Minmax Methods

in Geometric Analysis

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Part 4 : Minmax Operations

on the Area of Surfaces

The general Landscape for Minmax of the Area

- The *Ambient Space* $(M^m, g) \hookrightarrow \mathbb{R}^n$.

- The *Configuration Space*

$$\mathcal{M} = \{u \ : \ \Sigma^2 \to M^m \text{ immersion } \}$$

- $\mathcal{A}$ *Admissible family* : i.e. $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ s.t. " $\forall$ " homeo. Ξ of $\mathcal{M}$ isot. to identity

$$\Xi(\mathcal{A}) \subset \mathcal{A}$$

- The *Width* associated to $\mathcal{A}$

$$W_{\mathcal{A}} := \inf_{A \in \mathcal{A}} \sup_{u \in A} \quad \text{Area}(u) > 0$$

Goal of this last Part of the Minicourse

Theorem[R. Pub. IHES (2017), Pigati, R. CPAM (2020), Pigati, R. Duke (2020)] Let $\mathcal{A}$ be an admissible $n-$homological family of $\mathcal{M}$ s.t.

$$W_{\mathcal{A}} := \inf_{A\in\mathcal{A}} \max_{u\in A} \text{Area}(u) > 0 \ .$$

Then there exists a smooth minimal branched immersion v of a closed surface S s.t.

i)
$$\text{genus}(S) \le \text{genus}(\Sigma^2)$$

ii)
$$W = \text{Area}(v)$$

iii)
$$\text{index}(v) \le n$$

General Framework and Strategy

- Prove that any Minmax on the area of surfaces is achieved by a minimal surface.
- Call upon Palais Deformation Theory to perform Minmax for the area of Surfaces.
- Perturb the Area Lagrangian for Palais-Smale Condition to be fullfiled.
- Avoid the Sacks-Uhlenbeck Parametric Approach (Problem with the Conformal Class).
- Avoid Geometric Measure Theory Approach (Regularity Theory undevelopped).
- Use Lagrangians which are invariant under reparametrization.

A Viscosity Approach for the Minmax of the Area of Surfaces.

Let Σ^2 oriented closed surface. u immersion of Σ^2 in $M^m \hookrightarrow \mathbb{R}^n$.

Let $\vec{n}_u \in G_{n-2}(\mathbb{R}^n)$ be the Gauss Map $\underline{\text{in } \mathbb{R}^n}$.

Consider for $p > 1$

$$E_p^\sigma(u) := \text{Area}(u) + \sigma^2 \int_\Sigma (1 + |d\vec{n}_u|^2)^p \; dvol_g$$

on the **Finsler Manifold** $W^{2,2p}_{imm}(\Sigma^2, M^m)$.

The Closure of $W^{2,p}$ Immersions $p > 2$.

Theorem [Langer 1985, Breuning 2015] Let u_k be a sequence in $W^{2,p}_{imm}(\Sigma, M^m)$ such that

$$\limsup_{k\to+\infty} \int_\Sigma \left[1 + |d\vec{n}_k|^2_{g_{u_k}}\right]^p \, dvol_{g_{u_k}} < +\infty$$

Then there exists k' and $\phi_{k'} \in W^{2,p}_{diff}(\Sigma, \Sigma)$ s. t.

$$v_{k'} := u_{k'} \circ \phi_{k'} \quad \rightharpoonup \quad v_\infty \quad \text{weakly in } W^{2,p}(\Sigma, M^m)$$

moreover $v_\infty \in W^{2,p}_{imm}(\Sigma, M^m)$. □

Step 1 : Control of the Conformal Class.

Theorem Assume genus$(\Sigma) > 0$. Let u_k be a sequence in $W^{2,p}_{imm}(\Sigma, M^m)$ such that

$$\limsup_{k\to+\infty} \int_\Sigma \left[1 + |d\vec{n}_k|^2_{g_{u_k}}\right]^p \, dvol_{g_{u_k}} < +\infty$$

then the sequence of underlying constant curvature metric h_k of volume 1 such that

$$g_{u_k} = e^{2\alpha_k}\, h_k$$

is pre-compact in any $C^l(\Sigma)$ topology ($l \in \mathbb{N}$) modulo the pull-back action by diffeomorphisms. $\square$

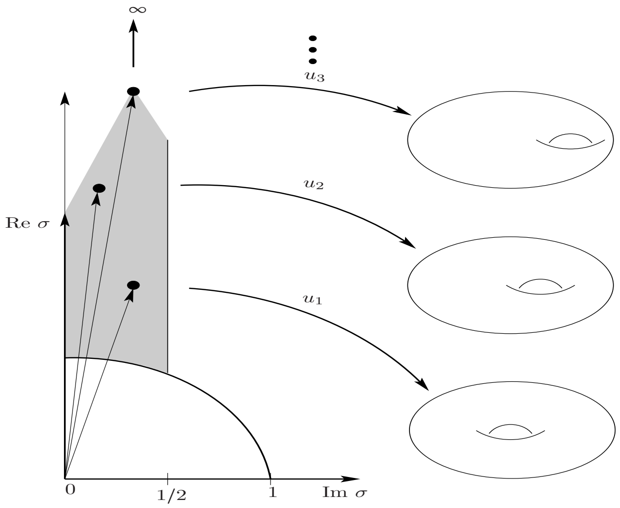


Figure 12: Sequence of Flat tori conformal Immersions with degenerating classes

Proof of Step 1.

The case when $\Sigma = T^2$ and Consider u_k conformal from $\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}$ into $\mathbb{R}^n$ ($\tau_k \geq 1$) pause such that

$$\limsup_{k\rightarrow+\infty}\int_{T^2}\left[1+|d\vec{n}_k|^2_{g_{u_k}}\right]^p\ dvol_{g_{u_k}} < +\infty$$

Let λ_k, $\vec{e}_k$ and $\vec{f}_k$ such that

$$u_k^\ast g_{\mathbb{R}^n} = e^{2\alpha_k}\ dx_1^2+dx_2^2\ ,\ \vec{e}_k := e^{-\alpha_k}\,\partial_{x_1}u_k \ \text{ and }\ \vec{f}_k := e^{-\alpha_k}\,\partial_{x_2}u_k \quad .$$

Fenchel Theorem gives for any x_2

$$2\pi \leq \int_0^1\ |\partial_{x_1}\vec{e}_k|\ dx_1 = \int_0^1\ \sqrt{\left|\vec{f}_k\cdot\partial_{x_1}\vec{e}_k\right|^2+|\vec{n}_k\llcorner\partial_{x_1}\vec{e}_k|^2}\ dx_1$$

We have

$$\vec{f}_k\cdot\partial_{x_1}\vec{e}_k = -\partial_{x_2}\alpha_k \quad \text{and} \quad \vec{n}_k\llcorner\partial_{x_1}\vec{e}_k = -\partial_{x_1}\vec{n}_k\llcorner\vec{e}_k$$

Proof continued.

