B_{s_o(L_g o)}(D_{L_g o}s_o(D_o L_g v_o), D_{L_g o}s_o(D_o L_g w_o)) \\ &= B_{s_o(L_g o)}(D_o(s_o L_g)v_o, D_o(s_o L_g)w_o) \\ &= B_{L_{\sigma(g)}o}(D_o(s_o L_g)v_o, D_o(s_o L_g)w_o) \\ &= B_{L_{\sigma(g)}o}(D_o L_{\sigma(g)}(D_o s_o v_o), D_o L_{\sigma(g)}(D_o s_o w_o)) \\ &= B_{L_{\sigma(g)}o}(D_o L_{\sigma(g)}(v_o), D_o L_{\sigma(g)}w_o) \\ &= B_o(v_o, w_o). \end{aligned}$$

As to the geodesic symmetry about an arbitrary point $p \in M$, the natural definition $s_p := L_g s_o L_g^{-1}$ works. □

Remark 2.34. Let M be a symmetric space and let (G, K) be the Riemannian symmetric pair associated to M. Then the involution $\sigma : G \to G$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$ is unique. A priori, given G and K, there may be several σ with this property. However, if the pair (G, K) comes from a symmetric space there is just one: Indeed, let s_o be the geodesic symmetry about $o \in M$. If $\sigma_1, \sigma_2 : G \to G$ are involutions of G such that $(G^{\sigma_i})^\circ \subseteq K \subseteq G^{\sigma_i}$ $(i \in \{1, 2\})$, then by the proof of Theorem 2.32 we have $L_{\sigma_1(g)} = s_o L_g s_o^{-1} = L_{\sigma_2(g)}$ for all $g \in G$. Hence $\sigma_1(g) = \sigma_2(g)$ for all $g \in G$ since G is the isometry group of M, see Lemma 2.3.

Definition 2.35. Let (G, K, σ) be a Riemannian symmetric pair. Further, denote $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. The map $\Theta := D_e\sigma : \mathfrak{g} \to \mathfrak{g}$ is the *Cartan involution* associated to (G, K, σ) and $\mathfrak{g} = \mathrm{E}_1(\Theta) \oplus \mathrm{E}_{-1}(\Theta) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathrm{E}_{-1}(\Theta)$, is the *Cartan decomposition* of $\mathfrak{g}$ with respect to Θ.

Proposition 2.36. Retain the notations of Definition 2.35. Then $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$.

Proof. This is a consequence of the eigenspace decomposition of $\mathfrak{g}$ into $\mathfrak{k}$ and $\mathfrak{p}$ with respect to Θ. Namely, if $\Theta X = \lambda X$ and $\Theta Y = \mu Y$, then

$$\Theta[X,Y] = [\Theta X, \Theta Y] = \lambda\mu[X,Y]$$

Hence the assertion. □

2.5. **Exponential Map and Geodesics.** Given that symmetric spaces come from Riemannian symmetric pairs and conversely, in this section we ask what the relation between the Riemannian and the Lie group exponential is.

In the following, we therefore let (G,K,σ) be a Riemannian symmetric pair with associated symmetric space $M = G/K$, basepoint $o \in M$ and $\pi : G \to M$, $g \mapsto g_*o$ the natural projection.

Theorem 2.37. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Further, denote by $\exp : \mathfrak{g} \to G$ the Lie group exponential and by $\mathrm{Exp}_o : \mathrm{T}_oM \to M$ the Riemannian exponential at $o \in M$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi|_{\mathfrak{p}}} & \mathrm{T}_oM \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \mathrm{Exp}_o} \\ G & \xrightarrow[\pi]{} & M. \end{array}$$

In particular, for $Y \in \mathfrak{p}$, the map $\mathbb{R} \to M$, $t \mapsto \exp(tY)_*o$ is the geodesic through o in the direction $D_e\pi Y$ and every geodesic through o is of this form.

Proof. For $X \in \mathfrak{p}$, consider the geodesic $\gamma : \mathbb{R} \to M$, $t \mapsto \mathrm{Exp}_o(tD_e\pi(X))$ and let $(\mathcal{T}_t)_t$ be the associated one-parameter group of transvections, given by $\mathcal{T}_t = s_{\gamma(t/2)}s_{\gamma(0)} \in \mathrm{Iso}(M)^\circ = G$. Recall that it satisfies $\mathcal{T}_t\,\gamma(s) = \gamma(s+t)$ for all $s,t \in \mathbb{R}$. Since $(\mathcal{T}_t)_t$ is a one-parameter group in G, there is $Y \in \mathfrak{g}$ such that $\mathcal{T}_t = \exp tY$ for all $t \in \mathbb{R}$. We show that actually $Y \in \mathfrak{p} \subseteq \mathfrak{g}$: To this end, compute

$$\begin{aligned}\sigma(\exp tY) =& \sigma(\mathcal{T}_t) = \sigma(s_{\gamma(t/2)}s_o) = s_os_{\gamma(t/2)}s_os_o = s_os_{\gamma(t/2)} = \\ &= s_o^{-1}s_{\gamma(t/2)}^{-1} = (s_{\gamma(t/2)}s_o)^{-1} = (\mathcal{T}_t)^{-1} = \exp(tY)^{-1} = \exp(-tY).\end{aligned}$$

Overall, this is $\exp(tD_e\sigma(Y)) = \sigma(\exp(tY)) = \exp(-tY)$ and hence $D_e\sigma Y = -Y$ by taking the derivative $d/dt|_{t=0}$. Thus $Y \in \mathfrak{p}$. Furthermore, we have for all $t \in \mathbb{R}$:

$$\pi(\exp tY) = \exp(tY)_*o = \mathcal{T}_t\,\gamma(0) = \gamma(t) = \mathrm{Exp}_0\, tD_e\pi(X).$$

Passing to tangent vectors yields $D_e\pi(Y) = D_e\pi(X)$ and hence $X = Y$ since $X, Y \in \mathfrak{p}$ and $D_e\pi|_{\mathfrak{p}}$ is injective. This implies the assertion. □

We finish this section about the Lie group viewpoint by constructing a connection on a given Lie group G whose geodesics through $e \in G$ are exactly the one-parameter subgroups of G; therefore the exponential of this connection coincides with the Lie group exponential. This will be useful later on.

First of all, we introduce the following model for vector fields on G. The map

$$X : C^\infty(G,\mathfrak{g}) \to \mathrm{Vect}(G),\ F \mapsto (X(F) : g \mapsto D_eL_gF(g))$$

is an isomorphism of $C^\infty(G)$-modules. Now, any connection

$$\nabla : \mathrm{Vect}(G) \times \mathrm{Vect}(G) \to \mathrm{Vect}(G)$$

gives rise to a map $D : C^\infty(G,\mathfrak{g}) \times C^\infty(G,\mathfrak{g}) \to C^\infty(G,\mathfrak{g})$ via

$$(F_1,F_2) \mapsto (D_{F_1}F_2 : g \mapsto (D_eL_g)^{-1}(\nabla_{XF_1}XF_2))$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Vect}(G)\times\mathrm{Vect}(G) & \xrightarrow{\nabla} & \mathrm{Vect}(G) \\ {\scriptstyle X\times X}\uparrow & & \uparrow{\scriptstyle X} \\ C^\infty(G,\mathfrak{g})\times C^\infty(G,\mathfrak{g}) & \xrightarrow[D]{} & C^\infty(G,\mathfrak{g}) \end{array}$$

and the subsequent three properties hold:

(D1) D is $\mathbb{R}$-linear in the second variable.
(D2) D is $C^\infty(G)$-linear in the first variable.
(D2) (Leibniz rule) For all $f\in C^\infty(G)$ $\forall F_1,F_2\in C^\infty(G,\mathfrak{g})$:

$$D_{F_1}(fF_2)(g)=D_e(f\circ L_g)(F_1(g))F_2(g)+f(g)D_{F_1}F_2(g)$$

The first two properties follow from the fact that X is an isomorphism of $C^\infty(G)$-modules. The Leibniz rule may be verified as follows:

$$\begin{aligned} D_{F_1}(fF_2)(g) &= (D_eL_g)^{-1}(\nabla_{XF_1}X(fF_2)(g)) \\ &= (D_eL_g)^{-1}(\nabla_{XF_1}fX(F_2)(g)) \\ &= (D_eL_g)^{-1}(X(F_1)(f)(g)X(F_2)(g)+f(g)\nabla_{XF_1}XF_2(g)) \\ &= \underbrace{X(F_1)(f)(g)}_{\text{see below}}\underbrace{(D_eL_g)^{-1}X(F_2)(g)}_{F_2(g)}+f(g)\nabla_{XF_1}XF_2(g). \end{aligned}$$

The Leibniz rule now follows from the computation

$$X(F_1)(f)(g)=D_gf(X(F_1)(g))=D_gf(D_eL_g(F_1(g)))=D_e(f\circ L_g)F_1(g).$$

Conversely, any map $D:C^\infty(G,\mathfrak{g})\times C^\infty(G,\mathfrak{g})\to C^\infty(G,\mathfrak{g})$ satisfying the above properties (D1), (D2) and (D2) gives rise to a connection ∇ on G via

$$\nabla_{XF_1}XF_2(g)=D_eL_g(D_{F_1}F_2(g)).$$

The point of this new model for connections on G is that it facilitates writing down actual connections. In fact, define a map $D:C^\infty(G,\mathfrak{g})\times C^\infty(G,\mathfrak{g})\to C^\infty(G,\mathfrak{g})$ by

$$(F_1,F_2)\mapsto\left(D_{F_1}F_2:g\mapsto\left.\frac{d}{dt}\right|_{t=0}F_2(g\exp tF_1(g))=D_e(F_2\circ L_g)(F_1(g))\right).$$

Lemma 2.38. The map D above satisfies properties (D1), (D2) and (D2). Let ∇ be the connection on G associated to D. Then

(i) ∇ is G-invariant,
(ii) $\nabla_{\widetilde{X}}\widetilde{Y}=0$ for all left-invariant vector fields $\widetilde{X},\widetilde{Y}\in\mathrm{Vect}(M)^G$, and
(iii) the exponential map associated to ∇ is the Lie group exponential.

Proof. It is readily checked that D satisfies the properties (D1), (D2) and (D2). To show (i), let $F\in C^\infty(G,\mathfrak{g})$ and $g\in G$. We reduce to showing that D is G-invariant in the suitable sense. First, compute

$$\begin{aligned} (L_{g*}X(F))(gh) = (D_hL_g)(X(F)(h)) = D_eL_h(F(h)) = \\ = D_e(L_{gh}(F(h))) = D_e(L_h(F(g^{-1}h))). \end{aligned}$$

Thus defining an element of $\mathrm{GL}(C^\infty(G,\mathfrak{g}))$ by $(\lambda(g)F)(h)=F(g^{-1}h)$, we have $L_{g*}X(F)=X(\lambda(g)F)$. Now we can verify the invariance property of ∇ using the corresponding invariance property of D:

$$\begin{aligned}\left(D_{\lambda(g)F_1}\lambda(g)F_2\right)(h) &= \left.\frac{d}{dt}\right|_{t=0} (\lambda(g)F_2)\left(h\exp t\lambda(g)F_1(h)\right)\\ &= \left.\frac{d}{dt}\right|_{t=0} F_2\left(g^{-1}h\exp F_1(g^{-1}h)\right)\\ &= D_{F_1}F_2(g^{-1}h)\\ &= \lambda(g)(D_{F_1}F_2)(h).\end{aligned}$$

For assertion (ii), observe that $X(F)(g) = (D_eL_g)F(g)$ is left-invariant if and only if F is constant. Hence left-invariant vector fields correspond to constant maps. Hence the assertion.

As to (iii), note that $\nabla_{\widetilde{X}}\widetilde{X} = 0$ for all $X \in \mathfrak{g}$ by (ii). Hence if $\gamma_X : \mathbb{R} \to G$ is the one-parameter group associated to X, then $\nabla_{\dot\gamma_X(t)}\dot\gamma_X(t) = 0$. Therefore, γ_X is the unique ∇-geodesic through $e \in G$ satisfying $\dot\gamma_X(0) = X$. That is, $\mathrm{Exp}_\nabla(X) = \gamma_X(1) = \exp(X)$. □

Now, for the theory of totally geodesic submanifolds, we will need the following formula for the derivative of the Lie group exponential. It follows readily from the formula for general analytic connections, see [Hel79, Thm. I.6.5].

Theorem 2.39. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Further, let $X \in \mathfrak{g}$. Upon identifying $\mathrm{T}_X\,\mathfrak{g}$ with $\mathfrak{g}$, the derivative of exp at $X \in \mathfrak{g}$ is given by the following diagram:

$$\begin{array}{ccc} \mathrm{T}_X\,\mathfrak{g} & \xrightarrow{\;D_X \exp\;} & \mathrm{T}_{\exp X}G \\ \uparrow & & \uparrow{\scriptstyle D_eL_{\exp X}} \\ \mathfrak{g} & \xrightarrow[\sum_{n=0}^\infty \frac{\mathrm{ad}(X)^n}{(n+1)!}]{} & \mathfrak{g}, \end{array}$$

that is, $D_X \exp = (D_eL_{\exp X}) \circ \left(\sum_{n=0}^\infty \frac{(\mathrm{ad}\,X)^n}{(n+1)!}\right)$.

We now apply this formula in the context of symmetric spaces: Let M be a symmetric space, $o \in M$, $G := \mathrm{Iso}(M)^\circ$, $K := \mathrm{stab}_G(o)$ and $\mathrm{Lie}(G) =: = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Further, let $\pi : G \to M,\ g \mapsto g_*o$ be the orbit map. Recall that $D_e\pi|_\mathfrak{p} : \mathfrak{p} \to \mathrm{T}_oM$ is an isomorphism. We use it to define $\mathrm{Exp} : \mathfrak{p} \to \mathrm{T}_oM \to M$ and record the following Corollary to Theorem 2.39.

Corollary 2.40. Retain the above notation. Let $X \in \mathfrak{p}$. Then

$$D_X\,\mathrm{Exp} = D_oL_{\exp X} \circ D_e\pi \left(\sum_{n=0}^\infty \frac{\mathrm{ad}(X)^2}{(2n+1)!}\right).$$

Proof. We have

$$D_X\,\mathrm{Exp} = D_X(\mathrm{Exp}_o \circ D_e\pi|_\mathfrak{p}) = D_X(\pi \circ \exp|_\mathfrak{p})$$

and therefore, temporarily ignoring the restriction to $\mathfrak{p}$,

$$\begin{aligned} D_X(\pi \circ \exp) &= D_{\exp X}\pi \circ D_X \exp \\ &= D_{\exp X}\pi \circ D_e L_{\exp X} \circ \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^n}{(n+1)!} \right) \\ &= D_e(\pi \circ L_{\exp X}) \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^n}{(n+1)!} \right) \\ &= D_e(L_{\exp X} \circ \pi) \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^n}{(n+1)!} \right) \\ &= D_o L_{\exp X} \circ D_e\pi \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^n}{(n+1)!} \right). \end{aligned}$$

We now examine the last expression in the above computation for $X \in \mathfrak{p}$. Given $Y \in \mathfrak{p}$, we have $\mathrm{ad}(X)^n(Y) \in \mathfrak{k}$ whenever n is odd because of Proposition 2.36. Since $D_e\pi(\mathfrak{k}) = 0$ we therefore obtain

$$D_e\pi \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^n}{(n+1)!} \right) = D_e\pi \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(X)^{2n}}{(2n+1)!} \right)$$

which is the assertion. □

2.6. **Totally Geodesic Submanifolds.** In this section we present an application of the formula for the derivative of the Lie exponential to the characterization of *totally geodesic submanifolds* of a symmetric space. First we recall this notion.

Definition 2.41. Let M be a Riemannian manifold and let N be a submanifold. Let $p \in N \subseteq M$. The submanifold N is *geodesic at* p if for all $v \in \mathrm{T}_pN$, the M-geodesic with initial data (p, v) is contained in N. The submanifold N is *totally geodesic* if it is geodesic at all $p \in N$.

