Minmax Methods

in Geometric Analysis

Tristan Rivière

ETH Zürich

Part 3 : Viscous Approximations

of Minmax Operations

Part 3.1 : Examples of Viscous Approximations

of Minmax Operations

Viscous Relaxation of the Dirichlet Energy

Let (M^m, g) and (N^n, h) be two closed oriented Riemmanian Manifolds

$$\int_{M^m} |\nabla u|^2 \; dvol_{M^m} + \sigma^2 \int_{M^m} |\nabla^l u|^p \; dvol_{M^m}$$

is Palais Smale for $l\, p > m$ in $W^{1,p}(M^m, N^n)$

Let $F \in C^\infty(\mathbb{R}^n)$ s.t. $F^{-1}(\{0\}) = N^n \hookrightarrow \mathbb{R}^K$ then

$$\int_{M^m} |\nabla u|^2 \; dvol_{M^m} + \frac{1}{\varepsilon} \int_{M^m} F(u) \; dvol_{M^m}$$

is Palais Smale in $W^{1,2}(M^m, \mathbb{R}^K)$

Part 3.2 : The Difficulty of Smoothing

Minmax Operations

A Viscous Approximation of the Length.

N^n closed sub-manifold of $\mathbb{R}^m$. On

$$\mathcal{M} := W^{2,2}_{imm}(S^1, N^n)$$

consider

$$E^\sigma(u) := \int_{S^1} \left[1 + \sigma^2\, |\vec{\kappa}_u|^2\right]\ dl_u$$

where $\vec{\kappa}_u$ is the curvature of u. For $v \in W^{2,2}(S^1, \mathbb{R}^m)$ with $v \in T_u N^n$ consider

$$\|v\|_u := \left[\int_{S^1} \left[|\nabla^2 v|^2_{g_u} + |\nabla v|^2_{g_u} + |v|^2\right]\ dvol_{g_u}\right]^{1/2}$$

Proposition $(\mathcal{M}, \|\cdot\|)$ *defines a* **complete Finsler manifold**. E^σ *is* C^1 *on* $\mathcal{M}$

Palais Smale modulo "gauge change".

Proposition Let $\sigma > 0$ and $u_k \in \mathcal{M} := W^{2,2}_{imm}(S^1, N^n)$, s.t.

$$E^\sigma(u_k) \longrightarrow \beta(\sigma) \quad \text{and} \quad DE^\sigma_{u_k} \longrightarrow 0 \quad ,$$

then $\exists\ u_{k'}$ and $\psi_{k'}$, $W^{2,2}-$diffeomorphisms of S^1, such that

$$u_{k'} \circ \psi_{k'} \longrightarrow \ u \quad \text{for } d_{\mathbf{P}}$$

$\square$

Let $\mathcal{A}$ admissible in $\mathcal{P}(\mathcal{M})$ and

$$\beta_\sigma := \inf_{A \in \mathcal{A}} \max_{u \in A} E^\sigma(u)$$

Palais Minmax Principle gives u_σ

$$E^\sigma(u_\sigma) = \beta_\sigma \quad , \quad DE^\sigma_{u_\sigma} = 0 \quad \text{and} \quad u_{\sigma_k} \rightharpoonup u_0 \quad \text{weak. in } (W^{1,\infty})^*$$

Do we have $\beta_0 = L(u_0)$ and u_0 is a geodesic ?

A first difficulty

Proposition

There exists $u_\sigma : S^1 \to S^2$ critical point of

$$E^\sigma(u) := \int_{S^1} 1 + \sigma^2 \kappa_u^2 \, dl_u$$

in normal parametrization s.t. as $\sigma \to 0$

$$\frac{du_\sigma}{dt} \rightharpoonup \frac{du_0}{dt} \quad \text{weakly in } (L^\infty)^*$$

but

$$\frac{du_\sigma}{dt} \text{ nowhere strongly converge in } L^1$$

and

$$u_0 \text{ is not a geodesic !}$$

A counter exemple

Precisely, let $f(\sigma) := \sqrt{1 - 2\sigma^2}$

$$u_\sigma(t) := \frac{\sigma}{f(\sigma)} \left(\cos\left(\frac{f(\sigma)}{\sigma} t\right), \sin\left(\frac{f(\sigma)}{\sigma} t\right), \frac{f(\sigma)}{\sigma} \sqrt{\frac{1 - 3\sigma^2}{1 - 2\sigma^2}} \right)$$

$$\lim_{\sigma \to 0} u_\sigma = (0, 0, 1)$$

$$\kappa_{u_\sigma}(t) \equiv -\frac{f(\sigma)}{\sigma}$$

$$E^\sigma(u_\sigma) = 2\, L(u_\sigma)\, (1 - \sigma^2) \;\longrightarrow\; \pi$$