Integrating w.r.t. x_2 gives for any $s \in (0,1)$

$$2\pi\,\tau_k \le \int_{\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}} \sqrt{|\nabla\alpha_k|^2 + |\nabla\vec{n}_k|^2}\ dx^2$$

$$\le\ \tau_k^{(1-s)/2}\ \left[\int_{\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}} e^{-2s\alpha_k}|\nabla\alpha_k|^2\ dx^2\right]^{1/2}\ \left[\int_{\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}} e^{2\,\alpha_k}\right]^{s/2}$$

$$+\sqrt{\tau_k}\ \left[\int_{\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}} |\nabla\vec{n}_k|^2\ dx^2\right]^{1/2}$$

Liouville equation reads

$$-\Delta\alpha_k = e^{2\alpha_k}\ K_k \qquad (\text{Recall}\ \ |K_k| \le |d\vec{n}_k|^2_{g_{u_k}})$$

Take $s = 1 - 1/p$, multiply Liouville by $-e^{-2\,s\,\alpha_k}$, integrate by parts and obtain

$$2\,s\int_{\mathbb{R}^2/\mathbb{Z}\times\tau_k\mathbb{Z}} e^{-2s\alpha_k}|\nabla\alpha_k|^2\ dx^2 \le \tau_k^s\ \left[\int_{T^2} |d\vec{n}_k|^{2p}_{g_{u_k}}\ dvol_{g_{u_k}}\right]^{1/p}$$

$\cdots$ and obtain finally

$$\tau_k \leq C \, \frac{p}{p-1} \, \left[\int_{T^2} |d\vec{n}_k|^{2p}_{g_{\vec{u}_k}} \, dvol_{g_{u_k}} \right]^{1/p} \, \left[\int_{T^2} dvol_{g_{u_k}} \right]^{1-1/p}$$

$\square$

Step 2 : Control of the Conformal Factor

Theorem Assume genus$(\Sigma) > 0$. Let u_k be a sequence in $W^{2,p}_{imm}(\Sigma,\mathbb{R}^n)$ such that

$$\limsup_{k\to+\infty}\int_\Sigma \left[1+|d\vec{n}_k|^2_{g_{u_k}}\right]^p \, dvol_{g_{u_k}} < +\infty$$

Let h_k of volume 1 such that

$$g_{u_k} = e^{2\alpha_k}\, h_k$$

Assume

$$h_k \to h_\infty \quad \text{in } C^l(\Sigma) \quad \forall\, l\in\mathbb{N}$$

Then

$$\limsup_{k\to+\infty} \|\alpha_k\|_{L^\infty(\Sigma)} < +\infty$$

Proof of Step 2 (Sketch for $g(\Sigma) > 1$) - Page 1

Willmore inequality gives

$$4\pi \le \int_\Sigma |\vec{H}_k|^2 \, dvol_{u_k} \le \frac{1}{2}\int_\Sigma |d\vec{n}_k|^2_{g_{u_k}} \, dvol_{u_k}$$

$$\le \frac{1}{2}\left[\int_\Sigma |d\vec{n}_k|^{2p}_{g_{u_k}} \, dvol_{g_{u_k}}\right]^{1/p} \left[\int_\Sigma dvol_{g_{u_k}}\right]^{1-1/p} .$$

$$\Longrightarrow \; 0 < \liminf_{k\to+\infty} \int_\Sigma e^{2\alpha_k} \, dvol_{h_k} \le \limsup_{k\to+\infty} \int_\Sigma e^{2\alpha_k} \, dvol_{h_k} < +\infty .$$

Liouville equation implies

$$-\Delta_{h_k}\alpha_k = e^{2\alpha_k} K_{g_k} - K_{h_k} .$$

Multiply $e^{-2\,s\,\alpha_k}$ $(s = 1 - 1/p)$ and integrate

$$2s \int_\Sigma e^{-2s\alpha_k} \left[|d\alpha_k|^2_{h_k} + 1\right] dvol_{h_k} \le \int_\Sigma e^{2\,\alpha_k/p} K_{g_k} \, dvol_{h_k}$$

$$\le \left(\int_\Sigma |K_{g_k}|^p \, dvol_{g_k}\right)^{1/p} \left(\int_\Sigma dvol_{g_k}\right)^s$$

Proof of Step 2 - L^q Estimate on α_k

Hence in particular, using Sobolev embeddings,

$$\forall\ q < +\infty \qquad \limsup_{k\to+\infty} \left\| e^{-\alpha_k} \right\|_{L^q_{h_k}(\Sigma)} < +\infty$$

We deduce using Jensen twice for e^{α_k} and $e^{-\alpha_k}$ we deduce

$$\limsup_{k\to+\infty} \left| \fint_\Sigma \alpha_k \ dvol_{h_k} \right| < +\infty\ .$$

Since for any $q < +\infty$

$$\limsup_{k\to+\infty} \|\alpha_k - \overline{\alpha}_k\|_{L^q_{h_k}(\Sigma)} \le C\ \limsup_{k\to+\infty} \|d\alpha_k\|_{L^{2,\infty}_{h_k}(\Sigma)} < +\infty$$
$$\le C\ \limsup_{k\to+\infty} \|\Delta_{h_k}\alpha_k\|_{L^1_{h_k}(\Sigma)} < +\infty$$

We deduce

$$\forall\ q < +\infty \qquad \limsup_{k\to+\infty} \|\alpha_k\|_{L^q_{h_k}(\Sigma)} < +\infty\ .$$

L^∞ estimates on α_k in Bubbles

Theorem [Hélein 92] There exists $\varepsilon > 0$ and $C > 0$ depending on (Σ, h_k) and n only s.t.

$$\int_{B_r^{h_k}(x_0)} |d\vec{n}_k|^2_{h_k}\ dvol_{h_k} \le \varepsilon \quad \Longrightarrow$$

$$\|\alpha_k - c_k\|_{L^\infty(B_{r/2}^{h_k}(x_0))} < C\ \left[1 + \|d\alpha_k\|_{L^{2,\infty}_{h_k}(B_r^{h_k}(x_0))}\right]$$

where $c_k \in \mathbb{R}$. □

$$\text{Let} \qquad r_k = \inf\left\{r > 0\ ;\ \int_{B_r^{h_k}(x_0)} |d\vec{n}_k|^2_{h_k}\ dvol_{h_k} = \varepsilon\right\}$$

then from the previous slide, using covering argument, if $\liminf r_k > 0$

$$\limsup_{k\to+\infty} \|\alpha_k\|_{L^\infty(\Sigma)} < +\infty\ .$$

L^∞ estimates on α_k in Necks

Assume now $\lim r_k = 0$.