As an exercise, assume that N is totally geodesic in M and prove that any N-geodesic is an M-geodesic and that any M-geodesic contained in N is an N-geodesic. This is far from true for submanifolds that are not totally geodesic, think e.g. of $S^2 \subset \mathbb{R}^3$. We also record the following proposition. For a systematic treatment of totally geodesic submanifolds, we refer to [Hel79, Ch. I.14].

Proposition 2.42. Let M be a Riemannian manifold and let N be a totally geodesic submanifold of M. Then the M-parallel transport along curves in N preserves the tangent space distribution $\{\mathrm{T}_pN \mid p \in N\}$.

Remark 2.43.

The Lie-theoretic concept that comes with total geodesicity is the following.

Definition 2.44. Let $\mathfrak{g}$ be a Lie algebra. A subset $\mathfrak{n} \subseteq \mathfrak{g}$ is a *Lie triple system* if $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subseteq \mathfrak{n}$.

Example 2.45. If $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition associated to a symmetric space, then $\mathfrak{p}$ is a Lie triple system in $\mathfrak{g}$.

In fact there is going to be a one-to-one correspondence between totally geodesic submanifolds of a symmetric space and Lie triple systems in $\mathfrak{p}$.

Lemma 2.46. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{n} \subseteq \mathfrak{g}$ be a Lie triple system. Then

(i) $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{g}$ is a subalgebra, and
(ii) $\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{g}$ is a subalgebra.

Proof. For (i), let $X, Y, Z, W \in \mathfrak{n}$. By Jacobi's identity

$$[[X,Y],[Z,W]] + [\underbrace{[[Z,W],X]}_{\in\mathfrak{n}}, \underbrace{Y}_{\in\mathfrak{n}}] + [\underbrace{[Y,[Z,W]]}_{\in\mathfrak{n}}, \underbrace{X}_{\in\mathfrak{n}}] = 0$$

and therefore $[[X,Y],[Z,W]] \in [\mathfrak{n},\mathfrak{n}]$.

For (ii), we consider the following possibilities: If $X, Y \in \mathfrak{n}$, then $[X,Y] \in [\mathfrak{n},\mathfrak{n}]$. If $X \in \mathfrak{n}$ and $Y \in [\mathfrak{n},\mathfrak{n}]$, then $[X,Y] \in [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] \subseteq \mathfrak{n}$. Finally, if $X, Y \in [\mathfrak{n},\mathfrak{n}]$, then $[X,Y] \in [\mathfrak{n},\mathfrak{n}]$ by part (i). □

We now prove the announced correspondence between Lie triple systems and totally geodesic submanifolds.

Theorem 2.47. Let M be a symmetric space, $o \in M$, $G := \mathrm{Iso}(M)^\circ$, $K := \mathrm{stab}_G(o)$, $\pi : G \to M,\ g \mapsto g_* o$ and $\mathrm{Lie}(G) =: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Then the following statements hold.

(i) If $\mathfrak{n} \subseteq \mathfrak{p}$ is a Lie triple system, then $N := \mathrm{Exp}_0(D_e\pi(\mathfrak{n})) \subseteq M$ is a totally geodesic submanifold of M.

(ii) If N is a totally geodesic submanifold of M containing $o \in M$, then $\mathfrak{n} := (D_e\pi)^{-1}\mathrm{T}_oN \subseteq \mathfrak{p}$ is a Lie triple system.

Remark 2.48. Retain the notation of Theorem 2.47. If the totally geodesic submanifold $N \subseteq M$ does not pass through $o \in M$, then transitivity of $\mathrm{Iso}(M)$ on M implies that there exists an isometry of M which takes N to the point $o \in M$, and this isometric translate of N is again a totally geodesic submanifold of M.

Proof. (Theorem 2.47). For (i), let G' be the Lie subgroup of G associated to the subalgebra $\mathfrak{g}' := \mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$. Put $M' := G'_* o$ and $K' := \mathrm{stab}_{G'}(o) = G' \cap K$. Then K' is a closed subgroup of G, hence a Lie subgroup. Therefore G'/K' has a smooth manifold structure. Further, the map $G'/K' \to M,\ g'K' \mapsto g'_* o$ is smooth and immersive which follows from the same properties of the map $G' \to G$. Therefore, $D_e\pi(\mathfrak{g}' \cap \mathfrak{p}) = \mathrm{T}_oM'$. Now observe that $\mathfrak{g}' = \mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$ implies $\mathfrak{g}' \cap \mathfrak{p} = \mathfrak{n}$. Thus actually $D_e\pi(\mathfrak{n}) = \mathrm{T}_oM'$. Now recall that every geodesic through $o \in M$ is of the form $t \mapsto (\exp tX)_* o$ for some $X \in \mathfrak{p}$. Hence such a geodesic is tangent to M' at $o \in M$ if and only if $X \in \mathfrak{n}$. In this case, $\exp tX \in G'$ for all $t \in \mathbb{R}$ and therefore $\exp(tX)_* o \in M'$ for all $t \in \mathbb{R}$. Hence M' is totally geodesic at $o \in M$. Using the G'-action on M', one sees that M' is geodesic at all $p \in M'$, hence totally geodesic.

For (ii), let $\mathfrak{n} := (D_e\pi)^{-1}\mathrm{T}_oN \subseteq \mathfrak{p}$. Recall the map $\mathrm{Exp} = \mathrm{Exp}_o \circ D_e\pi|_{\mathfrak{p}} : \mathfrak{p} \to \mathrm{T}_oM \to M$. If $X, Y \in \mathfrak{n}$, then $\mathrm{Exp}(tX)$ and $\mathrm{Exp}(tY)$ are M-geodesics which are contained in N by the definition of total geodesicity. Now, consider the restriction $\mathrm{Exp} : \mathfrak{n} \to N$ and its differential $D_{tY}\,\mathrm{Exp} : \mathfrak{n} \to \mathrm{T}_{\mathrm{Exp}(tY)}N$. By Corollary 2.40, this is

$$D_{tY}(\mathrm{Exp}(X) = D_oL_{\exp tY})D_e\pi\left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(tY)^{2n}(X)}{(2n+1)!}\right).$$

Since the M-parallel transport preserves $\{\mathrm{T}_pN \mid p \in M\}$ by Proposition 2.42 and is implemented by transvections by 2.27 we conclude

$$D_e\pi\left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(tY)^{2n}(X)}{(2n+1)!}\right) = D_{tY}(\mathrm{Exp}(X)D_oL_{\exp -tY}) \in \mathrm{T}_0N.$$

Thus, by definition of $\mathfrak{n}$, we have

$$\sum_{n=0}^{\infty} \frac{\mathrm{ad}(tY)^{2n}(X)}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{t^{2n}\,\mathrm{ad}(Y)^{2n}(X)}{(2n+1)!} \in \mathfrak{n}$$

which in turn implies $\operatorname{ad}(Y)^2(X) \in \mathfrak{n}$, that is, $[Y,[Y,X]] \in \mathfrak{n}$ for all $X, Y \in \mathfrak{n}$. In particular, for $X, Z, Y \in \mathfrak{n}$ we have $\operatorname{ad}(Y+Z)^2(X) \in \mathfrak{n}$. Making this explicit using

$$\operatorname{ad}(Y+Z)^2 = (\operatorname{ad}(Y)+\operatorname{ad}(Z))^2 = \operatorname{ad}(Y)^2+\operatorname{ad}(Y)\operatorname{ad}(Z)+\operatorname{ad}(Z)\operatorname{ad}(Y)+\operatorname{ad}(Z)^2 \in \mathfrak{n}$$

yields $\operatorname{ad}(Y)\operatorname{ad}(Z)(X) + \operatorname{ad}(Z)\operatorname{ad}(Y)(X) \in \mathfrak{n}$, that is

$$[Y,[Z,X]] + [Z,[Y,X]] \in \mathfrak{n} \tag{L}$$

Applying Jacobi's identity to the second term yields

$$[Z,[Y,X]]+[Y,[X,Z]]+[X,[Z,Y]] = 0 \quad \Leftrightarrow \quad [Z,[Y,X]] = -[X,[Z,Y]]-[Y,[X,Z]].$$

and by performing the same computation on the first term of (L) leads to

$$2[Y,[Z,X]] + [X,[Y,Z]] \in \mathfrak{n} \quad \text{and} \quad 2[X,[Z,Y]] + [Y,[X,Z]] \in \mathfrak{n}$$

and thus finally by subtraction

$$3[Y,[Z,X]] + 3[X,[Y,Z]] = -3[Z,[X,Y]] \in \mathfrak{n},$$

i.e., $\mathfrak{n}$ is a Lie triple system in $\mathfrak{g}$. □

Remark 2.49. It is an exercise to show that if M is a (locally) symmetric space and N is a totally geodesic submanifold of M, then N is a (locally) symmetric space in its own right and to identify the corresponding Riemannian symmetric pair in the global case, including the involution.

2.7. **Examples.** In this section we illustrate the developed theory with several guiding examples.

2.7.1. *The Riemannian Symmetric Pair* $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(n))$. Let $G = \mathrm{SL}(n,\mathbb{R})$. An involution on G is given by $\sigma(g) := (g^{-1})^T$. We have

$$K := \operatorname{stab}_G(\sigma) = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid g = (g^{-1})^T\} = \mathrm{SO}(n).$$

Since K is itself compact, (G, K, σ) is a Riemannian symmetric pair. Therefore, $M := G/K = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a symmetric space for any G-invariant Riemannian metric on M. The Lie algebra $\mathfrak{g} := \operatorname{Lie}(G) = \{X \in M_{n,n}(\mathbb{R}) \mid \operatorname{tr} X = 0\}$ is well-known. To determine the Cartan decomposition we compute the Cartan involution $\Theta = D_e\sigma : \mathfrak{g} \to \mathfrak{g}$:

$$\Theta(X) = \frac{d}{dt}\bigg|_{t=0} \sigma(\exp tX) = \frac{d}{dt}\bigg|_{t=0} \exp(-tX^T) = -X^T.$$

Thus

$$\mathfrak{k} = \mathrm{E}_1(\Theta) = \{X \in M_{n,n}(\mathbb{R}) \mid \operatorname{tr} X = 0,\ X + X^T = 0\}$$

and

$$\mathfrak{p} = \mathrm{E}_{-1}(\Theta) = \{X \in M_{n,n}(\mathbb{R}) \mid \operatorname{tr} X = 0,\ X = X^T\}$$

In words, the Cartan decomposition in this case is just the decomposition of a matrix into its symmetric and its antisymmetric part.

In order to give a G-invariant Riemannian metric on M, it suffices to provide an $\operatorname{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ by (T). To this end, recall that $\operatorname{Ad} : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(\mathfrak{sl}(n,\mathbb{R}))$ is given by $g \mapsto (\operatorname{Ad}(g) : X \mapsto gXg^{-1})$. Then $\langle X, Y\rangle := \operatorname{tr}(XY)$ is clearly an $\operatorname{Ad}(K)$-invariant scalar product on $\mathfrak{p}$. As a model for M we have

$$\mathrm{Sym}_1^+(n) = \{S \in M_{n,n}(\mathbb{R}) \mid S^T = S,\ S \gg 0,\ \det S = 1\}.$$

Viewed as quadratic forms, $\mathrm{Sym}_1^+(n)$ may be identified with the set of all ellipsoids in $\mathbb{R}^n$ centered at $0 \in \mathbb{R}^n$ with volume 1. This manifold M arises as a homogeneous space of G through the action $g_*S := gSg^T$, which is transitive by a linear algbra argument and has $\operatorname{stab}_G(\mathrm{Id}) = K$. The Riemannian exponential map $\operatorname{Exp}_{\mathrm{Id}} : \mathfrak{p} \to \operatorname{Sym}_1^+(n)$ can be computed using Theorem 2.37:

$$\begin{aligned}\operatorname{Exp}(X) &= \exp_G(X)_* \,\mathrm{Id} = \exp(X)\,\mathrm{Id}\exp(X)^T = \\ &= \exp_G(X)\exp_G(X) = \exp_G(2X) \in \operatorname{Sym}_1^+(n)\end{aligned}$$

Part of the beauty of the theory of symmetric spaces indeed lies in the fact that it facilitates otherwise difficult computations as the one above.

We end this example by making explicit the relationship between $\operatorname{Sym}_1^+(2)$ and $\mathbb{H}^2$. To every

$$S = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

we associate the quadratic form $q_S(x, y) := (x, y)S(x, y)^T = ax^2 + 2bxy + dy^2$. Since $ad - b^2 = 1$ we have $ad \neq 0$ and therefore

$$q(x, y) = ay^2 \left(\left(\frac{x}{y}\right)^2 + \frac{2b}{a}\left(\frac{x}{y}\right) + \frac{d}{a} \right) = ay^2 \left(\frac{x}{y} - z\right)\left(\frac{x}{y} - \overline{z}\right)$$

where $z, \overline{z}$ are given by

$$\frac{-\frac{2b}{a} \pm \sqrt{4\left(\frac{b}{a}\right)^2 - 4\frac{d}{a}}}{2} = \frac{-\frac{2b}{a} \pm \sqrt{\frac{4(b^2 - ad)}{a^2}}}{2} = \frac{-b \pm i}{a}.$$

Since S is positive definite, $a > 0$ and hence $z := (-b + i)/a \in \mathbb{H}$. In this way, we obtain a map $\operatorname{Sym}_1^+(2) \to \mathbb{H}^2$, $S \mapsto z_S$ which has the following equivariance property: $z_{g_* S} = (g^{-1})_*^T z_S$ where the action on right hand side is by fractional linear transformations. This map is very important and was probably already known by Gauss. Possibly, it is a natural way to discover fractional linear transformations.

2.7.2. *Closed Adjoint Subgroups.* Let $G \leq \mathrm{SL}(n, \mathbb{R})$ be a closed connected subgroup which is adjoint, i.e. closed under transposition. Then $\sigma(g) := (g^{-1})^T$ is an involution of G and $K := G^\sigma = G \cap \mathrm{SO}(n)$ is compact. Therefore, (G, K, σ) is a Riemannian symmetric pair. It comes with a natural bijection $G/K \to G_* \,\mathrm{Id} \subseteq \operatorname{Sym}_1^+(n)$ and in fact is a totally geodesic submanifold of $\operatorname{Sym}_1^+(n)$ as can be shown using the chararcterization via Lie triple systems.

2.7.3. *Quadrics.* To produce an example of a closed connected adjoint subgroup of $\mathrm{SL}(n, \mathbb{R})$, let $Q(x, y)$ be the quadratic form on $\mathbb{R}^{p+q}$ of signature (p, q), i.e.

$$Q(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j = x^T S y \quad \text{where} \quad S = \begin{pmatrix} \mathrm{Id}_p & \\ & -\mathrm{Id}_q \end{pmatrix}.$$

Set $n = p + q$. The orthogonal group of Q is denoted by

$$\mathrm{O}(Q) = \mathrm{O}(p, q) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid g^T S g = S\}.$$

Due to the symmetry of S, the group $\mathrm{O}(p, q)$ is closed under transposition. Thus $G := \mathrm{SO}(p, q)^\circ$ is a closed connected adjoint subgroup of $\mathrm{SL}(n, \mathbb{R})$ and

$$K = \mathrm{SO}(p, q)^\circ \cap \mathrm{SO}(n) = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \middle| \, A \in \mathrm{SO}(p),\ D \in \mathrm{SO}(q) \right\}.$$

The Lie algbra of G is given by

$$\begin{aligned}\mathfrak{g} := \operatorname{Lie}(\mathrm{SO}(p, q)^\circ) &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid \exp tX \in \mathrm{SO}(p, q)\ \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid (\exp tX)^T S \exp tX = S\ \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T S + SX = 0\}\end{aligned}$$

where the last equality follows from taking derivatives. Writing $X \in \mathfrak{sl}(n,\mathbb{R})$ as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad A \in M_{p,p}(\mathbb{R}),\ B \in M_{p,q}(\mathbb{R}),\ C \in M_{q,p}(\mathbb{R}),\ D \in M_{q,q}(\mathbb{R}),$$

the above computation of $\mathfrak{g}$ continues as

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sl}(n,\mathbb{R}) \middle| A + A^T = 0,\ D + D^T = 0,\ C = B^T \right\}$$
$$= \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \middle| A + A^T = 0,\ D + D^T = 0 \right\}.$$

The Cartan involution $\Theta = D_e\sigma$ of $\mathfrak{g}$ is given by $\Theta(X) = -X^T$, as for any closed connected adjoint subgroup of $\mathrm{SL}(n,\mathbb{R})$. We therefore have

$$\mathfrak{k} = \mathrm{E}_1(\Theta) = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \middle| A + A^T = 0,\ D + D^T = 0 \right\},$$

$$\mathfrak{p} = \mathrm{E}_{-1}(\Theta) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \middle| B \in\ M_{p,q}(\mathbb{R}) \right\}.$$

We conclude for instance that $\dim(G/K) = \dim(\mathrm{T}_{eK}G/K) = \dim \mathfrak{p} = pq$.

