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Contributions to calibrated geometric analysis and analytical gauge theory in supercritical dimension

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RICCARDO CANIATO

MSc in Mathematics at University of Trieste
born on the 9th of November 1995
citizen of Italy

accepted on the recommendation of

Prof. Dr. Tristan Rivière, examiner
Prof. Dr. Costante Bellettini, co-examiner

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Abstract

This doctoral dissertation consists in a structured and organized presentation of three scientific papers ([22], [23] and [25]) co-signed by the author, along with some new contributions. The main focus of the discussion will be on several contributions to geometric analysis, specifically to the field of harmonic maps between manifolds (in particular we will focus on pseudo-holomorphic maps) and to analytical gauge theories in supercritical dimension.

In Chapter 2, the author discusses the uniqueness of tangent maps for weakly pseudo-holomorph-ic and locally approximable maps from an almost complex manifold to projective algebraic varieties, while also obtaining a new statement about uniqueness of tangent cones to a special class of non-rectifiable semicalibrated cycles.

In Chapter 3, the author identifies the strong L^p-closure of vector fields that have finitely many integer topological singularities on a domain, and provides a useful characterization of this class of objects in terms of the existence of a (minimal) connection for their singular set. Furthermore, the author establishes that such strong closure is also the weak closure in L^p of the same space.

Chapter 4 concerns the regularity of weak Yang-Mills fields in supercritical dimension, establishing Coulomb gauge extraction and ε-regularity type results for a special class of very weak L^2-connections.

Sommario

Questa tesi di dottorato consiste in una presentazione strutturata e organizzata di tre articoli scientifici ([22], [23] e [25]) co-firmati dall'autore, insieme ad alcuni nuovi risultati. Il focus principale della discussione sarà su diversi contributi all'analisi geometrica, più specificamente nel campo delle mappe armoniche tra varietà (in particolare verranno trattate le mappe pseudo-olomorfe) e riguardo le teorie di gauge in dimensione supercritica da un punto di vista analitico.

Nel capitolo 2, l'autore discute l'unicità delle mappe tangenti per mappe debolmente pseudo-olomorfe e localmente approssimabili da una varietà quasi complessa a varietà algebriche proiettive, ottenendo anche un nuovo risultato sull'unicità dei coni tangenti a una classe speciale di cicli semicalibrati non rettificabili .

Nel capitolo 3, l'autore identifica la chiusura forte in L^p dei campi vettoriali che hanno un numero finito di singolarità topologiche con grado intero su un dominio euclideo n-dimensionale e fornisce un'utile caratterizzazione di questa classe di oggetti in termini dell'esistenza di un connessione per il loro insieme singolare. Inoltre, l'autore stabilisce che tale chiusura forte è anche chiusura debole in L^p dello stesso spazio.

Il capitolo 4 verte sulla regolarità dei campi deboli di Yang-Mills in dimensione supercritica, stabilendo risultati di tipo ε-regolarità ed estrazione di Coulomb gauge per una classe speciale di connessioni deboli.

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easy on your part. And Moreno, I want to thank you for showing me that sometimes a simple joke can be just what you need to get through a difficult time in your life.

Contents

1. Introduction

1.1. Calculus of variations and the direct method

Calculus of variations is a branch of mathematics concerned with the existence, uniqueness, and regularity of extremal points of functionals, which are real-valued maps $\mathscr{F} : X \to \mathbb{R}$ defined on a topological function space X. The space X represents the set of configurations or states that a given system can assume, while the functional $\mathscr{F}$ represents the cost or profit that results from the system being in a particular configuration. Without losing generality, throughout this thesis we will assume that a functional represents a cost and our objective will be to minimize it. Moreover, we will frequently use standard terminology from physics and refer to a cost function $\mathscr{F}$ as a *lagrangian* on the configuration space X.

Calculus of variations is distinguished from discrete optimization, a method frequently used in operations research, by the nature of the configuration space and the method of analysis. As the name suggests, calculus of variations examines the response of the cost function to infinitesimal variations in the system state with respect to a starting configuration $u \in X$. Therefore, to apply calculus of variations effectively, the problem must be formulated in a space X whose topology permits infinitesimal variations. Real vector spaces, of either finite or infinite dimension, are commonly used for this purpose, and in this thesis, we will exclusively analyze functionals defined on real vector spaces of infinite dimension.

Optimization problems can be classified into two primary categories. In unconstrained optimization, the objective is to find the minimum of the cost function over the entire state space. In constrained optimization, the minimum is sought within a specific subspace $K \subset X$ of the ambient space X. The subspace K is said to be a *constraint* for the problem. Linear constraints, where K is a vector subspace of X, are particularly common. However, interesting more general constraints are often given by convex subsets of the ambient space.

The most frequently used argument to prove the existence of extremal points is the so-called *direct method*: if F is sequentially lower semicontinuous and the subspace K where the minimization is to be performed is compact, then Weierstrass theorem guarantees that F will attain a global minimum over K. It is worth noting that the two conditions required for the application of the direct method are antagonistic to each other. In particular, as the topology of the configuration space becomes stronger, ensuring the continuity of the cost function becomes easier, but it becomes more challenging to ensure the compactness of the constraint. Thus, in the attempt of showing the existence of minimizers for certain variational problems there is a need of enlarging the ambient space X and of weakening its topology suitably in order to make the competition class weakly sequentially compact. This process typically leads to a dramatic loss of regularity: weak solutions to variational problems may a priori belong to very wild spaces containing functions that might be even everywhere discontinuous. Nevertheless, if the ambient space X and its topology were chosen carefully enough, then one could hope of exploiting the special properties of such weak solutions to recover their partial of full regularity. Indeed, any extremal point $u \in K \subset X$ of some smooth functional $\mathscr{F} : X \to \mathbb{R}$ over the competition class K satisfies

$$\frac{d}{dt}\bigg|_{t=0} \mathscr{F}(\gamma(t)) = 0,$$

for every smooth path $\gamma : [0,1] \to K$ such that $\gamma(0) = u$. The previous information is actually equivalent to saying that u satisfies, in a suitable weak sense to be specified, a set of partial differential equations (PDEs) known as *Euler–Lagrange equations* for the functional $\mathscr{F}$ under the constraint K. This is one of the main reasons why it is so important to develop a *regularity theory* for certain types of PDEs. The celebrated mathematician David Hilbert himself underlined the great need of investigating such issues in the formulation of his 19th problem, eventually solved by Ennio De Giorgi and John Forbes Nash independently.

Throughout our discussion, we will be concerned with minimizing functionals $\mathscr{F} = \mathscr{F}(u, du)$ depending just on a function u and on its first derivatives. The Euler-Lagrange equations for such functionals will turn into a set of second order PDEs and either they will or we will try to put them in the form

$$Lu = f(u, du)$$

where L represents an *elliptic* second order differential linear operator and f is a smooth function (we address the reader to the classical references [5], [39] and [40] for a detailed description of elliptic operators and their behavior). In this sense, all the issues that will be discussed further in this dissertation could be described as elliptic regularity problems arising from the variational analysis of some lagrangian $\mathscr{F}$.

1.2. The interface with geometry and the foundations of geometric analysis

The connection between calculus of variations and geometry is highly intuitive because of the inherent geometric content of partial differential equations that stem from variational principles. In fact, significant geometric structures can often be defined as critical points of certain "geometric functionals". For example, Yang–Mills fields (essential objects in high-energy and particle physics) can be characterized as critical points of the Yang–Mills lagrangian (see Section 4.1).

In light of all this, *geometric analysis* emerged as an interdisciplinary field that draws upon tools and techniques from a variety of mathematical disciplines, including differential geometry, geometric measure theory, partial differential equations and calculus of variations. Its aim is to tackle a diverse range of geometric problems that arise in areas such as mathematical physics, materials science, and other branches of mathematics. Key topics in geometric analysis include the study of minimal surfaces, harmonic maps, geometric flows and gauge theories. Among many significant achievements derived from the application of the methods of geometric analysis, we highlight the the work by T. Radó and J. Douglas on minimal surfaces, the results by K. Uhlenbeck and S. Donaldson in the 1980s about the Yang–Mills lagrangians in dimension 4 and their applications to the study of the differential invariants of 4-manifolds, the foundations of geometric measure theory and the development of essential tools such as currents and varifolds as weak versions of k-surfaces and the proof of the Poincarè conjecture by G. Perelman in 2006.

The primary focus of this thesis will be on a number of issues related to geometric analysis. Specifically, we will concentrate on investigating Yang–Mills fields and pseudo-holomorphic maps. These maps can be regarded as a very special class of harmonic maps that solve a “first-order version” of the non-linear harmonic maps equation.

1.3. Calibrated solutions to variational problems

Many of the the techniques that we will be utilizing in the following chapters belongs to the framework of *calibrated geometric analysis.* This branch of the mathematics was developed in the late 20th century to investigate geometric structures in higher dimensions is built upon the concept of *calibration* and *calibrated submanifolds*, which we recall here for the reader's convenience.

Definition 1.3.1 (Calibration)**.** Let (M^n, g) be an n-dimensional Riemannian smooth manifold and let $1 \leqslant k \leqslant n$. A differential k-form ω is said to be a *calibration* on M if it is closed and its comass equal to 1, i.e. $d\omega = 0$ and

$$\|\omega\|_* := \sup\left\{\langle \omega_x, \xi\rangle \text{ for every } x \in M,\, \xi \in \textstyle\bigwedge^k T_x M \text{ with } |\xi|_g = 1\right\} = 1.$$

Definition 1.3.2 (Calibrated currents and submanifolds)**.** Let (M^n, g) be an n-dimensional Riemannian manifold and let $1 \leqslant k \leqslant n$. A normal k-current $T \in \mathcal{N}_k(M)$ for some $k \in \mathbb{N}$ is said to be *calibrated* by a given calibration ω on M if one of the following equivalent conditions hold:

(1) the measure theoretic orientation $\vec{T}$ of T is a convex linear combination of k-vectors calibrated by ω (i.e. such that their duality with ω is unitary), $||T||$-a.e. on M where $||T||$ stands for the total variation of T;

(2) $\mathbb{M}(T) = \langle T, \omega\rangle$.

Let $\Sigma^k \subset M$ be an embedded, oriented k-submanifold of M with finite k-area and whose boundary has finite $(k-1)$-area. We say that Σ is *calibrated* by ω if the normal k-current $[\Sigma]$ is calibrated by ω, i.e. if and only if

$$\mathrm{Area}(\Sigma) = \mathbb{M}([\Sigma]) = \int_\Sigma \omega.$$

The notion of calibration has a long and rich history. The paper which gave its name to the corresponding general mathematical notion is the famous work of Harvey and Lawson [45] but complex analytic submanifolds and calibrated subvarieties had been introduced even before. We invite the reader to consult the works of F. Morgan [58] and [59] for a more complete introduction to this important object of geometry. The idea of calibration was first introduced in the context of minimal surfaces, but it was later generalized to a broader range of geometric structures, including special Lagrangian submanifolds.

One of the reasons why calibrated currents have been very much studied is that calibrated k-cycles are homologically mass-minimizing. Indeed, let $T \in \mathcal{D}_k(M)$ be a cycle (i.e. $\partial T = 0$) which is calibrated by some calibration ω. If we pick any other cycle $T' \in \mathcal{D}_k(M)$ in the same homology class of T, i.e. such that $T - T' = \partial S$ for some $S \in \mathcal{D}_{k+1}(M)$, we immediately get

$$\mathbb{M}(T) = \langle T, \omega\rangle = \left\langle T' + \partial S, \omega\right\rangle = \left\langle T', \omega\right\rangle \leqslant \mathbb{M}(T'), \tag{1.3.1}$$

where we used Stokes theorem, $d\omega = 0$ and $\|\omega\|_* = 1$. Unfortunately, it happens very often that the closeness requirement in the definition of calibration is too strong to suit certain problems, such as some of the ones we will be interested in. Therefore, as for example in [69], it is natural to study semicalibrations and semicalibrated cycles, being simply currents that are calibrated (in the sense of (1) or (2) in Definition 1.3.2) by possibily non-closed k-forms having unitary comass.

1.4. Organization of the thesis and main results

This doctoral dissertation is a structured elaboration of the results presented in three scientific papers ([22], [23], and [25]) co-signed by the author, along with some new contributions. Here, we provide a brief overview of the contents of each chapter. For more detailed explanations, we refer the reader to the individual introductions of the chapters themselves.

Chapter 2 collects the results of [25], where we establish the uniqueness of tangent maps for weakly pseudo-holomorphic and locally approximable maps from any almost complex manifold to projective algebraic varieties. As a byproduct, we obtain the unique tangent cone property for a special class of non-rectifiable semicalibrated pseudo-holomorphic cycles. This also provides a new proof of the main result by C. Bellettini in [7] concerning the uniqueness of tangent cones for positive integral (p, p)-cycles in any almost complex manifold.

Chapter 3 outlines the results of [22] and [23], where we identify the strong L^p-closure $L^p_{\mathbb{Z}}(D)$ of vector fields that have finitely many integer topological singularities on a domain D, which is either bi-Lipschitz equivalent to the open unit n-dimensional cube or to the boundary of the unit $(n + 1)$-dimensional cube, for any $p \in [1, +\infty)$ and $n \in \mathbb{N}$ with $n \geqslant 1$. Additionally, we prove that $L^p_{\mathbb{Z}}(D)$ is weakly sequentially closed for every $p \in (1, +\infty)$ and $n \in \mathbb{N}$ with $n \geqslant 2$, if D is an open domain in $\mathbb{R}^n$ that is bi-Lipschitz equivalent to the open unit cube. As a result of this analysis, we obtain a useful characterization of this class of objects in terms of the existence of a (minimal) connection for their singular set.

Chapter 4 concerns the most recent contributions of the author and T. Rivière to the regularity of weak Yang–Mills fields in supercritical dimension. In particular, Coulomb gauge extraction and ε-regularity type results à la Uhlenbeck are established for a special class of very weak L^2-connections on the open unit cube $Q^5_1(0)$.

2. The regularity of pseudo–holomorphic maps into projective algebraic varieties

2.1. Introduction

2.1.1. Weakly pseudo-holomorphic and locally approximable maps

For the purposes of this introduction, we always denote by M a connected smooth manifold without boundary and we will need to endow M with an arbitrarily chosen reference metric g. Since our results are local, all our discussion will be totally independent or, in any case, not essentially effected by such arbitrary choice.

Given a closed (i.e. connected, compact and without boundary) smooth manifold N and a smooth isometric embedding $N \hookrightarrow \mathbb{R}^k$, for some $k \in \mathbb{N}$ large enough, we let

$$W^{1,2}_{loc}(M,N) := \big\{u \in W^{1,2}_{loc}\big(M,\mathbb{R}^k\big) \text{ s.t. } u(x) \in N, \text{ for } \operatorname{vol}_g\text{-a.e. } x \in M\big\}.$$

Definition 2.1.1. Let M be any even-dimensional smooth manifold without boundary and let J be a Lipschitz almost complex structure on M. Let (N, J_N) be any closed smooth almost complex manifold. We say that a map $u \in W^{1,2}_{loc}(M,N)$ is *weakly* (J, J_N)*-holomorphic* if

$$du_x(JX) = J_N du_x(X), \qquad \text{for } \operatorname{vol}_g\text{-a.e. } x \in M, \, \forall\, X \in T_xM.$$

Whenever we don't need to specify which couple of complex structures is involved in the previous definition, we simply say that the map u is weakly pseudo-holomorphic or even just weakly holomorphic to lighten the notation.

Assume that M is any even-dimensional smooth manifold without boundary and let J be a Lipschitz almost complex structure on M which admits a compatible symplectic form Ω, meaning that the bilinear form $(X,Y) \mapsto \Omega(X, JY)$ defines a Lipschitz Riemannian metric on M. We will show (Lemma 2.3.2) that in this particular framework any weakly (J, J_N)-holomorphic map taking values into a closed smooth almost Kähler manifold N is weakly harmonic, i.e.

$$\frac{d}{dt}\int_M \big|d\big(\pi_N \circ (\Phi \circ u + tX)\big)\big|_g^2 \, d\operatorname{vol}_g\bigg|_{t=0} = 0, \qquad \forall\, X \in C^\infty\big(M,\mathbb{R}^k\big),$$

where $\pi_N : W \to N$ is the nearest-point projection into N, defined on a suitable tubular neighbourhood W of N, and $\Phi : N \hookrightarrow \mathbb{R}^k$ denotes a smooth, isometric embedding of N into $\mathbb{R}^k$. Nevertheless, it is well-known that no regularity is ensured for weakly harmonic maps when the dimension of the domain is larger than 2 (see [71]). Thus, we will need to prescribe some additional condition in order to get that the map u is at least stationary harmonic, i.e.

$$\frac{d}{dt}\int_M |d(u \circ \Phi_t)|_g^2 \, d\operatorname{vol}_g\bigg|_{t=0} = 0,$$

for any smooth one-parameter family of diffeomorphisms Φ_t of M with compact support. We will show (see Lemma 2.3.3) that imposing the following local, strong approximability property with respect to the $W^{1,2}$-norm sufficies to our purposes.

Definition 2.1.2. Let M be a smooth manifold without boundary and N be a closed smooth manifold. We say that a map $u \in W^{1,2}_{loc}(M,N)$ is *locally (strongly) approximable* with respect to the $W^{1,2}$-norm if for every open set $U \subset M$ such that U is diffeomorphic to some euclidean ball there exists a sequence of smooth maps $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(U,N)$ such that $u_j \to u$ as $j \to +\infty$, strongly in $W^{1,2}(U,N)$.

If a map u is locally approximable, then the following cohomological condition follows easily:

$$d(u^*\omega) = 0, \qquad \text{distributionally on } M, \tag{2.1.1}$$

for every closed 2-form $\omega \in \Omega^2(N)$. We refer the reader to [11] for further reading concerning the deep link between local approximability and (2.1.1).

In the most general case that we will address, i.e. when J doesn't admit a compatible symplectic form (even locally), weakly holomorphic and locally approximable maps are not stationary harmonic. Nevertheless, they are *almost stationary harmonic*, in the sense that they satisfy a perturbed version of the harmonic map equation and that there exists $C > 0$ such that

$$\frac{d}{dt}\int_M |d(u\circ\Phi_t)|_g^2 \, d\,\mathrm{vol}_g\bigg|_{t=0} \geqslant -C\int_M |X||du|^2,$$

with $X := \partial_t\Phi_t|_{t=0}$ for any smooth one-parameter family of diffeomorphisms Φ_t of M with compact support. We underline that such maps were also studied before by C. Bellettini and G. Tian in [9].

2.1.2. Statement of the main results and related literature

Section Given any $\rho \in (0,+\infty)$, we denote by $B_\rho := \rho\mathbb{B}^{2m} \subset \mathbb{R}^{2m}$ the open unit ball in $\mathbb{R}^{2m}$ centred at the origin and having radius ρ. From now on, for every $\rho \in (0,1)$ we let $\Phi_\rho : B_\rho \to B$ be given by $\Phi_\rho(x) := \rho^{-1}x$, for every $x \in B_\rho$.

Let M be a smooth manifold without boundary and N be a closed smooth manifold. Consider any map $u \in W^{1,2}_{loc}(M,N)$. Given $x_0 \in M$, pick any coordinate chart $\varphi : U \subset M \to \mathbb{B}^{2m}$ with relatively compact domain U at x_0, i.e. such that $x_0 \in U$ and $\varphi(x_0) = 0$. The family of the *blow-ups* of u at the point x_0, denoted by $\{u_\rho\}_{\rho\in(0,1)} \subset W^{1,2}(U,N)$, is given by $u_\rho := u\circ\varphi^{-1}\circ\Phi_\rho^{-1}\circ\varphi$, for every $\rho \in (0,1)$. If such family is bounded in $W^{1,2}(U,N)$, by standard compactness arguments it follows that for every sequence $\rho_k \to 0^+$ as $k \to +\infty$ there exists a subsequence $\{\rho_{k_j}\}_{j\in\mathbb{N}}$ such that $u_{\rho_{k_j}} \rightharpoonup u_\infty \in W^{1,2}(U,N)$, weakly in $W^{1,2}(U,N)$. We say that u_∞ is a *tangent map* for u at the point x_0. Any tangent map at x_0 is meant to represent a picture of the map u when one gets closer and closer to x_0. Such limiting configuration may very well depend on the sequence $\{\rho_{k_j}\}_{j\in\mathbb{N}}$ that we have chosen to approach x_0. If this is not the case, we say that the map u *has a unique tangent map* at the point x_0.

In the present chapter, we aim to give a complete and self-contained proof of the following theorem.

Theorem 2.1.1. *Let $m, n \in \mathbb{N}_0$ be such that $m \geqslant 2$. Let M be a smooth $2m$-dimensional manifold without boundary and let J be any Lipschitz almost complex structure on M. Let $N \subset \mathbb{CP}^n$ be* *a projective algebraic variety. Let $u \in W^{1,2}_{loc}(M,N)$ be weakly (J, j_n)-holomorphic and locally approximable, where j_n stands for the standard complex structure on $\mathbb{CP}^n$ (restricted to N). Then, u has a unique tangent map at every point.*

As we have noted before, if J admits a compatible symplectic form then weakly holomorphic and locally approximable maps are a special subclass in the much wider family of stationary harmonic maps (see previous Section 2.1.1). Hence, our main result relates with the whole set of well-know facts concerning stationary harmonic maps between manifolds (see e.g. [34], [14], [75]). In particular, both the existence of tangent maps at every point and Theorem 2.3.1 are immediate consequences of the general theory of stationary harmonic maps. However, nothing can be said a priori about uniqueness of tangent maps to general stationary harmonic maps, since B. White (see [94]) has shown that such property might fail even for energy-minimizing maps at their singular points. Nevertheless, whenever the target manifold is analytic, uniqueness of tangent maps was proved to hold for energy-minimizing harmonic maps by L. Simon in [84]. Hence, since any projective algebraic variety is analytic, if weakly holomorphic and locally approximable maps were energy-minimizing then Theorem 1.1 would be a direct consequence of Simon's result. Unfortunately, it's not hard to build sequences of even holomorphic maps that converge weakly but **not** strongly in $W^{1,2}_{loc}(\mathbb{R}^{2m}, N)$ (see Example 2.1.1). Since for energy-minimizing harmonic maps weak convergence implies strong convergence (see [79]), this sufficies to convince ourselves that weakly holomorphic and locally approximable maps are **not** energy-minimizing harmonic maps in general.

Example 2.1.1. Let $N = \mathbb{S}^2$, equipped with the standard Kähler structure. Let $S \in \mathbb{S}^2$ be the south-pole in and let $p_S : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection from the south-pole. For $\lambda > 0$, we define the map $u_\lambda : \mathbb{R}^2 \to \mathbb{S}^2$ as follows:

$$u_\lambda(x) := p_S^{-1}(\lambda x) \qquad \forall\, x \in \mathbb{R}^2.$$

For every $\lambda > 0$, u_λ is a finite-energy, orientation-preserving conformal map from $\mathbb{R}^2$ to $\mathbb{S}^2$. Hence, $\{u_\lambda\}_{\lambda>0}$ is a family of holomorphic maps in $W^{1,2}(\mathbb{R}^2, \mathbb{S}^2)$. An easy computation shows that $u_\lambda \rightharpoonup u_\infty \equiv S$ weakly in $W^{1,2}$ as $\lambda \to +\infty$. Nevertheless, the convergence cannot be strong because

$$\int_{\mathbb{R}^2} |du_\lambda|^2 \, d\mathcal{L}^2 \to 8\pi \neq 0 = \int_{\mathbb{R}^2} |du_\infty|^2 \, d\mathcal{L}^2 \qquad (\lambda \to +\infty).$$

Theorem 2.1.1 was already proven when the almost complex structure J on the domain is integrable by S. Sun and X. Chen in [28], thanks also to the previous contributions [49] and [46] who established the optimal bound for the Hausdorff measure of the singular set, namely

$$\mathscr{H}^{2m-4}\big(\mathrm{Sing}(u) \cap K\big) < +\infty, \qquad \forall\, K \subset M \text{ compact.} \tag{2.1.2}$$

Nevertheless, the proof provided by S. Sun and X. Chen in the integrable case makes an extensive use of complex holomorphic coordinates on the base manifold and of several algebraic tools that are not available in case we work in the almost complex framework.

As far as we know, the only available result concerning the non-integrable case that can be found in literature was achieved by the second author and G. Tian in [77]. In such paper, the case of a 4-dimensional domain M is completely solved, providing also the optimal size (2.1.2) for the singular set. Unfortunately, the proof that is given there strongly relies on positive intersection arguments that cannot be reproduced when $m > 2$.