An $\varepsilon-$neck is a sequence of degenerating annuli $A(\eta_k, \delta_k) := B_{\eta_k}(x_k) \setminus B_{\delta_k/\eta_k}(x_k)$ where

$$\lim_{k \to +\infty} \delta_k/\eta_k^2 = 0 \quad \text{and} \quad \int_{A(\eta_k,\delta_k)} |d\vec{n}_k|^2_{h_k} \, dvol_{h_k} < \varepsilon$$

Theorem [Bernard, R. 2014] There exists $\varepsilon > 0$ and $C > 0$ depending on (Σ, h_k) and n only s.t.

$$\|\alpha_k - d_k \log d(x, x_k)_{h_k} - A_k\|_{L^\infty(A(2^{-1}\eta_k,\delta_k))}$$

$$\leq C \left[1 + \|d\alpha_k\|_{L^{2,\infty}_{h_k}(A(\eta_k,\delta_k))}\right]$$

where $d_k, A_k \in \mathbb{R}$. □

Combining one Bubble with one Neck

The case of one bubble centred at x_k connected to the surface by one neck : $\delta_k = r_k$, $\eta_k = 1$. There holds for any $\eta < 1$

$$\limsup_{k\to+\infty} \|\alpha_k\|_{L^\infty(\Sigma\setminus B_\eta(x_k))} < +\infty \ .$$

Taking $d(x, x_k)_{h_k} = 1$ gives

$$\limsup_{k\to+\infty} |A_k| < +\infty$$

We have moreover

$$\begin{aligned}
\varepsilon = \int_{B^{h_k}_{\delta_k}(x_k)} |d\vec{n}_k|^2_{h_k}\ dvol_{h_k} &= \int_{B^{h_k}_{\delta_k}(x_k)} |d\vec{n}_k|^2_{g_{u_k}}\ dvol_{u_k} \\
&\le \left[\int_\Sigma |d\vec{n}_k|^{2p}_{g_{u_k}}\ dvol_{g_{u_k}}\right]^{1/p} \left[\int_{B^{h_k}_{\delta_k}(x_k)} e^{2\alpha_k}\ dvol_{h_k}\right]^{1-1/p} \quad .
\end{aligned}$$

$$\Longrightarrow \limsup_{k\to+\infty} \left\|\alpha_k - \log\frac{1}{\delta_k}\right\|_{L^\infty(B_{\delta_k}(x_k))} < +\infty$$

Proof of Step 2 - End

We deduce

$$d_k + 1 = O\left(\frac{1}{\log \delta_k^{-1}}\right)$$

We compute

$$\int_{A(1,\delta_k)} e^{2\alpha_k}\ dvol_{h_k} \geq C \int_{\delta_k}^{1} r^{2\,d_k+1}\ dr$$

$$\geq C \int_{\delta_k}^{1} r^{-1+K/\log \delta_k^{-1}}\ dr \geq \frac{\log \delta_k^{-1}}{2K} \to +\infty$$

Contradiction ! Hence there is no bubble, i.e. $\liminf r_k > 0$, and

$$\limsup_{k\to+\infty} \|\alpha_k\|_{L^\infty(\Sigma)} < +\infty\ .$$

The Euler Lagrange Equation

Let u be a critical point of

$$F^\sigma(u) := \int_\Sigma f_\sigma(|d\vec{n}_u|^2)\ dvol_{g_u}$$

Then it satisfies (in the case $M^m = \mathbb{R}^3$)

$$d^{*_g}\left[f_\sigma\ du - 2\, f'_\sigma\ d\vec{n} \cdot \otimes d\vec{n} \cdot_g du + 2\, d^{*_g}(f'_\sigma\ d\vec{n}) \cdot du\ \vec{n}\right] = 0$$

where $\vec{n} := \vec{n}_u$, $g := g_u$ and $f_\sigma := f_\sigma(|d\vec{n}_u|^2)$.

In conformal coordinates

$$g := e^{2\lambda}\ [dx_1^2 + dx_2^2]$$

this gives

$$\operatorname{div}\left[f_\sigma \nabla u -\ 2\ f'_\sigma\, e^{-2\lambda}\, \nabla\vec{n} \dot{\otimes}\, \partial_{x_j}\vec{n} \cdot \partial_{x_j} u + 2\, e^{-2\lambda} \operatorname{div}(f'_\sigma\ \nabla\vec{n}) \cdot \nabla u\, \vec{n}\right] = 0$$

The Palais Smale Condition

Theorem [Kuwert, Lamm, Li 2015] Critical points of the Functional F_p^σ are smooth.

Theorem [Kuwert, Lamm, Li 2015] For $p > 1$ the Functional is C^2 and **Palais Smale** modulo reparametrization

$$F_p^\sigma(u_k) \to \beta(\sigma) > 0 \quad \text{and} \quad DF_p^\sigma(u_k) \to 0$$

then there exists $u_{k'}$ and $\phi_{k'} \in \mathrm{Diff}(\Sigma^2)$ s.t.

$$v_{k'} := u_{k'} \circ \phi_{k'} \quad \longrightarrow \quad v_\infty \quad \text{strongly in} \quad W^{2,2p}(\Sigma, M^m) \quad ,$$

and

$$DF_p^\sigma(v_\infty) = 0$$

□

Palais Minmax Principle for the Relaxed Energies

Theorem Let $\mathcal{A}$ be an admissible Family in $\mathcal{M} := W^{2,2p}_{imm}(\Sigma, M^m)$. Let $\sigma > 0$. Assume

$$W_{\mathcal{A}}(\sigma) := \inf_{A\in\mathcal{A}} \sup_{\Phi\in A} F_p^\sigma(\Phi) \ > 0 \quad .$$

Then, $\exists\ u^\sigma \in C^\infty_{imm}(\Sigma, M^m)$ s. t.

$$F_p^\sigma(u^\sigma) = W_{\mathcal{A}}(\sigma) \quad \text{and} \quad DF_p^\sigma(u^\sigma) = 0 \quad .$$

$\square$

Passing to the Limit $\sigma \to 0$.