In particular

$$\lim_{\sigma \to 0} \sigma^2 \int_{S^1} \kappa_{u_\sigma}^2 \; dl_{u_\sigma} = \frac{\pi}{2}$$

Failure of $\epsilon-$Regularity

Conclusion : There is no $\epsilon-$regularity independent of σ. (Unlike Sacks Uhlenbeck relaxation).

That is $\nexists\, \varepsilon > 0$ s.t. in constant speed parametrisation

$$\int_{t_0-r}^{t_0+r} 1 + \sigma^2\, \kappa^2_{u_\sigma}\; dl_u < \varepsilon \quad \Longrightarrow \quad |\ddot{u}_\sigma|(t_0) \leq C\;\, r^{-1}$$

Another Case of Absence of $\varepsilon-$Regularity

Let $u_\sigma \neq Cte$ realising

$$H_\sigma := \min \left\{ \begin{array}{c} E^\sigma(u) := \int_{S^3} |du|^2 + \sigma^2 \, |du|^4 \, dvol_{S^3} \\ u \in W^{1,4}(S^3, S^2) \quad ; \quad \text{Hopf-deg}(u) = +1 \end{array} \right\}$$

One has

$$H_\sigma \to 0 \quad \text{Hence} \quad E^\sigma(u_\sigma) \to 0 \quad \Longrightarrow \quad du_\sigma \to 0 \quad \text{in } L^2(S^3)$$

But, since Hopf-deg$(u) = +1$

$$\liminf_{\sigma \to 0} \int_{S^3} |du_\sigma|^3_{S^3} \, dvol_{S^3} \geq C > 0$$

Hence we cannot have

$$E_\sigma(u_\sigma) < \varepsilon \quad \Longrightarrow \quad \|\nabla u_\sigma\|_\infty \leq C$$

This is due to a failure of the monotonicity formula in that case.

The Case of Vanishing Viscous Energy

Theorem

Let u_σ critical point of

$$E^\sigma(u) := \int_{S^1} 1 + \sigma^2\, \kappa_u^2 \; dl_u$$

in normal parametrization. Assume

$$\limsup_{\sigma\to 0} \int_{S^1} dl_{u_\sigma} < +\infty$$

and

$$\lim_{\sigma\to 0} \int_{S^1} \sigma^2\, \kappa_{u_\sigma}^2 \; dl_{u_\sigma} = 0$$

then $\exists\ \sigma_j \to 0$

$$\frac{du_{\sigma_j}}{dt} \longrightarrow \frac{du_0}{dt} \qquad \text{strongly in } L^1$$

and

$$u_0 \text{ is a geodesic}$$

A Proof of the Theorem. Page 1

Assume

$$|\dot{u}_{\sigma_j}| \equiv \frac{L_j}{2\pi} \qquad \text{and} \qquad u_{\sigma_j} \rightharpoonup u_0 \quad \text{weakly in } W^{1,\infty}(S^1)^* \ .$$

Denote $u_j := u_{\sigma_j}$ and $D_t := P_T(u_j)\frac{d}{dt}$. There holds

$$D_t\left[\dot{u}_j - \sigma_j^2\left[2D_t^2\dot{u}_j + 3\,\kappa_j^2\,\dot{u}_j\right]\right] + 2\,\sigma_j^2\,R(D_t\dot{u}_j, \dot{u}_j)\,\dot{u}_j = 0$$

We have

$$D_t\dot{u}_j = \left(\frac{L_j}{2\pi}\right)^2\,\vec{\kappa}_j$$

Denote

$$w_j := \dot{u}_j - \sigma_j^2\left[2D_t^2\dot{u}_j + 3\,\kappa_j^2\,\dot{u}_j\right]$$