To obtain a model for G/K, let $W_0 = \langle e_1, \dots, e_p\rangle \leq \mathbb{R}^{p+q}$. Then W_0 is a subspace of $\mathbb{R}^n$ on which Q is positive definite and it is maximal with respect to this property. Consider $W_0 \in \mathrm{Gr}_p(\mathbb{R}^n)$, the Grassmannian of p-dimensional subspaces of $\mathbb{R}^n$. Then

$$\mathrm{stab}_{\mathrm{SO}(p,q)^\circ}(W_0) = K = \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \middle| A \in \mathrm{SO}(p),\ D \in \mathrm{SO}(q) \right\}.$$

A model of G/K therefore is $X_{p,q} = \{W \in \mathrm{Gr}_p(\mathbb{R}^n) \mid Q|_W \gg 0\}$ which is an open subset of $\mathrm{Gr}_p(\mathbb{R}^n)$. For $p = 1$, the space $X_{1,q}$ is of dimension q and is the unique complete simply connected Riemannian manifold of constant sectional curvature -1. It is known as the *projective model* of $\mathbb{H}^n$. In particular, for $q = 2$, we obtain yet another model of the hyperbolic plane $\mathbb{H}^2$.

2.7.4. *Symplectic Groups.* Another example of a closed connected adjoint subgroup of $\mathrm{SL}(n,\mathbb{R})$ goes as follows. The standard symplectic form on $\mathbb{R}^{2n}$ is given by

$$\omega(x,y) = \sum_{i=1}^n x_i y_{n+1} - \sum_{j=1}^n x_{n+j} y_j = x^T J y \quad \text{where} \quad J = \begin{pmatrix} & \mathrm{Id}_n \\ -\mathrm{Id}_n & \end{pmatrix}.$$

The orthogonal group of ω is called symplectic group and denoted by

$$\mathrm{Sp}(2n,\mathbb{R}) := \mathrm{O}(\omega) = \{g \in \mathrm{GL}(2n,\mathbb{R}) \mid g^T J g = J\}.$$

Due to the symmetry of J, the group $\mathrm{Sp}(2n,\mathbb{R})$ is closed under transposition. It can be further shown, that $G := \mathrm{Sp}(2n,\mathbb{R})$ is indeed a closed connected adjoint subgroup of $\mathrm{SL}(2n,\mathbb{R})$. Then

$$K = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{SO}(2n) = \{g \in \mathrm{Sp}(2n,\mathbb{R}) \mid g^{-1} J g = J\} = Z_{\mathrm{Sp}(2n,\mathbb{R})}(J).$$

The Lie algebra of G is given by

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{Sp}(2n,\mathbb{R})) = \{X \in \mathfrak{sl}(2n,\mathbb{R}) \mid X^T J + J X = 0\}$$

as in the previous section. Writing $X \in \mathfrak{sl}(2n,\mathbb{R})$ as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad A, B, C, D \in M_{n,n}(\mathbb{R}),$$

the Lie algebra of $\mathrm{Sp}(2n,\mathbb{R})$ can be written more explicitly as

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sl}(n,\mathbb{R}) \middle| A^T + D = 0,\ B^T = B,\ C^T = C \right\}.$$

The Cartan involution $\Theta = D_e\sigma$ is again given by $\Theta(X) = (X^{-1})^T$ whence

$$\mathfrak{k} = \mathrm{E}_1(\Theta) = \left\{ \begin{pmatrix} A & B \\ -B^T & -A^T \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \,\middle|\, A^T + A = 0,\ B^T = B \right\},$$

$$\mathfrak{p} = \mathrm{E}_{-1}(\Theta) = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \,\middle|\, A = A^T,\ B = B^T(\mathbb{R}) \right\}.$$

In particular, we have $\dim(G/K) = \dim(\mathfrak{p}) = 2(n(n+1)/2) = n(n+1) \in 2\,\mathbb{Z}$.

To obtain a model for G/K, we investigate more closely the properties of J. It satisfies $J^2 = -\operatorname{Id}$ and the map $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R},\ (x,y) \mapsto \omega(Jx,y)$ is a symmetric positive definite bilinear form. Indeed,

$$\omega(Jx,y) = (Jx)^T Jy = -x^T J^2 y = x^T y.$$

A complex structure on $\mathbb{R}^{2n}$ is an endomorphism $M \in \operatorname{End}(\mathbb{R}^{2n})$ which satisfies $M^2 = -\operatorname{Id}$. The reason for this terminology is that the map

$$\mathbb{C} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ (x+iy, v) \mapsto xv + Myv$$

defines a $\mathbb{C}$-vector space structure on $\mathbb{R}^{2n}$. We say that a complex structure M on $\mathbb{R}^{2n}$ is ω-compatible if $\omega(M-,-)$ is a symmetric positive definite bilinear form on $\mathbb{R}^{2n}$. Let S_{2n} denote the set of all ω-compatible structures on $\mathbb{R}^{2n}$. Then $\mathrm{Sp}(2n,\mathbb{R})$ acts on S_{2n} by conjugation: It is readily checked that, given $g \in \mathrm{Sp}(2n,\mathbb{R})$ and $M \in S_{2n}$, the endomorphism gMg^{-1} again defines a complex structure on $\mathbb{R}^{2n}$. One also verifies that this action is transitive. Hence $G/K \cong S_{2n}$ where

$$\begin{aligned} K &= \operatorname{stab}_{\mathrm{Sp}(2n,\mathbb{R})}(J) \\ &= \{g \in \mathrm{Sp}(2n,\mathbb{R}) \mid gJ = Jg\} \\ &= \{g \in \mathrm{GL}(2n,\mathbb{R}) \mid gJ = Jg,\ (x,y) \mapsto \omega(Jx,y) + i\omega(x,y) \text{ is preserved}\} \\ &= \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g \text{ preserves a fixed positive definite hermitian form}\} \\ &= \mathrm{U}(n) \end{aligned}$$

The positive dimensional center of $\mathrm{U}(n)$ is responsible for the existence of a natural complex manifold structure on S_{2n} which is invariant under the action of $\mathrm{Sp}(2n,\mathbb{R})$. This is connected to abelian varieties in number theory.

2.8. **Decomposition of Symmetric Spaces.** In this section, we present a decomposition theorem for symmetric spaces based on curvature properties. This requires some knowledge about semisimple Lie algebras which we present below.

2.8.1. *The Killing Form.* Let $\mathfrak{g}$ be a Lie algebra over a field k. Recall the adjoint representation

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ X \mapsto [X,-]$$

which is a homomorphism of Lie algebras.

Definition 2.50. Let $\mathfrak{g}$ be a Lie algebra. The bilinear form

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to k,\ (X,Y) \mapsto \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$$

is the *Killing form* of $\mathfrak{g}$.

We now collect some properties of the Killing form of a Lie algebra.

Lemma 2.51. Let $\mathfrak{g}$ be a Lie algebra with Killing form B. Then

(i) $\forall D \in \operatorname{Der}(\mathfrak{g})\ \forall X,Y \in \mathfrak{g} :\ B(DX,Y) + B(X,DY) = 0$, and
(ii) $\forall \alpha \in \operatorname{Aut}(\mathfrak{g})\ \forall X,Y \in \mathfrak{g} :\ B(\alpha X, \alpha Y) = B(X,Y)$.

In other words, automorphisms of $\mathfrak{g}$ preserve $B_{\mathfrak{g}}$, and derivations of $\mathfrak{g}$ are anti-symmetric with respect to $B_{\mathfrak{g}}$, in particular this holds for $\operatorname{ad}(Z)$ $(Z \in \mathfrak{g})$.

Proof. For (i), recall that $D[X,Y] = [DX,Y] + [X,DY]$ for all $X,Y \in \mathfrak{g}$. Thus we have for all $X,Y,Z \in \mathfrak{g}$:

$$\begin{aligned}[DX,[Y,Z]] &= -[X,D[Y,Z]] + D[X,[Y,Z]] \\ &= [-X,[DY,Z]] - [X,[Y,DZ]] + D[X,[Y,Z]]\end{aligned}$$

In terms of ad this reads

$$\mathrm{ad}(DX)\,\mathrm{ad}(Y) = -\,\mathrm{ad}(X)\,\mathrm{ad}(DY) - \mathrm{ad}(X)\,\mathrm{ad}(Y)D + D\,\mathrm{ad}(X)\,\mathrm{ad}(Y)$$

and hence the result follows by taking the trace on both sides.

For (ii), recall that $\alpha[X,Y] = [\alpha X, \alpha Y]$ for all $X,Y \in \mathfrak{g}$. Thus $\alpha\,\mathrm{ad}(X)\,\mathrm{ad}(Y) = \mathrm{ad}(\alpha X)\,\mathrm{ad}(\alpha Y)$ and therefore $\alpha\,\mathrm{ad}(X) = \mathrm{ad}(\alpha X)\alpha$ or $\alpha\,\mathrm{ad}(X)\alpha^{-1} = \mathrm{ad}(\alpha X)$. We conclude

$$\begin{aligned}B(\alpha X, \alpha Y) = \mathrm{tr}(\mathrm{ad}(\alpha X), \mathrm{ad}(\alpha Y)) &= \mathrm{tr}(\alpha\,\mathrm{ad}(X)\alpha^{-1}\alpha\,\mathrm{ad}(Y)\alpha^{-1}) = \\ &= \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) = B(X,Y).\end{aligned}$$

□

Definition 2.52. Let $\mathfrak{g}$ be a real Lie algebra. Then $\mathfrak{g}$ is *semisimple* if $B_{\mathfrak{g}}$ is non-degenerate.

We will show later on that a Lie algebra is semisimple if and only if it is a direct sum of non-abelian simple ideals.

2.8.2. *Orthogonal Symmetric Pairs.* So far, we have seen that a symmetric space M with base point $o \in M$ leads to a Riemannian symmetric pair (G,K,σ) which in turn provides a pair $(\mathfrak{g},\Theta)$ of a Lie algebra $\mathfrak{g}$ and in involutive automorphism Θ of $\mathfrak{g}$ such that $\mathfrak{k} := \mathrm{E}_1(\Theta)$ is a subalgebra of $\mathfrak{g}$ with the property that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}) = \mathrm{Lie}\,\mathrm{Ad}_G(K)$ is the Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$. In this section, we study pairs $(\mathfrak{g},\Theta)$ of this kind and prove a decomposition theorem for them which we eventually globalize to a decomposition theorem for symmetric spaces.

Definition 2.53. Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{k} \leq \mathfrak{g}$ is *compactly embedded* if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ is the Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$.

Definition 2.54. An *orthogonal symmetric Lie algebra* is a pair $(\mathfrak{g},\Theta)$ consisting of a real Lie algebra $\mathfrak{g}$ and an involutive automorphism $\Theta \neq \mathrm{Id}$ of $\mathfrak{g}$ such that $\mathfrak{u} = \mathrm{E}_1(\Theta)$ is compactly embedded in $\mathfrak{g}$. An orthogonal symmetric Lie algebra is *effective* if $Z(\mathfrak{g}) \cap \mathfrak{u} = 0$.

The term "orthogonal" in Definition 2.54 is motivated by the following Lemma.

Lemma 2.55. Let $(\mathfrak{g},\Theta)$ be an orthogonal symmetric Lie algebra and let $\mathfrak{g} = \mathrm{E}_1(\Theta) \oplus \mathrm{E}_{-1}(\Theta) =: \mathfrak{u} \oplus \mathfrak{e}$ be the eigenspace decomposition of $\mathfrak{g}$ with respect to Θ. Then $\mathfrak{u}$ and $\mathfrak{e}$ are orthogonal with respect to the Killing form.

Proof. Let $X \in \mathfrak{u}$ and $Y \in \mathfrak{e}$. Then

$$B_{\mathfrak{g}}(X,Y) = B_{\mathfrak{g}}(\Theta X, \Theta Y) = B_{\mathfrak{g}}(X,-Y) = -B_{\mathfrak{g}}(X,Y)$$

and hence $B_{\mathfrak{g}}(X,Y) = 0$. □

The property of an orthogonal symmetric Lie algebra of being *effective* has the following consequence.

Lemma 2.56. Let $(\mathfrak{g},\Theta)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$. Then $B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite.

Proof. Let $U \leq \mathrm{GL}(\mathfrak{g})$ be a compact subgroup such that $\mathrm{Lie}(U) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$. With respect to a suitable basis, we have $U \leq \mathrm{SO}(n)$ where $n = \dim \mathfrak{g}$ and hence $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subseteq \mathrm{Lie}(\mathrm{SO}(n)) = \{X \in M_{n,n}(\mathbb{R}) \mid X + X^T = 0\}$. Hence we have for all $v \in \mathfrak{u}$,

$$B_{\mathfrak{g}}(v,v) = \mathrm{tr}(\mathrm{ad}(v)\,\mathrm{ad}(v)) - \mathrm{tr}(\mathrm{ad}(v)^T\,\mathrm{ad}(v)) \leq 0$$

with equality if and only if $\mathrm{ad}(v) = 0$ in which case $v \in Z(\mathfrak{g}) \cap \mathfrak{u} = 0$. □

Our decomposition theorem for effective orthogonal symmetric Lie algebras will distinguish the following three types.

Definition 2.57. Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ and Killing form $B_{\mathfrak{g}}$.

(i) If $B_{\mathfrak{g}} \ll 0$ then $(\mathfrak{g}, \Theta)$ is of *compact type.*
(ii) If $B_{\mathfrak{g}}$ is non-degenerate and $B_{\mathfrak{g}}|_{\mathfrak{e}\times\mathfrak{e}} \gg 0$ then $(\mathfrak{g}, \Theta)$ is of non-compact type.
(iii) If $\mathfrak{e}$ is an abelian ideal in $\mathfrak{g}$ then $(\mathfrak{g}, \Theta)$ is of Euclidean type.

Remark 2.58. The three types of effective orthogonal symmetric Lie algebras of Definition 2.57 are clearly mutually exclusive. However, they are not inclusive; in fact, as we will show later on, every orthogonal symmetric Lie algebra admits a decomposition as a direct sum of effective orthogonal symmetric Lie algebras of the above types.

By Lemma 2.56, we have $B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}} \ll 0$ for every effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$. Hence $(\mathfrak{g}, \Theta)$ is of compact type if and only if $B_{\mathfrak{g}}|_{\mathfrak{e}\times\mathfrak{e}} \ll 0$, and is of non-compact type if and only if $B_{\mathfrak{g}}|_{\mathfrak{e}\times\mathfrak{e}} \gg 0$.

Example 2.59. As an example of an orthogonal symmetric Lie algebra of Euclidean type, let $\mathfrak{g} = \mathbb{R}^n$ with the zero bracket and define $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ by $X \mapsto -X$. Then $\mathfrak{g} = \mathfrak{e} = Z(\mathfrak{g})$ and $\mathfrak{u} = 0$.

We now state and prove the announced decomposition theorem for effective orthogonal symmetric Lie algebras.

Theorem 2.60. Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is a direct sum of Θ-stable ideals such that:

(i) The $\mathfrak{g}_\mu$ $(\mu \in \{0, +, -\})$ are mutually orthogonal with respect to $B_{\mathfrak{g}}$.
(ii) The pairs $\left(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0}\right)$, $\left(\mathfrak{g}_+, \Theta|_{\mathfrak{g}_+}\right)$, $\left(\mathfrak{g}_-, \Theta|_{\mathfrak{g}_-}\right)$ are orthogonal symmetric Lie algebras of Euclidean, non-compact and compact type resepectively.