Theorem [R. 2017] Let $\sigma_k \to 0$ and u_k be a critical point of $F_p^{\sigma_k}$ s.t.

$$F_p^{\sigma_k}(u_k) \to W(0) > 0 \quad \text{and} \quad \partial_\sigma F_p^{\sigma_k}(u_k) = o\left(\frac{1}{\sigma_k \, \log \sigma_k^{-1}}\right)$$

then $\exists\, k'$ and

- (S, h) closed Riem. surface, $genus(S) \leq genus(\Sigma)$
- $v_\infty \in W_h^{1,2}(S, M^m)$ weakly conformal
- $N_\infty \in L^\infty(S, \mathbb{N})$

s.t.

$$(\Sigma, u_{k'}, 1) \rightharpoonup (S, v_\infty, N_\infty) \quad \text{varifold converges}$$

and (S, v_∞, N_∞) is locally stationary

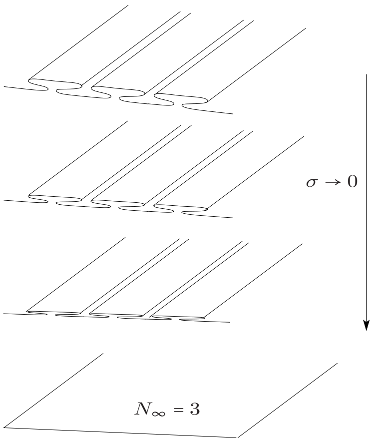


Figure 11: Weak Convergence in the Viscosity Method

Parametrized Varifold

Let $U \in C^0(G_2(TM^m))$. $G_2(TM^m) \hookrightarrow G_2(\mathbb{R}^n) \simeq \mathbb{R}^n \times Gr_2^n$

The varifold associated to $(\Sigma, u_k, 1)$

$$\mathbf{v}_k(U) := \int_\Sigma U(u_k(x), du_k(T_x\Sigma^2))\ dvol_{g_{u_k}}$$

i.e. "desintegration form"

$$d\mathbf{v}_k(z, P) := d\left(\mathcal{H}^2 \llcorner u_k(\Sigma)\right) \otimes \sum_{u_k(x)=z} \delta_{u_k(T_x\Sigma)}$$

The varifold associated to (S, v_∞, N_∞)

$$\mathbf{v}_\infty(U) := \int_S N_\infty\ U(v_\infty(x), dv_\infty(T_xS))\ dvol_{v_\infty}$$

i.e. "desintegration form"

$$d\mathbf{v}_\infty(z, P) := d\left(\mathcal{H}^2 \llcorner u_\infty(S)\right) \otimes \sum_{u_\infty(x)=z} N_\infty(x)\, \delta_{u_\infty(T_xS)}$$

More generally : Integer Rectifiable 2-Varifolds

K 2−rectifiable subset of $\mathbb{R}^n$,

$$N \ : \ K \to \mathbb{N}^* \qquad \mathcal{H}^2 \llcorner K \text{ measurable}$$

s.t.

$$N(z)\, d\left(\mathcal{H}^2 \llcorner K\right) \qquad \text{is locally finite}$$

The varifold $\mathbf{v} = (K, N)$ is given by

$$d\mathbf{v}(z, P) := N(z)\, d\left(\mathcal{H}^2 \llcorner K\right) \otimes \delta_{T_z K}$$

where

$$T_z K \text{ is the approximate tangent plane to } K \text{ at } z$$

The "weight" of $\mathbf{v}$ is given by

$$|\mathbf{v}| := \pi_* \mathbf{v} = N(z)\, \mathcal{H}^2 \llcorner K \qquad \text{where } \pi(z, P) := z$$

and the "mass"

$$\|\mathbf{v}\| := \mathbf{v}(G_2(M^m))$$

Varifold convergence

We have

$$\mathbf{v}_{k'} \rightharpoonup \mathbf{v}_\infty \quad \text{in Radon measure on } G_2(TM^m)$$

In particular the mass passes to the limit ! (Main reason why varifold theory was introduced)

First variation of Integer Rectifiable 2-Varifolds

Let $\Phi \in \text{Diff}(M^m)$. Define the varifold "push-forward" :

$$\Phi_*\mathbf{v}(U) := \int U(\Phi(z), d\Phi_z(P))\ J_\Phi(z,P)\ d\mathbf{v}(z,P)$$

where $J_\Phi(z,P)$ is the "natural" dilation factor (Jacobian)

$$J_\Phi(z,P) := \sqrt{\det(A^T A)} \quad \text{where } A := d\Phi_z \llcorner P$$

X C^1 vectorfield of M^m and Φ_t^X the associated flow. Define

$$\delta\mathbf{v}(X) := \frac{d}{dt}\|\Phi_*\mathbf{v}\| = \frac{d}{dt}\int J_{\Phi_t^X}(z,P)\ d\mathbf{v}(z,P)$$

If $\mathbf{v} = (K, N)$ is integral

$$\delta\mathbf{v}(X) = \int N\ div^{T_z K} X\ d\mathcal{H}^2 \llcorner K$$

where

$$div^P X := \sum_{i=1}^{2} e_i \cdot \nabla^M_{e_i} X \quad \text{where } (e_1, e_2) \text{ orth. basis of } P$$

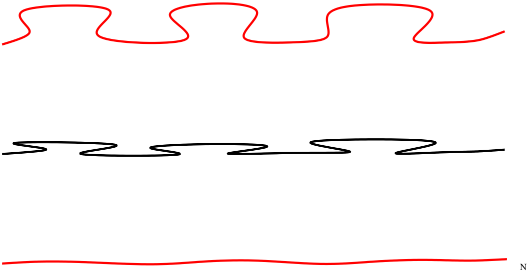
Sequential Weak closure of Stationary Integral Varifolds

Definition A varifold $\mathbf{v}$ is stationary if

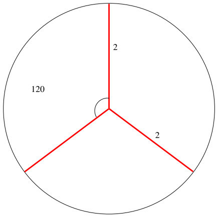
$$\delta \mathbf{v} = 0$$

Theorem [Allard 72] Let $\mathbf{v}^i$ be a sequence of stationary Integer Rectifiable Varifolds converging weakly to $\mathbf{v}^\infty$ then $\mathbf{v}^\infty$ is still a stationary Integer Rectifiable Varifolds.

Varifold Convergence to Integer Rectifiable Varifold



Integer Rectifiable Stationary Varifold can have singularities



classical stationary varifolds in S^2 : red lines are half geod. circles.

Regularity results for Stationary Varifolds.

Landmarks:

- Allard (1972) : regularity on a dense open set

- Allard - Almgren (1976) : structure of **1D** stationary varifolds

- Pitts (1981) : **almost minimizing** varifolds, **codim 1**

- Schoen - Simon (1981) : **stable** stationary varifolds, **codim. 1**

- Wickramasekera (2014) : **stable** stationary varifolds, **codim. 1**

- Bellettini - Wickramasekera (2017) : **stable** CMC varifolds, **codim. 1**

Open problem: almost everywhere regularity ?

Why is there is more hope in our case to get regularity ??

A naive question

Let $u \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{R}^n)$, weakly conformal s.t.

$$\forall\, X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \qquad \int_\Sigma \nabla u \cdot \nabla[X(u)]\; dx^2 = 0$$

Question : Should we expect

$$\Delta u = 0 \quad \text{and} \quad u \in C^\infty \quad ?$$

Answer : Obviously "no"

We are considering too few variations for the answer to be "yes".