2.1.3. Key ideas to face the non-integrable case in higher dimensions

In view of what we have seen in Section 2.1.2, we need to think of a completely new analytic approach in order to prove Theorem 2.1.1 in its full generality. From now on, we will denote by J_0 the standard complex structure on $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

Let M, N and u satisfy the hypotheses of Theorem 2.1.1. Given any point $x_0 \in M$ and a local chart $\varphi : U \subset M \to \mathbb{B}^{2m}$ with compact domain U at x_0 such that $J(0) = J_0$, it's clear that u has a unique tangent map at x_0 if and only if the local representative $\tilde{u} := u \circ \varphi^{-1} \in W^{1,2}(\mathbb{B}^{2m}, N)$ of u has a unique tangent map at the origin. Notice that $\tilde{u}$ is weakly $(\tilde{J}, j_n)$-holomorphic on $\mathbb{B}^{2m}$, where $\tilde{J} := d\varphi \circ J \circ d\varphi^{-1}$ is a Lipschitz almost complex structure on $\mathbb{B}^{2m}$. Moreover, a straightforward computation allows to conclude that $\tilde{u}$ is locally approximable on $\mathbb{B}^{2m}$. Hence, Theorem 2.1.1 will be proved if we just manage to show that the statement holds in case $M = \mathbb{B}^{2m} \subset \mathbb{R}^{2m}$, $x_0 = 0$, $J(0) = J_0$ and $u \in W^{1,2}(\mathbb{B}^{2m}, N)$.

The fact that we have reduced to prove the statement on the unit open ball leads to a key advantage. Since $\mathbb{B}^{2m}$ is contractible, we can find a Lipschitz almost symplectic form Ω on $\mathbb{B}^{2m}$ which is compatible with $\tilde{J}$, i.e. the symmetric bilinear form $(X, Y) \mapsto g(X, Y) := \Omega(X, JY)$ defines a Lipschitz metric g on $\mathbb{B}^{2m}$. Here and throughout the whole chapter, by "almost symplectic form" we mean any non-degenerate 2-form, even if not necessarily closed (the reader should be aware that the term "almost Hermitian structure" can also be found in literature). From now on, we will assume that the domain $\mathbb{B}^{2m}$ is endowed with such special metric g and all the computations involving scalar products will be referring to this specific choice.

Notice that $\Omega^k/k!$ is a semicalibration on $\mathbb{B}^{2m}$ with respect to the metric g, for every $k = 1, ..., m$ (see Definition 1.3.1. As we will see below, the proof of Theorem 2.1.1 will be further reduced to the problem of showing uniqueness of tangent cones for a special class of non-rectifiable, semicalibrated $(2m-2)$-cycles on B_2. Indeed, let u satisfy the assumptions of Theorem 2.1.1 with $M = \mathbb{B}^{2m}$. We can associate to the map u the $(2m-2)$-current $T_u \in \mathcal{D}_{2m-2}(\mathbb{B}^{2m})$ given by

$$\langle T_u, \alpha \rangle := \int_{\mathbb{B}^{2m}} u^* \omega_{\mathbb{CP}^n} \wedge \alpha, \qquad \forall\, \alpha \in \mathcal{D}^{2m-2}(\mathbb{B}^{2m}).$$

We can show (see Section 2.2) that T_u satisfies the following properties:

1. T_u is a cycle, i.e. $\partial T_u = 0$ in the sense of currents.
2. T_u is normal, with

$$\mathbb{M}(T_u) = \int_{\mathbb{B}^{2m}} \|u^* \omega_{\mathbb{CP}^n}\|_* \, d\,\mathrm{vol}_g = \frac{1}{2} \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\,\mathrm{vol}_g < +\infty.$$

3. T_u is semicalibrated by $\dfrac{\Omega^{m-1}}{(m-1)!}$.

For the reader's convenience, we recall at this point that we say that a current T on $\mathbb{B}^{2m}$ has a *unique tangent cone* at the origin if given any two sequences $\{\rho_k\}_{k \in \mathbb{N}} \subset (0,1)$ and $\{\rho'_k\}_{k \in \mathbb{N}} \subset (0,1)$ such that

1. $\rho_k \to 0^+$ and $\rho'_k \to 0^+$, as $k \to +\infty$,
2. $(\Phi_{\rho_k})_* T \rightharpoonup C_\infty$ and $(\Phi_{\rho'_k})_* T \rightharpoonup C'_\infty$, as $k \to +\infty$,

it follows that $C_\infty = C'_\infty$.

Substantial work on the uniqueness of tangent cones was carried out for special classes of **integral** semicalibrated cycles. In particular, such result was already obtained by C. Bellettini together with the second author in [8] for special legendrian integral cycles in $\mathbb{S}^5$ and C. Bellettini in [7] has proved uniqueness of tangent cone for positive integral (p,p)-cycles in arbitrary almost complex manifolds. The case of positive integral $(1,1)$-cycles in arbitrary almost Kähler manifolds was previously covered by the main regularity result obtained by the second author and G. Tian in [76]. Analogous results were obtained also by C. De Lellis, E. Spadaro and L. Spolaor in [29], by exploiting the fact that any integral semicalibrated k-cycle $T \in \mathcal{D}_k(\mathbb{B}^{2m})$ is *almost* homologically mass-minimizing, i.e. for every $x_0 \in \mathbb{B}^{2m}$ there are constants $C_0, r_0, \alpha_0 > 0$ such that

$$\mathbb{M}\big(T \llcorner B_\rho(x_0)\big) \leqslant \mathbb{M}\big((T + \partial S) \llcorner B_\rho(x_0)\big) + C_0 \rho^{k+\alpha_0}, \qquad \forall\, 0 < \rho < \rho_0$$

and for all $S \in \mathcal{D}_{k+1}(\mathbb{B}^{2m})$ such that $\operatorname{spt}(S) \subset B_\rho(x_0)$ (compare with the stronger property (1.3.1) that holds for calibrated cycles).

Nevertheless, very little is known so far concerning uniqueness of tangent cone when the rectifiability hypothesis is dropped. In general this is not true, counter-examples have been given initially by C. O. Kiselman in [50] and then generalized in [16]. On the other hand, a positive result on this matter was obtained by C. Bellettini in [6], where the author proves that the tangent cone to normal positive $(1,1)$-cycles is unique at any point where the density does not have a jump with respect to all of its values in a neighborhood. Thus, the present chapter is meant to be a contribution to this so far still fairly open class of problems.

In Section 2.6.3 we will show that uniqueness of tangent cone for the $(2m-2)$-dimensional cycle T_u and for its "localizations" in the target manifold can be used in order to achieve a full proof of the uniqueness of the tangent map for u at the origin. Therefore, most of our efforts will be devoted to the proof of the following statement.

Theorem 2.1.2. *Let $m, n \in \mathbb{N}_0$ be such that $m \geqslant 2$. Let J be a Lipschitz almost complex structure on $\mathbb{B}^{2m}$ such that $J(0) = J_0$. Assume that $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ is weakly (J, j_n)-holomorphic and locally approximable, where j_n stands for the standard complex structure on $\mathbb{CP}^n$.*
Then, the $(2m-2)$-cycle $T_u \in \mathcal{D}_{2m-2}(\mathbb{B}^{2m})$ has a unique tangent cone at the origin.

This last paragraph is dedicated to explain the main new ideas that we have introduced in order to prove Theorem 2.1.2. The whole proof is based on the fact that the level sets of any weakly (J, j_n)-holomorphic and locally approximable map are rectifiable, J-holomorphic cycles. This fact is proved in Section 2.5. By applying a slicing procedure on the right-hand-side of the monotonicity formula (2.2.3) (see Appendix 2.A), we get a foliation of the region of integration into rectifiable, almost J-holomorphic curves (see Definition 2.4.2 and Remark 2.4.1). By localizing properly in the target, integrating what we call the fundamental Morrey type estimate for almost J-holomorphic curves (see Section 2.4) and passing then to the limit as the localization sets invade the codomain, we finally get uniqueness of tangent cone for the 2-dimensional current $\big(T_u \llcorner \pi^* \omega^{m-2}_{\mathbb{CP}^{m-1}}\big)/(m-2)!$, where $\pi : \mathbb{C}^m \smallsetminus \{0\} \to \mathbb{CP}^{m-1}$ is the standard projection map (see the first paragraph of Section 2.2). Then, the statement of Theorem 2.1.2 follows as shown in the conclusion of Section 2.6.2.

2.1.4. Final comments and open problems

We would like to stress that our approach could also give an alternative proof of the uniqueness of tangent cone for integral (p,p)-cycles on almost complex manifolds, which was previously obtained by C. Bellettini in [7]. This could be achieved by considering maps u that are more and more concentrated on just one rectifiable pseudo-holomorphic set in the domain (see Remark 2.6.1).

We also believe that the method that we have developed in this work could be useful in order to proceed further in the analysis of the singular set of weakly holomorphic and locally approximable maps. In particular, we conjecture that the optimal bound (2.1.2) on the size of the singular set of such maps could be achieved as a further development, also exploiting Theorem 2.1.1. Furthermore, an interesting open problem concerns the generalization of our result to arbitrary almost Kähler target manifolds.

Taking a wider look and abandoning the framework of weakly holomorphic maps, there are plenty of other related problems that would deserve to be studied more deeply in view of recent developments in the field. In particular, the aim is to invent new analytic techniques that are robust enough to survive the non-availability of holomorphic coordinates in the almost complex non-integrable setting. Among all these problems, for sake of brevity we just mention uniqueness of tangent cone for triholomorphic maps in hyper-Kähler geometry (see e.g. [9]) and for Hermitian Yang-Mills connections (see [27], [26]).

2.2. Almost monotonicity formula and tangent cones

First, let us fix the notation that we will use throughout the present chapter, whenever it won't be differently specified. We let J be a Lipschitz almost complex structure on $\mathbb{B}^{2m}$ such that $J(0) = J_0$. We let Ω be a Lipschitz almost symplectic form on $\mathbb{B}^{2m}$ which is compatible with J and such that $\Omega(0) = \Omega_0$, where Ω_0 stands for the standard symplectic form on $\mathbb{R}^{2m} \cong \mathbb{C}^m$. We denote by g the Lipschitz metric on $\mathbb{B}^{2m}$ given by $g_x(v, w) := \Omega_x(v, Jw)$, for every $x \in \mathbb{B}^{2m}$ and $v, w \in \mathbb{R}^{2m}$. We indicate by $|\cdot|_g$ the norm induced by g and by $|\cdot|$ the standard euclidean norm. Whenever we use the musical isomorphisms "$\sharp$" and "$\flat$" or the Hodge $*$-operator, we always take as a reference metric g. Finally, $\pi : \mathbb{B}^{2m} \smallsetminus \{0\} \to \mathbb{CP}^{m-1}$ denotes the standard projection map given by

$$\pi(x_1, y_1, ..., x_m, y_m) := [x_1 + iy_1 : ... : x_m + iy_m]$$

for every $(x_1, y_1, ..., x_m, y_m) \in \mathbb{B}^{2m} \smallsetminus \{0\}$.

Remark 2.2.1. Notice that the fact that g is Lipschitz on $\mathbb{B}^{2m}$ guarantees that $|\cdot|_g$ is equivalent to the euclidean norm. Consider the function $f : \overline{\mathbb{B}^{2m}} \times \mathbb{S}^{2m-1} \to (0, +\infty)$ given by

$$f(x, v) := |v|^2_{g(x)},$$

where $\mathbb{S}^{2m-1} \subset \mathbb{R}^{2m}$ is the unit sphere in $\mathbb{R}^{2m}$ with respect to the euclidean norm. Since f is continuous on the compact set $\overline{\mathbb{B}^{2m}} \times \mathbb{S}^{2m-1}$, by Weierstrass theorem and by definition of $\mathbb{S}^{2m-1}$, there exists a constant $G > 0$ such that

$$\frac{1}{G}|v|^2 \leqslant f(x, v) = |v|^2_{g(x)} \leqslant G|v|^2, \qquad \forall\, x \in \overline{\mathbb{B}^{2m}},\, \forall\, v \in \mathbb{S}^{2m-1}.$$

By 2-homogeneity of the squared norm, it follows that

$$\frac{1}{G}|v|^2 \leqslant |v|^2_{g(x)} \leqslant G|v|^2, \qquad \forall\, x \in \overline{\mathbb{B}^{2m}},\, \forall\, v \in \mathbb{R}^{2m} \tag{2.2.1}$$

and our claim follows.

Since $|\cdot|_g$ and $|\cdot|$ are equivalent, when we refer to the Sobolev spaces on $\mathbb{B}^{2m}$ we don't need to specify which of these two norms we use in order to define them. In fact, we will use both of them according to what suits better in the context.

Lemma 2.2.1. *Let V be a $2m$-dimensional real vector space and J a linear complex structure on V. Let Ω be a symplectic form on V which is compatible with J.*
Then

$$\frac{\Omega^{m-1}}{(m-1)!} \wedge \xi \wedge J\xi = |\xi|_g^2 \frac{\Omega^m}{m!}, \qquad \text{for every } \xi \in V^*,$$

where $J\xi$ is given by $(J\xi)(v) := -\xi(Jv)$, for every $v \in V$.

Proof. Fix an g-orthonormal basis $\{e_j, Je_j\}_{j=1,\dots,m}$ of V, so that

$$\Omega = \sum_{j=1}^{m} e_j^* \wedge Je_j^*.$$

First of all, notice that

$$\begin{aligned}\frac{\Omega^m}{m!} &= \frac{1}{m!} \sum_{j_1,\dots,j_m=1}^{m} (e_{j_1}^* \wedge Je_{j_1}^*) \wedge \dots \wedge (e_{j_m}^* \wedge Je_{j_m}^*) \\ &= e_1^* \wedge Je_1^* \wedge \dots \wedge e_m^* \wedge Je_m^*.\end{aligned}$$

Fix any $\xi \in V^*$ and decompose it along the g-orthonormal basis denoted by $\{e_j^*, Je_j^*\}_{j=1,\dots,m}$ as

$$\xi = \sum_{j=1}^{m} (\xi_{j1} e_j^* + \xi_{j2} Je_j^*).$$

This in turn implies that

$$J\xi = \sum_{j=1}^{m} (\xi_{j1} Je_j^* - \xi_{j2} e_j^*)$$

and then

$$\begin{aligned}\xi \wedge J\xi = \sum_{j,k=1}^{m} (&\xi_{j1}\xi_{k1} e_j^* \wedge Je_k^* - \xi_{j1}\xi_{k2} e_j^* \wedge e_k^* \\ &+ \xi_{j2}\xi_{k1} Je_j^* \wedge Je_k^* - \xi_{j2}\xi_{k2} Je_j^* \wedge e_k^*).\end{aligned}$$

Since

$$\frac{\Omega^{m-1}}{(m-1)!} = \sum_{j_1,\dots,j_{m-1}=1}^{m} (e_{j_1}^* \wedge Je_{j_1}^*) \wedge \dots \wedge (e_{j_{m-1}}^* \wedge Je_{j_{m-1}}^*),$$

it's easy to see that

$$\frac{\Omega^{m-1}}{(m-1)!} \wedge e_j^* \wedge e_k^* = \frac{\Omega^{m-1}}{(m-1)!} \wedge Je_j^* \wedge Je_k^* = 0,$$

for every $j, k \in \{1, ..., m\}$. Moreover, we notice that

$$\frac{\Omega^{m-1}}{(m-1)!} \wedge \sum_{j,k=1}^{m} \xi_{j1}\xi_{k1} e_j^* \wedge Je_k^* = \left(\sum_{j=1}^{m} \xi_{j1}^2\right) e_1^* \wedge Je_1^* \wedge ... \wedge e_m^* \wedge Je_m^*$$

and

$$-\frac{\Omega^{m-1}}{(m-1)!} \wedge \sum_{j,k=1}^{m} \xi_{j2}\xi_{k2} Je_j^* \wedge e_k^* = \left(\sum_{j=1}^{m} \xi_{j2}^2\right) e_1^* \wedge Je_1^* \wedge ... \wedge e_m^* \wedge Je_m^*.$$

By adding all the contributions, we get

$$\begin{aligned}\frac{\Omega^{m-1}}{(m-1)!} \wedge \xi \wedge J\xi &= \left(\sum_{j=1}^{m} (\xi_{j1}^2 + \xi_{j2}^2)\right) e_1^* \wedge Je_1^* \wedge ... \wedge e_m^* \wedge Je_m^* \\ &= |\xi|_g^2 e_1^* \wedge Je_1^* \wedge ... \wedge e_m^* \wedge Je_m^*\end{aligned}$$

and the statement follows. □

Corollary 2.2.1. *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a compact almost Kähler manifold. Assume that $u \in W^{1,2}(\mathbb{B}^{2m}, N)$ is weakly (J, J_N)-holomorphic. Then*

$$u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \frac{|du|_g^2}{2} \frac{\Omega^m}{m!}, \qquad \mathcal{L}^{2m}\text{-a.e. on } \mathbb{B}^{2m}. \tag{2.2.2}$$

Proof. Let $E \subset \mathbb{B}^{2m}$ be the set of Lebesgue points of du. If $x \in E$ is such that $du_x = 0$, then equation (2.2.2) holds trivially. Assume then that $du(x) \neq 0$. Fix an ω_N-orthonormal basis $\{\xi_i, J_N\xi_i\}_{i=1}^n$ of $T_{u(x)}N$, so that

$$(\omega_N)_{u(x)} = \sum_{i=1}^{n} \xi_i^* \wedge J_N\xi_i^*.$$

Notice that, since u is weakly (J, J_N)-holomorphic, it holds that

$$(u^*\omega_N)_x = \sum_{i=1}^{n} u^*\xi_i^* \wedge u^*J_N\xi_i^* = \sum_{i=1}^{n} u^*\xi_i^* \wedge Ju^*\xi_i^*,$$

since $\big(u^*(J_N\xi^*)\big)(v) = \big(J(u^*\xi^*)\big)(v)$ for every $v \in T_x\mathbb{B}^{2m} \cong \mathbb{R}^{2m}$ follows from the definition of $J_N\xi^*$ and $J(u^*\xi^**)$ (see Lemma 2.2.1). Thus in particular,

$$(u^*\omega_N)_x \wedge \Omega_x^{m-1} = \sum_{i=1}^{n} u^*\xi_i^* \wedge Ju^*\xi_i^* \wedge \Omega_x^{m-1}.$$

By applying Lemma 2.2.1 with $\Omega = \Omega_x$ and $\xi = u^*\xi_i$ for every $i = 1, ..., n$, we get that

$$(u^*\omega_N)_x \wedge \frac{\Omega_x^{m-1}}{(m-1)!} = \left(\sum_{i=1}^{n} |u^*\xi_i^*|_g^2\right) \frac{\Omega_x^m}{m!} = \frac{|du_x|_g^2}{2} \frac{\Omega_x^m}{m!}.$$

The statement follows immediately. □

Lemma 2.2.2. *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a closed almost Kähler smooth manifold. Assume that $u \in W^{1,2}(\mathbb{B}^{2m}, N)$ is weakly (J, J_N)-holomorphic and locally approximable. Then, T_u is a normal $(2m-2)$-cycle on $\mathbb{B}^{2m}$ semicalibrated by $\dfrac{\Omega^{m-1}}{(m-1)!}$.*

Proof. First, we claim that T_u is a cycle, i.e. that $\partial T_u = 0$. Indeed, by Stokes theorem and since $d(u^*\omega_N) = 0$ holds distributionally on $\mathbb{B}^{2m}$ by local approximability of u, for every fixed $\alpha \in \mathcal{D}^{2m-3}(\mathbb{B}^{2m})$ we get

$$\langle \partial T_u, \alpha \rangle = \langle T_u, d\alpha \rangle = \int_B u^*\omega_N \wedge d\alpha = 0.$$

In order to conclude, we just need to show that

$$\left\langle T_u, \frac{\Omega^{m-1}}{(m-1)!} \right\rangle = \mathbb{M}(T) < +\infty.$$

Notice that, by Corollary 2.2.1, it holds that

$$\left\langle T_u, \frac{\Omega^{m-1}}{(m-1)!} \right\rangle = \int_{\mathbb{B}^{2m}} u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \frac{1}{2} \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\,\mathrm{vol}_g < +\infty,$$

since $du \in L^2(\mathbb{B}^{2m}; \mathbb{R}^{2m} \otimes u^*TN)$. We claim that

$$\mathbb{M}(T_u) = \frac{1}{2} \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\,\mathrm{vol}_g \, .$$

Indeed, fix a Lebesgue point $x \in \mathbb{B}^{2m}$ for du and let $\{e_1, Je_1, ..., e_m, Je_m\}$, $\{\xi_1^*, j\xi_1^*, ..., \xi_n^*, j\xi_n^*\}$ be orthonormal bases of $T_x\mathbb{B}^{2m}$ and $T_{u(x)}N$ respectively. Then, we have

$$\begin{aligned}
\langle (u^*\omega_N)_x, e_k \wedge Je_h \rangle &= \sum_{i=1}^n (u^*\xi_i^* \wedge Ju^*\xi_i^*)(e_k, Je_h) \\
&= \sum_{i=1}^n (u^*\xi_i^*)(e_k)(Ju^*\xi_i^*)(Je_h) - (u^*\xi_i^*)(Je_h)(Ju^*\xi_i^*)(e_k) \\
&= \sum_{i=1}^n (u^*\xi_i^*)(e_k)(u^*\xi_i^*)(e_h) + (u^*\xi_i^*)(Je_h)(u^*\xi_i^*)(Je_k) \\
&\leqslant \frac{1}{2} \sum_{i=1}^n \big(|u^*\xi_i^*(e_k)|^2 + |u^*\xi_i^*(e_h)|^2 \\
&\quad + |u^*\xi_i^*(Je_k)|^2 + |u^*\xi_i^*(Je_h)|^2 \big) \\
&\leqslant \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \big(|u^*\xi_i^*(e_j)|^2 + |u^*\xi_i^*(Je_j)|^2 \big) = \frac{|du_x|_g^2}{2},
\end{aligned}$$

for every $k, h = 1, ..., m$. Moreover, by exactly the same computation we get

$$\langle (u^*\omega_N)_x, e_k \wedge e_h \rangle = \langle (u^*\omega_N)_x, Je_k \wedge Je_h \rangle = 0, \qquad \forall \, k, h = 1, ..., m.$$

Thus, for every unit and simple 2-vector $v_1 \wedge v_2$ with $v_1, v_2 \in T_x\mathbb{B}^{2m}$ we have

$$\langle (u^*\omega_N)_x, v_1 \wedge v_2 \rangle \leqslant \frac{|du_x|_g^2}{2}.$$

Consider the unit vector

$$v := \frac{1}{\sqrt{m}} \sum_{\substack{i=1,\dots,m \\ i \text{ odd}}} e_i + \frac{1}{\sqrt{m}} \sum_{\substack{i=1,\dots,m \\ i \text{ even}}} Je_i \in T_x\mathbb{B}^{2m}$$

and notice that

$$\langle (u^*\omega_N)_x, v \wedge Jv \rangle = \left\langle (u^*\omega_N)_x, \frac{\Omega_x}{m} \right\rangle = *\left((u^*\omega_N)_x, \wedge \frac{\Omega_x^{m-1}}{(m-1)!} \right) = \frac{|du_x|_g^2}{2},$$

By definition of comass norm and since $x \in \mathbb{B}^{2m}$ was any arbitrary Lebesgue point of du, we conclude that

$$\|u^*\omega_N\|_* = \frac{|du|_g^2}{2}, \qquad \mathrm{vol}_g\text{-a.e. on } \mathbb{B}^{2m}.$$

Moreover, since T_u is the integration current induced by $u^*\omega_N$, it holds that

$$\mathbb{M}(T_u) = \int_{\mathbb{B}^{2m}} \|u^*\omega_N\|_* \, d\,\mathrm{vol}_g, \qquad \text{for every open set } U \subset\subset \mathbb{B}^{2m}.$$

The statement then follows. $\square$

Before stating the following fundamental proposition, we recall the following notation. Given any $x_0 \in \mathbb{B}^{2m}$, we define

$$\Omega_{t,x_0} := (dR_{x_0} \wedge \Omega) \llcorner \nu_{x_0}$$

where $R_{x_0} := |\cdot - x_0|$, $\nu_{x_0} := (dR_{x_0})^\sharp$ and with "$\llcorner$" we denote the interior product. We call Ω_{t,x_0} the *tangential part of* Ω *with respect to* x_0. Such notation is analogous to the one used in [69].