For the proof, let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ be the Cartan decomposition of $\mathfrak{g}$. Recall that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ preserves $\mathfrak{e}$ and that hence so does $K \leq \mathrm{GL}(\mathfrak{g})$ which satisfies $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) = \mathrm{Lie}(K)$. Given that K is compact by assumption, we may choose a K-invariant scalar product $\langle -, - \rangle$ on $\mathfrak{e}$ and write $B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle$ for all $X, Y \in \mathfrak{e}$ and some symmetric endomorphism A of $\mathfrak{e}$. In addition, A commutes with K and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$. Let $(f_1, \ldots, f_n)$ be a $\langle -, - \rangle$-orthonormal (and hence $B_{\mathfrak{g}}$-orthogonal) basis of $\mathfrak{e}$ consisting of eigenvectors of A, let $(\beta_1, \ldots, \beta_n)$ be the corresponding eigenvalues and define

$$\mathfrak{e}_0 := \langle f_i \mid \beta_i = 0 \rangle, \quad \mathfrak{e}_+ := \langle f_i \mid \beta_i > 0 \rangle, \quad \mathfrak{e}_- := \langle f_i \mid \beta_i < 0 \rangle.$$

These spaces are going to be the $\mathfrak{e}$'s of the decomposition of $\mathfrak{g}$. The following properties, which are consequences of the above, should be kept in mind: $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ and K leave $\mathfrak{e}_0$, $\mathfrak{e}_+$ and $\mathfrak{e}_-$ invariant; and we have $B_{\mathfrak{g}}|_{\mathfrak{e}_+\times\mathfrak{e}_+} \gg 0$ as well as $B_{\mathfrak{g}}|_{\mathfrak{e}_-\times\mathfrak{e}_-} \ll 0$.

Lemma 2.61. Retain the above notation. Then the following holds.

(i) The space $\mathfrak{e}_0 = \{X \in \mathfrak{g} \mid B_{\mathfrak{g}}(X, Y) = 0\ \forall Y \in \mathfrak{g}\}$ is the kernel of the Killing form. In particular, it is an ideal in $\mathfrak{g}$.
(ii) We have $[\mathfrak{e}_0, \mathfrak{e}] = 0$. In particular, $\mathfrak{e}_0$ is abelian.
(iii) We have $[\mathfrak{e}_+, \mathfrak{e}_-] = 0$.

Proof. For part (i), let $\mathfrak{n} = \{X \in \mathfrak{g} \mid B_\mathfrak{g}(X,Y) = 0\,\forall Y \in \mathfrak{g}\}$. Since $B_\mathfrak{g}(\mathrm{ad}(X)Y, Z) = -B_\mathfrak{g}(Y, \mathrm{ad}(X)Z)$ for all $X, Y, Z \in \mathfrak{g}$, it follows that $\mathfrak{n} \trianglelefteq \mathfrak{g}$. Furthermore, since Θ preserves $B_\mathfrak{g}$, we have $\Theta(\mathfrak{n}) = \mathfrak{n}$. Therefore, $\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{u}) + (\mathfrak{n} \cap \mathfrak{e})$. Since $B_\mathfrak{g}$ is negative definite on $\mathfrak{u}$, we conclude $\mathfrak{n} \cap \mathfrak{u} = 0$. Thus $\mathfrak{n} \leq \mathfrak{e}$ and then $\mathfrak{n} \subseteq \mathfrak{e}_0$ since for $X \in \mathfrak{n}$ and $Y \in \mathfrak{e}$ we have $0 = B_\mathfrak{g}(X,Y) = \langle AX, Y\rangle$, terefore $AX = 0$, i.e. $X \in \ker A = \mathfrak{e}_0$. We now show that $\mathfrak{e}_0 \subseteq \mathfrak{n}$. First of all, we already know that $\mathfrak{e}_0$ is orthogonal to $\mathfrak{u}$ with respect to the Killing form. It therefore suffices to show that $\mathfrak{e}_0$ is orthogonal to $\mathfrak{e}$: For $X \in \mathfrak{e}_0$ and $Y \in \mathfrak{e}$ we have $B(X,Y) = \langle AX, Y\rangle = 0$ since $X \in \mathfrak{e}_0 = \ker A$.

For part (ii), note that $[\mathfrak{e}_0, \mathfrak{e}] \subseteq \mathfrak{u}$. By the fact that $B_\mathfrak{g}$ is negative definite on $\mathfrak{u}$ it therefore suffices to show $B_\mathfrak{g}([Y,X], Z) = 0$ for all $Y \in \mathfrak{e}$, $X \in \mathfrak{e}_0$ and $Z \in \mathfrak{u}$. This is indeed the case:

$$B_\mathfrak{g}([Y,X], Z) = -B_\mathfrak{g}(X, [Y,Z]) \overset{\text{(i)}}{=} 0.$$

In particular, $[\mathfrak{e}_0, \mathfrak{e}_0] \subseteq [\mathfrak{e}_0, \mathfrak{e}] = 0$, that is, $\mathfrak{e}_0$ is abelian.

Finally, for part (iii), we let $X \in \mathfrak{e}_+$, $Y \in \mathfrak{e}_-$ and $Z \in \mathfrak{u}$. Then

$$B_\mathfrak{g}([X,Y], Z) = -B_\mathfrak{g}(Y, [X,Z]) = \langle AY, \mathrm{ad}(Z)X\rangle = 0$$

since $AY \in \mathfrak{e}_-$, $\mathrm{ad}(Z)X \in \mathfrak{e}_+$ and these spaces are orthogonal with respect to $\langle -, -\rangle$. As in part (ii) this suffices to conclude that $[X,Y] = 0$, thus $[\mathfrak{e}_+, \mathfrak{e}_-] = 0$. □

It is now natural to define

$$\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+] \quad \text{and} \quad \mathfrak{u}_- := [\mathfrak{e}_-, \mathfrak{e}_-].$$

Lemma 2.62. Retain the above notation. Then $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are orthogonal with respect to $B_\mathfrak{g}$.

Proof. Let $X_+, Y_+ \in \mathfrak{e}_+$ and $X_-, Y_- \in \mathfrak{e}_-$. Using Jacobi's identity, we compute

$$\begin{aligned} B_\mathfrak{g}([X_+,Y_+],[X_-,Y_-]) &= -B_\mathfrak{g}(Y_+,[X_+,[X_-,Y_-]]) = \\ &= B_\mathfrak{g}(Y_+,[Y_-,[X_+,X_-]]) + B_\mathfrak{g}(Y_+,[X_-,[Y_-,X_+]]) = 0 \end{aligned}$$

by Lemma 2.61. □

Given the above lemma, we finally set

$$\mathfrak{u}_0 := \mathfrak{u} \ominus (\mathfrak{u}_+ \oplus \mathfrak{u}_-).$$

Then $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-$ is an orthogonal direct sum and there are the following commutator relations.

Lemma 2.63. Retain the above notation. The the following statements hold.

(i) $\mathfrak{u}_0$, $\mathfrak{u}_-$ and $\mathfrak{u}_+$ are ideals in $\mathfrak{u}$.
(ii) $[\mathfrak{u}_0, \mathfrak{e}_-] = [\mathfrak{u}_0, \mathfrak{e}_+] = 0$.
(iii) $[\mathfrak{u}_-, \mathfrak{e}_0] = [\mathfrak{u}_-, \mathfrak{e}_+] = 0$.
(iv) $[\mathfrak{u}_+, \mathfrak{e}_0] = [\mathfrak{u}_+, \mathfrak{e}_-] = 0$.

Proof. For part (i), recall that $[\mathfrak{u}, \mathfrak{u}_+] = [\mathfrak{u}, [\mathfrak{e}_+, \mathfrak{e}_+]]$. Let $X \in \mathfrak{u}$ and $Y, Z \in \mathfrak{e}_+$. Then

$$[X,[Y,Z]] = -[Z,[X,Y]] - [Y,[Z,X]] \in \mathfrak{u}_+,$$

that is, $[\mathfrak{u}, \mathfrak{u}_+] \subseteq \mathfrak{u}_+$. Similarly, $[\mathfrak{u}, \mathfrak{u}_-] \subseteq \mathfrak{u}_-$. Furthermore, since $\mathfrak{u}_0$ is the orthogonal complement of an ideal in $\mathfrak{u}$, it is an ideal itself.

For the second assertion, we already know that $[\mathfrak{u}_0, \mathfrak{e}_-] \subseteq \mathfrak{e}_-$ since $\mathrm{ad}_\mathfrak{g}(\mathfrak{u})$ preserves $\mathfrak{e}_-$, and that $B_\mathfrak{g}|_{\mathfrak{e}_- \times \mathfrak{e}_-} \ll 0$. Thus, let $X \in \mathfrak{u}_0$ and $Y, Z \in \mathfrak{e}_-$. Then

$$B_\mathfrak{g}([X,Y], Z) = -B_\mathfrak{g}([Y,X], Z]) = B_\mathfrak{g}(X, [Y,Z]) = 0.$$

From negative definiteness of $B_\mathfrak{g}$ on $\mathfrak{e}_-$ we therefore conclude $[X,Y] = 0$. An analogous argument works for $[\mathfrak{u}_0, \mathfrak{e}_+]$.

For part (iii), Jacobi's identity again shows

$$[\mathfrak{u}_-, \mathfrak{e}_0] = [[\mathfrak{e}_-, \mathfrak{e}_-], \mathfrak{e}_0] \subseteq [[\mathfrak{e}_0, \mathfrak{e}_-], \mathfrak{e}_-] = 0$$

and similarly $[\mathfrak{u}_-, \mathfrak{e}_+] = 0$.

Part (iv) can be shown in the same manner. □

The following lemma summarizes several of the preceeding results.

Lemma 2.64. Retain the above notation. Let $\delta, \mu \in \{0, +, -\}$. If $\delta \neq \mu$ then

$$[\mathfrak{e}_\delta, \mathfrak{e}_\mu] = 0, \ [\mathfrak{u}_\delta, \mathfrak{u}_\mu] = 0, \text{ and } [\mathfrak{u}_\delta, \mathfrak{e}_\mu] = 0.$$

Furthermore, $[\mathfrak{u}_\delta, \mathfrak{e}_\delta] \subseteq \mathfrak{e}_\delta$ and $[\mathfrak{u}, \mathfrak{u}_\delta] \subseteq \mathfrak{u}_\delta$.

To complete the proof of Theorem 2.60, we would like to set

$$\mathfrak{g}_0 := \mathfrak{u}_0 \oplus \mathfrak{e}_0, \ \mathfrak{g}_+ := \mathfrak{u}_+ \oplus \mathfrak{e}_+, \ \mathfrak{g}_- := \mathfrak{u}_- \oplus \mathfrak{e}_- .$$

However, if $\mathfrak{e}_0 = 0$ we count $\mathfrak{u}_0$ to $\mathfrak{g}_-$ if $\mathfrak{e}_- \neq 0$ and otherwise to $\mathfrak{g}_+$ since in this case $\mathfrak{e}_+ \neq 0$. (At least one of the $\mathfrak{e}_\mu$ is non-zero since $\Theta \neq \mathrm{Id}$). Then $\mathfrak{g}_0$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are ideals in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$. In addition, $\mathfrak{g}_0$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Θ-invariant. We therefore set $\Theta_\mu := \Theta|_{\mathfrak{g}_\mu}$ for $\mu \in \{0, +, -\}$. It now remains to show that the Killing form of the $\mathfrak{g}_\mu$ ($\mu \in \{0, +, -\}$) coincides with the according restriction of $B_\mathfrak{g}$. This is a general fact: Let $\mathfrak{a} \trianglelefteq \mathfrak{g}$ be an ideal in $\mathfrak{g}$. Then $B_\mathfrak{a} = B_\mathfrak{g}|_{\mathfrak{a} \times \mathfrak{a}}$ (which in general is not true if $\mathfrak{a}$ is only a subalgebra of $\mathfrak{g}$). As to the proof, simply note that if $W \leq V$ are finite-dimensional vector spaces and $T \in \mathrm{End}(V)$ preserves W, then T canonically induces two endomorphisms, $T_W \in \mathrm{End}(W)$ and $T_{V/W} \in \mathrm{End}(V/W)$, which satisfy $\mathrm{tr}\, T = \mathrm{tr}\, T_W + \mathrm{tr}\, T_{V/W}$. In our case, if $\mathfrak{a} \trianglelefteq \mathfrak{g}$, then for $X, Y \in \mathfrak{a}$ we have

$$\begin{aligned} B_\mathfrak{g}(X, Y) &= \mathrm{tr}_\mathfrak{g}(\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)) \\ &= \mathrm{tr}_\mathfrak{a}(\mathrm{ad}_\mathfrak{a}(X)\,\mathrm{ad}_\mathfrak{a}(Y)) + \underbrace{\mathrm{tr}_{\mathfrak{g}/\mathfrak{a}}(\mathrm{ad}_{\mathfrak{g}/\mathfrak{a}}(X)\,\mathrm{ad}_{\mathfrak{g}/\mathfrak{a}}(Y))}_{0} = B_\mathfrak{a}(X, Y) \end{aligned}$$

where the second summand vanishes because $\mathrm{ad}_{\mathfrak{g}/\mathfrak{a}}(X) = 0 \in \mathrm{End}(\mathfrak{g}/\mathfrak{a})$ for all $X \in \mathfrak{a}$ since $\mathfrak{a}$ is an ideal.

To show that $(\mathfrak{g}_\mu, \Theta_\mu)$ ($\mu \in \{0, +, -\}$) is an orthogonal symmetric Lie algbra, it only remains to show that $\mathfrak{u}_\mu$ is compactly embedded. This follows from the following general lemma whose proof is an exercise.

Lemma 2.65. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further, let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Set $\mathfrak{k} := \mathfrak{k}_1 + \mathfrak{k}_2$. Then $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

We now turn to globalizing the Decomposition Theorem 2.60 to obtain a decomposition result for symmetric spaces. Recall that we have introduced the following three categories:

(i) Pointed symmetric spaces (M, o)
(ii) Riemannian symmetric pairs (G, K, σ)
(iii) (Effective) Orthogonal symmetric Lie algebras $(\mathfrak{g}, \Theta)$.

In order to give a decomposition theorem for (pointed) symmetric spaces we first have to see how the notion of effectiveness of orthogonal symmetric Lie algebras travels through these categories. Recall that an orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ is effective if $Z(\mathfrak{g}) \cap \mathfrak{u} = 0$, where $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is the Cartan decomposition of $(\mathfrak{g}, \Theta)$. Now, if (G, K, σ) is a Riemannian symmetric pair and $(\mathfrak{g}, \Theta) = (\mathrm{Lie}(G), D_e\sigma)$ is the associated orthogonal symmetric Lie algebra, then $Z(\mathfrak{g}) \cap \mathfrak{u} = \mathrm{Lie}(Z(G) \cap K)$.

Hence $(\mathfrak{g}, \Theta)$ is effective if and only if $Z(G) \cap K$ has a 0-dimensional Lie algebra. Being closed this is equivalent to $Z(G) \cap K$ being discrete. Accordingly, a Riemannian symmetric pair (G, K, σ) is *effective* if $Z(G) \cap K$ is discrete. The following lemma records that all Riemannian symmetric pairs coming from symmetric spaces are effective.

Lemma 2.66. Let M be a symmetric space, $o \in M$ and (G, K, σ) the associated Riemannian symmetric pair. If $N \trianglelefteq G$ with $N \subseteq K$, then N is trivial. In particular, $Z(G) \cap K = \{e\}$ and hence (G, K, σ) is effective.

Proof. Let $n \in N$. We show that n fixes every point of M. Let $p \in M$, then $p = g(o)$ for some $g \in G$ and hence

$$n(p) = ng(o) = g\underbrace{(g^{-1}ng)}_{\in N \subseteq K}(o) = g(o) = p.$$

Thus $n = \mathrm{id} \in \mathrm{Iso}(M)^\circ = G$. □

Definition 2.67.