Making the question a bit less naive

Let $u \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{R}^n)$, weakly conformal s.t.

$$\forall\, X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \quad , \quad \forall\, f \in C^1(\mathbb{C}, \mathbb{R})$$

then

$$\text{for a. e.} \quad t \in \mathbb{R} \quad \text{s. t.} \quad u\left(f^{-1}(\{t\})\right) \bigcap \text{supp}(X) = \emptyset$$

$$\int_{f>t} \nabla u \cdot \nabla[X(u)]\, dx^2 = 0$$

Question Should we expect

$$\Delta u = 0 \quad \text{and} \quad u \in C^\infty \quad ?$$

Theorem [R. 2017] The answer to the question is "yes".

Parametrized Stationary Integral Varifolds

Let $u \in W^{1,2}(\Sigma, \mathbb{R}^n)$, weakly conformal and $N \in L^\infty(\Sigma, \mathbb{N})$ s.t.

$$\forall X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \quad , \quad \forall f \in C^1(\Sigma, \mathbb{R})$$

then

$$\text{for a. e.} \quad t \in \mathbb{R} \quad \text{s. t.} \quad u\left(f^{-1}(\{t\})\right) \bigcap \text{supp}(X) = \emptyset$$

$$\int_{f>t} N\, \nabla u \cdot \nabla [X(u)]\ dx^2 = 0$$

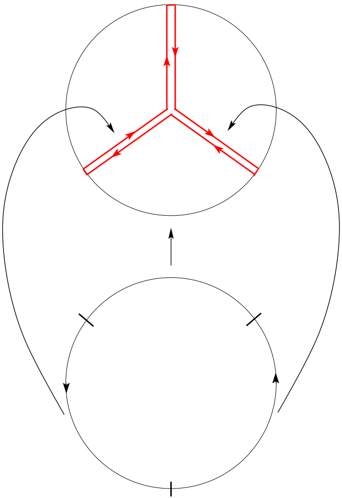
(Σ, u, N) : Parametrized Stationary Integral Varifolds.

Question Should we expect

$$N \equiv Cte \quad , \quad \Delta u = 0 \quad \text{and} \quad u \in C^\infty \quad ?$$

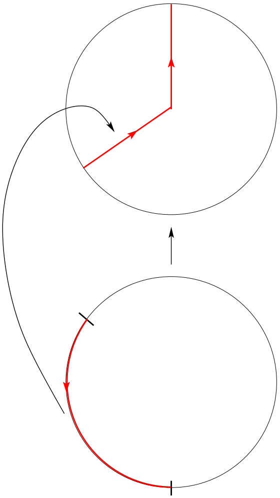
Theorem [Pigati, R. 2020] The answer to the question is "yes".

Parametrized Integer Rectifiable Varifolds



parametrized varifold

Parametrized Integer Rectifiable Varifolds



This is **not** a parametrized integer rectifiable stationary varifold

Stationarity of the Limiting Varifold .

Denote $u_k := u_{\sigma_k}$. We have for any $X \in \Gamma(TM^m)$

$$\int_\Sigma du_k \cdot d\left(X \circ u_k\right)\ dvol_\Sigma + o(1)$$

in local conf. chart

$$du_k \cdot d\left(X \circ u_k\right)\ dvol_\Sigma = \sum_{l=1}^{2}\sum_{i,j=1}^{n} \partial_{x_l} u_k^i\, \partial_{x_l} u_k^j\, \partial_{z_j} X^i \circ u_k\ dx^2$$

$$= \sum_{l=1}^{2} (e_l \cdot \nabla_{e_l} X) \circ u_k\ dvol_{u_k} = \mathrm{div}^{u_{k*}(T_x\Sigma)} X \circ u_k\ dvol_{u_k}$$

Hence since $d\mathbf{v}_k(z,P) := d\left(\mathcal{H}^2 \llcorner u_k(\Sigma)\right) \otimes \sum_{u_k(x)=z} \delta_{u_k(T_x\Sigma)}$

$$o(1) = \int_\Sigma \mathrm{div}^{u_{k*}(T_x\Sigma)} X \circ u_k\ dvol_{u_k} = \int_{G_2(TM^m)} \mathrm{div}^{\mathcal{P}} X(z)\ d\mathbf{v}_k(z,\mathcal{P})$$

$$\Longrightarrow \quad \int_{G_2(TM^m)} \mathrm{div}^{\mathcal{P}} X(z)\ d\mathbf{v}_\infty(z,\mathcal{P}) = 0\ .$$

Main Question to Solve

$$d\mathbf{v}_\infty(z,P) := d\left(\mathcal{H}^2 \llcorner u_\infty(S)\right) \otimes \sum_{u_\infty(x)=z} N_\infty(x)\, \delta_{u_\infty(T_x S)} \quad ?$$

The Monotonicity Formula : $\sigma = 0$, $M^m = \mathbb{R}^m$. Part 1

Let $u \in W^{1,2}(\Sigma, \mathbb{R}^m)$ weak. conf. and harmonic. Then

$$0 = \int_{u^{-1}(B_r(0))} u \cdot \Delta u \; dx^2$$
$$= - \int_{u^{-1}(B_r(0))} |\nabla u|^2 \; dx^2 + \frac{1}{2} \int_{u^{-1}(\partial B_r(0))} \frac{\partial |u|^2}{\partial \nu} \; dl$$

The co-area formula gives

$$\int_{u^{-1}(B_r(0))} |\nabla u|^2 \; dx^2 = \int_0^r ds \int_{|u|=s} \frac{|\nabla u|^2}{|\nabla |u||} \; dl$$

Hence

$$\frac{d}{dr}\left[\frac{1}{r^2} \int_{u^{-1}(B_r(0))} |\nabla u|^2 \; dx^2\right] = \frac{1}{r^2} \int_{u^{-1}(\partial B_r(0))} \frac{|\nabla u|^2}{|\nabla |u||} - 2 \left|\frac{\partial |u|}{\partial \nu}\right| \; dl$$

The Monotonicity Formula : $\sigma = 0$, $M^m = \mathbb{R}^m$. Part 2

We have $|\nabla |u|| = |\partial_\nu |u||$ and conformality of u implies

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial \nu}\right|^2 + \left|\frac{\partial u}{\partial \tau}\right|^2 = 2\left|\frac{\partial u}{\partial \nu}\right|^2$$

Hence

$$\frac{|\nabla u|^2}{|\nabla |u||} - 2\left|\frac{\partial |u|}{\partial \nu}\right| = 2\,\frac{\left|\frac{\partial u}{\partial \nu}\right|^2 - \left|\frac{\partial |u|}{\partial \nu}\right|^2}{|\nabla |u||}$$

which finally gives

$$2\,\frac{d}{dr}\left[\frac{Area(u(\Sigma)\cap B_r^m(q))}{r^2}\right] = \frac{d}{dr}\left[\frac{1}{r^2}\int_{u^{-1}(B_r(0))} |\nabla u|^2\, dx^2\right] \geq 0$$