Proposition 2.2.1 (Almost monotonicity formula)**.** *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a closed almost Kähler smooth manifold. Let $u \in W^{1,2}(\mathbb{B}^{2m}, N)$ be weakly (J, J_N)-holomorphic and locally approximable.*
Then, there exists $A = A(\mathrm{Lip}(\Omega)) \geqslant 0$ such that

$$\begin{aligned} e^{A\rho}(1+A\rho)\frac{\mathbb{M}\big(T_u \llcorner B_\rho(x_0)\big)}{\rho^{2m-2}} &- e^{A\sigma}(1+A\sigma)\frac{\mathbb{M}\big(T_u \llcorner B_\sigma(x_0)\big)}{\sigma^{2m-2}} \\ &\geqslant \int_{B_\rho(x_0)\smallsetminus B_\sigma(x_0)} \frac{1}{|\cdot - x_0|^{2m-2}} u^*\omega_N \wedge \frac{\Omega_{t,x_0}^{m-1}}{(m-1)!} \end{aligned} \tag{2.2.3}$$

and

$$\begin{aligned} e^{-A\rho}(1-A\rho)\frac{\mathbb{M}\big(T_u \llcorner B_\rho(x_0)\big)}{\rho^{2m-2}} &- e^{-A\sigma}(1-A\sigma)\frac{\mathbb{M}\big(T_u \llcorner B_\sigma(x_0)\big)}{\sigma^{2m-2}} \\ &\leqslant \int_{B_\rho(x_0)\smallsetminus B_\sigma(x_0)} \frac{1}{|\cdot - x_0|^{2m-2}} u^*\omega_N \wedge \frac{\Omega_{t,x_0}^{m-1}}{(m-1)!}, \end{aligned} \tag{2.2.4}$$

for every $x_0 \in \mathbb{B}^{2m}$ and $0 < \sigma < \rho \leqslant r_{x_0} := \mathrm{dist}(x_0, \partial\mathbb{B}^{2m})$.

Proof. A direct computations leads immediately to

$$\left(\Omega^{m-1}\right)_{t,x_0} = \Omega^{m-1}_{t,x_0}.$$

Hence, the statement follows directly by Lemma 2.2.2 and [69, Proposition 1] by simply noticing that exactly the same proof works when Ω is just Lipschitz. □

Remark 2.2.2. Fix any $x_0 \in \mathbb{B}^{2m}$. Notice that

$$\begin{aligned} *\left(u^*\omega_N(x) \wedge \frac{\Omega_{t,x_0}(x)^{m-1}}{(m-1)!}\right) &= \left\langle \frac{\Omega_{t,x_0}(x)^{m-1}}{(m-1)!}, *u^*\omega_N \right\rangle \\ &= \frac{1}{2}\left|\frac{\partial u}{\partial \nu_{x_0}}\right|^2_g, \qquad \text{for } \mathcal{L}^{2m}\text{-a.e. } x \in \mathbb{B}^{2m}, \end{aligned}$$

where $\nu_{x_0} := (dR_{x_0})^\sharp$ as above with $R_{x_0} := |\cdot - x_0|$. Hence, by equation (2.2.3) we conclude that the function

$$(0, r_{x_0}) \ni \rho \mapsto e^{A\rho}(1 + A\rho)\frac{\mathbb{M}(T_u \llcorner B_\rho)}{\rho^{2m-2}}$$

is non-decreasing. As

$$\lim_{\rho \to 0^+} e^{A\rho}(1 + A\rho) = 1,$$

we conclude that the limit

$$\theta(x_0, u) := \lim_{\rho \to 0^+} \frac{\mathbb{M}\big(T_u \llcorner B_\rho(x_0)\big)}{\rho^{2m-2}} \tag{2.2.5}$$

exists and is finite. We say that $\theta(x_0, u)$ is the *density of the map u at the point x_0*.

We conclude the section by discussing the existence and the structure of tangent cones for the current T_u. Let's pick any sequence $\rho_k \to 0^+$ as $k \to +\infty$ and the relative blow-up sequence $\{T_{\rho_k} := (\Phi_{\rho_k})_* T_u\}_{k \in \mathbb{N}}$. Since T_{ρ_k} is a cycle for every $k \in \mathbb{N}$ and

$$\mathbb{M}(T_{\rho_k}) = \frac{\mathbb{M}(T_u \llcorner B_{\rho_k})}{\rho^{2m-2}} \leqslant e^A(1 + A)\mathbb{M}(T_u) < +\infty,$$

by Federer-Fleming compactness theorem we know that there exists a subsequence $\{\rho_{k_j}\}_{j \in \mathbb{N}}$ of $\{\rho_k\}_{k \in \mathbb{N}}$ such that $T_{\rho_{k_j}} \rightharpoonup C_\infty$ as $j \to +\infty$ in the sense of currents. Moreover, by exploiting the almost monotonicity formula, we get that any tangent cone C_∞ is a $(2m-2)$-cycle calibrated by Ω_0 and invariant under dilations, i.e. $(\Phi_\rho)_* C_\infty = C_\infty$, for every $\rho \in (0, 1)$ (see e.g. [69, Section 3]).

2.3. Smoothness at points with small density

In the present section, we will assume that Ω is a symplectic form and we prove that weakly holomorphic and locally approximable maps are stationary harmonic in this particular case. Therefore, we conclude that such maps are smooth at points of small density via standard ε-regularity for stationary harmonic maps (Theorem 2.3.1).

The almost symplectic case is completely treated in [9, Propositions 1, 3, 4], where it is shown that similar results hold in the almost stationary scenario. Hence, the conclusions of the present section hold even if $d\Omega \neq 0$. We just present here a simplified case in order to deal with less technicalities and draw some light on the key ideas and concepts.

Lemma 2.3.1 (Wirtinger's inequality). *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a closed almost Kähler smooth manifold.*
Then, for every map $v \in W^{1,2}(\mathbb{B}^{2m}, N)$ it holds that

$$*\left(v^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!}\right) \leqslant \frac{|dv|_g^2}{2}, \qquad \mathrm{vol}_g\text{-}a.e.\ on\ \mathbb{B}^{2m} \tag{2.3.1}$$

Proof. Let $E \subset \mathbb{B}^{2m}$ be the set of the Lebesgue points of dv. If $x \in E$ is such that $dv_x = 0$, then (2.3.1) holds trivially. Assume then that $x \in E$ is such that $dv_x \neq 0$. Fix a g-orthonormal basis $\{e_{2k-1}, e_{2k} := Je_{2k-1}\}_{k=1}^m$ of $T_x\mathbb{B}^{2m}$ and an ω_N-orthonormal basis $\{\xi_{2i-1}, \xi_{2i} := j\xi_{2i-1}\}_{i=1}^n$ of $T_{v(x)}N$, so that

$$\Omega_x = \sum_{k=1}^m e_{2k-1}^* \wedge e_{2k}^*$$

and

$$(\omega_N)_{v(x)} = \sum_{i=1}^n \xi_{2i-1}^* \wedge \xi_{2i}^*.$$

Then, we compute

$$\begin{aligned}
&*\left((v^*\omega_N)_x \wedge \frac{\Omega_x^{m-1}}{(m-1)!}\right)\\
&= *\sum_{k=1}^m\sum_{i=1}^n v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^* \wedge e_1^* \wedge e_2^* \wedge ... \wedge \widehat{e_{2k-1}^*} \wedge \widehat{e_{2k}^*} \wedge ... \wedge e_{2m-1} \wedge e_{2m}\\
&= \sum_{k=1}^m\sum_{i=1}^n (v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*)(e_{2k-1} \wedge e_{2k})\\
&= \sum_{k=1}^m\sum_{i=1}^n (v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*)(e_{2k-1}, e_{2k})\\
&\leqslant \sum_{k=1}^m\sum_{i=1}^n \left|(v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*)(e_{2k-1}, e_{2k})\right|_g\\
&= \sum_{k=1}^m\sum_{i=1}^n \left|v^*\xi_{2i-1}^*(e_{2k-1})v^*\xi_{2i}^*(e_{2k}) - v^*\xi_{2i-1}^*(e_{2k})v^*\xi_{2i}^*(e_{2k-1})\right|_g\\
&\leqslant \frac{1}{2}\sum_{k=1}^{2m}\sum_{i=1}^n \left(|v^*\xi_{2i-1}^*(e_k)|_g^2 + |v^*\xi_{2i}^*(e_k)|_g^2\right) = \frac{|dv_x|_g^2}{2}.
\end{aligned}$$

Thus, the statement follows. □

Lemma 2.3.2 (Weakly holomorphic maps are weakly harmonic)**.** *Let $n \in \mathbb{N}_0$ and assume that (N^{2n}, J_N, ω_N) is a closed almost Kähler smooth manifold.*
If $u \in W^{1,2}(\mathbb{B}^{2m}, N)$ is weakly (J, J_N)-holomorphic, then u is weakly harmonic.

Proof. Recall that we always identify N as a smooth submanifold of $\mathbb{R}^k$, for k large enough, through the smooth isometric embedding $\Phi : N \hookrightarrow \mathbb{R}^k$ (see Section 2.1).
Let the map $\pi_N : W \subset \mathbb{R}^k \to N$ be the nearest point projection from a tubular neighbourhood W of $\Phi(N)$ onto N. Fix any vector field $X \in C_c^\infty(\mathbb{B}^{2m}, \mathbb{R}^k)$. As for every $t \in \mathbb{R}$ the map $\pi_N \circ (\Phi \circ u + tX)$ belongs to $W^{1,2}(\mathbb{B}^{2m}, N)$, by exploiting Lemma 2.3.1 we get that

$$\int_{\mathbb{B}^{2m}} \big|d\big(\pi_N \circ (\Phi \circ u + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \geqslant 2 \int_{\mathbb{B}^{2m}} (\Phi \circ u + tX)^* \pi_N^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!}, \tag{2.3.2}$$

for $t \in \mathbb{R}$ such that $|t| < \delta$ with $\delta > 0$ sufficiently small. Moreover, the equality holds for $t = 0$ by virtue of equation (2.2.2). We claim that

$$\int_{\mathbb{B}^{2m}} (\Phi \circ u + tX)^* \pi_N^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \int_{\mathbb{B}^{2m}} u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!}, \tag{2.3.3}$$

for every $|t| < \delta$. Indeed, it holds that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{B}^{2m}} (\Phi \circ u + tX)^* \pi_N^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right) \\
&= \int_{\mathbb{B}^{2m}} \frac{d}{dt}\big((\Phi \circ u + tX)^* \pi_N^* \omega_N\big) \wedge \frac{\Omega^{m-1}}{(m-1)!} \\
&= \left\langle d\big(u^*(\pi_N^* \omega_N) \llcorner X\big), * \frac{\Omega^{m-1}}{(m-1)!} \right\rangle \\
&= -\int_{\mathbb{B}^{2m}} u^*(\pi_N^* \omega_N) \llcorner X \wedge d\left(\frac{\Omega^{m-1}}{(m-1)!}\right) = 0,
\end{aligned}$$

where "$\llcorner$" stands for the interior product. Hence, equation (2.3.3) follows. By using together equation (2.3.2) and (2.3.3), we get

$$\int_{\mathbb{B}^{2m}} \big|d\big(\pi_N \circ (\Phi \circ u + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \geqslant \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\,\mathrm{vol}_g \qquad \text{for every } |t| < \delta,$$

and the equality holds for $t = 0$. Thus, $t = 0$ is a global minimum for the differentiable function

$$t \mapsto \int_{\mathbb{B}^{2m}} \big|d\big(\pi_N \circ (\Phi \circ u + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \,.$$

Hence, we conclude that

$$\frac{d}{dt} \int_{\mathbb{B}^{2m}} \big|d\big(\pi_N \circ (\Phi \circ u + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \Big|_{t=0} = 0.$$

Since our choice of $X \in C_c^\infty(\mathbb{B}^{2m}, \mathbb{R}^k)$ was arbitrary, the statement follows. □

Lemma 2.3.3 (Weakly holomorphic and locally approximable maps are stationary harmonic)**.** *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a closed almost Kähler smooth manifold.*
If $u \in W^{1,2}(\mathbb{B}^{2m}, N)$ is weakly (J, J_N)-holomorphic and locally approximable, then u is stationary harmonic.

Proof. Fix any vector field $X \in C_c^\infty(\mathbb{B}^{2m},\mathbb{R}^{2m})$. Notice that the map $u \circ (\mathrm{Id} + tX)$ belongs to $W^{1,2}(\mathbb{B}^{2m}, N)$ for $t \in \mathbb{R}$ such that $|t| < \delta$ with $\delta > 0$ sufficiently small. Then, by Lemma 2.3.1, it holds that

$$\int_{\mathbb{B}^{2m}} \big|d\big(u \circ (\mathrm{Id} + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \geqslant 2 \int_{\mathbb{B}^{2m}} (\mathrm{Id} + tX)^* u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \tag{2.3.4}$$

for every $|t| < \delta$ and the equality holds for $t = 0$ by virtue of equation (2.2.2). We claim that

$$\int_{\mathbb{B}^{2m}} (\mathrm{Id} + tX)^* u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \int_{\mathbb{B}^{2m}} u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!}, \tag{2.3.5}$$

for every $|t| < \delta$. Indeed, as $d(u^*\omega_N) = 0$ distributionally on B, it holds that

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\mathbb{B}^{2m}} (\mathrm{Id} + tX)^* u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right) &= \int_{\mathbb{B}^{2m}} \frac{d}{dt}\big((\mathrm{Id} + tX)^* u^* \omega_N\big) \wedge \frac{\Omega^{m-1}}{(m-1)!} \\
&= \left\langle \mathcal{L}_X(u^*\omega_N), * \frac{\Omega^{m-1}}{(m-1)!} \right\rangle \\
&= \left\langle d\big(u^*\omega_N \llcorner X\big), * \frac{\Omega^{m-1}}{(m-1)!} \right\rangle \\
&= - \int_{\mathbb{B}^{2m}} u^* \omega_N \llcorner X \wedge d\left(\frac{\Omega^{m-1}}{(m-1)!} \right) = 0.
\end{aligned}$$

Hence, equation (2.3.5) follows. By using together equation (2.3.4) and (2.3.5) we get that

$$\int_{\mathbb{B}^{2m}} \big|d\big(u \circ (\mathrm{Id} + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \geqslant \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\,\mathrm{vol}_g, \quad \text{for every } |t| < \delta. \tag{2.3.6}$$

and the equality holds for $t = 0$. Thus, $t = 0$ is a global minimum for the differentiable function

$$t \mapsto \int_{\mathbb{B}^{2m}} \big|d\big(u \circ (\mathrm{Id} + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \,.$$

Hence, we conclude that

$$\frac{d}{dt} \int_{\mathbb{B}^{2m}} \big|d\big(u \circ (\mathrm{Id} + tX)\big)\big|_g^2 \, d\,\mathrm{vol}_g \bigg|_{t=0} = 0.$$

Since our choice of $X \in C_c^\infty(\mathbb{B}^{2m}, \mathbb{R}^k)$ was arbitrary, the statement follows. ☐

The following ε-regularity statement follows immediately by Lemma 2.3.3 and [75, Theorem 2.1].

Theorem 2.3.1 (ε-regularity for weakly holomorphic and locally approximable maps)**.** *Let $n \in \mathbb{N}_0$ and let (N^{2n}, J_N, ω_N) be a closed almost Kähler smooth manifold.*
Let $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ be weakly (J, J_N)-holomorphic and locally approximable. Then, there exists a threshold $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that whenever $\theta(x_0, u) < \varepsilon_0$ there exists ball $B_\rho(x_0) \subset B$ such that u is Hölder continuous (and hence smooth) on $B_\rho(x_0)$.

We define

$$\mathrm{Sing}(u) := \{x_0 \in \mathbb{B}^{2m} \text{ s.t. } \theta(x_0, u) \geqslant \varepsilon_0\}$$

and we say that $\mathrm{Sing}(u)$ is the *singular set* of u. Moreover, by stationarity of u, it follows that

$$\mathscr{H}^{2m-2}\big(\mathrm{Sing}(u)\big) = 0.$$

2.4. The fundamental Morrey type estimate

We aim to collect here the proofs of the (mostly technical) tools and estimates that will be used in Section 2.6. Throughout the present chapter, given any $\mathscr{H}^k$-rectifiable subset $\Sigma \subset \mathbb{B}^{2m}$ equipped with an orienting $\mathscr{H}^k$-measurable field of unit and simple k-vectors $\vec{\Sigma}$ we denote by $[\Sigma]$ the current of integration on Σ, i.e. the k-dimensional current given by

$$\langle [\Sigma], \alpha \rangle := \int_\Sigma \langle \alpha, \vec{\Sigma} \rangle \, d\mathscr{H}^k, \qquad \forall\, \alpha \in \mathcal{D}^k(\mathbb{B}^{2m}).$$

2.4.1. Some technical lemmata

Definition 2.4.1 (J-holomorphic curves)**.** A locally $\mathscr{H}^2$-rectifiable subset $\Sigma \subset \mathbb{B}^{2m}$ equipped with an orienting $\mathscr{H}^2$-measurable field of unit and simple 2-vectors $\vec{\Sigma}$ is a *J-holomorphic curve* if $\vec{\Sigma}(x)$ is J-invariant for $\mathscr{H}^2$-a.e. $x \in \Sigma$.

Moreover, if $\partial[\Sigma] = 0$ we say that Σ is *closed.*

Definition 2.4.2 (Almost J-holomorphic curves)**.** A locally $\mathscr{H}^2$-rectifiable subset $\Sigma \subset \mathbb{B}^{2m}$ equipped with an orienting $\mathscr{H}^2$-measurable field of unit and simple 2-vectors $\vec{\Sigma}$ is a *almost J-holomorphic curve* if there exists some $\mathscr{H}^2$-measurable and J-invariant field of 2-vectors $\vec{\Sigma}_J : \Sigma \to \wedge_2 \mathbb{R}^{2m}$ such that for some $\gamma \in (0, 1]$, $\ell \geqslant 0$ it holds that

$$|\vec{\Sigma}(x) - \vec{\Sigma}_J(x)| \leqslant \ell |x|^\gamma, \qquad \text{for } \mathscr{H}^2\text{- a.e. } x \in \Sigma. \tag{2.4.1}$$

Moreover, if $\partial[\Sigma] = 0$ we say that Σ is *closed.*

Remark 2.4.1. Given an almost J-holomorphic curve in B, we can build a 2-dimensional varifold on $\mathbb{B}^{2m}$ associated to it in the following way.

Let $\mathscr{G}_2(\mathbb{B}^{2m}) := \mathbb{B}^{2m} \times \mathrm{Gr}(2, \mathbb{R}^{2m})$, where $\mathrm{Gr}(2, \mathbb{R}^{2m})$ is the Grassmannian of the real 2-planes in $\mathbb{R}^{2m}$. Notice that $\mathscr{G}_2(\mathbb{B}^{2m})$ can be given the structure of a smooth manifold, since it is the product of two smooth manifolds. Following the notation by W.K. Allard and L. Simon (see [85, Chapter 8], [3]), a general 2-dimensional varifold on $\mathbb{B}^{2m}$ is simply a Radon measure on $\mathscr{G}_2(\mathbb{B}^{2m})$. Then, we may associate to an almost J-holomorphic curve $\Sigma \subset \mathbb{B}^{2m}$ the Radon measure on $\mathscr{G}_2(\mathbb{B}^{2m})$ given by

$$\mathscr{H}^2 \llcorner \Sigma \otimes \delta_{\mathrm{span}\{\vec{\Sigma}_J\}},$$

where by $\otimes$ we denote the usual tensor product of measures and $\mathrm{span}\{\vec{\Sigma}_J\}$ denotes the field of 2-planes associated with the field of 2-vectors $\vec{\Sigma}_J$.

Such objects are very close to being rectifiable varifolds but the almost tangent space of Σ is "tilted", conveniently with respect to the purposes that will be clear in the forthcoming discussion.

We point out explicitely that the form of these new objects is built (and therefore meaningful) just to work around the origin. We would need to consider a "shifted" version of almost J-holomorphic curves in order to work around an arbitrary point $x_0 \in \mathbb{B}^{2m}$.

Remark 2.4.2. All the estimates and the results that will be presented in this section concerning closed almost J-holomorphic curves in B are still valid for closed J-holomorphic curves. Indeed, any J-holomorphic curve is trivially almost J-holomorphic (just pick $\vec{\Sigma}_J = \vec{\Sigma}$, $\ell = 0$ and $\gamma = 1$ in Definition 2.4.2).

Hence, in order to get the corresponding estimates for closed J-holomorphic curves it will always be sufficient to set $\vec{\Sigma}_J = \vec{\Sigma}$, $\ell = 0$ and $\gamma = 1$ in what follows.

From now on, we will denote by ν_0 the vector field on $\mathbb{B}^{2m} \setminus \{0\}$ given by $\nu_0(x) = x/|x|$. Moreover, we notice that since Ω is Lipschitz and $\Omega(0) = \Omega_0$, there exists a constant $\tilde{L} > 0$ depending only on $\mathrm{Lip}(\Omega)$ such that

$$|\nu - \nu_0| \leqslant \tilde{L}|\cdot|.$$

Proposition 2.4.1 (Almost monotonicity formula). *Let Σ be a closed almost J-holomorphic curve in $\mathbb{B}^{2m}$, according to Definition 2.4.2. Then, there exists a positive constant $A \geqslant 0$ depending only on the Lipschitz constant of Ω such that*

$$\begin{aligned} e^{A\rho + \ell\frac{\rho^\gamma}{\gamma}}(1 + A\rho)\frac{\mathscr{H}^2(\Sigma \cap B_\rho)}{\rho^2} &- e^{A\sigma + \ell\frac{\sigma^\gamma}{\gamma}}(1 + A\sigma)\frac{\mathscr{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \\ &\geqslant \int_{\Sigma \cap (B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} \big|\vec{\Sigma}_J \wedge \nu\big|_g^2 \, d\mathscr{H}^2 \end{aligned} \tag{2.4.2}$$

and

$$\begin{aligned} e^{-\left(A\rho + \ell\frac{\rho^\gamma}{\gamma}\right)}(1 - A\rho)\frac{\mathscr{H}^2(\Sigma \cap B_\rho)}{\rho^2} &- e^{-\left(A\sigma + \ell\frac{\sigma^\gamma}{\gamma}\right)}(1 - A\sigma)\frac{\mathscr{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \\ &\leqslant \int_{\Sigma \cap (B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} \big|\vec{\Sigma}_J \wedge \nu\big|_g^2 \, d\mathscr{H}^2, \end{aligned} \tag{2.4.3}$$

Proof. Throughout this proof, R will denote the smooth radial vector field on $\mathbb{B}^{2m}$ given by $R(x) := x$, for every $x \in \mathbb{B}^{2m}$. We denote by Ω_0 the standard symplectic form on B^{2m} and we define $\Omega_1 := \Omega - \Omega_0$. Moreover, given any arbitrary form $\alpha \in \Omega^2(\mathbb{B}^{2m})$ we denote by α_t the tangential part of a form with respect to the vector field ν, given by

$$\alpha_t := (dr \wedge \alpha) \llcorner \nu,$$

according to the notation used in Proposition 2.2.1.

Define the normal 2-current on $\mathbb{B}^{2m}$ given by

$$\langle [\Sigma]_J, \alpha \rangle := \int_\Sigma \langle \alpha, \vec{\Sigma}_J \rangle \, d\mathscr{H}^2, \qquad \forall\, \alpha \in \mathcal{D}^2(B).$$

As $[\Sigma]_J$ is semicalibrated by Ω, we will apply the same method that is used in [69, Proposition 1]. Nevertheless, we need to take into account the fact that the 2-current $[\Sigma]_J$ is not a cycle (though not far from being one).

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be smooth, non-increasing and such that:

1. $\varphi \equiv 1$ on $[0, 1/2]$;
2. $|\varphi'| \leqslant 4$ on $[0, +\infty)$.
3. $\varphi \equiv 0$ on $[1, +\infty)$.