(i) An effective Riemannian symmetric pair is of *Euclidean (compact, non-compact) type* if the corresponding orthogonal symmetric Lie algebra is.
(ii) A Riemannian symmetric space is of *Euclidean (compact, non-compact) type* if the corresponding (effective) Riemannian symmetric pair is.

The announced globalization of Theorem 2.60 now reads as follows.

Theorem 2.68. Let M be a simply connected symmetric space. Then $M = M_0 \times M_+ \times M_-$ is the Riemannian product of symmetric spaces of Euclidean, non-compact and compact type respectively.

The assumption that M be simply connected in Theorem 2.68 is necessary. For instance the space $\langle\sigma\rangle\backslash(S^2 \times S^2)$, where $\sigma : (x, y) \mapsto (-x, -y)$, covers and is covered by a product but itself is not a product.

Proof. (Theorem 2.68). Let o be a basepoint in M. Further, let (G, K, σ) be the associated effective Riemannian symmetric pair and $(\mathfrak{g}, \Theta)$ the associated effective orthogonal symmetric Lie algebra. By Theorem 2.60, we obtain a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ such that in particular $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_+ \oplus \mathfrak{k}_-$ and $\Theta_\mu = \Theta|_{\mathfrak{g}_\mu}$ ($\mu \in \{0, +, -\}$).

Now, let G_0, G_+ and G_- be the Lie subgroups of G corresponding to the subalgebras $\mathfrak{g}_0, \mathfrak{g}_+$ and $\mathfrak{g}_-$, and consider the isomorphism of Lie algebras

$$\psi : \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_- \to \mathfrak{g},\ (x, y, z) \mapsto x + y + z.$$

Let $\widetilde{G}_0$, $\widetilde{G}_+$, $\widetilde{G}_-$ and $\widetilde{G}$ be the universal covering groups of G_0, G_+, G_- and G, and let $\Psi : \widetilde{G}_0 \times \widetilde{G}_+ \times \widetilde{G}_- \to \widetilde{G}$ be the isomorphism of Lie groups whose derivative is ψ. Now, let $p : \widetilde{G} \to G$ be the canonical covering homomorphism. Then the map $\widetilde{G}/p^{-1}(K)^\circ \to \widetilde{G}/p^{-1}(K) \cong M$ is a covering map because $p^{-1}(K)^\circ$ is an open subgroup of $p^{-1}(K)$. As in Remark 2.23, we conclude that $p^{-1}(K)$ is connected. Let now K_0, K_+ and K_- be the Lie subgroups of $\widetilde{G}_0$, $\widetilde{G}_+$ and $\widetilde{G}_-$ corresponding to $\mathfrak{k}_0$, $\mathfrak{k}_+$ and $\mathfrak{k}_-$. Since $\psi = D_e\Psi : \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_- \to \mathfrak{g}$ maps $\mathfrak{k}_0 \times \mathfrak{k}_+ \times \mathfrak{k}_-$ isomorphically to $\mathfrak{k}$, we have that Ψ maps $K_0 \times K_+ \times K_-$ bijectively to $p^{-1}(K)$ by connectivity of the latter. Since Ψ is bicontinuous, we deduce that K_0, K_+ and K_- are closed subgroups of $\widetilde{G}_0$, $\widetilde{G}_+$ and $\widetilde{G}_-$ respectively. One verifies that $(\widetilde{G}_0, K_0)$, $(\widetilde{G}_+, K_+)$ and $(\widetilde{G}_-, K_-)$ are Riemannian symmetric pairs of the according type. From the above it is furthermore clear that Ψ induces an equivariant diffeomorphism

$$\overline{\Psi} : \widetilde{G}_0/K_0 \times \widetilde{G}_+/K_+ \times \widetilde{G}_-/K_- \to \widetilde{G}/p^{-1}(K) = M$$

which implies that the Riemannian metrics correspond.

□

Remark 2.69. Let (G, K) be a Riemannian symmetric pair. In general, the cover $\widetilde{G} \to G$ may be infinite, as is the case for e.g. $(G, K) = (\mathrm{SL}(2, \mathbb{R}), \mathrm{SO}(2))$ since $\pi_1(\mathrm{SL}(2, \mathbb{R})) = \pi_1(\mathrm{SO}(2)) \cong \pi_1(S^1) \cong \mathbb{Z}$,

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow \{e\}$$

$$\mathbb{Z} \longrightarrow p^{-1}(K) \longrightarrow K$$

as can be seen from the Iwasawa decomposition. This motivates the funny-looking assumption that $\mathrm{Ad}_G(K)$ rather than K be compact in Definition 2.31 of a Riemannian symmetric pair. We will see later that if G/K is of non-compact type, then G has the same homotopy type as K.

2.8.3. *Irreducible Symmetric Spaces.* We have obtained a decomposition theorem for symmetric spaces. It remains the natural question whether there are "smallest pieces" into which such a space can be decomposed and which do not admit any further decomposition. For instance, the symmetric space $\mathbb{E}^n$ admits many ways to decompose it further into a Riemannian product. For a general symmetric space, there is the notion of irreducibility so that a general such space is a canonical product of irreducible ones.

Definition 2.70. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $(\mathfrak{g}, \theta)$ is *irreducible* if

(i) $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$, and
(ii) the Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

Definition 2.71. A Riemannian symmetric pair (G, K, σ) is *irreducible* if the associated orthogonal symmetric Lie algebra is. A symmetric space M is *irreducible* if the associated Riemannian symmetric pair is.

Before we state a decomposition theorem regarding irreducible orthogonal symmetric Lie algebras, here are some remarks.

Remark 2.72.

(i) Any irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ is effective since $Z(\mathfrak{g}) \cap \mathfrak{k}$ is an ideal in $\mathfrak{k}$ and hence vanishes.
(ii) Let (G, K, σ) be an irreducible Riemannian symmetric pair. Then there is up to scaling a unique G-invariant Riemannian metric on $M := G/K$: To show this, it suffices to see that there is up to scaling a unique $\mathrm{Ad}_G(K)$-invariant scalar product on $\mathfrak{p}$ where $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, see the proof of Theorem 2.32. Since (G, K, σ) is irreducible, $\mathrm{Ad}_G : K \to \mathrm{GL}(\mathfrak{p})$ is an irreducible representation. As in the proof of Theorem 2.60, let $\langle -, - \rangle$ be an $\mathrm{Ad}_G(K)$-invariant scalar product on $\mathfrak{p}$. Then there is a symmetric endomorphism $A \in \mathrm{End}(\mathfrak{p})$ such that $B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle$ for all $X, Y \in \mathfrak{p}$. Hence A commutes with $\mathrm{Ad}_G(K)$ and therefore has a unique eigenvalue given that $\mathrm{Ad}_G(K)$ acts irreducibly (Schur's Lemma); $A = \lambda \mathrm{Id}$. Therefore, any two $\mathrm{Ad}_G(K)$-invariant scalar products on $\mathfrak{p}$ are scalar multiples of each other; and we may take $-B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ respectively $B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ in the compact and non-compact case as such a product.

Theorem 2.73. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. Then there are ideals $(\mathfrak{g}_i)_{i \in I}$ of $\mathfrak{g}$ such that

(i) $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$,
(ii) the $(\mathfrak{g}_i)_{i\in I}$ are pairwise orthogonal with respect to $B_{\mathfrak{g}}$ and θ-invariant, and
(iii) $(\mathfrak{g}_i, \theta|_{\mathfrak{g}_i})$ is an irreducible orthogonal symmetric Lie algebra.

As before, this decomposition passes from effective orthogonal symmetric Lie algebras to symmetric spaces, see [Hel79, Prop. VIII.5.5].

2.9. **Curvature of symmetric spaces.** In this section, we will see that the distinction of symmetric spaces into the three types has a fundamental geometric meaning.

Theorem 2.74. Let (G, K, σ) be a Riemannian symmetric pair and let $M := G/K$ be the associated symmetric space (equipped with an arbitrary G-invariant Riemannian metric). Then the following statements hold.

(i) If (G, K, σ) is of compact type, then M has sectional curvature everywhere bigger than or equal to zero.
(ii) If (G, K, σ) is of non-compact type, then M has sectional curvature everywhere less than or equal to zero.
(iii) If (G, K, σ) is of Euclidean type, then M has sectional curvature everywhere equal to zero.

First of all, we recall the *Riemannian curvature tensor* and *sectional curvature.* Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The Riemannian curvature operator is the multilinear mapping $R : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ given by

$$(X, Y, Z) \mapsto R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Using the Leibniz rule, one verifies that $(R(X, Y)Z)_p$ only depends on the values X_p, Y_p and Z_p and hence is a tensor invariant of the Riemannian metric. We refer to [Ber03, Sec. 4.4] for a discussion of various aspects of the Riemannian curvature tensor.

Another curvature notion, which is "equivalent" to the knowledge of Riemannian curvature is the sectional curvature $(\sigma_p)_{p\in M}$ defined for all $p \in M$ on the Grassmannian 2-planes in T_pM as follows:

$$\sigma_p : \mathrm{Gr}_2(\mathrm{T}_pM) \to \mathbb{R},\ P \mapsto \langle R(u, v)u, v\rangle_{g_p},$$

where (u, v) is an orthonormal basis of $P \leq \mathrm{T}_pM$.

Note that if $\dim M = 2$, then sectional curvature may be viewed as a function on M and as such equals Gaussian curvature. Recall in this context, that $\mathbb{S}^2$, $\mathbb{H}^2$ and $\mathbb{E}^2$ are the unique simply connected Riemannian symmetric spaces of dimension two of constant sectional curvature one, minus one and zero respectively.

To illustrate the significance of sectional curvature, we recall a theorem essentially due to Hadamard which will be essential for the study of the metric properties of Riemannian symmetric spaces of non-compact type.

Theorem 2.75. Let M be a complete Riemannian manifold of sectional curvature everywhere less than or equal to zero. Further, let $p \in M$, $v \in \mathrm{T}_pM$ and identify $\mathrm{T}_v(\mathrm{T}_pM)$ with T_pM. Then

$$D_v \exp_p : \mathrm{T}_pM \to \mathrm{T}_{\exp_p v}M$$

is norm-increasing in the sense that $\|D_v \exp_p(\xi)\| \geq \|\xi\|$ for all $\xi \in \mathrm{T}_pM$. This has in particular the following consequences:

(i) $\forall\sigma : [0, 1] \to \mathrm{T}_pM$: $\mathrm{length}(\sigma) \leq \mathrm{length}(\exp_p \circ\sigma)$.
(ii) $\exp_p : \mathrm{T}_pM \to M$ is a covering map.

(iii) If M is simply connected, $\exp_p : \mathrm{T}_pM \to M$ is a diffeomorphism and

$$d(\exp_p v, \exp_p w) \geq \|v - w\|$$

for all $v, w \in \mathrm{T}_pM$.

The proof of Theorem 2.75 is based on a non-trivial computation which we nevertheless skip. Instead we turn to proving Theorem 2.74 by establishing the following simple formula to compute the Riemannian curvature tensor of a symmetric space.

Theorem 2.76. Let (G, K, σ) be a Riemannian symmetric pair and R the Riemannian curvature tensor of the associated (pointed) symmetric space (M, o) with respect to any G-invariant Riemannian metric. Further let $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. With the usual isomorphism $D_e\pi : \mathfrak{p} \to \mathrm{T}_oM$ we then have

$$R_o(\overline{X}, \overline{Y})\overline{Z} = \overline{[[X, Y], Z]}$$

where $\overline{T} := D_e\pi(T)$ for $T \in \mathfrak{p}$.

There are different proofs of this theorem; one at [Hel79, Thm. IV.4.2] using Theorem 2.39, and another in [Jos05] using the theory of Jacobi fields. Theorem 2.74 now follows rather easily.

Proof. (Theorem 2.74). Let $(\mathfrak{g}, \theta)$ be the orthogonal symmetric Lie algbera with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to (G, K). To compute sectional curvatures of $M = G/K$, we compute for $X_1, X_2 \in \mathfrak{p}$:

$$B_{\mathfrak{g}}([[X_1, X_2], X_1], X_2) = -B_{\mathfrak{g}}([X_1, [X_1, X_2]], X_2) = B_{\mathfrak{g}}([X_1, X_2], [X_1, X_2]).$$

By Remark 2.58, we may take $-B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ as the Riemannian metric of M at $o \in M$ if (G, K) is of compact type, and $B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ if (G, K) is of non-compact type. Thus, if $\overline{X}_1, \overline{X}_2$ are orthonormal in T_oM, we compute

$$\sigma_o(\langle \overline{X}_1, \overline{X}_2 \rangle) = \|\overline{[X_1, X_2]}\|^2 \quad \text{respectively} \quad \sigma_o(\langle \overline{X}_1, \overline{X}_2 \rangle) = -\|\overline{[X_1, X_2]}\|^2$$

if (G, K) is of compact respectively non-compact type. This proves the assertion. □

These simple formulas will be used later on to study totally geodesic flat subspaces in symmetric spaces $M = G/K$.

2.10. **Semisimple Lie Groups: Basics.** We have seen that the theory of Riemannian symmetric spaces leads to the study of a class of objects called *semisimple Lie algebras*. The corresponding Lie groups are then closely related to isometry groups of Riemannian symmetric spaces. In this short introductory section we will establish an important global property of those with negative definite Killing form.

Definition 2.77. A real Lie group G is *semisimple* if its Lie algebra $\mathrm{Lie}(G)$ is.

Besides their importance in the theory of Riemannian symmetric spaces, semisimple Lie groups, together with the solvable ones, are the basic building blocks of general real Lie groups and afford a theory of their own. For the sake of culture, we will just indicate one decomposition theorem which explains the name "semisimple".

Definition 2.78. A Lie algebra is *simple* if it is not abelian and contains no proper, non-zero ideals.

Theorem 2.79. A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a direct sum of simple ideals.

The proof of the only if direction of theorem 2.79 proceeds by taking an ideal of positive dimension, showing that $B_{\mathfrak{g}}$ restricted to it is non-degenerate and then proceeding with the orthogonal complement.

Now, let G be a real connected Lie group with Lie algebra $\mathfrak{g}$. Consider its adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}),\ g \mapsto D_e\mathrm{int}(g)$. Clearly, $\mathrm{Ad}(G)$ is contained in the closed subgroup

$$\begin{aligned}\mathrm{Aut}(\mathfrak{g}) &:= \{g \in \mathrm{GL}(\mathfrak{g}) \mid g \text{ is an automorphism of } \mathfrak{g}\} \\ &= \{g \in \mathrm{GL}(\mathfrak{g}) \mid g[X,Y] = [gX, gY]\ \forall X, Y \in \mathfrak{g}\}.\end{aligned}$$

of $\mathrm{GL}(\mathfrak{g})$. The first global result that we establish about semisimple Lie groups is the following.

Theorem 2.80. Let G be a connected, semisimple Lie group. Then $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^\circ$. In particular, $\mathrm{Ad}(G)$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$.

The proof of Theorem 2.80 will use the following purely algebraic result.

Proposition 2.81. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every derivation $D \in \mathrm{Der}(\mathfrak{g})$ is inner, i.e. there is $X \in \mathfrak{g}$ such that $D = \mathrm{ad}(X)$.