The Perturbed Monotonicity Formula : $\sigma \neq 0$, $M^m = \mathbb{R}^3$. Part 1

Recall

$$\mathrm{div}\left[f_\sigma\,\nabla u - \ 2\ f'_\sigma\, e^{-2\lambda}\,\nabla\vec{n}\,\dot{\otimes}\,\partial_{x_j}\vec{n}\cdot\partial_{x_j}u + 2\,e^{-2\lambda}\,\mathrm{div}(f'_\sigma\ \nabla\vec{n})\cdot\nabla u\,\vec{n}\right] = 0$$

where $2\,e^{2\lambda} = |\nabla u|^2$ and

$$f_\sigma(y) := 1 + \sigma^2\,y^2 \quad \Longrightarrow \quad f'_\sigma(y) = 2\,\sigma^2\,y$$

for $y = |d\vec{n}_u|^2 = e^{-2\lambda}\,|\nabla n|^2$. We have

$$0 = \Delta u + \sigma^2 \cdots$$

Apply the same strategy as before and one gets...

The "Perturbed" Monotonicity Formula : $\sigma \neq 0$

... $\forall\ r > 0$ and $q \in M^m$

$$\frac{d}{dr}\left[e^{C\,r}\,\frac{Area(u(\Sigma)\cap B_r^m(q))}{r^2} - \frac{C_0}{r^2}\sigma^2\ \int_{\Sigma\cap B_r^m(q)}|d\vec{n}_u|^4\ dvol_{g_u}\right]$$

$$\geq -\frac{C_0}{r^2}\ \sigma^2\ \int_{\Sigma\cap\partial B_r^m(q)}|d\vec{n}_u|^4\ dvol_{g_u}$$

A substitute to the $\epsilon-$Regularity.

Energy Quantization Lemma There exists $Q_0 > 0$ s.t. for $\sigma_k \to 0$

- $$D\left[\text{Area} + \sigma_k^2 \int_\Sigma |d\vec{n}|^4 \ dvol_g\right](u_k) = 0 \ ,$$
- $$\limsup_{k\to+\infty} \text{Area}(u_k) < +\infty \ ,$$
- $$f(\sigma_k) := \frac{\sigma_k^2 \displaystyle\int_\Sigma |d\vec{n}_{u_k}|^4 \ dvol_{g_{u_k}}}{\text{Area}(u_k)} = o\left(\frac{1}{\log(1/\sigma_k)}\right) \ .$$

then

$$\liminf_{k\to+\infty} \text{Area}(u_k) \geq Q_0 > 0$$

□

Proof of the Energy Quantization Lemma

Let $\delta > 0$ and denote E_δ the set of $q \in u(\Sigma)$ such that

i)

$$\sup_{r>\sigma} \frac{\sigma^2 \displaystyle\int_{\Sigma\cap B^n_r(q)} |d\vec{n}_u|^4 \; dvol_{g_u}}{r^2} \leq \frac{\delta}{\log(1/\sigma)}$$

ii)

$$\frac{\mathrm{Area}(u(\Sigma) \cap B^n_\sigma(q))}{\sigma^2} > \frac{\pi}{3}$$

A Besicovitch covering argument gives

$$\mathrm{Area}(u(\Sigma) \setminus E_\delta) \leq \delta^{-1} f(\sigma) \log \sigma^{-1} \mathrm{Area}(u(\Sigma)) = o(1) \; \mathrm{Area}(u(\Sigma))$$

Integrate the perturbed monotonicity between σ and 1 centred at $p \in E_\delta$...

Subsequence extraction

We assume $(\Sigma, h_k := [g_{u_k}])$ is pre-compact in the Moduli Space of Riemann Surfaces

$$\nu_k := |du_k|^2_{h_k} \; dvol_{h_k} \rightharpoonup \nu_\infty$$

and w.l.o. generality we assume

$$[g_{u_k}] \equiv h_\infty$$

Energy Quantization $\Longrightarrow$ Dichotomy Lemma

Dichotomy Lemma There exists $\varepsilon_0 > 0$ and $C_0 > 0$ s.t. if

$$\begin{cases} u_k \longrightarrow u_\infty \quad \text{in } C^0(\partial B_t(x_0)) \quad \text{and} \\ s := \text{ diam}(u_\infty(\partial B_t(x_0))) \le \varepsilon_0 \end{cases}$$

then

- either
$$\nu_\infty(\overline{B_t(x_0)}) \ge \frac{Q_0}{2} \ ,$$
- or
$$\begin{cases} (u_k \llcorner B_t(x_0))_* \nu_k \llcorner \ \mathbb{R}^n \setminus B_{4s}(u_\infty(x_0)) \ \rightharpoonup \ 0 \\ \text{diam } (u_\infty(\overline{B_t(x_0)})) \le C_0 \ \text{ diam } (u_\infty(\partial B_t(x_0))) \\ \sqrt{\nu_\infty(B_t(x_0))} \le C_0 \ \text{ diam}(u_\infty(\partial B_r(x_0))) \end{cases}$$

The continuity of u_∞ away from finitely many atoms

Lemma There exist $p_1 \cdots p_Q \in \Sigma$ s.t.

$$u_\infty \in C^0_{loc}(\Sigma \setminus \{p_1 \cdots p_Q\})$$

Proof of the Partial Continuity of u_∞

Proof Let $x_0 \in \Sigma$ s.t. $\nu_\infty(\{x_0\}) < Q_0/2$. Let $\varepsilon > 0$,

$$\exists r > 0 \quad \text{s. t.} \quad \begin{cases} \nu_\infty(\overline{B_{2r}(x_0)}) < Q_0/2 \quad \text{and} \\ \displaystyle\int_{B_{2r}(x_0)} |\nabla u_\infty|^2 \, dx^2 < \varepsilon^2 \end{cases}$$

Then there exists $t \in (r, 2r)$ and k' s.t

$$\begin{cases} \mathrm{diam}(u_\infty(\partial B_t(x_0))) \leq \displaystyle\int_{\partial B_t(x_0)} |\nabla u_\infty| dl \leq 4\sqrt{\int_{B_{2r}(x_0)} |\nabla u_\infty|^2 \, dx^2} \\ \displaystyle\limsup_{k' \to +\infty} \int_{\partial B_t(x_0)} |\nabla u_{k'}|^2 dl \leq r^{-1} \, \nu_{k'}(B_{2r}(x_0)) \end{cases}$$

$$\Longrightarrow \quad u_{k'} \longrightarrow u_\infty \quad \text{in } C^0(\partial B_t(x_0))$$

Dichotomy Lemma implies

$$\mathrm{diam}(u_\infty(B_t(x_0))) \leq C \;\, \mathrm{diam}(u_\infty(\partial B_t(x_0))) \leq C \, \varepsilon$$

A preliminary Structure Result for ν_∞

Lemma 1 There exist $f \in L^1(\Sigma)$ and $p_1 \cdots p_Q \in \Sigma$ s.t.

$$\nu_\infty = f \; dvol_{h_\infty} + \sum_{i=1}^{Q} c_i \; \delta_{p_i} \; .$$

where $c_i > 0$.