For every $0<\rho<1$, define $\varphi_\rho(x):=\varphi(|x|/\rho)$, for every $x\in\mathbb{R}^{2m}$. Notice that $\varphi_\rho\equiv 1$ on $B_{\rho/2}$, $\varphi_\rho\equiv 0$ on $\mathbb{R}^{2m}\smallsetminus B_\rho$ and $|\nabla\varphi_\rho|\leqslant 4/\rho$ on $\mathbb{R}^{2m}$. Define

$$\begin{aligned}
I(\rho) &:= \int_\Sigma \varphi_\rho\langle\Omega,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 = \int_\Sigma \varphi_\rho\, d\mathscr{H}^2,\\
J(\rho) &:= \int_\Sigma \varphi_\rho\langle\Omega_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2.
\end{aligned}$$

Recall that $\mathcal{L}_R\Omega_0=2\Omega_0$ (where $\mathcal{L}_R\Omega_0$ denotes the Lie derivative of Ω_0 with respect to the vector field R) and compute

$$\begin{aligned}
2I(\rho) &= 2\int_\Sigma \varphi_\rho\langle\Omega_0,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \int_\Sigma \langle\varphi_\rho d(\Omega_0 \llcorner R),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 - \int_\Sigma \langle d\varphi_\rho\wedge(\Omega_0 \llcorner R),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner (\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner \nu),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner (\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega_0-(\Omega_0)_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega-\Omega_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner (\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 - \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega_1-(\Omega_1)_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&= \rho I'(\rho)-\rho J'(\rho) + \int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad + \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner (\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&\quad - \rho\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega_1-(\Omega_1)_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 + 2\int_\Sigma \varphi_\rho\langle\Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2,
\end{aligned}$$

which leads to

$$-2\frac{I(\rho)}{\rho^3}+\frac{I'(\rho)}{\rho^2}-\frac{J'(\rho)}{\rho^2} = -\frac{1}{\rho^3}\int_\Sigma \langle d\big(\varphi_\rho(\Omega_0 \llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2$$

$$
\begin{aligned}
&+\frac{1}{\rho^2}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0 \llcorner (\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&+\frac{1}{\rho^2}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega_1-(\Omega_1)_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\\
&-\frac{2}{\rho^3}\int_\Sigma \varphi_\rho\langle \Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2.
\end{aligned}
$$

Notice that

$$
\left|\frac{1}{\rho^3}\int_\Sigma \langle d\big(\varphi_\rho(\Omega_0\llcorner R)\big),\vec{\Sigma}-\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\right| \leqslant \ell\rho^{\gamma-1}\left(\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right),
$$

$$
\left|\frac{1}{\rho^2}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle \Omega_1-(\Omega_1)_t,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\right| \leqslant \frac{2\operatorname{Lip}(\Omega)}{\rho}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\, d\mathscr{H}^2 = 2\operatorname{Lip}(\Omega)\frac{I'(\rho)}{\rho},
$$

$$
\left|\frac{1}{\rho^2}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\langle dr\wedge(\Omega_0\llcorner(\nu_0-\nu)),\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\right| \leqslant \frac{\tilde{L}}{\rho}\int_\Sigma \frac{\partial\varphi_\rho}{\partial\rho}\, d\mathscr{H}^2 = \tilde{L}\frac{I'(\rho)}{\rho}
$$

and

$$
\left|\frac{2}{\rho^3}\int_\Sigma \varphi_\rho\langle \Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2\right| \leqslant \frac{2\operatorname{Lip}(\Omega)}{\rho^2}\int_\Sigma \varphi_\rho\langle \Omega_1,\vec{\Sigma}_J\rangle\, d\mathscr{H}^2 = 2\operatorname{Lip}(\Omega)\frac{I(\rho)}{\rho^2}.
$$

Hence, we conclude

$$
\begin{aligned}
\left|\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^2}\right)-\frac{J'(\rho)}{\rho^2}\right| &= \left|-2\frac{I(\rho)}{\rho^3}+\frac{I'(\rho)}{\rho^2}-\frac{J'(\rho)}{\rho^2}\right|\\
&\leqslant \big(2\operatorname{Lip}(\Omega)+\tilde{L}\big)\frac{I'(\rho)}{\rho}+2\operatorname{Lip}(\Omega)\frac{I(\rho)}{\rho^2}\\
&\quad+\ell\rho^{\gamma-1}\left(\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right)\\
&\leqslant A\frac{I(\rho)}{\rho^2}+A\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho}\right)+\ell\rho^{\gamma-1}\left(\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right),
\end{aligned}
$$

where $A := 2\operatorname{Lip}(\Omega)+\tilde{L}$. From the previous estimate, we immediately conclude that

$$
\begin{aligned}
\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^2}\right)+(A+\ell\rho^{\gamma-1})\frac{I(\rho)}{\rho^2} &\geqslant \frac{J'(\rho)}{\rho^2}-\frac{d}{d\rho}\left(A\rho\frac{I(\rho)}{\rho^2}\right)\\
&\quad+\ell\rho^{\gamma-1}\left(\frac{I(\rho)}{\rho^2}-\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right)
\end{aligned}
\tag{2.4.4}
$$

and

$$
\begin{aligned}
\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^2}\right)-(A+\ell\rho^{\gamma-1})\frac{I(\rho)}{\rho^2} &\leqslant \frac{J'(\rho)}{\rho^2}+\frac{d}{d\rho}\left(A\rho\frac{I(\rho)}{\rho^2}\right)\\
&\quad+\ell\rho^{\gamma-1}\left(\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}-\frac{I(\rho)}{\rho^2}\right).
\end{aligned}
\tag{2.4.5}
$$

By letting φ increase to the characteristic function of the interval $[0,1]$ in (2.4.4), the above estimate passes to the limit in the sense of distributions and we obtain

$$\frac{d}{d\rho}\left(\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right)+(A+\ell\rho^{\gamma-1})\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}$$
$$\geqslant \frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho}\frac{\langle\Omega_t,\vec{\Sigma}_J\rangle}{|\cdot|^2}\,d\mathscr{H}^2\right)-\frac{d}{d\rho}\left(A\rho\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right).$$

Multiplying both of the last inequality sides by the factor $e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}$ and taking into account the fact that the first term on the right-hand-side is non-negative, we get

$$\frac{d}{d\rho}\left(e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right)\geqslant\frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho}\frac{\langle\Omega_t,\vec{\Sigma}_J\rangle}{|\cdot|^2}\,d\mathscr{H}^2\right)$$
$$-\frac{d}{d\rho}\left(e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}A\rho\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right),$$

which turns into

$$\frac{d}{d\rho}\left(e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\right)\geqslant\frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^2}\langle\Omega_t,\vec{\Sigma}_J\rangle\,d\mathscr{H}^2\right).$$

By integration of the previous inequality, we get

$$e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}-e^{A\sigma+\ell\frac{\sigma^\gamma}{\gamma}}(1+A\sigma)\frac{\mathscr{H}^2(\Sigma\cap B_\sigma)}{\sigma^2}$$
$$\geqslant\int_{\Sigma\cap(B_\rho\setminus B_\sigma)}\frac{1}{|\cdot|^2}\langle\Omega_t,\vec{\Sigma}_J\rangle\,d\mathscr{H}^2,$$

for every $0<\sigma<\rho<1$. Since

$$\langle\Omega_t,\vec{\Sigma}_J\rangle=|\vec{\Sigma}_J\wedge\nu|_g^2,$$

the estimate (2.4.2) follows.

By applying the same techniques to (2.4.5), we get (2.4.3) and the statement follows. □

Remark 2.4.3. Proposition 2.4.1 immediately implies that the function

$$(0,1)\ni\rho\mapsto e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}$$

is non-decreasing. In particular, the limit

$$\theta(0,\Sigma):=\lim_{\rho\to0^+}\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}=\lim_{\rho\to0^+}e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}$$

exists and it is finite.

Lemma 2.4.1. *Let Σ be a closed almost J-holomorphic curve in $\mathbb{B}^{2m}$, according to Definition 2.4.2. Then, there exist an $\mathscr{H}^2$-measurable field of unit and simple 2-vectors $\vec{\Sigma}_0\in\wedge_2\mathbb{R}^{2m}$ on Σ and a constant $L>0$ depending only on ℓ and on* $\mathrm{Lip}(\Omega)$ *such that for $\mathscr{H}^2$-a.e. $x\in\Sigma$ the following facts hold:*

(1) $\vec{\Sigma}_0(x)$ *is a unit simple* 2*-vector calibrated by* Ω_0*;*

(2) $|\vec{\Sigma}(x) - \vec{\Sigma}_0(x)| \leqslant L|x|^{\gamma/2}$*;*

(3) if $\vec{\Sigma}(x)$ *is* J_0*-invariant, then* $\vec{\Sigma}_0(x) = \vec{\Sigma}(x)$*;*

(4) if $\vec{\Sigma}(x)$ *is not* J_0*-invariant, then*

$$|\vec{\Sigma}_0(x) \wedge \nu_0(x) \wedge J_0\nu(x)| = \max_{v\in S_x} |v \wedge J_0 v \wedge \nu_0(x) \wedge J_0\nu_0(x)|,$$

where S_x *denotes the unit sphere in the approximate tangent space* $T_x\Sigma$*.*

Proof. Recall the definition of the vector field ν_0, given at the beginning of the present section. If $x \in \Sigma$ is such that $\vec{\Sigma}(x)$ is J_0-invariant, we set $\vec{\Sigma}_0(x) := \vec{\Sigma}(x)$ and all the required properties are satisfied. Otherwise, if $x \in \Sigma$ is such that $\vec{\Sigma}(x)$ is not J_0-invariant, we first claim that there exists some Ω_0-orthonormal basis

$$\big\{e_1(x), J_0e_1(x), ..., e_m(x), J_0e_m(x)\big\}$$

of $\mathbb{R}^{2m}$ such that

$$|e_1(x) \wedge J_0e_1(x) \wedge \nu_0(x) \wedge J_0\nu_0(x)| = \max_{v\in S_x} |v \wedge J_0 v \wedge \nu_0(x) \wedge J_0\nu_0(x)|$$

and we can write $\vec{\Sigma}(x)$ as

$$\vec{\Sigma}(x) = \cos\phi(x)e_1(x) \wedge J_0e_1(x) + \sin\phi(x)e_1(x) \wedge e_2(x).$$

for some angle $\phi(x) \in [0, 2\pi]$. Indeed, since S_x is compact, there exists $e_1(x) \in S_x$ maximizing the continuous function $v \mapsto |v \wedge J_0v \wedge \nu_0(x) \wedge J_0\nu_0(x)|$. We complete $\{e_1(x)\}$ to an Ω_0-orthonormal basis $\{e_1(x), \xi(x)\}$ of $T_x\Sigma$ and we write $\vec{\Sigma}(x) = e_1(x) \wedge \xi(x)$.

Notice that the set $\{e_1(x), J_0e_1(x), \xi(x) - J_0e_1(x)\}$ is linearly independent, otherwise $\vec{\Sigma}(x)$ would be J_0-invariant. We define $e_2(x)$ as the unique vector such that $\{e_1(x), J_0e_1(x), e_2(x)\}$ is an orthonormal set and

$$\text{span}\{e_1(x), J_0e_1(x), e_2(x)\} = \text{span}\{e_1(x), J_0e_1(x), \xi(x) - J_0e_1(x)\}.$$

Moreover, notice that

$$\vec{\Sigma}(x) = e_1(x) \wedge \xi(x) \in \text{span}\big\{e_1(x) \wedge J_0e_1(x), e_1(x) \wedge e_2(x)\big\}$$

and our initial claim follows since

$$|e_1(x) \wedge J_0e_1(x)| = |e_1(x) \wedge e_2(x)| = 1.$$

Then, we define $\vec{\Sigma}_0(x) := e_1(x) \wedge J_0e_1(x)$. Clearly, $\vec{\Sigma}_0(x)$ satisfies (1) and (4). For what concerns (2), notice that $|\Omega_x - \Omega_0| \leqslant \text{Lip}(\Omega)|x|$. In particular, since $\langle\vec{\Sigma}_J, \Omega_x\rangle = 1$ and $|\vec{\Sigma}_J - \vec{\Sigma}| \leqslant \ell|\cdot|^\gamma$, it follows that $|\big\langle\Omega_0, \vec{\Sigma}(x)\big\rangle - 1| \leqslant C|x|^\gamma$, for some constant $C > 0$ depending only on ℓ and $\text{Lip}(\Omega)$. Then,

$$\begin{aligned} 1 + C|x|^\gamma &\geqslant \big\langle\Omega_0, \vec{\Sigma}(x)\big\rangle \\ &= \big\langle\Omega_0, \cos\phi(x)\, e_1(x) \wedge J_0e_1(x) + \sin\phi(x)\, e_1(x) \wedge e_2(x)\big\rangle \end{aligned}$$

$$= \cos\phi(x) \geqslant 1 - C|x|^\gamma.$$

Hence, we eventually obtain

$$\big|\vec{\Sigma}(x) - \vec{\Sigma}_0(x)\big|^2 = (1-\cos\phi(x))^2 + \sin^2\phi(x) = 2(1-\cos\phi) \leqslant 2C|x|^\gamma$$

and the statement follows with $L := \sqrt{2C}$. □

For the purposed of the following lemma, we recall that with $\pi : \mathbb{C}^m \smallsetminus \{0\} \to \mathbb{CP}^{m-1}$ we denote the standard projection to the quotient (see Section 2.1.3).

Lemma 2.4.2. *Under the same hypothesis and notation of Lemma 2.4.1, let $\tilde{r} \in (0,1)$. Then, there exists some constant $\Xi = \Xi\big(m, \tilde{r}, \mathrm{Lip}(\Omega), \ell, \gamma\big) > 0$ such that*

(1) for every $0 < \rho \leqslant \tilde{r}$ and every open set $U \subset B_\rho$ it holds that

$$\left|\mathbb{M}\big(\pi_*([\Sigma] \llcorner U)\big) - \int_{\Sigma\cap U} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2\right| \leqslant \Xi \mathscr{H}^2(\Sigma \cap B_{\tilde{r}})\rho^{\gamma/2}.$$

(2) there exists constants $G > 0$ (depending only on the metric) and $K_m > 0$ (obtained as C_{2m} in [69, Lemma 1]) such that in for every $0 < \rho \leqslant \tilde{r}$, we have

$$\int_{\Sigma\cap B_\rho} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant 2K_m G\bigg(e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2} - \theta(0,\Sigma)\bigg) + \Xi\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\rho^\gamma$$

Proof. First, we aim to prove (1). By Lemma 2.4.1 and by the definition of almost J-holomorphic curve, it follows that

$$\begin{aligned}\big||\wedge_2 d\pi_x(\vec{\Sigma}(x))| - |\wedge_2 d\pi_x(\vec{\Sigma}_0(x))|\big| &\leqslant |\wedge_2 d\pi_x(\vec{\Sigma}(x)) - \wedge_2 d\pi_x(\vec{\Sigma}_0(x))| \\ &\leqslant \frac{1}{|x|^2}|\vec{\Sigma}(x) - \vec{\Sigma}_0(x)| \leqslant \frac{L}{|x|^{2-\gamma/2}},\end{aligned}$$

for $\mathscr{H}^2$-a.e. every $x \in \Sigma \smallsetminus \{0\}$. By integrating on U both sides in the previous inequality, we get

$$\begin{aligned}\left|\mathbb{M}\big(\pi_*([\Sigma] \llcorner U)\big) - \int_{\Sigma\cap U} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2\right| &\leqslant L\int_{\Sigma\cap U} \frac{1}{|x|^{2-\gamma/2}}\, d\mathscr{H}^2 \\ &\leqslant L\int_{\Sigma\cap B_\rho} \frac{1}{|x|^{2-\gamma/2}}\, d\mathscr{H}^2.\end{aligned} \tag{2.4.6}$$

Notice that, by exploiting (2.4.2), we get

$$\begin{aligned}\int_{\Sigma\cap(B_\rho\smallsetminus B_{\rho/2})} \frac{1}{|x|^{2-\gamma/2}}\, d\mathscr{H}^2(x) &\leqslant 2^{2-\gamma/2}\,\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^{2-\gamma/2}} \\ &\leqslant 4e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}\frac{\rho^{\gamma/2}}{2^{\gamma/2}} \\ &\leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\frac{\rho^{\gamma/2}}{2^{\gamma/2}},\end{aligned}$$

for every $0 < \rho \leqslant \tilde{r}$. By iteration, we obtain

$$\int_{\Sigma\cap(B_\rho\setminus B_{\rho/2^n})} \frac{1}{|x|^{2-\gamma/2}}\, d\mathscr{H}^2(x) \leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\left(\sum_{j=0}^{n-1}\frac{1}{(2^{\gamma/2})^j}\right)\rho^{\gamma/2}$$

and, by passing to the limit as $n\to+\infty$, we have

$$\int_{\Sigma\cap B_\rho} \frac{1}{|x|^{2-\gamma/2}}\, d\mathscr{H}^2(x) \leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2(1-2^{-\gamma/2})}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\rho^{\gamma/2}, \tag{2.4.7}$$

for every $0 < \rho \leqslant \tilde{r}$. By combining the previous estimate with (2.4.6), estimate (1) follows with

$$\Xi_1 := \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)L}{\tilde{r}^2(1-2^{-\gamma/2})}.$$

For what concerns (2), we simply notice that by (2.4.2), by point (2) in Lemma 2.4.1 and by [69, Lemma 1], we get

$$\begin{aligned}
\int_{\Sigma\cap B_\rho} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 &\leqslant K_m \int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^2}|\vec{\Sigma}_0\wedge\nu_0|^2\, d\mathscr{H}^2\\
&\leqslant 4K_m \int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^2}|\vec{\Sigma}_J\wedge\nu|^2\, d\mathscr{H}^2\\
&\quad + 4K_m \int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^2}|\nu-\nu_0|^2\, d\mathscr{H}^2\\
&\quad + 4K_m \int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^2}|(\vec{\Sigma}_J-\vec{\Sigma}_0)\wedge\nu_0|^2\, d\mathscr{H}^2\\
&\leqslant 4K_m G\left(e^{A\rho+\ell\rho^\gamma}(1+A\rho)\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2}-\theta(0,\Sigma)\right)\\
&\quad + 4K_m\tilde{L}\mathscr{H}^2(\Sigma\cap B_\rho)\\
&\quad + 4K_m(\ell^2+L^2)\int_{\Sigma\cap B_\rho}\frac{1}{|\cdot|^{2-\gamma}}\, d\mathscr{H}^2.
\end{aligned}$$

By using the same method that we have used in order to prove the decay in (2.4.7), we can show that

$$\int_{\Sigma\cap B_\rho} \frac{1}{|x|^{2-\gamma}}\, d\mathscr{H}^2(x) \leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\rho^{\gamma}, \tag{2.4.8}$$

for very $0 < \rho \leqslant \tilde{r}$. Moreover, we clearly have that

$$\begin{aligned}
\frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^\gamma} \leqslant \frac{\mathscr{H}^2(\Sigma\cap B_\rho)}{\rho^2} &\leqslant \frac{4e^{A\rho+\ell\frac{\rho^\gamma}{\gamma}}(1+A\rho)}{\rho^2}\mathscr{H}^2(\Sigma\cap B_\rho)\\
&\leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}}),
\end{aligned}$$

for very $0 < \rho \leqslant \tilde{r}$. Thus, we get that (2) holds with

$$\Xi_2 := \frac{4K_m e^{A+\frac{\ell}{\gamma}}(1+A)(\tilde{L}+\ell^2+L^2)}{\tilde{r}^2}.$$

Hence, the statement follows with $\Xi := \max\{\Xi_1, \Xi_2\}$. □

Remark 2.4.4. A first remarkable consequence of Lemma 2.4.2 and Proposition 2.4.1 is that

$$\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_\rho)\big) \to 0 \qquad \text{as } \rho \to 0^+. \tag{2.4.9}$$

Lemma 2.4.3 (Good slicing). *Under the same hypotheses and notation of Lemma 2.4.1, let $\tilde{r} \in (0,1)$. Then, for every $r \in (0,\tilde{r}]$ there exist $\tilde{\rho} \in [r/2, r]$ and $\Theta = \Theta\big(m, \tilde{r}, \mathrm{Lip}(\Omega), \ell, \gamma\big) > 0$ such that:*

(1) $\mathscr{H}^1(\Sigma \cap \partial B_{\tilde{\rho}}) \leqslant \Theta \mathscr{H}^2(\Sigma \cap B_{\tilde{r}})\tilde{\rho}$;

(2) $\displaystyle\int_{\Sigma\cap\partial B_{\tilde{\rho}}} |\wedge_2 d\pi(\vec{\Sigma})|\, d\mathscr{H}^1 \leqslant \frac{\Theta}{\tilde{\rho}} \int_{\Sigma\cap(B_r \smallsetminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma})|\, d\mathscr{H}^2$;

(3) $\displaystyle\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_{\tilde{\rho}})\big) \leqslant \Theta\sqrt{K_m} \fint_{r/2}^{r} \frac{1}{\rho} \int_{\Sigma\cap\partial B_\rho} |\vec{\Sigma}_0 \wedge \nu_0|\, d\mathscr{H}^1\, d\mathcal{L}^1(\rho)$.

Proof. First, we notice that, by the coarea formula and the monotonicity formula (2.4.2), it holds that

$$\begin{aligned}\int_{r/2}^{r} \frac{\mathscr{H}^1(\Sigma\cap\partial B_\rho)}{\rho}\, d\mathcal{L}^1(\rho) &\leqslant \frac{2}{r}\mathscr{H}^2(\Sigma\cap B_r)\\ &\leqslant 2e^{Ar+\ell\frac{r^\gamma}{\gamma}}(1+Ar)\frac{\mathscr{H}^2(\Sigma\cap B_r)}{r^2} r\\ &\leqslant \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})\frac{r}{2}.\end{aligned}$$

Hence,

$$\fint_{r/2}^{r} \frac{1}{\Theta_1 \mathscr{H}^2(\Sigma\cap B_{\tilde{r}})} \frac{\mathscr{H}^1(\Sigma\cap\partial B_\rho)}{\rho}\, d\mathcal{L}^1(\rho) \leqslant 1, \tag{2.4.10}$$

with

$$\Theta_1 := \frac{4e^{A+\frac{\ell}{\gamma}}(1+A)}{\tilde{r}^2}.$$

Moreover, again by the coarea formula, we get

$$\begin{aligned}&\fint_{r/2}^{r} \rho \int_{\Sigma\cap\partial B_\rho} |\wedge_2 d\pi(\vec{\Sigma})|\, d\mathscr{H}^1\, d\mathcal{L}^1(\rho)\\ &\leqslant 2\int_{r/2}^{r}\int_{\Sigma\cap\partial B_\rho} |\wedge_2 d\pi(\vec{\Sigma})|\, d\mathscr{H}^1\, d\mathcal{L}^1(\rho)\\ &= 2\int_{\Sigma\cap(B_r\smallsetminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma})|\, d\mathscr{H}^2 =: a,\end{aligned}$$

which leads to

$$\fint_{r/2}^{r} \frac{1}{a}\rho \int_{\Sigma\cap\partial B_\rho} |\wedge_2 d\pi(\vec{\Sigma})| \, d\mathscr{H}^1 \, d\mathcal{L}^1(\rho) \leqslant 1. \tag{2.4.11}$$

Lastly, by [54, Lemma 7.6.1], we know that for a.e. $\rho \in (0,1)$ the slice $\Sigma \cap \partial B_\rho$ is a 1-rectifiable subset of $\mathbb{B}^{2m}$ and the vector field $\vec{\Sigma}_\rho$ orienting its approximate tangent space at x belongs to S_x (see notation of Lemma 2.4.1). Then, by [69, Lemma 1] and by points (3) and (4) of Lemma 2.4.1, it follows that

$$\begin{aligned}
\big|\vec{\Sigma}_\rho \wedge \nu_0 \wedge J_0\nu_0\big|^2 &= \big|\vec{\Sigma}_\rho \wedge J_0\vec{\Sigma}_\rho \wedge \nu_0 \wedge J_0\nu_0\big| \\
&\leqslant \big|\vec{\Sigma}_0 \wedge \nu_0 \wedge J_0\nu_0\big| \\
&\leqslant K_m\big|\vec{\Sigma}_0 \wedge \nu_0\big|^2, \qquad \mathscr{H}^1\text{-a.e. on } \Sigma\cap\partial B_\rho.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho)\big) &\leqslant \int_{\Sigma\cap\partial B_\rho} \big|d\pi(\vec{\Sigma}_\rho)\big| \, d\mathscr{H}^1 \\
&= \frac{1}{\rho}\int_{\Sigma\cap\partial B_\rho} \big|\vec{\Sigma}_\rho \wedge \nu_0 \wedge J\nu_0\big| \, d\mathscr{H}^1 \\
&\leqslant \frac{\sqrt{K_m}}{\rho}\int_{\Sigma\cap\partial B_\rho} \big|\vec{\Sigma}_0 \wedge \nu_0\big| \, d\mathscr{H}^1.
\end{aligned}$$

Thus, by averaging the previous inequality on $[r/2, r]$, we obtain

$$\fint_{r/2}^{r} \mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho) \, d\mathcal{L}^1(\rho) \leqslant \sqrt{K_m}\fint_{r/2}^{r} \frac{1}{\rho}\int_{\Sigma\cap\partial B_\rho} \big|\vec{\Sigma}_0 \wedge \nu_0\big| \, d\mathscr{H}^1 =: b,$$

which leads to

$$\fint_{r/2}^{r} \frac{1}{b}\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho) \, d\mathcal{L}^1(\rho) \leqslant 1. \tag{2.4.12}$$

By summing up the three inequalities (2.4.10), (2.4.11) and (2.4.12) we obtain

$$\begin{aligned}
&\fint_{r/2}^{r} \bigg(\frac{1}{\Theta_1\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})}\frac{\mathscr{H}^1(\Sigma\cap\partial B_\rho)}{\rho} + \frac{1}{a}\rho\int_{\Sigma\cap\partial B_\rho} |\wedge_2 d\pi(\vec{\Sigma})| \, d\mathscr{H}^1 \\
&+ \frac{1}{b}\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho)\big)\bigg) \, d\mathcal{L}^1(\rho) \leqslant 3.
\end{aligned}$$

Then, we conclude that there exists $\rho \in [r/2, r]$ such that

$$\begin{aligned}
&\frac{1}{\Theta_1\mathscr{H}^2(\Sigma\cap B_{\tilde{r}})}\frac{\mathscr{H}^1(\Sigma\cap\partial B_\rho)}{\rho} + \frac{1}{a}\rho\int_{\Sigma\cap\partial B_\rho} |\wedge_2 d\pi(\vec{\Sigma})| \, d\mathscr{H}^1 \\
&+ \frac{1}{b}\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho) \leqslant 3
\end{aligned}$$

and the statement follows with $\Theta := \max\{\Theta_1, 6\}$. □

Lemma 2.4.4 (Controlling the mass of the projected boundaries)**.** *Under the same hypotheses and notation of Lemma 2.4.1, let $\tilde{r} \in (0,1)$. Let $r \in (0,\tilde{r}]$ be such that*

$$\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_r)\big) < 2K_m^2 \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{r/2})\big) \tag{2.4.13}$$

and

$$\int_{\Sigma \cap B_r} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2 > \zeta^{-1} \Xi \mathscr{H}^2(\Sigma \cap B_{\tilde{r}}) r^{\gamma/2}, \tag{2.4.14}$$

for some $\zeta \in (0,1)$. If $\rho_j \in [r/2, r]$ is such that $\Sigma \cap \partial B_{\rho_j}$ is a good slice of Σ in the sense of Lemma 2.4.3, then

$$\begin{aligned} \mathbb{M}\big(\pi_* \partial([\Sigma] \llcorner B_{\rho_j})\big) &\leqslant \Lambda \sqrt{\mathscr{H}^2(\Sigma \cap B_{\tilde{r}})} \sqrt{\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j})\big)} \\ &\quad + \Lambda \mathscr{H}^2(\Sigma \cap B_{\tilde{r}}) \rho_j^{\gamma/4}, \end{aligned} \tag{2.4.15}$$

for a constant $\Lambda = \Lambda\big(m, \tilde{r}, \mathrm{Lip}(\Omega), \ell, \gamma\big) > 0$.