Proof. Consider the inclusion of spaces

$$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g}).$$

On $\mathrm{End}(\mathfrak{g})$, we have the *non-degenerate* symmetric bilinear form $(A, B) \mapsto \mathrm{tr}(AB)$. Its restriction to $\mathrm{ad}(\mathfrak{g})$ is the Killing form $(X,Y) \mapsto B_{\mathfrak{g}}(X,Y) = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ which is non-degenerate since $\mathfrak{g}$ is assumed to be semisimple. In order to show that $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$ it therefore suffices to show that if $D \in \mathrm{Der}(\mathfrak{g})$ is orthogonal to $\mathrm{ad}(\mathfrak{g})$, then it vanishes. Indeed, we have for all $X, Y \in \mathfrak{g}$:

$$\begin{aligned}B_{\mathfrak{g}}(DX, Y) &= \mathrm{tr}(\mathrm{ad}(DX)\,\mathrm{ad}(Y)) \\ &= \mathrm{tr}(D\,\mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(X) D\,\mathrm{ad}(Y)) \qquad (D \text{ derivation}) \\ &= \mathrm{tr}(D\,\mathrm{ad}(X)\,\mathrm{ad}(Y) - D\,\mathrm{ad}(Y)\,\mathrm{ad}(X)) \\ &= \mathrm{tr}(D[\mathrm{ad}(X), \mathrm{ad}(Y)]) = \mathrm{tr}(D\,\mathrm{ad}([X,Y])) = 0\end{aligned}$$

since by assumption, D is orthogonal to $\mathrm{ad}(\mathfrak{g})$. Overall, $B_{\mathfrak{g}}(DX, Y) = 0$ for all $X, Y \in \mathfrak{g}$ which implies $D = 0$ by the opening remarks. □

The proof of Theorem 2.80 now follows with the help of the following Lemma which is left as an exercise.

Lemma 2.82. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathrm{Aut}(\mathfrak{g})$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ and its Lie algebra coincides with $\mathrm{Der}(\mathfrak{g})$.

Proof. (Theorem 2.80). Let G be a connected semisimple Lie group. Then $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathfrak{g})$ as we have noted before, and $\mathrm{Lie}(\mathrm{Ad}(G)) = \mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ by Propositon 2.81 and hence $\mathrm{Lie}(\mathrm{Ad}(G)) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ which proves the assertion. □

We now turn to the main result of this section which is the following.

Theorem 2.83. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that the Killing form $B_{\mathfrak{g}}$ is negative definite. Then G is compact.

Corollary 2.84. Let (G, K) be a Riemannian symmetric pair of compact type. Then G is compact and hence so is $M := G/K$.

Note that symmetric spaces of non-compact type are non-compact by Hadamard's Theorem 2.75.

Proof. (Theorem 2.83). Since $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ preserving a negative definite symmetric bilinear form form, it is compact. Thus $\mathrm{Aut}(\mathfrak{g})^\circ$ is compact as well. Since G is semisimple, it follows from Theorem 2.80 that $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^\circ$ is compact as well. Consider now the exact sequence of topological groups

$$e \to \ker \mathrm{Ad} = Z(G) \to G \to \mathrm{Ad}(G) \to e.$$

Since $\mathfrak{g}$ is semisimple, $Z(\mathfrak{g}) = 0$ and hence $Z(G)$, being closed, has to be discrete. Set $U := \mathrm{Ad}(G)$. Then G is a connected covering group of U and therefore is a quotient the universal covering group $\widetilde{U}$ of U.

We claim that if U is a compact connected Lie group with negative definite Killing form, then $\pi_1(U)$ is finite. In this case, $\widetilde{U}$ is compact as a finite extension of U. Hence also G is compact being covered by $\widetilde{U}$.

Now for the claim: We know that $\pi_1(U)$ is abelian (being a discrete and normal, hence central subgroup of $\widetilde{U}$). By Hurewicz' theorem,

$$\pi_1(U)_{\mathrm{ab}} = \mathrm{H}_1(U, \mathbb{Z})$$

and hence $\pi_1(U) \cong \mathrm{H}_1(U, \mathbb{Z})$. Since U is a compact manifold, $\mathrm{H}_1(U, \mathbb{Z})$ is finitely generated and hence, being an abelian group, isomorphic to $\mathbb{Z}^l \oplus F$ for some $l \in \mathbb{N}$ and a finite abelian group F. By the universal coefficient theorem, we thus have

$$\mathrm{H}^1(U, \mathbb{R}) = \hom(\mathrm{H}_1(U, \mathbb{Z}), \mathbb{R}) \cong \mathbb{R}^l \,.$$

Now we employ de Rham's theorem $\mathrm{H}^1(U, \mathbb{R}) \cong \mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R})$ and proceed to show that $\mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R}) = 0$ in which case $l = 0$ and we are done. Recall that

$$\mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R}) = \{\text{closed 1-forms on } U\}/\{\text{exact 1-forms on } U\}.$$

Let ω be a 1-form with $d\omega = 0$. Now, the fundamental observation to make is that $L_g^* R_h^* \omega - \omega$ is exact for all $g, h \in U$ where L_g and R_h denote left- and right multiplication in U with the respective element. This can be seen using the connectedness of U to construct a primitive directly, or by observing that since L_g and R_h are homotopic to id_U, they induce the identity in $\mathrm{H}_1(U, \mathbb{Z})$ and hence in $\mathrm{H}^1(U, \mathbb{R}) \cong \mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R})$ by duality. Thus ω and

$$\alpha := \int_U \int_U L_g^* R_h^* \omega \;\mu(g)\;\mu(h)$$

represent the same element in $\mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R})$. But now α is a bi-invariant 1-form on U. In particular, $\alpha_e : U = \mathrm{T}_e U \to \mathbb{R}$ is an $\mathrm{Ad}(U)$-invariant linear form. Let $a \in \mathfrak{u}$ be the vector determined by $\alpha_e(x) = B_{\mathfrak{u}}(a, x)$ for all $x \in \mathfrak{u}$. Then a is $\mathrm{Ad}(U)$-fixed and hence $\mathbb{R}\, a \subseteq Z(\mathfrak{u})$. Thus $a = 0$ by semisimplicity and hence $\alpha_e \equiv 0$. By invariance, we therefore have $\alpha = 0 \in \mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R})$. Hence also $\omega = 0 \in \mathrm{H}^1_{\mathrm{dR}}(U, \mathbb{R})$. □

Remark. There are many different proofs of Theorem 2.83, none of which is easy.

2.11. **Duality Theory.** There is a remarkable duality between symmetric spaces of compact and of non-compact type which we outline in this section.

Let $\mathfrak{g}$ be a real Lie algebra. On the $\mathbb{R}$-vector space $\mathfrak{g} \times \mathfrak{g}$ define an $\mathbb{R}$-linear map by $J : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $(X, Y) \mapsto (-Y, X)$. It allows us to equip $\mathfrak{g} \times \mathfrak{g}$ with the structure of a $\mathbb{C}$-vector space, denoted $\mathfrak{g}_{\mathbb{C}}$, as follows: Given $z = x + iy \in \mathbb{C}$ and $Z = (X, Y) \in \mathfrak{g} \times \mathfrak{g}$ define $z \cdot Z := (xX, xY) + J(yX, yY) = (xX - yX, xY + yX)$. One checks that the $\mathbb{C}$-vector space $\mathfrak{g}_{\mathbb{C}}$ is endowed with the structure of a complex Lie algebra via the bracket

$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2] - [Y_1, Y_2], [Y_1, X_2] + [X_1, Y_2]).$$

The map $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$, $X \mapsto (X, 0)$ identifies $\mathfrak{g}$ with an $\mathbb{R}$-subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Under this identification, we have $i\,\mathfrak{g} = \{(0, Y) \mid Y \in \mathfrak{g}\}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\,\mathfrak{g}$, that is, every element Z of $\mathfrak{g}_{\mathbb{C}}$ can be written uniquely in the form $X + iY$ where $X, Y \in \mathfrak{g}$. We now have

$$[Z_1, Z_2] = [X_1 + iY_1, X_2 + iY_2] = [X_1, X_2] - [Y_1, Y_2] + i([Y_1, X_2] + [X_1, Y_2])$$

as above. We remark that $\mathfrak{g}_{\mathbb{C}}$ need not be an orthogonal symmetric Lie algebra if $\mathfrak{g}$ is; however, we always have the map

$$\zeta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}},\ X + iY \mapsto X - iY$$

which is an $\mathbb{R}$-linear involutory automorphism of the real Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Definition 2.85. Let $\mathfrak{h}$ be a complex Lie algebra. A *real form* of $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{h}$ as a complex Lie algebra.

Of course, a real Lie algebra is a real form of its complexification but in general, there may be many pairwise non-isomorphic real forms. This corresponds to the fact that all non-degenerate symmetric bilinear forms are equivalent over $\mathbb{C}$ but may not be so over $\mathbb{R}$.

Now, let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there are the following immediate bracketing relations between the real subspaces $\mathfrak{k}, i\,\mathfrak{k}, \mathfrak{p}$ and $i\,\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$.

(i) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$
(ii) $[\mathfrak{k}, i\,\mathfrak{p}] = i[\mathfrak{k}, \mathfrak{p}] \subseteq i\,\mathfrak{p}$
(iii) $[i\,\mathfrak{p}, i\,\mathfrak{p}] = -[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$

In particular, $\mathfrak{g}^* := \mathfrak{k} \oplus i\,\mathfrak{p}$ is an $\mathbb{R}$-Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In addition, ζ preserves $\mathfrak{g}^*$. Then $\zeta|_{\mathfrak{g}^*}$ is an involutory automorphism of $\mathfrak{g}^*$ and $\mathfrak{g}^* = \mathfrak{k} \oplus i\,\mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}^*$ with respect to $\zeta|_{\mathfrak{g}^*}$.

Lemma 2.86. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. Then so is $(\mathfrak{g}^*, \zeta|_{\mathfrak{g}^*})$.

As to the proof, it remains to show that $\mathfrak{k} \leq \mathfrak{g}^*$ is compactly embedded which we leave as an exercise; it boils down to the fact the continuous maps preserve compacta.

Definition 2.87. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. The *dual orthogonal symmetric Lie algebra* of $(\mathfrak{g}, \theta)$ is $(\mathfrak{g}^*, \zeta|_{\mathfrak{g}^*})$.

In the following, we shall write $(\mathfrak{g}^*, \zeta)$ instead of $(\mathfrak{g}^*, \zeta|_{\mathfrak{g}^*})$. It is an exercise to show that the bidual of $(\mathfrak{g}, \theta)$ is isomorphic to $(\mathfrak{g}, \theta)$.

Proposition 2.88.

(i) Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. Then $(\mathfrak{g}^*, \zeta)$ is of compact (non-compact) type if and only if $(\mathfrak{g}, \theta)$ is of non-compact (compact) type.
(ii) Let $(\mathfrak{g}_1, \theta_1)$ and $(\mathfrak{g}_2, \theta_2)$ be orthogonal symmetric Lie algebras. Then $(\mathfrak{g}_1, \theta_1)$ and $(\mathfrak{g}_2, \theta_2)$ are isomorphic if and only if $(\mathfrak{g}_1^*, \zeta_1)$ and $(\mathfrak{g}_2^*, \zeta_2)$ are.

Proof. We only prove the first assertion. To this end, we establish a relation between the respective Killing forms: Note that there is the isomorphism of real vector spaces $\Psi : \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g} \to \mathfrak{k} \oplus i\,\mathfrak{p} = \mathfrak{g}^*$ given by $X + Y \mapsto X + iY$. Let $Z_1, Z_2 \in \mathfrak{p}$ and compute

$$\begin{aligned}
\mathrm{ad}_{\mathfrak{g}^*}(iZ_1)\,\mathrm{ad}_{\mathfrak{g}^*}(iZ_2)(X + iY) &= [iZ_1, [iZ_2, X + iY]] \\
&= -[Z_1, [Z_2, X]] - i[Z_1, [Z_2, Y]] \\
&= -\Psi([Z_1, [Z_2, X + Y]]) \\
&= -\Psi(\mathrm{ad}_{\mathfrak{g}}(Z_1)\,\mathrm{ad}_{\mathfrak{g}}(Z_2)(X + Y)).
\end{aligned}$$

That is, $\mathrm{ad}_{\mathfrak{g}^*}(iZ_1)\,\mathrm{ad}_{\mathfrak{g}^*}(iZ_2)\Psi = -\Psi\,\mathrm{ad}_{\mathfrak{g}}(Z_1)\,\mathrm{ad}_{\mathfrak{g}}(Z_2)$. Therefore we have the equality $B_{\mathfrak{g}^*}(iZ_1, iZ_2) = -B_{\mathfrak{g}}(Z_1, Z_2)$ which proves the assertion. □

Example 2.89. Consider the orthogonal symmetric Lie algebra $(\mathfrak{sl}(n,\mathbb{R}),\theta)$ of non-compact type where $\theta : X \mapsto -X^T$. Its Cartan decomposition is given by

$$\mathfrak{k} = \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X^T + X = 0\},\ \mathfrak{p} = \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X = X^T\}.$$

Then $\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$ and

$$\begin{aligned}\mathfrak{k} + i\,\mathfrak{p} &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) \mid Z = X + iY,\ X^T + X = 0,\ Y^T = Y\}\\ &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) \mid Z + \overline{Z}^T = 0\}\\ &= \mathfrak{su}(n).\end{aligned}$$

In this way, the symmetric spaces $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $\mathrm{SU}(n)/\mathrm{SO}(n)$ are dual to each other. For $n = 2$, we obtain that $\mathbb{H}^2$, "a sphere of radius i", is dual to $\mathbb{S}^2$ since $\mathrm{SU}(2)$ is the universal cover of $\mathrm{SO}(3)$.

Example 2.90. Whereas in the non-compact type case, the involution of an orthogonal symmetric Lie algebra is unique, there may be many in the compact type case. For instance, let

$$\mathfrak{g} := \mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^T + X = 0\}.$$

Then for p, q such that $p + q = n$, define

$$\theta_{p,q} : \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R}),\ X \mapsto I_{p,q} X I_{p,q} \quad \text{where} \quad I_{p,q} = \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix}.$$

Then $\theta_{p,q}$ is an involutive automorphism of $\mathfrak{gl}(n,\mathbb{R})$ preserving $\mathfrak{so}(n)$. One shows

$$(\mathfrak{so}(n), \theta_{p,q})^* = (\mathfrak{so}(p,q), \theta_{p,q}).$$

In particular, the spaces $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n = \mathrm{SO}(n,1)^\circ/\mathrm{SO}(n)$ are dual.

3. Symmetric Spaces of Non-Compact Type

We now study the geometry of symmetric spaces of non-compact type.

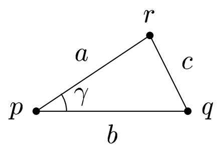
3.1. **Cartan's Fixed Point Theorem.**

To begin with, we collect certain geometric facts about Cartan-Hadamard manifolds.

Definition 3.1. A connected, simply connected, complete Riemannian manifold of non-positive sectional curvature is a *Cartan-Hadamard* manifold.

Observe that on a Cartan-Hadamard manifold M, there is a unique geodesic segment between a pair of points: Since the exponential map increases distance by Theorem 2.75 there cannot be different starting vectors. Thus, for $p, q, o \in M$ where $o \notin \{p, q\}$, we may define the angle $\sphericalangle_o(p,q) := \sphericalangle_{\mathrm{T}_o M}\{u, v\}$ where $u, v \in \mathrm{T}_o(M)$ are such that $\mathrm{Exp}_o(u) = p$ and $\mathrm{Exp}_o(v) = q$.

Proposition 3.2. Let M be a Cartan-Hadamard manifold. Consider the following geodesic triangle inside M.



Then $c^2 \geq a^2 + b^2 - 2ab\cos\gamma$.

Recall that in the Euclidean case we have equality in Proposition 3.2.

Proof. Consider the exponential map $\exp_p : \mathrm{T}_pM \to M$ and let $u, v \in \mathrm{T}_pM$ with $\exp_p(u) = r$ and $\exp_p(v) = q$. Then $\|u\| = a$ and $\|v\| = b$ as well as $\sphericalangle_0(u,v) = \gamma$. Thus, by Theorem 2.75 we have

$$\begin{aligned} c^2 = d(\exp_p(u), \exp_p(v)) &\geq \|u - v\|^2 \\ &= \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle = a^2 + b^2 - 2ab\cos\gamma. \end{aligned}$$

□

Proposition 3.2 can be utilized to show the following general fact about non-positive curvature.

Proposition 3.3. Let M be a Cartan-Hadamard manifold and let $S \subseteq M$ be a bounded subset. Then there exists a unique closed ball $B(z, r)$, $z \in M$ of minimal radius containing S.