Proof of the preliminary Structure Result for ν_∞

There can be only finitely many atoms where $\nu_\infty(\{p_i\}) \geq \frac{Q_0}{2}$ ..
Let $K \subset \Sigma \setminus \{p_1 \cdots p_Q\}$ compact s. t. $\mathcal{H}^2(K) = 0$. Let $r > 0$ and $(x_i)_{i\in I}$ s.t.

$$K \subset \bigcup_{i\in I} B_r(x_i) \quad \text{and} \quad \sum_{i\in I} \mathbf{1}_{B_{2r}(x_i)} \leq C$$

Hence $\forall i \in I \ \exists t_i \in (r, 2r)$

$$\nu_\infty(B_r(x_i)) \leq C_0 \ \ \text{diam}^2(u_\infty(\partial B_{t_i}(x_0))) \leq C_0 \left[\int_{\partial B_{t_i}(x_0)} |\nabla u_\infty| \ dl\right]^2$$

$$\leq C_0 \ \int_{B_{2r}(x_i)} |\nabla u_\infty|^2 \ dx^2$$

$$\Longrightarrow \quad \nu_\infty(K) \leq C \ \int_{\text{dist}(x,K)<2r} |\nabla u_\infty|^2 \ dx^2 \ .$$

This holds $\forall r > 0$ hence $\nu_\infty(K) = 0$.

L^1 Convergence relative to ν_k

Lemma 2 Let $K \subset \Sigma \setminus \{p_1 \cdots p_Q\}$ compact. There holds

$$\lim_{k \to +\infty} \int_K |u_k - u_\infty| \; d\nu_k = 0 \ .$$

and

$$(u_k \llcorner K)_* \nu_k \ \rightharpoonup \ (u_\infty \llcorner K)_* \nu_\infty$$

Proof of the L^1 Convergence relative to ν_k

Let $r > 0$ and $(x_i)_{i\in 1\cdots L}$, s.t.

$$K \subset \bigcup_{i\in I} B_r(x_i) \quad \text{and} \quad \sum_{i\in I} \mathbf{1}_{B_{2r}(x_i)} \leq C$$

Hence $\exists t_i \in (r, 2r)$ and a subsequence k'

$$u_{k'} \longrightarrow u_\infty \quad \text{in } C^0\left(\bigcup_{i=1}^{L} \partial B_{t_i}(x_i)\right)$$

Let $s_i :=\ \mathrm{diam}(u_\infty(\partial B_{t_i}(x_0)))$. Dichotomy Lemma $\Longrightarrow$

$$\int_{B_{t_i}(x_0)} |u_{k'} - u_\infty|\ d\nu_{k'} = \int_{B_{t_i}(x_0)\cap u_{k'}^{-1} B_{4\,s_i}(u_\infty(x_0))} |u_{k'} - u_\infty|\ d\nu_{k'} + o(1)$$

$$\leq C\, s_i\, \nu_{k'}(B_{t_i}(x_0)) + o(1)$$

$$\Longrightarrow \limsup_{k'\to+\infty} \int_K |u_{k'} - u_\infty|\ d\nu_{k'} \leq C \max\, s_i\, \nu_\infty(\{\mathrm{dist}(x, K) \leq r\}) = o_r(1)$$

The final Structure Result for ν_∞

Lemma 3 There exists $N \in L^\infty(\Sigma, \mathbb{N})$ and $p_1 \cdots p_Q \in \Sigma$ s.t.

$$\nu_\infty = N \, |du_\infty \wedge du_\infty|_{h_\infty} \, dvol_{h_\infty} + \sum_{i=1}^{Q} c_i \, \delta_{p_i} \ .$$

where $c_i > 0$.

In local coordinates

$$|du_\infty \wedge du_\infty|_{h_\infty} \, dvol_{h_\infty} = |\partial_{x_1} u_\infty \wedge \partial_{x_2} u_\infty| \, dx_1 \wedge dx_2$$

Proof of the final Structure Result for ν_∞ - page 1

Choose a Lebesgue point p for f, u_∞ and ∇u_∞ and (x_1, x_2) s. t.

$$h_\infty = e^{2\lambda}\, dx_1^2 + dx_2^2 \quad , \quad x(p) = 0 \quad \text{and} \quad f(0) > 0 \ .$$

Hence

- $$\frac{\nu_\infty(B_r(0))}{\pi\, r^2} = f(0)\ e^{2\lambda(0)} + o_r(1)$$
- $$\fint_{B_r(0)} |\nabla u_\infty(x) - \nabla u_\infty(0)|^2\ dx^2 = o_r(1)$$
- $$\fint_{B_r(0)} \frac{|u_\infty(x) - u_\infty(0) - \nabla u_\infty(0)\cdot x|^2}{r^2}\ dx^2 = o_r(1)$$

$$\Longrightarrow \quad \forall t > 0 \quad \text{small enough} \quad \exists r \in (t/2, t)$$

$$\|u_\infty(x) - u_\infty(0) - \nabla u_\infty(0)\cdot x\|_{L^\infty(\partial B_r(0))} = o(r)$$

Proof of the final Structure Result for ν_∞ - page 2

A diagonal arg. gives $r_j \to 0$ and $k_j \to +\infty$ s.t.

i)

$$\left|\nu_{k_j}(B_{r_j}(0)) - \nu_\infty(B_{r_j}(0))\right| = o(r_j^2)$$

ii)

$$\fint_{B_{r_j}(0)} |u_{k_j} - u_\infty|^2 \, dx^2 = o(r_j^2)$$

iii)

$$\fint_{B_{r_j}(0)} |u_{k_j} - u_\infty| \, d\nu_{k_j} = o(r_j)$$

iv)

$$r_j^{-1} \sqrt{\sigma_{k_j}} = o(1) \quad \left(\Longrightarrow \log \sigma_{k_j} \simeq \log \sigma_{k_j}/r_j\right)$$

v)

$$\frac{\sigma_{k_j}^2}{r_j^2} \log \sigma_j^{-1} \int_{B_{r_j}(0)} |d\vec{n}_{k_j}|^4 \, dvol_{k_j} = o(1)$$

Proof of the final Structure Result for ν_∞ - page 3

Assuming $u_{k_j}(0) = 0 = u_\infty(0)$ introduce

$$x := r_j\, y \quad , \quad \hat{u}_{k_j}(y) := r_j^{-1}\, u_{k_j}(r_j\, y) \quad \text{and } \hat{n}_j := \vec{n}_{k_j}(r_j\, y)$$