Proof. Let $\rho_j \in [r/2, r]$ be such that $\Sigma \cap \partial B_{\rho_j}$ is a good slice of Σ. We apply twice the Cauchy-Schwarz inequality in the right-hand side of (3) in Lemma 2.4.3 and the coarea formula to get

$$\begin{aligned} \mathbb{M}\big(\pi_* \partial([\Sigma] \llcorner B_{\rho_j})\big) &\leqslant \Theta \sqrt{K_m} \fint_{r/2}^{r} \frac{1}{\rho} \int_{\Sigma \cap \partial B_\rho} |\vec{\Sigma}_0 \wedge \nu_0| \, d\mathscr{H}^1 \, d\mathcal{L}^1(\rho) \\ &= \Theta \sqrt{K_m} \fint_{r/2}^{r} \int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho} |\vec{\Sigma}_0 \wedge \nu_0| \, d\mathscr{H}^1 \, d\mathcal{L}^1(\rho) \\ &\leqslant \Theta \sqrt{K_m} \fint_{r/2}^{r} \sqrt{\mathscr{H}^1(\Sigma \cap \partial B_\rho)} \\ &\qquad \cdot \sqrt{\int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho^2} |\vec{\Sigma}_0 \wedge \nu_0|^2 \, d\mathscr{H}^1} \, d\mathcal{L}^1(\rho) \\ &= \Theta \sqrt{K_m} \frac{2}{r} \int_{r/2}^{r} \sqrt{\mathscr{H}^1(\Sigma \cap \partial B_\rho)} \\ &\qquad \cdot \sqrt{\int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho^2} |\vec{\Sigma}_0 \wedge \nu_0|^2 \, d\mathscr{H}^1} \, d\mathcal{L}^1(\rho) \\ &\leqslant \Theta \sqrt{K_m} \frac{2}{r} \sqrt{\int_{r/2}^{r} \mathscr{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \\ &\qquad \cdot \sqrt{\int_{r/2}^{r} \int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho^2} |\vec{\Sigma}_0 \wedge \nu_0|^2 \, d\mathscr{H}^1 \, d\mathcal{L}^1(\rho)} \\ &= \Theta \sqrt{K_m} \sqrt{\frac{2}{r}} \sqrt{\fint_{r/2}^{r} \mathscr{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \\ &\qquad \cdot \sqrt{\int_{\Sigma \cap (B_r \smallsetminus B_{r/2})} \frac{1}{|\cdot|^2} |\vec{\Sigma}_0 \wedge \nu_0|^2 \, d\mathscr{H}^2}. \end{aligned}$$

We notice that, by point (1) in Lemma 2.4.2 and by our assumption (2.4.14), it holds that

$$\begin{aligned}\left|\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_r)\big)-\int_{\Sigma\cap B_r}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2\right| &\leqslant \Xi\mathscr{H}^2(\Sigma\cap B_{\tilde r})r^{\gamma/2}\\ &< \zeta\int_{\Sigma\cap B_r}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2\end{aligned}$$

which implies

$$\int_{\Sigma\cap B_r}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2 \leqslant \frac{1}{1-\zeta}\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_r)\big)$$

Hence, by [69, Lemma 1] we have

$$\begin{aligned}\int_{\Sigma\cap(B_r\setminus B_{r/2})}\frac{1}{|\cdot|^2}|\vec{\Sigma}_0\wedge\nu_0|^2\,d\mathscr{H}^2 &\leqslant K_m\int_{\Sigma\cap(B_r\setminus B_{r/2})}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2\\ &\leqslant \frac{K_m}{1-\zeta}\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_r)\big).\end{aligned}$$

Moreover, by (2.4.10), it follows that

$$\begin{aligned}\sqrt{⨍_{r/2}^{r}\mathscr{H}^1(\Sigma\cap\partial B_\rho)\,d\mathcal{L}^1(\rho)} &\leqslant \sqrt{r}\sqrt{⨍_{r/2}^{r}\frac{\mathscr{H}^1(\Sigma\cap\partial B_\rho)}{\rho}\,d\mathcal{L}^1(\rho)}\\ &\leqslant \sqrt{2\Theta}\sqrt{\mathscr{H}^2(\Sigma\cap B_{\tilde r})}\sqrt{\frac{r}{2}}.\end{aligned}$$

Thus,

$$\mathbb{M}\big(\pi_*\partial([\Sigma]\llcorner B_{\rho_j})\big)\leqslant\frac{\sqrt{2}\Theta^{3/2}K_m}{\sqrt{1-\zeta}}\sqrt{\mathscr{H}^2(\Sigma\cap B_{\tilde r})}\sqrt{\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_r)\big)}.$$

By our hypothesis (2.4.13) and since $\rho_j > r/2$, we obtain that

$$\begin{aligned}\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_r)\big) &< 2K_m^2\,\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_{r/2})\big)\\ &\leqslant 2K_m^2\,\mathbb{M}\big(\pi_*([\Sigma]\llcorner B_{\rho_j})\big)+4K_m^2\Xi\mathscr{H}^2(\Sigma\cap B_{\tilde r})\rho_j^{\gamma/2}.\end{aligned}$$

We point out that the last inequality follows a direct application of point (1) in Lemma 2.4.2. Then, the statement follows with

$$\Lambda := \max\left\{\frac{2\Theta^{3/2}K_m^2}{\sqrt{1-\zeta}},\frac{2\sqrt{2}\Theta^2K_m^2}{\sqrt{1-\zeta}}\right\}.$$

□

We recall the following general fact about integral 2-currents on $\mathbb{CP}^{m-1}$ with small mass which are ζ-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$, whose proof can be found in [69, Lemma 11]. Recall that a current $T\in\mathcal{D}^2(\mathbb{CP}^{m-1})$ is said to be ζ-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$ for some constant $\zeta\in(0,1)$ if

$$(1-\zeta)|\langle T\llcorner U,\omega_{\mathbb{CP}^{m-1}}\rangle|\leqslant\mathbb{M}(T\llcorner U)\leqslant(1+\zeta)|\langle T\llcorner U,\omega_{\mathbb{CP}^{m-1}}\rangle|,$$

for every open set $U\subset\mathbb{CP}^{m-1}$.

Lemma 2.4.5. *Let $\zeta \in (0,1)$. Given any couple of constants $\tilde{\Lambda} > 0$ and $\lambda > 0$, there exist $\delta > 0$ and $\varepsilon > 0$ satisfying what follows. For every integral 2-current $T \in \mathcal{D}^2(\mathbb{CP}^{m-1})$ such that*

1. *T is ζ-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$,*
2. *$\mathbb{M}(T) + \mathbb{M}(\partial T) < \delta$,*
3. *$\mathbb{M}(\partial T) \leqslant \tilde{\Lambda}\sqrt{\mathbb{M}(T)}$,*

there is a complex projective $(m-2)$-hyperplane $H \subset \mathbb{CP}^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{CP}^{m-1}$ of H with width ε such that

$$\frac{\mathbb{M}(T \llcorner H_\varepsilon)}{\varepsilon^2} \leqslant \lambda \mathbb{M}(T). \tag{2.4.16}$$

Lastly we need to establish that, given a complex projective $(m-2)$-hyperplane $H \subset \mathbb{CP}^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{CP}^{m-1}$ of H with width ε, we can approximate the symplectic form $\omega_{\mathbb{CP}^{m-1}}$ on $\mathbb{CP}^{m-1}$ with an exact form $d\alpha$ that coincides with $\omega_{\mathbb{CP}^{m-1}}$ on the complement of H_ε and vanishes on $H_{\varepsilon/2}$. We achieve this approximation through the following lemma, whose proof is again in [69, Lemma 6].

Lemma 2.4.6. *Let $H \subset \mathbb{CP}^{m-1}$ be any complex projective $(m-2)$-hyperplane and let $H_\varepsilon \subset \mathbb{CP}^{m-1}$ be a tubular neighbourhood of H with width ε. Then there exists a 1-form $\alpha \in \Omega^1(\mathbb{CP}^{m-1})$ and a universal constant $\kappa > 0$ such that:*

1. *$\omega_{\mathbb{CP}^{m-1}} = d\alpha$ on $\mathbb{CP}^{m-1} \setminus H_\varepsilon$;*
2. *$\alpha = 0$ on $H_{\varepsilon/2}$;*
3. *$||\alpha||_* \leqslant \kappa$;*
4. *$||\omega_{\mathbb{CP}^{m-1}} - d\alpha||_* \leqslant \dfrac{\kappa}{\varepsilon^2}$.*

2.4.2. Proof of the fundamental Morrey type estimate

Recall that $\kappa, \Xi > 0$ are positive constants introduced in Lemma 2.4.6 and Lemma 2.4.2 respectively.

Fix any $j_0 \in \mathbb{N} \setminus \{0\}$ and let $\ell \geqslant 0$ be a constant depending only on $\mathrm{Lip}(\Omega)$. Let $\delta > 0$ and $\varepsilon > 0$ be the constants given by applying Lemma 2.4.5 with $\tilde{\Lambda} = \Lambda\big(m, 2^{-j_0}, \mathrm{Lip}(\Omega), \ell, 1/2\big)$ from Lemma 2.4.4 and $\lambda := 5(24\kappa)^{-1}$. Let $\delta' > 0$ be such that

$$\Lambda\sqrt{\delta'} + \frac{\delta'}{2} + \delta' < \delta$$

and choose $\tilde{r} \in \big(0, \min\{2^{-j_0}, \delta'(2\Xi)^{-1}\}\big)$ such that

$$\max\{\Lambda, \Xi\}\big(e^{A+2\ell}(1+A)\big)^{-1}\tilde{r}^{1/4} < \frac{\delta'}{2}.$$

Assume that $\mathscr{F}$ is a family of closed almost J-holomorphic curves in $\mathbb{B}^{2m}$ such that every element $\Sigma \in \mathscr{F}$ satisfies the following properties:

(1) $|\vec{\Sigma}(x) - \vec{\Sigma}_J(x)| \leqslant \ell |x|^{1/2}$, for $\mathscr{H}^2$-a.e. $x \in \Sigma$,

(2) $\mathscr{H}^2(\Sigma \cap B_{2^{-j_0}}) < \big(e^{A+2\ell}(1+A)\big)^{-1}$,

(3) $\displaystyle\int_{\Sigma\cap B_{2^{-j_0}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 < \frac{\delta'}{2}.$

Remark 2.4.5. For every $\Sigma \in \mathscr{F}$, the hypotheses (2) and (3) combined with point (1) in Lemma 2.4.2 imply that:

1. $\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_\rho)\big) + \mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho)\big) < \delta,$
2. $\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_\rho)\big) \leqslant \Lambda\sqrt{\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_\rho)\big)},$

for every good slice $\Sigma \cap \partial B_\rho$ of Σ, where $\rho \in [r/2, r]$ with $r \in (0, \tilde{r}]$ satisfying the hypotheses (2.4.13) and (2.4.14) of Lemma 2.4.4.

We want to show that for every $\Sigma \in \mathscr{F}$ there exist constants $C > 0$ and $0 < \alpha < 1$ depending on m, j_0, $\mathrm{Lip}(\Omega)$ such that

$$\bigg|\int_{\Sigma\cap B_\rho} \pi^*\omega_{\mathbb{CP}^{m-1}}|_\Sigma\bigg| \leqslant C\rho^\alpha, \qquad \forall\, \rho \in (0, 2^{-j_0}). \tag{2.4.17}$$

By definition of mass it holds that

$$\begin{aligned}\bigg|\int_{\Sigma\cap B_\rho} \pi^*\omega_{\mathbb{CP}^{m-1}}|_\Sigma\bigg| &= |\,\langle \pi_*([\Sigma] \llcorner B_\rho), \omega_{\mathbb{CP}^{m-1}}\rangle\,| \\ &\leqslant \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_\rho)\big),\end{aligned} \tag{2.4.18}$$

for every $\rho \in (0, 1)$. Hence, in order to prove (2.4.17) it is enough to show that

$$\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_\rho)\big) \leqslant C\rho^\alpha, \qquad \text{for every } \rho \in (0, \tilde{r}). \tag{2.4.19}$$

Moreover, by exploiting point (1) in Lemma 2.4.2, we realize that we if we show

$$\int_{\Sigma\cap B_\rho} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant \tilde{C}\rho^{\tilde{\alpha}}, \qquad \text{for every } \rho \in (0, \tilde{r}), \tag{2.4.20}$$

then (2.4.19) will follow with $C := \tilde{C} + \Xi$ and $\alpha := \min\{\tilde{\alpha}, 1/4\}$. Thus, we just need to show (2.4.20).

Fix any $\Sigma \in \mathscr{F}$. Let

$$E(\rho) := \int_{\Sigma\cap B_\rho} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2, \qquad \forall\, \rho \in (0, 1)$$

and define

$$\begin{aligned} I &:= \Big\{ j \in \mathbb{N} \text{ s.t. } 2^{-j} \leqslant \tilde{r} \text{ and } E\big(2^{-(j+1)}\big) \leqslant \frac{1}{2}E\big(2^{-j}\big)\Big\}, \\ J &:= \big\{ j \in \mathbb{N} \text{ s.t. } 2^{-j} \leqslant \tilde{r} \text{ and } E\big(2^{-(j+1)}\big) \leqslant 5\Xi 2^{-(j+1)/4}\big\}.\end{aligned}$$

First, we claim that there exists $\theta = \theta\big(m, j_0, \mathrm{Lip}(\Omega)\big) \in (0, 1)$ such that

$$E\big(2^{-(j+1)}\big) \leqslant \theta\Big(E\big(2^{-j}\big) + 2^{-j/4}\Big), \tag{2.4.21}$$

for every $j \in (I \cup J)^c$. Fix $j \in (I \cup J)^c$ and set $r := 2^{-j}$. Pick a radius $\rho_j \in [r/2, r]$ such that $\Sigma \cap \partial B_{\rho_j}$ is a good slice of Σ (see Lemma 2.4.3). By Lemma 2.4.2 and Lemma 2.4.4, it follows that we can choose a sequence of radii $\{s_k\}_{k\in\mathbb{N}} \in (0, r/2)$ such that $s_k \to 0^+$ as $k \to +\infty$ and

$$\mathbb{M}\big(\pi_*\partial([\Sigma] \llcorner B_{s_k})\big) \leqslant \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_r \smallsetminus B_{r/2})\big), \qquad \forall k \in \mathbb{N}. \tag{2.4.22}$$

Since $\Sigma \in \mathscr{F}$, by Remark 2.4.5 and since $j \in (I \cup J)^c$ it follows that $T = \pi_*([\Sigma] \llcorner B_{\rho_j})$ satisfies the hypotheses of Lemma 2.4.5 with $\zeta = 1/2$. Thus, there exists a complex projective $(m-2)$-hyperplane $H \subset \mathbb{CP}^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{CP}^{m-1}$ of H with width $\varepsilon > 0$ such that

$$\frac{\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j}) \llcorner H_\varepsilon\big)}{\varepsilon^2} \leqslant \lambda \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j})\big).$$

We let $\alpha \in \Omega^1(\mathbb{CP}^{m-1})$ be a smooth 1-form given by Lemma 2.4.6 relatively to H, H_ε. Following the proof of point (1) in Lemma 2.4.2, we notice that

$$\begin{aligned}
&\int_{\Sigma\cap(B_{r/2}\smallsetminus B_{s_k})} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \\
&\leqslant \int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \\
&= 2\left| \int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})} \left\langle \pi^*\omega_{\mathbb{CP}^{m-1}}, \vec{\Sigma} \right\rangle d\mathscr{H}^2 \right| \\
&= 2\left| \int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})} \left\langle \pi^* d\alpha, \vec{\Sigma} \right\rangle d\mathscr{H}^2 \right| \\
&\quad + 2\left| \int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})} \left\langle \pi^*(\omega_{\mathbb{CP}^{m-1}} - d\alpha), \vec{\Sigma} \right\rangle d\mathscr{H}^2 \right|.
\end{aligned} \tag{2.4.23}$$

For what concerns the second term in the last sum, by Lemmas 2.4.5, 2.4.6 and (1) in Lemma 2.4.2 we see that

$$\begin{aligned}
&\left| \int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})} \left\langle \pi^*(\omega_{\mathbb{CP}^{m-1}} - d\alpha), \vec{\Sigma} \right\rangle d\mathscr{H}^2 \right| \\
&\leqslant ||\omega_{\mathbb{CP}^{m-1}} - d\alpha||_* \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j}) \llcorner H_\varepsilon\big) \\
&\leqslant \kappa \frac{\mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j}) \llcorner H_\varepsilon\big)}{\varepsilon^2} \\
&\leqslant \kappa\lambda\, \mathbb{M}\big(\pi_*([\Sigma] \llcorner B_{\rho_j})\big) \\
&\leqslant \frac{1}{4} \int_{\Sigma\cap B_{\rho_j}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 + \frac{\Xi}{4} \mathscr{H}^2(\Sigma \cap B_{\rho_j}) \rho_j^{1/4} \\
&\leqslant \frac{1}{4} \int_{\Sigma\cap B_r} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 + \tilde{\Xi} r^{1/4},
\end{aligned} \tag{2.4.24}$$

with

$$\tilde{\Xi} := \frac{\Xi}{4}\big(e^{A+2\ell}(1+A)\big)^{-1}.$$

Now we want to estimate

$$\int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})}\left\langle\pi^*d\alpha,\vec{\Sigma}\right\rangle\,d\mathscr{H}^2.$$

Since π is a smooth map on $B_{\rho_j}\smallsetminus B_{s_k}$, by Stokes theorem we get that

$$\begin{aligned}&\left|\int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})}\left\langle\pi^*d\alpha,\vec{\Sigma}\right\rangle\,d\mathscr{H}^2\right|\\&=\left|\int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})}\pi^*d\alpha|_\Sigma\right|=\left|\int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})}d(\pi^*\alpha)|_\Sigma\right|\\&=\left|\int_{\Sigma\cap\partial B_{\rho_j}}\pi^*\alpha|_{\Sigma\cap\partial B_{\rho_j}}-\int_{\Sigma\cap\partial B_{s_k}}\pi^*\alpha|_{\Sigma\cap\partial B_{s_k}}\right|\\&\leqslant\left|\int_{\Sigma\cap\partial B_{\rho_j}}\pi^*\alpha|_{\Sigma\cap\partial B_{\rho_j}}\right|+\left|\int_{\Sigma\cap\partial B_{s_k}}\pi^*\alpha|_{\Sigma\cap\partial B_{s_k}}\right|.\end{aligned}$$

Since $j\in(I\cup J)^c$, by (2.4.22) we get that

$$\begin{aligned}\left|\int_{\Sigma\cap\partial B_{s_k}}\pi^*\alpha|_{\Sigma\cap\partial B_{s_k}}\right|&=|\langle\pi_*\partial([\Sigma]\llcorner B_{s_k}),\alpha\rangle|\leqslant||\alpha||_*\mathbb{M}\big(\pi_*\partial([\Sigma]\llcorner B_{s_k})\big)\\&\leqslant\kappa\mathbb{M}\big(\pi_*([\Sigma]\llcorner(B_r\smallsetminus B_{r/2}))\big)\\&\leqslant\frac{3\kappa}{2}\int_{\Sigma\cap(B_r\smallsetminus B_{r/2})}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2.\end{aligned}$$

Thus, we have obtained

$$\begin{aligned}\left|\int_{\Sigma\cap(B_{\rho_j}\smallsetminus B_{s_k})}\left\langle\pi^*d\alpha,\vec{\Sigma}\right\rangle\,d\mathscr{H}^2\right|\leqslant&\left|\int_{\Sigma\cap\partial B_{\rho_j}}\pi^*\alpha|_{\Sigma\cap\partial B_{\rho_j}}\right|\\&+\frac{3\kappa}{2}\int_{\Sigma\cap(B_r\smallsetminus B_{r/2})}|\wedge_2 d\pi(\vec{\Sigma}_0)|\,d\mathscr{H}^2\end{aligned}\tag{2.4.25}$$

and we just need to bound

$$\left|\int_{\Sigma\cap\partial B_{\rho_j}}\pi^*\alpha|_{\Sigma\cap\partial B_{\rho_j}}\right|.$$

To do this, we write the 1-rectifiable closed curve $\Sigma\cap\partial B_{\rho_j}$ as

$$\Sigma\cap\partial B_{\rho_j}=\bigcup_{i=0}^{\infty}\Gamma_i,$$

where Γ_i is a Lipschitz connected closed curve in B. We let $\gamma_i:[0,\mathscr{H}^1(\Gamma_i)]\to\mathbb{B}^{2m}$ be the parametrization of Γ_i through its arc-length, so that $|\gamma_i'|\equiv 1$ a.e. on $[0,\mathscr{H}^1(\Gamma_i)]$. First, fix $i\in\mathbb{N}$ and notice that for every smooth function $f:\mathbb{B}^{2m}\smallsetminus\{0\}\to\mathbb{R}$ such that $\bar{f}^i=0$, where

$$\bar{f}^i:=\int_{\Gamma_i}f\,d\mathscr{H}^1,$$

the following Poincaré type inequality holds for Γ_i:

$$
\begin{aligned}
\left(\int_{\Gamma_i} |f|^2 \, d\mathscr{H}^1\right)^{1/2} &= \left(\int_0^{\mathscr{H}^1(\Gamma_i)} |f\circ\gamma_i|^2 |\gamma_i'| \, d\mathcal{L}^1\right)^{1/2} \\
&= \left(\int_0^{\mathscr{H}^1(\Gamma_i)} |f\circ\gamma_i|^2 \, d\mathcal{L}^1\right)^{1/2} \\
&\leqslant \mathscr{H}^1(\Gamma_i)\left(\int_0^{\mathscr{H}^1(\Gamma_i)} |(f\circ\gamma_i)'|^2 \, d\mathcal{L}^1\right)^{1/2} \\
&= \mathscr{H}^1(\Gamma_i)\left(\int_0^{\mathscr{H}^1(\Gamma_i)} |df_{\gamma_i}(\gamma_i')|^2 \, d\mathcal{L}^1\right)^{1/2} \\
&= \mathscr{H}^1(\Gamma_i)\left(\int_0^{\mathscr{H}^1(\Gamma_i)} |df_{\gamma_i}(\vec{\Sigma}_{\rho_j})|^2 |\gamma_i'| \, d\mathcal{L}^1\right)^{1/2} \\
&= \mathscr{H}^1(\Gamma_i)\left(\int_{\Gamma_i} |df|_{\Sigma_{\rho_j}}|^2 \, d\mathscr{H}^1\right)^{1/2}. \qquad (2.4.26)
\end{aligned}
$$