Proof. Since S is bounded, the set $E = \{(x, r) \in M \times \mathbb{R}_{\geq 0} \mid B_{\leq}(x, r) \supseteq S\}$ is non-empty. Set $R := \inf\{r \mid \exists x \in M : (x, r) \in E\}$. Take any sequence $(x_n, r_n) \in E$ such that $r_n \to R$. We claim that in this case $(x_n)_n$ is a Cauchy sequence which readily implies the asserted uniqueness (think of a sequence with $(x_n)_n$ alternating between different centers). Now, let $\varepsilon > 0$ and $N = N(\varepsilon)$ such that $r_n < R + \varepsilon$ for all $n \geq N$; pick $n, m \geq N$ and an arbitrary point $p \in S$. Consider the geodesic triangle (p, x_n, x_m) and let q be the midpoint of the segment $\overline{x_n x_m}$. Then $\sphericalangle_q(p, x_m) + \sphericalangle_q(p, x_n) = \pi$. Without loss of generality, we may assume that $\sphericalangle_q(p, x_n) \geq \pi/2$. Then $\cos\sphericalangle_q(p, x_n) \leq 0$. By Proposition 3.2 we thus have

$$(R + \varepsilon)^2 \geq r_n^2 \geq d(x_n, p)^2 \geq \frac{d(x_n, x_m)^2}{4} + d(p, q)^2.$$

Observe that $\sup_{p \in S} d(q, p) \geq R$ by definition of R. Therefore,

$$(R + \varepsilon)^2 \geq \frac{d(x_n, x_m)^2}{4} + R^2 \quad \Rightarrow \quad 4(2R\varepsilon + \varepsilon^2) \geq d(x_n, x_m)^2$$

In particular, we conclude that the sequence $(x_n)_n$ is Cauchy. Let $z = \lim_n x_n$. Then for all $n \in \mathbb{N}$ we have

$$S \subseteq B(x_n, r_n) \subseteq B(z, r_n + d(z, x_n))$$

which implies that $S \subseteq B(z, R)$. □

Theorem 3.4. Let M be a symmetric space of non-compact type and let $\mathrm{Iso}(M)^\circ \leq G \leq \mathrm{Iso}(M)$. Then the following statements hold.

(i) $K := \mathrm{stab}_G(p)$ meets every connected component of G.
(ii) Any compact subgroup of G fixes a point.
(iii) The set of all maximal compact subgroups of G is $\{\mathrm{stab}_G(p) \mid p \in M\}$.

Part (i) of Theorem 3.4 says that K is as disconnected as G is; part (iii) implies that we can recover M from G as opposed to the compact-type case. Also, observe that any two maximal compact subgroups are conjugate.

Proof. We only sketch the proofs.

(i) (Idea). The map $G/K^\circ \to G/K = M$ is a Galois covering of M with Galois group K/K° acting from the right. Since M is simply connected, G/K° is diffeomorphic to $M \times K/K^\circ$.
(ii) Let K be a compact subgroup of G and let $p \in M$. Then $C = Kp \subseteq M$ is a compact K-invariant subset of M. Its circumcenter $z \in M$ is fixed by K.
(iii) This is immediate from part (ii).

□

3.2. **Flats and Rank.** In this section, we study the flat subspaces of a symmetric space. They organize themselves into a combinatorial object called root system which together with its associated Weyl group is used to classify symmetric spaces.

Definition 3.5. A Riemannian manifold is *flat* if its sectional curvature vanishes identically.

This is equivalent to saying that the Riemannian curvature tensor vanishes. Observe that if F is a flat, complete Riemannian manifold, then given $p \in F$, the map $\exp_p : \mathrm{T}_pF \to F$ is a covering and a local isometry. In particular, if F is simply connected, it is isometric to the Euclidean space T_pM via $\exp_p$.

Definition 3.6. Let M be a symmetric space. The *rank* of M, denoted $\mathrm{rk}(M)$, is the maximal dimension of a flat totally geodesic submanifold of M.

We are now going to concentrate on the non-compact type case.

Remark 3.7. Let M be a Riemannian manifold. If $F \subseteq M$ is a totally geodesic submanifold then the Riemannian objects of F are the restriction from M. This applies in particular to the Riemannian connection, the sectional curvature and the exponential map. It is far from true for submanifolds which are not totally geodesic: Consider for instance $\mathbb{S}^2 \subset \mathbb{R}^3$.

Note that if M is a symmetric space then $\mathrm{rk}(M) \geq 1$ since geodesics are flats. In fact, a symmetric space of non-compact type has rank one if and only if its sectional curvature is strictly negative. The full list of rank one symmetric spaces of non-compact type goes as follows:

Space	Symmetric Pair	Remark
$\mathbb{H}^n_{\mathbb{R}},\ n \geq 1$	$(\mathrm{SO}(n,1), \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(1))$	constant sectional curvature
$\mathbb{H}^n_{\mathbb{C}},\ n \geq 1$	$(\mathrm{SU}(n,1), \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$	non-constant sectional curvature
$\mathbb{H}^n_{\mathbb{K}},\ n \geq 1$		non-constant sectional curvature
$\mathbb{H}^2_{\mathbb{O}}$	$G = F_{4(-20)}$	non-constant sectional curvature

Here, $F_{4(-20)}$ is an exceptional Lie group. Note that, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{K}$ are exactly the division algebras over $\mathbb{R}$. They are related to the group structures on $S^0 = \{\pm 1\}$, $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ and $S^3 \cong \{x \in \mathbb{K} \mid \|x\| = 1\}$. If an associativity condition is dropped, one also obtains $\mathbb{O}$. In this case, S^7 is an H-group, i.e. it satisfies the group axioms up to homotopy.

Lemma 3.8. Let (G, K) be a symmetric pair of non-compact type, $M = G/K$, $o \in M$, $\mathrm{Lie}(G) =: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathrm{Exp} : \mathfrak{p} \to M,\ X \mapsto (\exp_G(X)).o$. There is a one-to-one correspondence between flat submanifolds of M through $o \in M$ and abelian subspaces of $\mathfrak{g}$ contained in $\mathfrak{p}$.

An *abelian subspace* is one whose Lie bracket with itself vanishes.

Proof. By Theorem 2.47 there is a one-to-one correspondence between Lie triple systems in $\mathfrak{p}$ and totally geodesic submanifolds of M going through $o \in M$. Now, given an orthonormal frame (X_1, X_2) in $\mathfrak{p}$ we have $\sigma_o(\langle \overline{X}_1, \overline{X}_2 \rangle) = B_{\mathfrak{g}}([X_1, X_2], [X_1, X_2])$ which vanishes if and only if $[X_1, X_2]$ since $B_{\mathfrak{g}}$ is (negative) definite. □

Example 3.9. In this example we compute the rank of several symmetric spaces.

(i) Let $M = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$. Then $\mathfrak{p} = \{X \in M_{n,n}(\mathbb{R}) \mid \mathrm{tr}\, X = 0,\ X = X^T\}$. In this case,

$$\mathfrak{a} := \{X \in \mathfrak{p} \mid X \text{ is diagonal}\}$$

is a maximal abelian subspace of dimension $n-1$: The computation

$$\left[\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, X\right] = (t_j - t_i)X_{ij}$$

shows that $\mathfrak{a}$ is maximal for inclusion. Furthermore, every abelian subspace of $\mathfrak{p}$ can be conjugated into $\mathfrak{a}$ via K, hence $\mathfrak{a}$ is also maximal in terms of dimension. Thus $\mathrm{rk}(M) = n-1$.

Similar arguments in linear algebra show the following.

(ii) $\mathrm{rk}(\mathrm{SO}(p,q)/\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))) = \min(p,q)$
(iii) $\mathrm{rk}(\mathrm{Sp}(2n)/\mathrm{U}(n)) = n$.

We now move towards the existence and "uniqueness" of maximal flat subspaces. Let $(\mathfrak{g},\theta)$ be an orthogonal symmetric Lie algebra (of compact or non-compact type). Recall that the centralizer of $X \in \mathfrak{g}$ is $\mathrm{Z}_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y,X] = 0\} \leq \mathfrak{g}$. For $X \in \mathfrak{p}$, the involution θ stabilizes $\mathrm{Z}_{\mathfrak{g}}(X)$, hence $\mathrm{Z}_{\mathfrak{g}}(X) = (\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{k})\oplus(\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p})$. Now, let $\mathfrak{a}\subseteq\mathfrak{p}$ be an abelian subspace and let $X\in\mathfrak{a}$. Then $\mathfrak{a}\subseteq\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p}$. Thus $\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p}$ is maximal abelian as soon as it is abelian.

Definition 3.10. Let $(\mathfrak{g},\theta)$ be an orthogonal symmetric Lie algebra and let $X\in\mathfrak{p}$. Then X is *regular* if $\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p}$ is abelian.

Theorem 3.11. Let M be a symmetric space of compact or non-compact type with associated Cartan decomposition $\mathrm{Lie}(\mathrm{Iso}(M)^\circ) =: \mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. Then for every maximal abelian subspace $\mathfrak{a}\subseteq\mathfrak{p}$, there exists $X\in\mathfrak{p}$ such that $\mathfrak{a} = \mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p}$. In particular, X is regular.

Proof. We will first prove this for symmetric spaces of compact type and then deduce the result for the non-compact type case using duality.

(i) (Compact type case). Let (U,K,σ) be a symmetric pair of compact type and let $(\mathfrak{u},\theta)$ be the associated orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{u} = \mathfrak{k}\oplus\mathfrak{e}$. Now, let $\mathfrak{a}\subseteq\mathfrak{p}$ be a maximal abelian subspace. Then $\exp\mathfrak{a}\subseteq U$ is a connected abelian subgroup. Then $\overline{\exp\mathfrak{a}} =: A$ is a closed connected abelian subgroup and hence $\mathfrak{u}\supseteq\mathrm{Lie}(A)\supseteq\mathfrak{a}$ is an abelian subalgebra. Since

$$\sigma(\exp X) = \exp(\theta X) = \exp(-X) = \exp(X)^{-1}$$

for all $X\in\mathfrak{a}$, we have $\sigma(a) = a^{-1}$ for all $a\in A$. Thus $\mathrm{Lie}(A)\subseteq\mathfrak{p}$ and hence $\mathrm{Lie}(A) = \mathfrak{a}$ by maximality of $\mathfrak{a}$. Therefore, $\exp(\mathfrak{a})$ is a closed (connected abelian) subgroup of U, namely a torus (Kronecker). Hence there exists $X\in\mathfrak{a}$ such that $\{\exp tX \mid t\in\mathbb{R}\}$ is dense in $\exp\mathfrak{a}$. If now $Y\in\mathrm{Z}_{\mathfrak{u}}(X)$ then $[Y,X]=0$. Hence $\{\exp sY \mid s\in\mathbb{R}\}$ commutes with $\{\exp tX \mid t\in\mathbb{R}\}$ and hence commutes with $\exp\mathfrak{a}$ by density of $\{\exp tX \mid t\in\mathbb{R}\}$. In other words, $[Y,\mathfrak{a}]=0$ and hence $\mathrm{Z}_{\mathfrak{u}}(X) = \mathrm{Z}_{\mathfrak{u}}(\mathfrak{a})$ whence $\mathrm{Z}_{\mathfrak{u}}(X)\cap\mathfrak{p} = \mathrm{Z}_{\mathfrak{u}}(\mathfrak{a})\cap\mathfrak{p} = \mathfrak{a}$.

(ii) (Non-compact type case). Let $(\mathfrak{g},\theta)$ be an orthogonal symmetric Lie algebra of non-compact type with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. Furthermore, let $\mathfrak{g}^* = \mathfrak{k}\oplus i\,\mathfrak{p}$ be its compact dual. For every $\mathfrak{n}\subseteq\mathfrak{p}$ we have $[[i\,\mathfrak{n}, i\,\mathfrak{n}], i\,\mathfrak{n}] = i[[\mathfrak{n},\mathfrak{n}],\mathfrak{n}]$ and $[i\,\mathfrak{n}, i\,\mathfrak{n}] = [\mathfrak{n},\mathfrak{n}]$. Therefore $\mathfrak{n}$ is a Lie triple system (abelian) if and only if $i\,\mathfrak{n}$ is. Also, $i(\mathrm{Z}_{\mathfrak{g}}(X)\cap\mathfrak{p}) = \mathrm{Z}(iX)\cap i\,\mathfrak{p}$ for every $X\in\mathfrak{p}$; that is, X is regular if and only if iX is.

□

Next, we will show that any two maximal abelian subspaces of $\mathfrak{p}$ are $\mathrm{Ad}(K)$-conjugate. To this end, let (G,K) be a symmetric pair of non-comact type with associated Cartan decomposition $\mathrm{Lie}(G) =: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then if $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian with respect to inclusion then so is $\mathrm{Ad}_G(k)(\mathfrak{a}) \subseteq \mathfrak{p}$.

Theorem 3.12. Let (G, K) be a symmetric pair of non-compact type with associated Cartan decomposition $\mathrm{Lie}(G) =: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Further, let $\mathfrak{a}, \mathfrak{a}' \subseteq \mathfrak{p}$ be maximal abelian subspaces. Then there is $k \in K$ such that $\mathrm{Ad}_G(k)(\mathfrak{a}) = \mathfrak{a}'$

In particular, Theorem 3.12 implies that all maximal abelian subspaces have the same (maximal) dimension.

Proof. Let $H, H' \in \mathfrak{p}$ be regular elements for $\mathfrak{a}$ and $\mathfrak{a}'$, i.e. $\mathfrak{a}^{(')} = \mathrm{Z}_{\mathfrak{g}}(H^{(')}) \cap \mathfrak{p}$. Consider the smooth map

$$K \to \mathbb{R},\ k \mapsto B_{\mathfrak{g}}(\mathrm{Ad}(k)H, H').$$

Since K is a compact manifold without boundary, it has a critical point $k_0 \in K$; that is:

$$\forall z \in \mathfrak{k} : \left.\frac{d}{dt}\right|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}(k_0 \exp(tZ)H, H') = 0.$$

Expanding the expression yields

$$\mathfrak{g}(\underbrace{\mathrm{Ad}(k_0)\,\mathrm{ad}(Z)H}_{\mathrm{Ad}(k_0)[Z,H]}, H') = 0 \quad \Rightarrow \quad B_{\mathfrak{g}}(\underbrace{\mathrm{Ad}(k_0)Z}_{\text{generic in } \mathfrak{k}}, [\mathrm{Ad}(k_0)H, H']) = 0$$

for all $Z \in \mathfrak{k}$. Therefore, $[\mathrm{Ad}(k_0)H, H'] = 0$ whence $\mathrm{Ad}(k_0)H \in \mathrm{Z}_{\mathfrak{g}}(H') \cap \mathfrak{p} = \mathfrak{a}'$. This in turn implies that $\mathfrak{a}' \subseteq \mathrm{Z}_{\mathfrak{g}}(\mathrm{Ad}(k_0)H) \cap \mathfrak{p} = \mathrm{Ad}(k_0)(\mathrm{Z}_{\mathfrak{g}}(H) \cap \mathfrak{p}) = \mathrm{Ad}(k_0)\,\mathfrak{a}$. By the same argument, there is some $k_1 \in K$ such that $\mathfrak{a} \subseteq \mathrm{Ad}(k_1)\,\mathfrak{a}'$ and hence $\mathfrak{a}' \subseteq \mathrm{Ad}(k_0)\,\mathfrak{a} \subseteq \mathrm{Ad}(k_0 k_1)\,\mathfrak{a}'$ which implies equality everywhere by the assumption that $\mathfrak{a}'$ is maximal. □

The following is an immediate consequence of Theorem 3.12.

Corollary 3.13. Let $M = G/K$ be a symmetric space of non-compact type. Then any to two maximal flat subspaces F, F' of M are G-congruent, i.e. there is $g \in G$ such that $gF = F'$.

3.3. **Roots and Root Spaces.** Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of non-compact type. On $\mathfrak{g}$ we define a symmetric positive definite bilinear form by $\langle X, Y\rangle := -B_{\mathfrak{g}}(X, \theta(Y))$: Whereas bilinearity is immediate, symmetry follows from the fact that $B_{\mathfrak{g}}$ is invariant under θ (in fact any automorphism of $\mathfrak{g}$). To see positive definiteness, decompose vectors according to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and use that $(\mathfrak{g}, \theta)$ is of non-compact type.