Hence

$$D_t w_j \longrightarrow 0 \quad \text{in } L^2$$

A Proof of the Theorem. Page 2

In a local chart (u_j is pre-compact in C^0)

$$(D_t w_j)^k = \frac{dw_j^k}{dt} + \Gamma^k_{lm}\, w_j^l\, \dot u_j^m$$

$$= \frac{dw_j^k}{dt} + (1 - 3\,\sigma_j^2\,\kappa_j^2)\,\Gamma^k_{lm}\,\dot u_j^l\,\dot u_j^m - 2\,\sigma_j^2\,\Gamma^k_{lm}\,(D_t^2\,\dot u_j)^l\,\dot u_j^m$$

and

$$\sigma_j^2\,\Gamma^k_{lm}\,(D_t^2\,\dot u_j)^l\,\dot u_j^m = \left(\frac{L_j}{2\pi}\right)^2\,\sigma_j^2\,\Gamma^k_{lm}\,\frac{d}{dt}\kappa_j^l\,\dot u_j^m$$

$$+\left(\frac{L_j}{2\pi}\right)^2\,\sigma_j^2\,\Gamma^k_{lm}\,\Gamma^l_{\alpha\beta}\,\kappa_j^\alpha\,\dot u_j^\beta\,\dot u_j^m \quad \longrightarrow 0 \text{ strongly in } L^1 + H^{-1}$$

Hence

$$w_j := (1 - 3\,\sigma_j^2\,\kappa_j^2)\,\dot u_j - 2\,\left(\frac{L_j}{2\pi}\right)^2\,\sigma_j^2\,D_t\vec\kappa_j$$

is pre-compact in L^2

A Proof of the Theorem. Page 3

$$w_j := (1 - 3\,\sigma_j^2\,\kappa_j^2)\,\dot{u}_j - 2\,\left(\frac{L_j}{2\pi}\right)^2\,\sigma_j^2 D_t\vec{\kappa}_j$$

is pre-compact in L^2 and

$$w_j \longrightarrow \dot{u}_0 \quad \text{in } \mathcal{D}'(S^1)\ .$$

Thus

$$\int_{S^1} w_j\cdot\dot{u}_j\ dt \quad\longrightarrow\quad \int_{S^1} |\dot{u}_0|^2\ dt$$

But

$$\int_{S^1} w_j\cdot\dot{u}_j\ dt = \int_{S^1} |\dot{u}_j|^2\ dt + o(1)$$

Thus

$$\dot{u}_j \ \longrightarrow\ \dot{u}_0 \quad \text{strongly in } L^p(S^1) \quad \forall\, p < +\infty\ \ldots$$

Part 3.3 : Modifying The Pseudo-Gradient

of Viscous Palais-Smale Approximations

Struwe Monotonicity Trick

Theorem Let $(\mathcal{M}, \|\cdot\|)$ be a complete Finsler manifold. Let $E^\sigma \in C^1(\mathcal{M})$ for $\sigma \in [0,1]$ s.t.

$$\forall\ \Phi \in \mathcal{M} \quad \sigma \longrightarrow E^\sigma(\Phi) \quad \text{and} \quad \sigma \longrightarrow \partial_\sigma E^\sigma(\Phi)$$

are **increasing** and continuous functions with respect to σ. Assume

$$\|DE^\sigma_\Phi - DE^\tau_\Phi\|_\Phi \le C(\sigma)\ \delta(|\sigma-\tau|)\ f(E^\sigma(\Phi))$$

where

$$C(\sigma) \in L^\infty_{loc}((0,1))\ ,\ \delta \in L^\infty_{loc}(\mathbb{R}_+)\ ,\ \lim_{s\to 0}\delta(s) = 0 \ \text{ and } f \in L^\infty_{loc}(\mathbb{R}).$$

Assume E^σ satisfies (PS). Let $\mathcal{A}$ admissible

$$\beta(\sigma) := \inf_{A\in\mathcal{A}}\ \sup_{\Phi\in A}\ E^\sigma(\Phi)$$

Then $\exists\ \sigma_j \to 0$ and $\Phi_j \in \mathcal{M}$ s.t.

$$E^{\sigma_j}(\Phi_j) = \beta(\sigma_j)\ ,\ DE^{\sigma_j}(\Phi_j) = 0 \text{ and } \ \partial_{\sigma_j} E^{\sigma_j}(\Phi_j) = o\left(\frac{1}{\sigma_j\ \log\left(\frac{1}{\sigma_j}\right)}\right)$$

Another Proof of Birkhoff Existence Result.