Secondly, since $\operatorname{spt}(\alpha) \subset \mathbb{CP}^{m-1} \smallsetminus H_{\varepsilon/2}$ and $\mathbb{CP}^{m-1} \smallsetminus H_{\varepsilon/2}$ is diffeomorphic to $\mathbb{R}^{2m-2}$ we can write α in coordinates $\{y_1, ...y_{2m-2}\}$ on $\mathbb{CP}^{m-1}$ as

$$
\alpha = \sum_{a=1}^{2m-2} \alpha_a dy_a
$$

in order to get the expansion

$$
\pi^*\alpha|_{\Gamma_i} = \left\langle \pi^*\alpha, \gamma_i' \right\rangle = \sum_{a=1}^{2m-2} (\alpha_a \circ \pi) \left\langle dy_a, d\pi|_{\Gamma_i} \right\rangle .
$$

Moreover, we notice that

$$
\begin{aligned}
\left|d(\alpha_a \circ \pi)|_{\Sigma_{\rho_j}}\right| &\leqslant \left|d\alpha_a \circ \pi\right|^2 \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 \leqslant \left|d\alpha \circ \pi\right|^2 \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 \\
&\leqslant \max\left\{ ||\omega_{\mathbb{CP}^{m-1}}||_\infty, \frac{\kappa}{\varepsilon^2}\right\} \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 \\
&\leqslant \max\left\{ ||\omega_{\mathbb{CP}^{m-1}}||_\infty, \frac{\kappa}{\varepsilon^2}\right\} \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 \\
&= M_m \left|d\pi|_{\Sigma_{\rho_j}}\right|^2,
\end{aligned}
$$

where

$$
M_m := \max\left\{ ||\omega_{\mathbb{CP}^{m-1}}||_\infty, \frac{\kappa}{\varepsilon^2}\right\}
$$

depends only on m. Then, by (2.4.26), Hölder's inequality and point (3) in Lemma 2.4.1, we estimate

$$
\begin{aligned}
&\left|\int_{\Gamma_i} \pi^*\alpha|_{\Gamma_i}\right| \\
&= \left|\sum_{a=1}^{2m-2} \int_{\Gamma_i} (\alpha_a \circ \pi) \left\langle dy_a, d\pi(\gamma_i')\right\rangle\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \sum_{a=1}^{2m-2} \int_{\Gamma_i} (\alpha_a \circ \pi - \bar{\alpha}_k^i \circ \pi) \left\langle dy_a, d\pi(\gamma_i') \right\rangle \right| \\
&\leqslant \sum_{a=1}^{2m-2} \int_{\Gamma_i} \left|\alpha_a \circ \pi - \bar{\alpha}_k^i \circ \pi\right| \left|d\pi|_{\Gamma_i}\right|^2 \\
&\leqslant \sum_{a=1}^{2m-2} \left(\int_{\Gamma_i} \left|\alpha_a \circ \pi - \bar{\alpha}_k^i \circ \pi\right|^2 d\mathscr{H}^1 \right)^{1/2} \left(\int_{\Gamma_i} \left|d\pi|_{\Sigma_{\rho_j}}\right| d\mathscr{H}^1 \right)^{1/2} \\
&\leqslant \mathscr{H}^1(\Gamma_i) \sum_{a=1}^{2m-2} \left(\int_{\Gamma_i} \left|d(\alpha_a \circ \pi)|_{\Sigma_{\rho_j}}\right|^2 d\mathscr{H}^1 \right)^{1/2} \left(\int_{\Gamma_i} \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 d\mathscr{H}^1 \right)^{1/2} \\
&\leqslant \tilde{M}_m \mathscr{H}^1(\Gamma_i) \int_{\Gamma_i} \left|d\pi|_{\Sigma_{\rho_j}}\right|^2 d\mathscr{H}^1 \\
&\leqslant \tilde{M}_m \mathscr{H}^1(\Sigma \cap \partial B_{\rho_j}) \int_{\Gamma_i} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^1,
\end{aligned}
$$

where $\tilde{M}_m := (2m-2)M_m$ and the last inequality follows by our choice of $\vec{\Sigma}_0$ (see point (4) in Lemma 2.4.1). Summing up over $i \in \mathbb{N}$ in the previous inequality, using the properties of good slices established in Lemma 2.4.3 and since $\Sigma \in \mathscr{F}$, we eventually get

$$
\begin{aligned}
&\left| \int_{\Sigma \cap \partial B_{\rho_j}} \pi^* \alpha|_{\Sigma \cap \partial B_{\rho_j}} \right| \\
&\leqslant \tilde{M}_m \mathscr{H}^1(\Sigma \cap \partial B_{\rho_j}) \int_{\Sigma \cap \partial B_{\rho_j}} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^1 \\
&\leqslant \tilde{M}_m \Theta^2 \mathscr{H}^2(\Sigma \cap B_{2^{-j_0}}) \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2 \\
&\leqslant \tilde{M}_m \Theta^2 \big(e^{A+j_0}(1+A)\big)^{-1} \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2
\end{aligned}
\tag{2.4.27}
$$

Plugging (2.4.27) in (2.4.25) we get

$$
\begin{aligned}
&\left| \int_{\Sigma \cap (B_{\rho_j} \setminus B_{s_k})} \left\langle \pi^* d\alpha, \vec{\Sigma} \right\rangle d\mathscr{H}^2 \right| \\
&\leqslant \left(\tilde{M}_m \Theta^2 \big(e^{A+2\ell}(1+A)\big)^{-1} + \frac{3\kappa}{2} \right) \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2.
\end{aligned}
\tag{2.4.28}
$$

Combining (2.4.28), (2.4.24) and (2.4.23) and setting

$$
\hat{C} := \max \left\{ \frac{2\tilde{M}_m \Theta^2}{e^{A+2\ell}(1+A)} + 3\kappa, 1, 2\tilde{\Xi} \right\},
$$

we obtain

$$
\begin{aligned}
\int_{\Sigma \cap (B_{r/2} \setminus B_{s_k})} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2 \leqslant \hat{C} \bigg(&\int_{\Sigma \cap (B_r \setminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2 + r^{1/4} \bigg) \\
&+ \frac{1}{2} \int_{\Sigma \cap B_r} |\wedge_2 d\pi(\vec{\Sigma}_0)| \, d\mathscr{H}^2.
\end{aligned}
$$

By letting $k \to +\infty$ in the previous inequality, we obtain

$$\int_{\Sigma\cap B_{r/2}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant \hat{C}\bigg(\int_{\Sigma\cap(B_r\smallsetminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 + r^{1/4}\bigg) + \frac{1}{2}\int_{\Sigma\cap B_r} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2.$$

and by subtracting from both sides the quantity

$$\frac{1}{2}\int_{\Sigma\cap B_{r/2}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2,$$

we get

$$\int_{\Sigma\cap B_{r/2}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant \bar{C}\bigg(\int_{\Sigma\cap(B_r\smallsetminus B_{r/2})} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 + r^{1/4}\bigg),$$

where $\bar{C} > 0$ is chosen big enough so that $\bar{C} \geqslant 2\hat{C}$ and

$$\frac{\bar{C}}{\bar{C}+1} \geqslant 2^{-1/4}.$$

By the hole filling technique and recalling that $r = 2^{-j}$, we obtain

$$\int_{\Sigma\cap B_{2^{-(j+1)}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant \theta \int_{\Sigma\cap B_{2^{-j}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 + \theta^{j+1},$$

with $\theta \in (0,1)$ given by $\theta := \bar{C}/(\bar{C}+1) \geqslant 2^{-1/4}$ and our claim (2.4.21) follows.
By (2.4.21) we obtain that

$$\begin{aligned} E\big(2^{-j}\big) &\leqslant \theta E\big(2^{-(j-1)}\big) + \theta^j \leqslant \theta^2 E\big(2^{-(j-2)}\big) + 2\theta^j \\ &\leqslant \ldots \leqslant \theta^{j_0} E\big(2^{-j_0}\big)\theta^{-j} + \big((j-j_0)\theta^{-j/2}\big)\theta^{-j/2} \\ &\leqslant \hat{\Xi}\tilde{\theta}^{-j} \end{aligned}$$

with $\tilde{\theta} := \theta^{1/2}$ and

$$\hat{\Xi} := \frac{\delta' \theta^{j_0}}{2} + \sup_{j\geqslant j_0}\big\{(j-j_0)\tilde{\theta}^{-j}\big\} < +\infty.$$

In order to get (2.4.20), we notice that if $j \in I \cup J$ then either $j \in I$ or $j \in J \smallsetminus I$. In the first case, we have

$$\begin{aligned} \int_{\Sigma\cap B_{2^{-(j+1)}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 &\leqslant \frac{1}{2}\int_{\Sigma\cap B_{2^{-j}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \\ &\leqslant \theta \int_{\Sigma\cap B_{2^{-j}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2. \end{aligned}$$

In the second case, by definition of J, it holds that

$$\int_{\Sigma\cap B_{2^{-(j+1)}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant 5\Xi\, 2^{-(j+1)/4}.$$

By setting $\tilde{\alpha} = \min\{-\log_2 \tilde{\theta}, 1/4\} \in (0,1)$, we get

$$\int_{\Sigma\cap B_{2^{-j}}} |\wedge_2 d\pi(\vec{\Sigma}_0)|\, d\mathscr{H}^2 \leqslant 5\Xi\bigg(\frac{1}{2^j}\bigg)^{\alpha},$$

which leads to (2.4.20) with $\tilde{C} := \max\{5\theta^{1/4}\Xi, \hat{\Xi}\}$.

2.5. Almost pseudo-holomorphic foliations

Lemma 2.5.1. *Let $m \geqslant 2$ and let (X, J_X, ω_X) be a closed, almost Kähler, smooth $(2m-2)$-dimensional manifold. Assume that $v \in W^{1,2}(\mathbb{B}^{2m}, X)$ satisfies $v^* \operatorname{vol}_X \in L^1(\mathbb{B}^{2m})$ and*

$$d(v^* \operatorname{vol}_X) = 0$$

in $\mathcal{D}'(\mathbb{B}^{2m})$. Then, there exists a representative of v such that the coarea formula holds. Moreover, given such a representative, for vol_X-a.e. $z \in X$ the following facts hold:

1. *$v^{-1}(z)$ is a countably $\mathscr{H}^2$-rectifiable subset of $\mathbb{B}^{2m}$;*
2. *$(v^* \operatorname{vol}_X)_x \neq 0$, for $\mathscr{H}^2$-a.e. $x \in v^{-1}(z)$;*
3. *$[v^{-1}(z)]$ is a cycle of finite mass.*

Proof. By [41, Theorem 11, Theorem 12] there exists a representative of v such that both (1) and the co-area formula hold. Moreover, if we denote by $E \subset \mathbb{B}^{2m}$ the set of all the $x \in \mathbb{B}^{2m}$ such that $(v^* \operatorname{vol}_X)_x = 0$, by the coarea formula we get

$$0 = \int_E |v^* \operatorname{vol}_X|_g \, d\operatorname{vol}_g = \int_X \mathscr{H}^2\big(v^{-1}(z) \cap E\big) \, d\operatorname{vol}_X(z),$$

which implies that for vol_X-a.e. $z \in X$ the set $v^{-1}(z) \cap E$ has vanishing $\mathscr{H}^2$-measure. Thus, (2) immediately follows.

We are just left to prove (3). By the coarea formula, it follows that

$$\int_X \mathscr{H}^2\big(v^{-1}(z)\big) \, d\operatorname{vol}_X(z) = \int_{\mathbb{B}^{2m}} |v^* \operatorname{vol}_X| \, d\mathcal{L}^{2m} < +\infty.$$

Hence, the function $X \ni z \overset{f}{\mapsto} \mathscr{H}^2\big(v^{-1}(z)\big)$ belongs to $L^1(X)$ and we know that a.e. $z \in X$ is a Lebesgue point for f such that $f(z) < +\infty$. Fix any such point $z \in X$. By our choice of z, it holds that $\mathbb{M}([v^{-1}(z)]) = \mathscr{H}^2\big(v^{-1}(z)\big) = f(z) < +\infty$. Hence, just need to show that $[v^{-1}(z)]$ is a cycle. Let $\exp_z : \mathbb{R}^{2m-2} \to X$ be the exponential map of X at the point z. Denote by $\rho_0 \in (0, +\infty)$ the injectivity radius of X at z and we define

$$B_\varepsilon(z) := \exp_z\big(B_\varepsilon(0)\big), \qquad \text{for every } \varepsilon \in (0, \rho_0).$$

For every $\varepsilon \in (0, \rho_0)$, we let $\{\varphi_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset C^\infty(X)$ be a sequence of smooth functions on X such that:

1. $\varphi_{\varepsilon,k} \equiv 0$ on $X \smallsetminus B_\varepsilon(z)$;
2. $0 < \varphi_{\varepsilon,k} \leqslant \big(\operatorname{vol}_X(B_\varepsilon(z))\big)^{-1}$ on $B_\varepsilon(z)$;
3. it holds that

$$\varphi_{\varepsilon,k} \xrightarrow{k\to\infty} \frac{1}{\operatorname{vol}_X\big(B_\varepsilon(z)\big)} \chi_{B_\varepsilon(z)}, \qquad \operatorname{vol}_X\text{-a.e. on } X.$$

Fix any $\alpha \in \mathcal{D}^1(\mathbb{B}^{2m})$. By the coarea formula, it follows that

$$\int_{\mathbb{B}^{2m}} d\alpha \wedge v^*(\varphi_{\varepsilon,k} \operatorname{vol}_X) = \int_X \varphi_{\varepsilon,k}(z) \bigg(\int_{v^{-1}(z)} d\alpha|_{v^{-1}(z)} \bigg) d\operatorname{vol}_X(z).$$

Hence, by dominated convergence, we get

$$\lim_{k\to+\infty}\int_{\mathbb{B}^{2m}} d\alpha\wedge v^*(\varphi_{\varepsilon,k}\operatorname{vol}_X)=\fint_{B_\varepsilon(z)}\left(\int_{v^{-1}(z)} d\alpha|_{v^{-1}(z)}\right)d\operatorname{vol}_X(z).$$

Since z is a Lebesgue point for f, we obtain

$$\lim_{\varepsilon\to0^+}\lim_{k\to+\infty}\int_{\mathbb{B}^{2m}} d\alpha\wedge v^*(\varphi_{\varepsilon,k}\operatorname{vol}_X)=\int_{v^{-1}(z)} d\alpha|_{v^{-1}(z)}=:\langle[v^{-1}(z)],d\alpha\rangle \tag{2.5.1}$$

Moreover, since v is such that $d\big(v^*\operatorname{vol}_X\big)=0$ distributionally on $\mathbb{B}^{2m}$ and by the upper bound on $\varphi_{\varepsilon,k}$, it holds that

$$\begin{aligned}\left|\int_{\mathbb{B}^{2m}} d\alpha\wedge v^*(\varphi_{\varepsilon,k}\operatorname{vol}_X)\right|&=\left|\int_{\mathbb{B}^{2m}} v^*\varphi_{\varepsilon,k}\big(d\alpha\wedge v^*\operatorname{vol}_X\big)\right|\\&\leqslant\frac{1}{\operatorname{vol}_X(B_\varepsilon(z))}\left|\int_{\mathbb{B}^{2m}} d\alpha\wedge v^*\operatorname{vol}_X\right|=0,\end{aligned}\tag{2.5.2}$$

for every $\varepsilon\in(0,\rho_0)$ and $k\in\mathbb{N}$.

By (2.5.1) and (2.5.2) we get that $\big\langle[v^{-1}(x)],d\alpha\big\rangle=0$ and, by arbitrariness of $\alpha\in\mathcal{D}^1(\mathbb{B}^{2m})$, it follows that $\partial[v^{-1}(x)]=0$ in the sense of currents. The statement follows. □

Lemma 2.5.2. *Let $v\in W^{1,2}(\mathbb{B}^{2m},X)$ be a weakly (J,J_X)-holomorphic map such that we have $v^*\operatorname{vol}_X\in L^1(\mathbb{B}^{2m})$ and $d(v^*\operatorname{vol}_X)=0$ in $\mathcal{D}'(\mathbb{B}^{2m})$. Then, there exist a representative of u and a full measure set $\operatorname{RegVal}(v)\subset X$ such that:*

1. *the coarea formula holds for v;*
2. *for every $z\in\operatorname{RegVal}(u)$, the level set $v^{-1}(z)$ is a closed J-holomorphic curve in $\mathbb{B}^{2m}$.*

Proof. By Lemma 2.5.1, it follows immediately that there exists a representative of v such that the coarea formula holds and, for such a representative, $v^{-1}(z)$ is an $\mathscr{H}^2$-rectifiable subset of $\mathbb{B}^{2m}$ with $\partial[v^{-1}(z)]=0$, for vol_X-a.e. $z\in X$. Thus, we are just left to show that $v^{-1}(z)$ is J-holomorphic, for a.e. $z\in X$. By the coarea formula and since v is weakly (J,J_X)-holomorphic, for vol_X-a.e. $z\in X$ the form $v^*\operatorname{vol}_X$ is non-vanishing on $v^{-1}(z)$ and $dv(Jw)=J_Xdv(w)$ for every $w\in\mathbb{R}^{2m}$, up to some $\mathscr{H}^2$-negligible set. For such $z\in X$, the orienting vector field to $v^{-1}(z)$ is given by

$$\vec{\Sigma}:=\frac{*v^*\operatorname{vol}_X}{|v^*\operatorname{vol}_X|_g}.$$

We claim that $\vec{\Sigma}$ is J-invariant for $\mathscr{H}^2$-a.e. $x\in v^{-1}(z)$. Indeed, given any $x\in v^{-1}(z)$ such that $(v^*\operatorname{vol}_X)_x\neq0$, we pick an orthonormal basis $\{\xi_1,J_X\xi_1,...,\xi_{m-1},J_X\xi_{m-1}\}$ of $T^*_{v(x)}X$ and we notice that

$$\begin{aligned}(v^*\operatorname{vol}_X)_x&=\frac{1}{(m-1)!}v^*(\xi_1\wedge J_X\xi_1\wedge...\wedge\xi_{m-1}\wedge J_X\xi_{m-1})\\&=\frac{1}{(m-1)!}v^*\xi_1\wedge v^*J_X\xi_1\wedge...\wedge v^*\xi_{m-1}\wedge v^*J_X\xi_{m-1}\\&=\frac{1}{(m-1)!}v^*\xi_1\wedge J(v^*\xi_1)\wedge...\wedge v^*\xi_{m-1}\wedge J(v^*\xi_{m-1}).\end{aligned}$$

This clearly implies that $\vec{\Sigma}$ is J-invariant and the statement follows. □

In the following lemma, which generalises the model situation presented in Lemma 2.5.2, we will adopt the notation developed in Appendix 2.A Moreover, we will denote by X the product space $X := \mathbb{CP}^1 \times \mathbb{CP}^{m-2}$ and by $p_1 : X \to \mathbb{CP}^1$ and $p_2 : X \to \mathbb{CP}^{m-2}$ the canonical projections on the first and on the second factor respectively. We will endow X with the complex structure $J_X := p_1^* j_1 + p_2^* j_{m-2}$ and with symplectic form $\omega_X := p_1^*\omega_{\mathbb{CP}^1} + p_2^*\omega_{\mathbb{CP}^{m-2}}$ in order to obtain the Kähler manifold (X, J_X, ω_X).

Lemma 2.5.3. *Let $m, n \in \mathbb{N}_0$ be such that $m \geqslant 3$. Let $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ be weakly (J, j_n)-holomorphic and locally approximable. If $n \geqslant 2$, then for a.e. $(q_1, ..., q_{n-1}, p) \in \mathbb{CP}^n \times \mathbb{CP}^{n-1} \times ... \times \mathbb{CP}^2 \times \mathbb{CP}^{m-1}$ the map $v_{q_1,...,q_{n-1},p} := \big(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u, F_p \circ \pi\big) : \mathbb{B}^{2m} \to X$ has the following properties:*

(1) $v_{q_1,...,q_{n-1},p} \in W^{1,2}(\mathbb{B}^{2m}, X)$;

(2) $v^*_{q_1,...,q_{n-1},p} \operatorname{vol}_X \in L^1(\mathbb{B}^{2m})$;

(3) there exists a set $\operatorname{RegVal}(v_{q_1,...,q_{n-1},p}) \subset X$ such that

$$\operatorname{vol}_X \big((X \smallsetminus \operatorname{RegVal}(v_{q_1,...,q_{n-1},p})\big) = 0$$

and for every $(y, z) \in \operatorname{RegVal}(v_{q_1,...,q_{n-1},p})$ the $\mathscr{H}^2$-rectifiable set $v^{-1}_{q_1,...,q_{n-1},p}(y, z)$ is a closed almost J-holomorphic curve in $\mathbb{B}^{2m}$, in the sense of Definition 2.4.2. Moreover, the constants $\ell > 0$ and $\gamma \in (0, 1]$ can be chosen as $\ell = 2\sqrt{2\operatorname{Lip}(\Omega)}$ and $\gamma = 1/2$.

If $n = 1$, analogous properties hold for the map $v_p := (u, F_p \circ \pi) : \mathbb{B}^{2m} \to X$ and for a.e. $p \in \mathbb{CP}^{m-1}$.

Proof. Since the techniques are identical both in the case $n = 1$ and $n \geqslant 2$, we just focus on the second one.

Let $Y := \mathbb{CP}^n \times ... \times \mathbb{CP}^2$. First, we want to prove (1). By Lemma 2.A.1, we know that $p_2 \circ v_{q_1,...,q_{n-1},p}$ belongs to $W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^{m-2})$, for every $p \in \mathbb{CP}^{m-1}$. We claim that the map

$$p_1 \circ v_{1_1,...,q_{n-1},p} = F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u$$

belongs to $W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^1)$ for a.e. $(q_1, ..., q_{n-1}) \in Y$. Indeed, notice that the map $F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u$ is weakly (J, j_1)-holomorphic, for every $(q_1, ..., q_{n-1}) \in Y$. Thus, by Corollary 2.2.1 we have that

$$\begin{aligned}
&\int_{\mathbb{B}^{2m}} |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_g \\
&= 2\int_{\mathbb{B}^{2m}} (F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)^*\omega_{\mathbb{CP}^1} \wedge \frac{\Omega^{m-1}}{(m-1)!}.
\end{aligned}$$

Hence, by Lemma 2.A.2 we obtain

$$\begin{aligned}
&\int_{\mathbb{B}^{2m}} \varphi\bigg(\int_Y |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_Y(q_1, ..., q_{n-1})\bigg) \, d\operatorname{vol}_g \\
&= 2\int_{\mathbb{B}^{2m}} \varphi\bigg(\int_Y (F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)^*\omega_{\mathbb{CP}^1} \, d\operatorname{vol}_Y(q_1, ..., q_{n-1})\bigg) \wedge \frac{\Omega^{m-1}}{(m-1)!} \\
&= 2D\int_{\mathbb{B}^{2m}} \varphi\, u^*\omega_{\mathbb{CP}^n} \wedge \frac{\Omega^{m-1}}{(m-1)!} = D\int_{\mathbb{B}^{2m}} \varphi |du|_g^2 \, d\operatorname{vol}_g < +\infty,
\end{aligned}$$

where $D := B_{n+1} \cdot ... \cdot B_3$ (be careful to the bad notation, the letter "B" in the definition of the constant D refers to the constants that are determined by Lemma 2.A.2), for every $\varphi \in C_c^\infty(\mathbb{B}^{2m})$. Thus, we get that

$$\int_Y |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_Y(q_1, ..., q_{n-1}) = D|du|_g^2, \tag{2.5.3}$$

for a.e. $x \in \mathbb{B}^{2m}$. By integrating both sides of (2.5.3) on $\mathbb{B}^{2m}$ and by Fubini's theorem, we get

$$\begin{aligned}
&\int_{\mathbb{B}^{2m}} \left(\int_Y |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_Y(q_1, ..., q_{n-1}) \right) d\mathcal{L}^{2m} \\
&= \int_Y \left(\int_{\mathbb{B}^{2m}} |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_g \right) d\operatorname{vol}_Y(q_1, ..., q_{n-1}) \\
&= D \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\operatorname{vol}_g < +\infty.
\end{aligned} \tag{2.5.4}$$

Since (2.5.4) directly implies that

$$\int_{\mathbb{B}^{2m}} |d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u)|_g^2 \, d\operatorname{vol}_g < +\infty$$

for vol_Y-a.e. $(q_1, ..., q_{n-1}) \in Y$, point (1) follows.