Lemma 3.14. Retain the above notation. For all $X \in \mathfrak{p}$, the map $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathfrak{gl}(\mathfrak{g})$ is symmetric with respect to $\langle -, -\rangle$.

Proof. Let $X \in \mathfrak{p}$ and $Y, Z \in \mathfrak{g}$. Then

$$\begin{aligned} -B_{\mathfrak{g}}([X,Y], \theta(Z)) &= B_{\mathfrak{g}}(Y, [X, \theta(Z)]) \\ &= -B_{\mathfrak{g}}(Y, [\theta(X), \theta(Z)]) = -B_{\mathfrak{g}}(Y, \theta([X, Z])). \end{aligned}$$

□

The crucial observation now is that for an abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, the operators $\{\mathrm{ad}(H) \mid H \in \mathfrak{a}\}$, which are symmetric with respect to $\langle -, -\rangle$, commute and hence are simultaneously diagonalizable. For $\lambda \in \mathfrak{a}^*$ we thus define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \lambda(H)X\ \forall H \in \mathfrak{a}\}$$

Then $\mathfrak{g}$ decomposes into subspaces of the form $\mathfrak{g}_\lambda$.

Definition 3.15. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of non-compact type with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Further, let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subspace. A *root of* $\mathfrak{a}$ *in* $\mathfrak{g}$ is a linear form $\alpha \in \mathfrak{a}^* - \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$, which is the associated *root space.*

Retain the above notation and let Σ denote the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\,.$$

In particular, Σ is finite.

Lemma 3.16. Retain the notation of Definition 3.15. Then the following hold.

(i) $\forall \alpha, \beta \in \mathfrak{a}^* : \ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
(ii) $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

Proof. For (i), let $H \in \mathfrak{a} \subseteq \mathfrak{p}$, $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then

$$0 = [H,[X,Y]] + [X, \underbrace{[Y,H]}_{-\beta(H)Y}] + [Y, \underbrace{[H,X]}_{\alpha(H)X}] \quad \Rightarrow \quad [H,[X,Y]] = (\alpha(H)+\beta(H))[X,Y].$$

For part (ii), compute

$$[H, \theta(X)] = [-\theta(H), \theta(H)] = -\theta(\underbrace{[H,X]}_{\alpha(H)X}) = -\alpha(H)(X).$$

□

Now, for every root $\alpha \in \sigma \subseteq \mathfrak{a}^* - \{0\}$, there is a vector H_α uniquely determined by requiring $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{a}$.

Lemma 3.17. Retain the notation of Definition 3.15. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be maximal abelian. Further, let $\alpha \in \Sigma$. Then for all $X \in \mathfrak{g}_\alpha$ we have

$$[X, \theta(X)] = -B(X, \theta(X))H_\alpha = \langle X, X \rangle H_\alpha.$$

Proof. Let $X_+ \in \mathfrak{g}_\alpha$, $X_- \in \mathfrak{g}_{-\alpha}$ and $Y \in \mathfrak{a}$. Then

$$\begin{aligned}\langle [X_+, X_-], Y \rangle = -B([X_+, X_-], \theta(Y)) = B([X_+, X_-], Y) = \\ &= -B(X_-, [X_+, Y]) = B(X_-, [Y, X_+]) = \alpha(Y)B(X_-, X_+) \\ &= \alpha(Y)B(X_+, X_-) = B(H_\alpha, Y)B(X_+, X_-) \\ &= -B(H_\alpha, \theta(Y))B(X_+, X_-) = \langle H_\alpha, Y \rangle B(X_+, X_-).\end{aligned}$$

Therefore, $[X_+, X_-] - H_\alpha B(X_+, X_-)$ is orthogonal to $\mathfrak{a}$ with respect to $\langle -, - \rangle$. Applying this to $X_+ = X$ and $X_- = \theta(X)$ yields

$$\langle [X, \theta(X)] - H_\alpha B(X, \theta(X)), Y \rangle = 0 \quad \forall Y \in \mathfrak{a},$$

i.e. $Z := [X, \theta(X)] - H_\alpha B(X, \theta(X)) \in \mathfrak{g}_0$. Now, by Lemma 3.16, $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$ and hence $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p})$. Consider $\mathfrak{g}_0 \cap \mathfrak{p}$ and let $R \in \mathfrak{a}$ be a regular element for $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{a} = Z_\mathfrak{g}(R) \cap \mathfrak{p} \supseteq \mathfrak{g}_0 \cap \mathfrak{p} \supseteq \mathfrak{a}$. Therefore, $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$ whence $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \oplus \mathfrak{a}$ and hence $Z \in \mathfrak{g}_0 \cap \mathfrak{k}$. However, one computes $\theta(Z) = -Z$, i.e. $Z \in \mathfrak{p}$. Overall, $Z = 0$ which is the assertion. □

Note that the proof of Lemma 3.17 crucially uses maximality of $\mathfrak{a}$ through the existence of regular elements. We conclude that for all $X \in \mathfrak{g}_\alpha - \{0\}$ we have

$$-H_\alpha = \left[\frac{X}{\sqrt{\langle X, X \rangle}}, \frac{\theta(X)}{\sqrt{\langle X, X \rangle}} \right].$$

Define $h_\alpha = 2H_\alpha/B_{\mathfrak{g}}(H_\alpha, H_\alpha)$. Then for $X \in \mathfrak{g}_\alpha$ we have

$$[h_\alpha, X] = \frac{2[H_\alpha, X]}{B(H_\alpha, H_\alpha)} = \frac{2\alpha(H_\alpha)X}{B(H_\alpha, H_\alpha)} = 2X$$

and similarly, $[h_\alpha, X] = -2X$ for $X \in \mathfrak{g}_{-\alpha}$. Pick $X \in \mathfrak{g}_\alpha$ with $\langle X, X\rangle = 2/B(H_\alpha, H_\alpha)$. Then

$$[X, \theta(X)] = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)} = h_\alpha.$$

The upshot of these computations is that $s_X := \mathbb{R}X + \mathbb{R}h_\alpha + \mathbb{R}\theta(X)$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Example 3.18. Consider the orthogonal symmetric Lie algebra $(\mathfrak{sl}(2, \mathbb{R}), \theta)$ where $\theta : X \mapsto -X^T$. Choose

$$\mathfrak{a} = \left\{ \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}.$$

Then we have the following commutator relations:

$$\left[\begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}, \begin{pmatrix} & \mu \\ & \end{pmatrix} \right] = 2\lambda \begin{pmatrix} & \mu \\ & \end{pmatrix} \quad , \quad \left[\begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}, \begin{pmatrix} & \\ \mu & \end{pmatrix} \right] = -2\lambda \begin{pmatrix} & \\ \mu & \end{pmatrix}.$$

Defining $\alpha \in \mathfrak{a}^*$ by $\alpha(\mathrm{diag}(\lambda, -\lambda)) = 2\lambda$ we then have

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} & \mu \\ & \end{pmatrix} \middle| \mu \in \mathbb{R} \right\}, \ \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} & \\ \mu & \end{pmatrix} \middle| \mu \in \mathbb{R} \right\} \text{ and } \mathfrak{g}_0 = \mathfrak{a}.$$

Furthermore, one computes

$$B\left(\begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}, \begin{pmatrix} \mu & \\ & -\mu \end{pmatrix} \right) = 8\lambda\mu$$

whence

$$H_\alpha = \begin{pmatrix} \frac{1}{4} & \\ & -\frac{1}{4} \end{pmatrix} \quad \text{and} \quad h_\alpha = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Let $X := \mu E_{1,2}$. Then $\theta(X) = -\mu E_{2,1}$ and $[X, \theta(X)] = \mathrm{diag}(-\mu^2, \mu^2) = -\mu^2 h_\alpha$. Thus setting

$$e_+ := \begin{pmatrix} & 1 \\ & \end{pmatrix}, \ e_- = \theta(X) = \begin{pmatrix} & \\ -1 & \end{pmatrix} \text{ and } h := h_\alpha = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

we have the commutator relations found above, namely

$$[h, e_+] = 2e_+, \ [h, e_-] = -2e_- \text{ and } [e_+, e_-] = -h.$$

By the computations preceding Example 3.18 we have for every $X \in \mathfrak{g}_\alpha$ a copy of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{g}$, summarized by the following lemma.

Lemma 3.19. Retain the notation of Lemma 3.17. Let $X \in \mathfrak{g}_\alpha$ be such that $\langle X, X\rangle = 2/\langle H_\alpha, H_\alpha\rangle$, then

$$[h_\alpha, X] = 2X, \ [h_\alpha, \theta(X)] = -2\theta(X) \text{ and } [X, \theta(X)] = -h_\alpha.$$

Hence the map $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})_X := \mathbb{R}X + \mathbb{R}h_\alpha + \mathbb{R}\theta(X)$ which sends e_+ to X, h to h_α and e_- to $\theta(X)$ is a Lie algebra isomorphism onto its image.

Given Lemma 3.19, we need to understand the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ as in $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})_X \subseteq \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{g})$.

Theorem 3.20. Every finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ is a direct sum of irreducible ones. Up to isomorphism, every irreducible, finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ is classified by its dimension. If $\varrho : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(V)$ is an irreducible represenation then $\varrho(h)$ is diagonalizable with simple eigenvalues $\{\dim V - 1, \dim V - 3, \dim V - 5, \ldots, 1 - \dim V\}$.

Example 3.21. We now illustrate Theorem 3.20 with some examples.

(i) The standard representation $\varrho : \mathfrak{sl}(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$ is irreducible. In this case, $\varrho(h) = h$ with eigenvalues $\{1,-1\} = \{2-1, 1-2\}$.
(ii) The adjoint representation $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(2,\mathbb{R})$ is irreducible as well and $\mathrm{ad}(h)$ has eigenvalues $\{2,0,-2\} = \{3-1, 3-3, 1-3\}$.
(iii) The general irreducible n-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ can be described as follows: Let $V_n = \{f(X,Y) = \sum_{k=0}^n a_k X^k Y^{n-k} \mid a_k \in \mathbb{R}\}$ be the vector space of real homogeneous polynomials of degree n. It has the basis $\{Y^n, XY^{n-1}, \dots, X^n\}$ and admits the irreducible representation $\varrho_n : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(V_n)$, $\varrho_n(g)(f)(X,Y)^T = f(g^{-1}(X,Y)^T)$ of $\mathrm{SL}(2,\mathbb{R})$. Its derivative $D_{\mathrm{Id}}\varrho_n : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V_n)$ is the n-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$.

Let now $\alpha, \beta \in \Sigma$ and $X \in \mathfrak{g}_\alpha$ as above. We consider now the adjoint action of $\mathfrak{sl}(2,\mathbb{R})_X = \mathbb{R}X + \mathbb{R}h_\alpha + \mathbb{R}\theta(X)$ on $\mathfrak{g}$. Consider the space $W := \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$. Observe that for $y \in \mathfrak{g}_{\beta+n\alpha}$ we have

$$[h_\alpha, y] = (\beta + n\alpha)(h_\alpha)y = \beta((h_\alpha) + 2n)y$$

as well as the relations

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{\beta+n\alpha}] \subseteq \mathfrak{g}_{\beta+(n+1)\alpha} \quad \text{and} \quad [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\beta+n\alpha}] \subseteq \mathfrak{g}_{\beta+(n-1)\alpha}\,.$$

In particular, W is invariant under the action of $\mathfrak{sl}(2,\mathbb{R})_X$. We may thus consider $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{R})_X \to \mathfrak{gl}(W)$. Let $W = \bigoplus_{l\in I} V_l$ be a decomposition of W into a direct sum of irreducible subspaces. If $W(\lambda) = E_{\mathrm{ad}(h_\alpha)}(\lambda)$ then $W(\lambda) = \bigoplus_{l\in I} V_l(\lambda)$. Now, since $\beta \in \Sigma$, we have $\mathfrak{g}_\beta \neq 0$ and $\mathfrak{g}_\beta = W(\beta(h_\alpha)) \neq 0$. Therefore, $V_l(\beta(h_\alpha)) \neq 0$ for some $l \in I$. That is, there is an irreducible $V \subseteq W$ with $V \cap \mathfrak{g}_\beta \neq 0$. By the structure of irreducible representations of $\mathfrak{sl}(2,\mathbb{R})_X$, we deduce that there are integers $k \leq 0 \leq s$ such that $V \cap \mathfrak{g}_{\beta+n\alpha} \neq 0$ for all $k \leq n \leq s$. And these k, s are determined by $1 - \dim V = \beta(h_\alpha) + 2k$ and $\dim V - 1 = \beta(h_\alpha) + 2s$. Adding these equations we obtain $-\beta(h_\alpha) = k + s$ where $k \leq k + s \leq s$. In particular, $V \cap \mathfrak{g}_{\beta+(k+s)\alpha} \neq 0$. Thus $\beta + (k+s)\alpha = \beta - \beta(h_\alpha)\alpha \in \Sigma$. We have thus constructed a new root out of α, β which will turn out to be very strong information. We summarize this in the following lemma.

Lemma 3.22. Retain the above notation. If $\alpha, \beta \in \Sigma$, then there exist $k \leq 0 \leq s$ with $\beta + n\alpha \in \Sigma$ for all $k \leq n \leq s$, $k + s = -\beta(h_\alpha)$ and $\beta - \beta(h_\alpha)\alpha \in \Sigma$.

Lemma 3.23. Retain the above notation. Assume that $\alpha, \beta \in \Sigma$ with $\beta = \lambda\alpha$ for some $\lambda \in \mathbb{R}$. Then $\lambda \in \{\pm 1/2, \pm 1, \pm 2\}$.

Proof. The equation $\beta = \lambda\alpha$ implies the assertion since $\beta(h_\alpha) \in \mathbb{Z}$ and $\alpha(h_\beta) \in \mathbb{Z}$ by Lemma 3.22. □

3.4. **Root Systems.** Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of noncompact type with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be an abelian subspace which we equip with the inner product $-B_{\mathfrak{g}}|_{\mathfrak{a}\times\mathfrak{a}}$. In this context, we set $R := \{H_\alpha \mid \alpha \in \Sigma\}$ and define for $\alpha \in \Sigma$ the reflection

$$\sigma_\alpha : \mathfrak{a} \to \mathfrak{a},\ v \mapsto v - \frac{2B(H_\alpha, v)}{B(H_\alpha, H_\alpha)} H_\alpha$$

about the hyperplane $(\mathbb{R}H_\alpha)^\perp$.

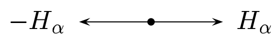
Corollary 3.24. Retain the above notation. Then the following hold.

(i) $\mathfrak{a} = \langle R \rangle$.
(ii) $\forall \alpha \in \Sigma :\ \sigma_\alpha(R) = R$.
(iii) $\forall \alpha, \beta \in \Sigma :\ 2B(H_\alpha, H_\beta)/B(H_\alpha, H_\alpha) = \beta(h_\alpha) \in \mathbb{Z}$

The conditions that Corollary 3.24 imposes on the finite set of vectors R in the Euclidean space $\mathfrak{a}$ are very strong. The set R is called a *root system* and is used to classify orthogonal symmetric Lie algebras of non-compact type. The group generated by the reflections $\{\sigma_\alpha \mid \alpha \in \Sigma\}$ is termed *Weyl group*.

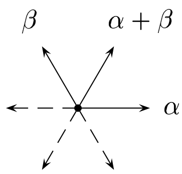
Example 3.25. Here are some examples of root systems. For more information on root systems and their Weyl groups, see e.g. [Bro89].

(i) The root system of $\mathfrak{sl}(2, \mathbb{R})$ is



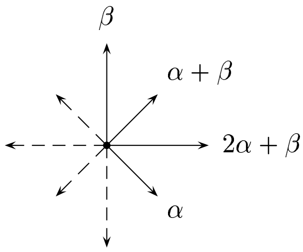
The associated Weyl group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(ii) The root system of $\mathfrak{sl}(3, \mathbb{R})$ is



The associated Weyl group is isomorphic to S_3.

(iii) The root system of $\mathfrak{sp}(4)$ is



The associated Weyl group is isomorphic to D_4.

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