Let $\mathcal{A}$ admissible in $\mathcal{P}(W^{2,2}_{imm}(S^1, N^n))$ and

$$\beta_\sigma := \inf_{A\in\mathcal{A}} \max_{\Phi\in A} E^\sigma(\Phi) := \text{Length}(\Phi(S^1)) + \sigma^2 \int_{S^1} \kappa_\Phi^2 \; dl_\Phi$$

Struwe Monotonicity gives $\sigma_j \to 0$, Φ_{σ_j} s.t.

$$E^{\sigma_j}(\Phi_{\sigma_j}) = \beta_{\sigma_j} \quad , \quad DE^{\sigma_j}_{\Phi_{\sigma_j}} = 0$$

and

$$\sigma_j^2 \int_{S^1} \kappa^2_{\Phi_{\sigma_j}} \; dl_{\Phi_{\sigma_j}} = o\left(\frac{1}{\log\left(\frac{1}{\sigma_j}\right)}\right) \quad .$$

then $\exists \; \sigma_{j'} \to 0$

$$\frac{d\Phi_{\sigma_j}}{dt} \to \frac{d\Phi_0}{dt} \quad \text{strongly in } L^1$$

and

$$\Phi_0 \text{ is a geodesic with } L(\Phi_0) = \beta_0$$

The Proof of Struwe Monotonicity Trick - page 1

$$\beta(\sigma) \searrow \beta(0) \quad \Longrightarrow \quad \beta \text{ is diff. a.e.}$$

and

$$D\beta(\sigma) = \beta'(\sigma)\ d\mathcal{L}^1 \llcorner [0,1] + \mu \quad \text{where} \quad \mu \perp d\mathcal{L}^1 \llcorner [0,1]$$

$$\int_0^\sigma \beta'(s)\ ds \leq \beta(\sigma) - \beta(0)$$

Hence $\exists\ \sigma_j \to 0$

$$\beta'(\sigma_j) = o\left(\frac{1}{\sigma_j\ \log \sigma_j^{-1}}\right)$$

The Proof of Struwe Monotonicity Trick - page 2

Let σ be a point of differentiability

$$\sigma < \tau < \sigma + \delta \quad \Longrightarrow \quad \beta(\tau) \le \beta(\sigma) + [\beta'(\sigma) + \varepsilon]\ (\tau - \sigma)$$

$A \in \mathcal{A}$ and $\Phi \in A$ s.t.

$$\left\{ \begin{array}{l} \beta(\sigma) \le E^\sigma(\Phi) + \varepsilon\ (\tau - \sigma) \\ \\ E^\tau(\Phi) \le \beta(\tau) + \varepsilon(\tau - \sigma) \quad . \end{array} \right. \quad \left(\Longrightarrow \partial_\sigma E^\sigma(\Phi) \le \beta'(\sigma) + 3\,\varepsilon\right)$$

Replace the original **pseudo-gradient** X_τ for E^τ by X_τ^σ

$$X_\tau^\sigma(\Phi) := \chi\left(\frac{E^\sigma(\Phi) - \beta(\sigma) + \varepsilon\,(\tau - \sigma)}{\varepsilon\,(\tau - \sigma)}\right)\ X_\tau(\Phi)$$

where $\chi \equiv 1$ in $[1, +\infty]$ and $\chi \equiv 0$ in $[0, 1/2]$.

The Proof of Struwe Monotonicity Trick - page 3

Assume $\exists\,\delta > 0$ (indep. of $\tau \searrow \sigma$)

$$\left\{\begin{array}{l} \beta(\sigma) \le E^\sigma(\Phi) + \varepsilon\ (\tau - \sigma) \\ \\ E^\tau(\Phi) \le \beta(\tau) + \varepsilon(\tau - \sigma) \quad . \end{array}\right. \quad \Longrightarrow \quad \|DE^\tau_\Phi\| > \delta$$

Let $A \in \mathcal{A}$ s. t.

$$\sup_{\Phi \in A} E^\tau(\Phi) \le \beta(\tau) + \varepsilon(\tau - \sigma)$$

Since the flow is active **only** if $\beta(\sigma) \le E^\sigma(\Phi) + \varepsilon\ (\tau - \sigma)$

$$\text{Hypothesis above} \ \Rightarrow \quad \forall\ \Phi \in A \qquad t^\Phi_{max} = +\infty$$

$$E^\sigma(\Phi) - \beta(\sigma) \ge 0 \quad \Rightarrow \quad \frac{d}{dt} E^\sigma(\phi_t(\Phi))\Big|_{t=0} \le -\,C\,\delta^2 \quad \Rightarrow \quad \text{Contrad. !}$$