Next, we turn to show (2). By (2.5.3), Lemma 2.A.2, (2.2.1), (2.A.1) and by Fubini's theorem, we have

$$\begin{aligned}
&\int_Y \left(\int_{\mathbb{B}^{2m}} \left| v^*_{q_1,...,q_{n-1},p} \operatorname{vol}_X \right|_g d\operatorname{vol}_g \right) d\operatorname{vol}_{Y \times \mathbb{CP}^{m-1}}(q_1, ..., q_{n-1}, p) \\
&\leqslant \int_{\mathbb{B}^{2m}} \left(\int_Y \left| d(F_{q_{n-1}} \circ ... \circ F_{q_1} \circ u) \right|_g^2 d\operatorname{vol}_{Y \times \mathbb{CP}^{m-1}}(q_1, ..., q_{n-1}, p) \right) \\
&\quad \cdot |d(F_p \circ \pi)|_g^{2m-4} \, d\operatorname{vol}_g \\
&= DG^{m-2} \int_{\mathbb{B}^{2m}} |du|_g^2 \cdot |d(F_p \circ \pi)|^{2m-4} \, d\operatorname{vol}_g \\
&\leqslant DG^{m-2} \int_{\mathbb{B}^{2m}} \frac{|du|_g^2}{\operatorname{dist}(\,\cdot\,, L_p)^{2m-4}} \, d\operatorname{vol}_g,
\end{aligned} \tag{2.5.5}$$

where L_p is defined as in Appendix A. For any $\rho \in (0,1)$, define $L_p^\rho := (L_p + B_\rho) \cap \mathbb{B}^{2m}$. By the almost monotonicity formula (2.2.3), we get that

$$\begin{aligned}
\int_{L_p^\rho \setminus L_p^{\rho/2}} \frac{|du|_g^2}{\operatorname{dist}(\,\cdot\,, L_p)^{2m-4}} \, d\operatorname{vol}_g &\leqslant \frac{2^{2m-4}}{\rho^{2m-4}} \int_{B_\rho} |du|_g^2 \, d\operatorname{vol}_g \\
&\leqslant 2^{2m-2} \frac{e^{A\rho}(1 + A\rho)}{\rho^{2m-2}} \int_{B_\rho} |du|_g^2 \, d\operatorname{vol}_g \, \frac{\rho^2}{4} \\
&\leqslant \left(2^{2m-2} e^A (1 + A) \int_{\mathbb{B}^{2m}} |du|_g^2 \, d\operatorname{vol}_g \right) \frac{\rho^2}{4},
\end{aligned}$$

for every $\rho \in (0,1)$. By iteration (see also the proof of Lemma 2.4.2) we get

$$\int_{L_p^\rho} \frac{|du|_g^2}{\operatorname{dist}(\,\cdot\,, L_p)^{2m-4}} \, d\operatorname{vol}_g \leqslant \frac{2^{2m} e^A (1 + A)}{3} \left(\int_{\mathbb{B}^{2m}} |du|_g^2 \, d\operatorname{vol}_g \right) \rho^2. \tag{2.5.6}$$

Combining (2.5.5) and (2.5.6), we obtain

$$\int_{Y\times\mathbb{CP}^{m-1}}\left(\int_B\left|v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\right|_g d\operatorname{vol}_g\right)d\operatorname{vol}_{Y\times\mathbb{CP}^{m-1}}(q_1,\dots,q_{n-1},p)$$
$$\leqslant C\int_{\mathbb{B}^{2m}}|du|_g^2\,d\operatorname{vol}_g<\infty, \tag{2.5.7}$$

where $C>0$ is a constant depending on m, n and $\operatorname{Lip}(\Omega)$. Again, point (2) follows by Fubini's theorem.

We are left to prove (3). First, we claim that for a.e. $(q_1,\dots,q_{n-1},p)\in Y\times\mathbb{CP}^{m-1}$ it holds that $d\big(v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\big)=0$ in the sense of distributions. Indeed, we already know that for vol_Y-a.e. $(q_1,\dots,q_{n-1})\in Y$ the estimate (2.5.4) holds. Fix any $\alpha\in\mathcal{D}^1(\mathbb{B}^{2m})$. Notice that by estimates (2.5.4) and (2.5.6) we obtain

$$\left|\int_{\mathbb{B}^{2m}}v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\wedge d\alpha\right|\leqslant C|d\alpha|_*\int_{L_p^\rho}\frac{|du|_g^2}{\operatorname{dist}(\,\cdot\,L_p)^{2m-4}}\,d\operatorname{vol}_g$$
$$\leqslant C|d\alpha|_*\left(\int_{\mathbb{B}^{2m}}|du|_g^2\,d\operatorname{vol}_g\right)\rho^2,\qquad\forall\,\rho\in(0,1),$$

where $C>0$ is a constant depending only on m, n and $\operatorname{Lip}(\Omega)$. By letting $\rho\to0^+$, we get

$$\int_{Y\times\mathbb{CP}^{m-1}}\left|\int_B v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\wedge d\alpha\right|d\operatorname{vol}_{Y\times\mathbb{CP}^{m-1}}(q_1,\dots,q_{n-1},p)=0.$$

By arbitrariness of $\alpha\in\mathcal{D}^1(\mathbb{B}^{2m})$, our claim follows.

Let $E\subset Y\times\mathbb{CP}^{m-1}$ be the set of all the n-tuples $(q_1,\dots,q_{n-1},p)\in Y\times\mathbb{CP}^{m-1}$ such that

1. (1) and (2) hold;
2. $d\big(v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\big)=0$ in the sense of distributions.

By what we have shown so far, we have $\operatorname{vol}_{Y\times\mathbb{CP}^{m-1}}(E^c)=0$. Fix any $(q_1,\dots,q_{n-1},p)\in E$. We fix the representative of the map $v_{q_1,\dots,q_{n-1},p}$ given by Lemma 2.5.1. Thus, we know that the following facts hold for vol_X-a.e. $(y,z)\in X$:

1. the set $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$ is $\mathscr{H}^2$-rectifiable;
2. $(v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X)_x\neq0$, for $\mathscr{H}^2$-a.e. $x\in v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$ and the rectifiable set $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$ is oriented by the $\mathscr{H}^2$-measurable and unitary field 2-vectors given by:
$$\vec{\Sigma}:=\frac{\big(*\,(v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X)\big)^\sharp}{\left|v^*_{q_1,\dots,q_{n-1},p}\operatorname{vol}_X\right|_g}$$
3. $\partial[v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)]=0$.

Hence, we just need to show that $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$ is almost J-holomorphic according to Definition 2.4.2, i.e. we claim that there exists some J-invariant and $\mathscr{H}^2$-measurable field of g-unitary 2-vectors $\vec{\Sigma}_J:v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)\to\wedge_2\mathbb{R}^{2m}$ such that

$$\big|\vec{\Sigma}-\vec{\Sigma}_J\big|\leqslant\ell|\cdot|^\gamma, \tag{2.5.8}$$

for some $\ell > 0$ and $\gamma \in (0,1]$. In order to prove our claim, consider the following $\mathscr{H}^2$-measurable and g-unitary fields respectively of 2-vectors and 4-vectors on $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$:

$$\vec{\Sigma}^1 := \frac{\big((p_1 \circ v_{q_1,\dots,q_{n-1},p})^* \omega_{\mathbb{CP}^1}\big)^\sharp}{\big|(p_1 \circ v_{q_1,\dots,q_{n-1},p})^* \omega_{\mathbb{CP}^1}\big|_g}$$

$$\vec{\Sigma}^2 := \frac{\big(*\,(p_2 \circ v_{q_1,\dots,q_{n-1},p})^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big)^\sharp}{\big|(p_2 \circ v_{q_1,\dots,q_{n-1},p})^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big|_g} = \frac{\big(*\,(F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big)^\sharp}{\big|(F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big|_g}.$$

Notice that such fields are both well defined $\mathscr{H}^2$-a.e. on $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$, since

$$(v^*_{q_1,\dots,q_{n-1},p} \operatorname{vol}_X)_x \neq 0$$

for $\mathscr{H}^2$-a.e. $x \in v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$. Fix $x \in v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$ such that $(v^*_{q_1,\dots,q_{n-1},p} \operatorname{vol}_X)_x \neq 0$, so that the subspace $W_1 := \operatorname{span}\{\vec{\Sigma}^1(x)\}$ is a J-holomorphic 2-plane and $W_2 := \operatorname{span}\{\vec{\Sigma}^2(x)\}$ is a J_0-holomorphic 4-plane. Let $W := \operatorname{span}\{\vec{\Sigma}(x)\}$ and notice that we have $W = W_1 \cap W_2$, $W_2 = (W_1^\perp \cap W_2) \oplus W$ and $\dim(W) = 2$. Moreover,

$$4 = \dim(W_2) = \dim(W_1^\perp \cap W_2) + \dim(W) = \dim(W_1^\perp \cap W_2) + 2,$$

which implies $\dim(W_1^\perp \cap W_2) = 2$. We let $\{e_1, e_3\}$ be an g-orthonormal basis of W and let $\{v, w\}$ be an g-orthonormal basis of $W_1^\perp \cap W_2$. By construction, $\{e_1, e_3, v, w\}$ is an Ω_0-orthonormal basis of W_2 and we can write

$$\vec{\Sigma}^2(x) := e_1 \wedge e_3 \wedge v \wedge w.$$

If $\vec{\Sigma}^2(x)$ is J-invariant, we set $\vec{\Sigma}^2_J(x) := \vec{\Sigma}^2(x)$.

If not, notice that $\{e_1, Je_1, e_3, Je_3, v - Je_1, w - Je_3\}$ is a linearly independent set. Let e_2, e_4 be the unique unitary vectors such that $\{e_1, Je_1, e_3, Je_3, e_2, e_4\}$ is an g-orthonormal set such that

$$\operatorname{span}\{e_1, Je_1, e_3, Je_3, v - Je_1, w - Je_3\} = \operatorname{span}\{e_1, Je_1, e_3, Je_3, e_2, e_4\}.$$

Exactly as in the proof of Lemma 2.4.1, it follows that there exist two angles $\phi_1, \phi_2 \in [0, 2\pi]$ such that

$$\vec{\Sigma}^2(x) := e_1 \wedge (\cos\phi_1 Je_1 + \sin\phi_1 e_2) \wedge e_3 \wedge (\cos\phi_2 Je_3 + \sin\phi_2 e_4).$$

The same computation as in Lemma 2.4.1 leads to

$$1 - \operatorname{Lip}(\Omega)|x| \leqslant \cos\phi_1 \cos\phi_2 \leqslant 1 + \operatorname{Lip}(\Omega)|x|.$$

We set

$$\Sigma^2_J(x) = e_1 \wedge \cos\phi_1 Je_1 \wedge e_3 \wedge \cos\phi_2 Je_3$$

and we compute

$$\begin{aligned} \big|\vec{\Sigma}^2(x) - \vec{\Sigma}^2_J(x)\big|^2_g &= (1 - \cos\phi_1\cos\phi_2)^2 + \sin^2\phi_1 \\ &= 1 + \cos^2\phi_2 - 2\cos\phi_1\cos\phi_2 \end{aligned}$$

$$\leqslant 2(1-\cos\phi_1\cos\phi_2) \leqslant 2\,\mathrm{Lip}(\Omega)|x|,$$

which leads to

$$\big|\vec{\Sigma}^2(x) - \vec{\Sigma}^2_J(x)\big|_g \leqslant \sqrt{2\,\mathrm{Lip}(\Omega)}|x|^{1/2}. \tag{2.5.9}$$

Eventually, we define

$$\vec{\Sigma}_J := *\big(\vec{\Sigma}^1 \wedge *\vec{\Sigma}^2_J\big).$$

By construction, $\vec{\Sigma}_J$ is an $\mathscr{H}^2$-measurable and unitary field of 2-vectors on $v^{-1}_{q_1,\dots,q_{n-1},p}(y,z)$. Moreover, by (2.5.9), we have

$$\begin{aligned}
\big|\vec{\Sigma}^1 \wedge *\vec{\Sigma}^2\big|_g &= \big|\vec{\Sigma}^1 \wedge *(\vec{\Sigma}^2 - \vec{\Sigma}^2_J) + \vec{\Sigma}^1 \wedge *\vec{\Sigma}^2_J\big|_g \\
&\geqslant \big|\vec{\Sigma}^1 \wedge *\vec{\Sigma}^2_J\big|_g - \big|\vec{\Sigma}^1 \wedge *(\vec{\Sigma}^2 - \vec{\Sigma}^2_J)\big|_g \\
&\geqslant 1 - \big|\vec{\Sigma}^2 - \vec{\Sigma}^2_J\big|_g \geqslant 1 - \sqrt{2\,\mathrm{Lip}(\Omega)}|\cdot|^{1/2},
\end{aligned}$$

which leads to

$$\begin{aligned}
\big|\vec{\Sigma} - \vec{\Sigma}_J\big| \leqslant G\big|\vec{\Sigma} - \vec{\Sigma}_J\big|_g &= G\left|\frac{\vec{\Sigma}^1 \wedge *\vec{\Sigma}^2}{\big|\vec{\Sigma}^1 \wedge *\vec{\Sigma}^2\big|} - \vec{\Sigma}^1 \wedge *\vec{\Sigma}^2_J\right|_g \\
&\leqslant 2\sqrt{2\,\mathrm{Lip}(\Omega)}G|\cdot|^{1/2}.
\end{aligned}$$

Hence, (2.5.8) holds with $\ell = 2\sqrt{2\,\mathrm{Lip}(\Omega)}G$ and $\gamma = 1/2$. The statement follows. □

2.6. Proof of the main theorem

This section is entirely devoted to proof Theorems 2.1.1 and 2.1.2, whose local versions will be recalled now for the reader's convenience.

Theorem. *Let $m,n \in \mathbb{N}_0$ be such that $m \geqslant 2$. Assume that $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ is weakly (J,j_n)-holomorphic and locally approximable.*
Then, u has a unique tangent map at the origin.

Theorem. *Let $m,n \in \mathbb{N}_0$ be such that $m \geqslant 2$. Assume that $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ is weakly (J,j_n)-holomorphic and locally approximable.*
Then, the $(2m-2)$-cycle $T_u \in \mathcal{D}_{2m-2}(\mathbb{B}^{2m})$ has a unique tangent cone at the origin.

In the first two subsections, we treat the proof of Theorem 2.1.2. We will first address the easy case $m=2$, $n=1$ in Section 2.6.1, in order to clarify which will be the main ideas in order to proceed towards higher dimensions and codimensions. The general case will be discussed in Section 2.6.2. Lastly, in Section 2.6.3 we will show how Theorem 2.1.1 can be obtained as a consequence of Theorem 2.1.2.

2.6.1. A model problem

Let $u \in W^{1,2}(\mathbb{B}^4, \mathbb{CP}^1)$ be weakly (J, j_1)-holomorphic and locally approximable. As usual, $\pi : \mathbb{B}^4 \to \mathbb{CP}^1$ denotes the Hopf map.

If $\theta(0, u) < \varepsilon_0$, then u is smooth in a neeighbourhood of 0 by Theorem 2.3.1 and the statement follows. Assume then that $\theta(0, u) \geqslant \varepsilon_0$.

We use the same notations and labeling for the constants as in Sections 2.4 and 2.5. Since $u^*\omega_{\mathbb{CP}^1} \in L^1(\mathbb{B}^4)$, by using Lemma 2.5.2 with $X = \mathbb{CP}^1$, we get that there exists a representative of u and a full measure set $\mathrm{RegVal}(u) \subset \mathbb{CP}^1$ such that:

1. the coarea formula holds for u;
2. for every $y \in \mathrm{RegVal}(u)$, the level set $u^{-1}(y)$ is a closed J-holomorphic curve.

Hence, all the estimates in Section 2.4 will be used assuming $\vec{\Sigma}_J = \vec{\Sigma}$, $\ell = 0$ and $\gamma = 1$, as we stressed out in Remark 2.4.2.

For every $k \in \mathbb{N}_0$, we consider the set $E_k \subset \mathbb{CP}^1$ given by all the points y in $\mathrm{RegVal}(u)$ such that

(1) $\mathscr{H}^2(u^{-1}(y) \cap B_{2^{-k}}) < \big((e^A(1 + A))\big)^{-1}$;

(2) $\displaystyle\int_{u^{-1}(y)\cap B_{2^{-k}}} |\wedge_2 d\pi(\vec{\Sigma}_0^y)|\, d\mathscr{H}^2 < \frac{\delta'}{2}$,

where $\delta' > 0$ is the constant introduced in Section 2.4.2 and $\vec{\Sigma}_0^y$ is built as shown in Lemma 2.4.1 starting from the J-holomorphic field of 2-vectors given by

$$\vec{\Sigma}^y := \frac{*(u^*\omega_{\mathbb{CP}^1})^\sharp}{|u^*\omega_{\mathbb{CP}^1}|_g},$$

which orients the closed J-holomorphic curve $u^{-1}(y)$ for every $y \in \mathrm{RegVal}(u)$. We notice that $E_{k-1} \subset E_k$ for every $k \in \mathbb{N}_0$. Moreover, since $\mathrm{RegVal}(u) \subset \mathbb{CP}^1$ has full measure in $\mathbb{CP}^1$, we get

$$\mathrm{vol}_{\mathbb{CP}^1}\left(\mathbb{CP}^1 \smallsetminus \bigcup_{k=1}^{+\infty} E_k\right) = 0$$

For every $k \in \mathbb{N}_0$, we define the localized current $T_k := T_u \llcorner u^{-1}(E_k)$, i.e.

$$\langle T_k, \alpha\rangle := \int_{u^{-1}(E_k)} u^*\omega_{\mathbb{CP}^1} \wedge \alpha \qquad \forall\, \alpha \in \mathcal{D}^2(\mathbb{B}^4).$$

Claim. We claim that every T_k has a unique tangent cone at the origin. First, notice that T_k is a normal 2-cycle on $\mathbb{B}^4$ semicalibrated by Ω. By definition of E_k and by Proposition 2.4.1, for every $y \in E_k$ we get that

$$\begin{aligned} e^{A\rho}(1 + A\rho)\frac{\mathscr{H}^2\big(u^{-1}(y) \cap B_\rho\big)}{\rho^2} &- e^{A\sigma}(1 + A\sigma)\frac{\mathscr{H}^2\big(u^{-1}(y) \cap B_\sigma\big)}{\sigma^2} \\ &\geqslant \int_{u^{-1}(y)\cap(B_\rho \smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\big\langle \Omega_t, \vec{\Sigma}^y\big\rangle\, d\mathscr{H}^2 \end{aligned}$$

and

$$e^{-A\rho}(1 - A\rho)\frac{\mathscr{H}^2\big(u^{-1}(y) \cap B_\rho\big)}{\rho^2} - e^{-A\sigma}(1 - A\sigma)\frac{\mathscr{H}^2\big(u^{-1}(y) \cap B_\sigma\big)}{\sigma^2}$$

$$\leqslant \int_{u^{-1}(y)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\big\langle \Omega_t, \vec{\Sigma}^y_J\big\rangle \, d\mathscr{H}^2,$$

for every $0<\sigma<\rho<1$. Since a direct computation leads to

$$\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2} = \frac{1}{\rho^2}\int_{E_k} \mathscr{H}^2\big(u^{-1}(y)\cap B_\rho\big)\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y),$$

by integrating on E_k the two previous inequalities we get the following almost monotonicity formulas for the current T_k:

$$\begin{aligned} &e^{A\rho}(1+A\rho)\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2} - e^{A\sigma}(1+A\sigma)\frac{\mathbb{M}(T_k \llcorner B_\sigma)}{\sigma^2} \\ &\geqslant \int_{E_k}\bigg(\int_{u^{-1}(y)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\big\langle \Omega_t, \vec{\Sigma}^y\big\rangle \, d\mathscr{H}^2\bigg)\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y), \end{aligned} \tag{2.6.1}$$

$$\begin{aligned} &e^{-A\rho}(1-A\rho)\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2} - e^{-A\sigma}(1-A\sigma)\frac{\mathbb{M}(T_k \llcorner B_\sigma)}{\sigma^2} \\ &\leqslant \int_{E_k}\bigg(\int_{u^{-1}(y)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\big\langle \Omega_t, \vec{\Sigma}^y\big\rangle \, d\mathscr{H}^2\bigg)\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y), \end{aligned} \tag{2.6.2}$$

for every $0<\sigma<\rho<1$. Equation (2.6.1) immediately implies that function

$$(0,1)\ni\rho\mapsto e^{A\rho}(1+A\rho)\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2}$$

is monotonically non-decreasing. Thus, the density of the current T_k at zero, which is given by

$$\theta(T_k,0) := \lim_{\rho\to0^+}\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2} = \lim_{\rho\to0^+} e^{A\rho}(1+A\rho)\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2}$$

exists and is finite. Moreover, by (2.6.2), the coarea formula, (2.4.17) and the estimate (2.4.8), it follows that

$$\begin{aligned} &\bigg|\frac{\mathbb{M}(T_k \llcorner B_\rho)}{\rho^2} - \theta(0,T_k)\bigg| \\ &\leqslant C\bigg|\int_{u^{-1}(E_k)\cap B_\rho} u^*\omega_{\mathbb{CP}^1}\wedge\frac{\Omega_t}{|\cdot|^2}\bigg| \\ &\leqslant C\bigg|\int_{u^{-1}(E_k)\cap B_\rho} u^*\omega_{\mathbb{CP}^1}\wedge\pi^*\omega_{\mathbb{CP}^1}\bigg| + C\bigg|\int_{u^{-1}(E_j)\cap B_\rho} u^*\omega_{\mathbb{CP}^1}\wedge\frac{(\Omega-\Omega_0)_t}{|\cdot|^2}\bigg| \\ &\leqslant C\int_{E_k}\bigg|\int_{u^{-1}(y)\cap B_\rho}\pi^*\omega_{\mathbb{CP}^1}\bigg|\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y) \\ &\quad + C\int_{E_k}\int_{u^{-1}(y)\cap B_\rho}\frac{|\Omega-\Omega_0|}{|\cdot|^2}\, d\mathscr{H}^2\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y) \\ &\leqslant C\,\mathrm{vol}_{\mathbb{CP}^1}(\mathbb{CP}^1)\rho^\alpha + C\int_{E_k}\int_{u^{-1}(y)\cap B_\rho}\frac{1}{|\cdot|}\, d\mathscr{H}^2\, d\,\mathrm{vol}_{\mathbb{CP}^1}(y) \\ &\leqslant C\,\mathrm{vol}_{\mathbb{CP}^1}(\mathbb{CP}^1)\rho^\alpha + C\,\mathrm{vol}_{\mathbb{CP}^1}(\mathbb{CP}^1)\rho \leqslant C\,\mathrm{vol}_{\mathbb{CP}^1}(\mathbb{CP}^1)\rho^\alpha, \end{aligned} \tag{2.6.3}$$

for every $\rho \in (0, \tilde{r})$, where the constant $C > 0$ and $\alpha, \tilde{r} \in (0,1)$ all depend just on k and on $\mathrm{Lip}(\Omega)$. From the Morrey decay (2.6.3), uniqueness of tangent cone for T_k follows by standard arguments.