There holds

$$\int_{B_{r_j}} dvol_{u_{k_j}} + \sigma_{k_j}^2 \int_{B_{r_j}} |d\vec{n}_{k_j}|^4_{u_{k_j}}\ dvol_{u_{k_j}}$$

$$= r_j^2 \int_{B_1} dvol_{\hat{u}_j} + \frac{\sigma_{k_j}^2}{r_j^2} \int_{B_1} |d\hat{n}_j|^4_{\hat{u}_j}\ dvol_{\hat{u}_j}$$

Introduce $\hat{\sigma}_j := \sigma_{k_j}/r_j^2 = o(1)$. Hence $\hat{u}_j$ is a critical point of

$$F^{\hat{\sigma}_j}(\hat{u}) = \int_{B_1} dvol_{\hat{u}} + \hat{\sigma}_j^2 \int_{B_1} |d\hat{n}|^4_{\hat{u}}\ dvol_{\hat{u}}$$

satisfying

$$\lim_{j\rightarrow+\infty} F^{\hat{\sigma}_j}(\hat{u}_j) = \pi\, f(0)\ e^{2\lambda(0)} \quad \text{and} \quad \hat{\sigma}_j^2 \int_{B_1} |d\hat{n}_j|^4_{\hat{u}_j}\ dvol_{\hat{u}_j} = o\left(\frac{1}{\log \hat{\sigma}_j^{-1}}\right)$$

Proof of the final Structure Result for ν_∞ - page 4

Denote

$$\hat{u}_{j,\infty}(y) = r_j^{-1}\, u_\infty(r_j\, y) \;\longrightarrow\; \hat{u}_{\infty,\infty}(y) := \partial_{x_1} u_\infty(0)\, y_1 + \partial_{x_2} u_\infty(0)\, y_2$$

Assume

$$\mathrm{Span}(\partial_{x_1} u_\infty(0), \partial_{x_2} u_\infty(0)) \subset \mathrm{Span}((1,0\cdots 0),(0,1,0\cdots 0)) = \mathcal{P}_0$$

From ii) and iii) we have

$$\int_{B_1} |\hat{u}_j - \hat{u}_{j,\infty}|^2\, dy^2 = o(1) \quad \text{and} \quad \int_{B_1} |\hat{u}_j - \hat{u}_{j,\infty}|\, |\nabla_y \hat{u}_j|^2\, dy^2 = o(1)$$

Hence, multiplying the $l-th$ comp. of the PDE by $\hat{u}_j^l\, \chi(\hat{u}_j)$ where $\chi \in C_0^\infty(\mathbb{R}^n)$ gives

$$\sum_{l=3}^{n} \int_{\mathbb{C}} |\nabla \hat{u}_j^l|^2\, \chi(\hat{u}_j)\, dy^2 = o_j(1) \quad \Longrightarrow d\hat{\mathbf{v}}_j \rightharpoonup d\hat{\mathbf{v}}_\infty = \mu_\infty(y_1,y_2) \otimes \delta_{\mathcal{P}_0}$$

where $\mathrm{Supp}(\mu) \subset \mathcal{P}_0$

Proof of the final Structure Result for ν_∞ - page 5

The Stationarity of $\hat{\mathbf{v}}_\infty$ + Constancy Theorem imply $\exists\,\theta_0 > 0$ s.t.

$$d\hat{\mathbf{v}}_\infty = \theta_0 \; d\mathcal{H}^2 \llcorner \mathcal{P}_0 \otimes \delta_{\mathcal{P}_0}$$

The PDE implies, restricting to $a, b \in C^\infty_0(B_1(0))$, (one cannot test $(a \circ \hat{u}_j, b \circ \hat{u}^j, 0 \cdots 0)$ directly....some smoothing of (a, b) is required)

$$\sup_{\|\nabla_{z_1,z_2} a\|_\infty + \|\nabla_{z_1,z_2} b\|_\infty \le 1} \int_{\mathbb{C}} \nabla(a \circ \hat{u}_j) \cdot \nabla \hat{u}^1_j + \nabla(b \circ \hat{u}_j) \cdot \nabla \hat{u}^2_j \; dy^2 = o(1)\,.$$

The almost conformality of $(y_1, y_2) \to \tilde{u}_j = (\hat{u}^1_j, \hat{u}^2_j)$ + co-area formula

$$\sup_{\|\nabla_{z_1,z_2} a\|_\infty + \|\nabla_{z_1,z_2} b\|_\infty \le 1} \int_{\mathcal{P}_0} \mathrm{Card}(\tilde{u}_j^{-1}(z_1, z_2)) \; \partial_{z_1} a + \partial_{z_2} b = o(1)$$

Proof of the final Structure Result for ν_∞ - page 6

Lemma Let $N_j \in L^1(B_1^2(0))$ s.t.

$$\lim_{j\to+\infty} \sup_{\|\nabla_{z_1,z_2} a\|_\infty + \|\nabla_{z_1,z_2} b\|_\infty \le 1} \int_{B_1(0)} N_j \, \partial_{z_1} a + \partial_{z_2} b = 0$$

then

$$\left| N_j - \fint_{B_{1/2}(0)} N_j \right|_{L^{1,\infty}(B_{1/2}(0))} = o(1)$$

□

Proof of the final Structure Result for ν_∞ - page 7

Hence

$$\text{dist}\left(\fint_{B_{1/2}} \text{Card}\left(\tilde{u}_j^{-1}(z_1,z_2)\right), \mathbb{N}\right) = o(1)$$

$$\Longrightarrow \quad \lim_{j\to+\infty} N_j = N_\infty \in \mathbb{N}^*$$

Since

$$\hat{u}_j(y) \to \hat{u}_\infty(y) = \partial_{x_1} u_\infty(0)\, y_1 + \partial_{x_2} u_\infty(0)\, y_2 \quad \text{ in } C^0_{loc}(\mathbb{C})$$

then

$$\pi\ f(0)\, e^{2\lambda(0)} = \lim_{j\to+\infty} \int_{B_1(0)} 2^{-1}\, |\nabla \hat{u}_j|^2\ dy^2$$

$$= \lim_{j\to+\infty} \int_{\hat{u}_j(B_1(0))} 2^{-1} \sum_{\tilde{u}_j(y)=p} \frac{|\nabla \hat{u}_j|^2}{|\partial_{y_1}\tilde{u}_j \wedge \partial_{y_2}\tilde{u}_j|}\ dp^2$$

$$= \int_{\hat{u}_\infty(B_1(0))} N_\infty\ dp^2 = \pi\, N_\infty\, |\partial_{x_1} u_\infty \wedge \partial_{x_2} u_\infty|$$