Conclusion. For every $j \in \mathbb{N}_0$, consider the residual current $R_j := T_u - T_j$. By the same arguments that we have used in the proof of the previous claim, we conclude that R_j is a normal 2-cycle in $\mathbb{B}^4$ which is semicalibrated by Ω. In particular, the quantity

$$e^{A\rho}(1 + A\rho)\frac{\mathbb{M}(R_j \llcorner B_\rho)}{\rho^2} \tag{2.6.4}$$

is non-decreasing in $\rho \in (0,1)$. Therefore, the limit as $\rho \to 0^+$ of the quantity (2.6.4) exists and it is finite. Then, since the quantity (2.6.4) is also non-increasing in $j \in \mathbb{N}$ and going to 0 as $j \to +\infty$, we are allowed to exchange the limits in the following chain of equalities and we get

$$\begin{aligned} \lim_{j\to+\infty} \lim_{\rho\to 0^+} \frac{\mathbb{M}(R_j \llcorner B_\rho)}{\rho^2} &= \lim_{j\to+\infty} \lim_{\rho\to 0^+} e^{A\rho}(1 + A\rho)\frac{\mathbb{M}(R_j \llcorner B_\rho)}{\rho^2} \\ &= \lim_{\rho\to 0^+} e^{A\rho}(1 + A\rho) \lim_{j\to+\infty} \frac{\mathbb{M}(R_j \llcorner B_\rho)}{\rho^2} = 0. \end{aligned} \tag{2.6.5}$$

Fix any $\varepsilon > 0$. By (2.6.5), we can pick $j \in \mathbb{N}_0$ sufficiently large so that

$$\lim_{\rho\to 0^+} \frac{\mathbb{M}(R_j \llcorner B_\rho)}{\rho^2} < \frac{\varepsilon}{2}. \tag{2.6.6}$$

Now assume that $\{\rho_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\rho'_k\}_{k\in\mathbb{N}} \subset (0,1)$ are two sequences converging to 0 as $k \to +\infty$ and both

$$\begin{aligned} (\Phi_{\rho_k})_* T_u &\rightharpoonup C_\infty, \\ (\Phi_{\rho'_k})_* T_u &\rightharpoonup C'_\infty, \end{aligned}$$

where for every $\rho \in (0,1)$ the map Φ_ρ is defined as in subsection 1.2. By further extracting subsequences if needed, we assume also that the sequences $\{(\Phi_{\rho_k})_* T_j\}_{k\in\mathbb{N}}$ and $\{(\Phi_{\rho'_k})_* T_j\}_{k\in\mathbb{N}}$ converge weakly in the sense of currents. By our previous claim, they converge to the same limit and then we have

$$\begin{aligned} C'_\infty - C_\infty &= \lim_{k\to+\infty} \big((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j\big) + \lim_{k\to+\infty} (\Phi_{\rho'_k})_* T_j \\ &\quad - \lim_{k\to+\infty} (\Phi_{\rho_k})_* T_j \\ &= \lim_{k\to+\infty} \big((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j\big), \end{aligned}$$

in the sense of currents. By sequential lower semicontinuity of mass with the respect to weak convergence of currents, and by (2.6.6), we eventually get

$$\begin{aligned} \mathbb{M}(C'_\infty - C_\infty) &\leqslant \liminf_{k\to+\infty} \mathbb{M}\big((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j\big) \\ &\leqslant \liminf_{k\to+\infty} \mathbb{M}\big((\Phi_{\rho'_k})_* R_j\big) + \liminf_{k\to+\infty} \mathbb{M}\big((\Phi_{\rho_k})_* R_j\big) \\ &= \lim_{k\to+\infty} \frac{\mathbb{M}\big(R_j \llcorner B_{\rho'_k}\big)}{(\rho'_k)^2} + \lim_{k\to+\infty} \frac{\mathbb{M}\big(R_j \llcorner B_{\rho_k}\big)}{\rho_k^2} < \varepsilon. \end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we obtain that $\mathbb{M}(C'_\infty - C_\infty) = 0$ and the conclusion follows.

2.6.2. The general case

Let $m, n \in \mathbb{N}_0$ be such that $m \geqslant 3$. Let $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ be weakly (J, j_n)-holomorphic and locally approximable. As usual, $\pi : \mathbb{B}^{2m} \to \mathbb{CP}^{m-1}$ denotes the Hopf map.

If $\theta(0,u) < \varepsilon_0$, then u is smooth in a neeighbourhood of 0 by Theorem 2.3.1 and the statement follows. Assume then that $\theta(0,u) \geqslant \varepsilon_0$. Moreover, since the case $n = 1$ can be done exactly using the same method, we just focus on the case $n \geqslant 2$.

Let $T \in \mathcal{D}_2(\mathbb{B}^{2m})$ be the 2-current given by

$$\langle T, \alpha \rangle := \frac{1}{(m-2)!} \int_{\mathbb{B}^{2m}} u^*\omega_{\mathbb{CP}^n} \wedge \pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2} \wedge \alpha \qquad \forall\, \alpha \in \mathcal{D}^2(\mathbb{B}^{2m}).$$

Notice that T is well-defined and normal, since

$$\begin{aligned} |\langle T, \alpha \rangle| &\leqslant \frac{1}{(m-2)!} |\alpha|_* \int_{\mathbb{B}^{2m}} |du|_g^2 |\wedge_2 d\pi|_g^{2m-4}\, d\,\mathrm{vol}_g \\ &\leqslant C \frac{1}{(m-2)!} |\alpha|_* \int_{\mathbb{B}^{2m}} \frac{|du|_g^2}{|\cdot|^{2m-4}}\, d\,\mathrm{vol}_g \\ &\leqslant C \frac{1}{(m-2)!} |\alpha|_* \int_{\mathbb{B}^{2m}} |du|_g^2\, d\,\mathrm{vol}_g < +\infty, \qquad \forall\, \alpha \in \mathcal{D}^2(\mathbb{B}^{2m}), \end{aligned}$$

where $C = C\big(\mathrm{Lip}(\Omega)\big) > 0$ is a constant and the last inequality follows from (2.2.3) exactly in the same way as estimate (2.5.6). Let $Y := \mathbb{CP}^n \times ... \times \mathbb{CP}^2 \times \mathbb{CP}^{m-1}$ and $X := \mathbb{CP}^1 \times \mathbb{CP}^{m-2}$. Notice that by Lemma 2.A.2, Fubini's theorem, Lemma 2.5.3 and the coarea formula, we can write the action of T as

$$\begin{aligned} \langle T, \alpha \rangle &= \frac{1}{(m-2)!} \int_Y \left(\int_{\mathbb{B}^{2m}} v_y^* \,\mathrm{vol}_X \wedge \alpha \right) d\,\mathrm{vol}_Y(y) \\ &= \frac{1}{(m-2)!} \int_Y \int_X \left(\int_{v_y^{-1}(z)} \langle \alpha, \vec{\Sigma}^y \rangle\, d\mathscr{H}^2 \right) d\,\mathrm{vol}_X(z)\, d\,\mathrm{vol}_Y(y), \end{aligned}$$

for every $\alpha \in \mathcal{D}^2(\mathbb{B}^{2m})$, where $y := (q_1, ..., q_{n-1}, p) \in Y$ is any point in Y such that (1), (2) and (3) of Lemma 2.5.3 hold, $v_y := v_{q_1,...,q_{n-1},p}$ and

$$\vec{\Sigma}^y := \frac{\big(* \,(v_y^* \,\mathrm{vol}_X)\big)^\sharp}{|v_y^* \,\mathrm{vol}_X|_g},$$

following the notation that is used in Lemma 2.5.3, is the g-unitary field of 2 vectors orienting $v_y^{-1}(z)$, for every $z \in \mathrm{RegVal}(v_y) \subset X$. We define the "tilted current" $T_J \in \mathcal{D}_2(\mathbb{B}^{2m})$ by

$$\langle T_J, \alpha \rangle = \frac{1}{(m-2)!} \int_Y \int_X \left(\int_{v_y^{-1}(z)} \langle \alpha, \vec{\Sigma}_J^y \rangle\, d\mathscr{H}^2 \right) d\,\mathrm{vol}_X(z)\, d\,\mathrm{vol}_Y(y),$$

for every $\alpha \in \mathcal{D}^2(\mathbb{B}^{2m})$, where $\vec{\Sigma}_J^y$ is the J-holomorphic field of 2-vectors that we have built in the proof of Lemma 2.5.3.

Step 1. We want to show that that T_J has a unique tangent cone at the origin. First, for every $k \in \mathbb{N}$ we define the set $E_k \subset Y \times X$ given by

1. points (1), (2) and (3) in Lemma 2.5.3 hold for the map v_y and the level set $v_y^{-1}(z)$;

2. $\mathscr{H}^2\big(v_y^{-1}(z)\cap B_{2^{-k}}\big) < \big(e^{A+2\ell}(1+A)\big)^{-1}$;

3. $\displaystyle\int_{v_y^{-1}(z)\cap B_{2^{-k}}} |\wedge_2 d\pi(\vec{\Sigma}_0^y)|\, d\mathscr{H}^2 < \frac{\delta'}{2}$,

where we are using the notation of subsection 3.2 and $\ell = \ell\big(\operatorname{Lip}(\Omega)\big) > 0$ is the constant provided in Lemma 2.5.3. Notice that, by Lemma 2.5.3, Lemma 2.4.2 and Fubini's theorem, it holds that

$$\operatorname{vol}_{Y\times X}\left(Y\times X \smallsetminus \bigcup_{k\in\mathbb{N}} E_k\right).$$

Fix any $k\in\mathbb{N}$. Define the truncated current $T_J^k \in \mathcal{D}_2(\mathbb{B}^{2m})$ by

$$\left\langle T_J^k, \alpha\right\rangle = \int_{E_k}\left(\int_{v_y^{-1}(z)} \langle \alpha, \vec{\Sigma}_J^y\rangle\, d\mathscr{H}^2\right) d\operatorname{vol}_{Y\times X}(y,z), \qquad \forall\, \alpha\in\mathcal{D}^2(\mathbb{B}^{2m}).$$

Notice that, by Proposition 2.4.1 and by definition of E_k, for every $(y,z)\in E_k$ it holds that

$$\begin{aligned}
&e^{A\rho+\ell\rho^{1/2}}(1+A\rho)\frac{\mathscr{H}^2\big(v_y^{-1}(z)\cap B_\rho\big)}{\rho^2}\\
&\quad - e^{A\sigma+\ell\sigma^{1/2}}(1+A\sigma)\frac{\mathscr{H}^2\big(v_y^{-1}(z)\cap B_\sigma\big)}{\sigma^2}\\
&\geqslant \int_{v_y^{-1}(z)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\langle \Omega_t, \vec{\Sigma}_J^y\rangle\, d\mathscr{H}^2
\end{aligned}$$

and

$$\begin{aligned}
&e^{-(A\rho+\ell\rho^{1/2})}(1-A\rho)\frac{\mathscr{H}^2\big(v_y^{-1}(z)\cap B_\rho\big)}{\rho^2}\\
&\quad - e^{-(A\sigma+\ell\sigma^{1/2})}(1-A\sigma)\frac{\mathscr{H}^2\big(v_y^{-1}(z)\cap B_\sigma\big)}{\sigma^2}\\
&\leqslant \int_{v_y^{-1}(z)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\langle \Omega_t, \vec{\Sigma}_J^y\rangle\, d\mathscr{H}^2,
\end{aligned}$$

for every $0<\sigma<\rho<1$. Since a direct computation leads to

$$\frac{\mathbb{M}(T_J \llcorner B_\rho)}{\rho^2} = \frac{1}{\rho^2}\int_{E_k} \mathscr{H}^2\big(v_y^{-1}(z)\cap B_\rho\big)\, d\operatorname{vol}_{Y\times X}(y,z),$$

by integrating on E_k the two previous inequalities we get the following almost monotonicity formulas for the current T_J^k:

$$\begin{aligned}
&e^{A\rho+\ell\rho^{1/2}}(1+A\rho)\frac{\mathbb{M}(T_J^k\llcorner B_\rho)}{\rho^2} - e^{A\sigma+\ell\sigma^{1/2}}(1+A\sigma)\frac{\mathbb{M}(T_J^k\llcorner B_\sigma)}{\sigma^2}\\
&\geqslant \int_{E_k}\left(\int_{v_y^{-1}(z)\cap(B_\rho\smallsetminus B_\sigma)} \frac{1}{|\cdot|^2}\langle \Omega_t, \vec{\Sigma}_J^y\rangle\, d\mathscr{H}^2\right) d\operatorname{vol}_{Y\times X}(y,z),
\end{aligned}\qquad (2.6.7)$$

$$e^{-(A\rho+\ell\rho^{1/2})}(1-A\rho)\frac{\mathbb{M}(T_J^k\llcorner B_\rho)}{\rho^2} - e^{-(A\sigma+\ell\sigma^{1/2})}(1-A\sigma)\frac{\mathbb{M}(T_J^k\llcorner B_\sigma)}{\sigma^2}$$

$$\leqslant \int_{E_k} \left(\int_{v_y^{-1}(z)\cap(B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} \langle \Omega_t, \vec{\Sigma}_J^y \rangle \, d\mathscr{H}^2 \right) d\,\mathrm{vol}_{Y\times X}(y,z), \tag{2.6.8}$$

for every $0 < \sigma < \rho < 1$. The inequality (2.6.7) immediately implies that the function

$$(0,1) \ni \rho \mapsto e^{A\rho + \ell\rho^{1/2}}(1 + A\rho)\frac{\mathbb{M}(T_J \llcorner B_\rho)}{\rho^2}$$

is monotonically non-decreasing. Thus, the density of T_J^k at 0, given by

$$\theta(T_J^k, 0) := \lim_{\rho\to 0^+} \frac{\mathbb{M}(T_J^k \llcorner B_\rho)}{\rho^2} = \lim_{\rho\to 0^+} e^{A\rho+\ell\rho^{1/2}}(1 + A\rho)\frac{\mathbb{M}(T_J^k \llcorner B_\rho)}{\rho^2}$$

exists and is finite.

We claim that T_J^k has a unique tangent cone at the origin, for every given $k \in \mathbb{N}$. The fact that T_J itself has a unique tangent cone at the origin will follow directly by the same method that is used in the conclusion of the previous subsection. By using (2.6.8), the fact that Ω is Lipschitz, point (3) in Lemma 2.5.3, the estimates (2.4.17) and (2.4.8), we get

$$\begin{aligned}
&\left| \frac{\mathbb{M}(T_J^k \llcorner B_\rho)}{\rho^2} - \theta(T_J^k, 0) \right| \\
&\leqslant C \int_{E_k} \left(\int_{v_y^{-1}(z)\cap B_\rho} \frac{1}{|\cdot|^2} \langle \Omega_t, \vec{\Sigma}_J^y \rangle \, d\mathscr{H}^2 \right) d\,\mathrm{vol}_{Y\times X}(y,z) \\
&\leqslant C \int_{E_j} \int_{v_y^{-1}(z)\cap B_\rho} \frac{|\Omega - \Omega_0|}{|\cdot|^2} \, d\mathscr{H}^2 \, d\,\mathrm{vol}_{Y\times X}(y,z) \\
&\quad + C \int_{E_j} \left| \int_{v_y^{-1}(z)\cap B_\rho} \frac{|\vec{\Sigma}^y - \vec{\Sigma}_J^y|}{|\cdot|^2} \, d\mathscr{H}^2 \right| d\,\mathrm{vol}_{Y\times X}(y,z) \\
&\quad + C \int_{E_j} \left| \int_{v_y^{-1}(z)\cap B_\rho} \pi^*\omega_{\mathbb{CP}^{m-1}}|_{\Sigma^y} \, d\mathscr{H}^2 \right| d\,\mathrm{vol}_{Y\times X}(y,z) \\
&\leqslant C\,\mathrm{vol}_{Y\times X}(Y\times X)\rho^\alpha,
\end{aligned}$$

for every $\rho \in (0, \tilde{r})$, where the constant $C > 0$ and $\alpha, \tilde{r} \in (0,1)$ all depend just on k and on $\mathrm{Lip}(\Omega)$. The fact that T_J^k has a unique tangent cone at the origin than follows by standard arguments and step 1 is proved, due to the arbitrariness of $k \in \mathbb{N}$.

Step 2. We claim T has a unique tangent cone at the origin. A direct computation using the estimate in point (3) of Lemma 2.5.3 leads to

$$\mathbb{M}\big((T - T_J)\llcorner B_\rho\big) \leqslant \int_Y \int_X \int_{v_y^{-1}(z)\cap B_\rho} |\cdot|^{1/2} \, d\mathscr{H}^2 \, d\,\mathrm{vol}_X(z)\ \mathrm{vol}_Y(y).$$

Hence,

$$\frac{\mathbb{M}\big((T - T_J)\llcorner B_\rho\big)}{\rho^2} \leqslant \int_Y \int_X \int_{v_y^{-1}(z)\cap B_\rho} \frac{1}{|\cdot|^{3/2}} \, d\mathscr{H}^2 \, d\,\mathrm{vol}_X(z)\ \mathrm{vol}_Y(y). \tag{2.6.9}$$

By (2.4.8) (recall that $\gamma = 1/2$)), we get

$$\int_{v_y^{-1}(z)\cap B_\rho} \frac{1}{|\cdot|^{3/2}} \, d\mathscr{H}^2 \leqslant C\mathscr{H}^2(v_y^{-1}(z))\rho^{1/2},$$

for every $\rho \in (0,1)$ and for $\mathrm{vol}_{Y\times X}$-a.e. $(y,z) \in Y \times X$. By integrating the previous equality on $Y \times X$, (2.6.9) and (2.5.7), we get that

$$\begin{aligned}\frac{\mathbb{M}\big((T-T_J)\llcorner B_\rho\big)}{\rho^2} &\leqslant C\bigg(\int_Y\int_{\mathbb{B}^{2m}} |v_y^*\,\mathrm{vol}_X|_g\, d\,\mathrm{vol}_g\bigg)\rho^{1/2}\\ &\leqslant C\bigg(\int_{\mathbb{B}^{2m}} |du|_g^2\, d\,\mathrm{vol}_g\bigg)\rho^{1/2},\end{aligned}$$

where $C>0$ is a constant depending only on m, n and $\mathrm{Lip}(\Omega)$. This implies that the density of $T-T_J$ at 0, given by

$$\theta(T-T_J,0) := \lim_{\rho\to 0^+}\frac{\mathbb{M}\big((T-T_J)\llcorner B_\rho\big)}{\rho^2} = 0$$

and there is a Morrey decrease of the mass ratio to the limiting density zero. Thus, $T-T_J$ has a unique tangent cone at the origin. Since by step 1 we know that T_J has a unique tangent cone at the origin and $T = T_J + (T-T_J)$, our claim follows.

Conclusion. Notice that $T = (m-2)!\,T_u \llcorner \pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}$ and recall that $\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}$ is invariant under Φ_ρ^*. We address the reader to [54, Section 7.2] for the definition of the standard operations "$\llcorner$" and "$\wedge$" when the arguments are a current and a form. Pick any two sequences of radii $\{\rho_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\rho_k'\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\rho_k, \rho_k' \to 0^+$ as $k\to+\infty$ and

$$\begin{aligned}(\Phi_{\rho_k})_*T_u &\rightharpoonup C_\infty,\\ (\Phi_{\rho_k'})_*T_u &\rightharpoonup C_\infty'.\end{aligned}$$

Since

$$\begin{aligned}\langle(\Phi_\rho)_*T,\alpha\rangle = \langle T,(\Phi_\rho)^*\alpha\rangle &= (m-2)!\left\langle T_u\llcorner \pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}, (\Phi_\rho)^*\alpha\right\rangle\\ &= (m-2)!\left\langle T_u, \pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\wedge(\Phi_\rho)^*\alpha\right\rangle\\ &= (m-2)!\left\langle T_u, (\Phi_\rho)^*(\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\wedge\alpha)\right\rangle\\ &= (m-2)!\left\langle (\Phi_\rho)_*T_u, \pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\wedge\alpha\right\rangle,\end{aligned}$$

for every $\alpha \in \mathcal{D}^2(\mathbb{B}^{2m})$ and for every $\rho\in(0,1)$, we get that

$$\begin{aligned}(\Phi_{\rho_k})_*T &\rightharpoonup (m-2)!\,C_\infty\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\\ (\Phi_{\rho_k'})_*T &\rightharpoonup (m-2)!\,C_\infty'\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}.\end{aligned}$$

Since the tangent cone to T at the origin is unique, we conclude that

$$C_\infty\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2} = C_\infty'\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2},$$

which implies

$$\big(C_\infty\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\big)\wedge\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2} = \big(C_\infty'\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\big)\wedge\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}. \tag{2.6.10}$$

Notice that

$$C_\infty = \big(C_\infty\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\big)\wedge\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2} + \big(C_\infty\wedge\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2}\big)\llcorner\pi^*\omega_{\mathbb{CP}^{m-1}}^{m-2},$$

$$C'_\infty = \left(C'_\infty \llcorner \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) \wedge \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}} + \left(C'_\infty \wedge \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) \llcorner \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}.$$

Since $m \geqslant 3$ we have, by dimensional considerations, that

$$\pi_*\left(\left(C_\infty \wedge \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) \llcorner \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) = 0, \tag{2.6.11}$$

$$\pi_*\left(\left(C'_\infty \wedge \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) \llcorner \pi^*\omega^{m-2}_{\mathbb{CP}^{m-1}}\right) = 0. \tag{2.6.12}$$

Thus, by (2.6.10), (2.6.11) and (2.6.12), we get $\pi_* C_\infty = \pi_* C'_\infty$. Since C_∞ and C'_∞ are J_0-holomorphic cones, we get $C_\infty = C'_\infty$ and the statement of Theorem 2.1.2 follows.

2.6.3. Recovering uniqueness of tangent maps for u

The case $n = 1$. Let $m \geqslant 3$ and let $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^1)$ be weakly (J, j_1)-holomorphic and locally approximable. By the methods that we have introduced in the previous subsection, it follows that uniqueness of tangent cone holds for every $(2m-2)$-dimensional current $T_{u,\psi}$ of the form

$$\langle T_{u,\psi}, \alpha \rangle := \int_{\mathbb{B}^{2m}} u^*(\psi\, \omega_{\mathbb{CP}^1}) \wedge \alpha \qquad \forall\, \alpha \in \mathcal{D}^{2m-2}(B),$$

with $\psi \in C^\infty(\mathbb{CP}^1)$.

Pick any two sequences of radii $\{\rho_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\rho'_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\rho_k, \rho'_k \to 0^+$ as $k \to +\infty$ and

$$u_{\rho_k} \rightharpoonup u_\infty,$$
$$u_{\rho'_k} \rightharpoonup u'_\infty,$$

weakly in $W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^1)$. By uniqueness of tangent cone for $T_{u,\psi}$ we get immediately that

$$\int_{\mathbb{B}^{2m}} u^*_\infty(\psi\, \omega_{\mathbb{CP}^1}) \wedge (\varphi\Omega_0) = \int_{\mathbb{B}^{2m}} (u'_\infty)^*(\psi\, \omega_{\mathbb{CP}^1}) \wedge (\varphi\Omega_0),$$

for every $\psi \in C^\infty(\mathbb{CP}^1)$, $\varphi \in C^\infty_c(\mathbb{B}^{2m})$. As both u_∞ and u'_∞ are weakly (J_0, j_1)-holomorphic, by the coarea formula and by Corollary 2.1 we get

$$\int_{\mathbb{B}^{2m}} u^*_\infty(\varphi\, \omega_{\mathbb{CP}^1}) \wedge (\varphi\Omega_0) = \int_{\mathbb{CP}^1} \psi(y)\left(\int_{\mathbb{B}^{2m}} \varphi \chi_{u^{-1}_\infty(y)}\, d\mathscr{H}^{2m-2}\right) \mathrm{vol}_{\mathbb{CP}^1}(y)$$

$$\int_{\mathbb{B}^{2m}} (u'_\infty)^*(\varphi\, \omega_{\mathbb{CP}^1}) \wedge (\varphi\Omega_0) = \int_{\mathbb{CP}^1} \psi(y)\left(\int_{\mathbb{B}^{2m}} \varphi \chi_{(u'_\infty)^{-1}(y)}\, d\mathscr{H}^{2m-2}\right) \mathrm{vol}_{\mathbb{CP}^1}(y)$$

for every $\psi \in C^\infty(\mathbb{CP}^1)$, $\varphi \in C^\infty_c(\mathbb{B}^{2m})$. Hence,

$$\int_{\mathbb{CP}^1} \psi(y)\left(\int_{\mathbb{B}^{2m}} \varphi\big(\chi_{u^{-1}_\infty(y)} - \chi_{(u'_\infty)^{-1}(y)}\big)\, d\mathscr{H}^{2m-2}\right) \mathrm{vol}_{\mathbb{CP}^1}(y) = 0$$

for every $\psi \in C^\infty(\mathbb{CP}^1)$, $\varphi \in C^\infty_c(\mathbb{B}^{2m})$. This implies that for $\mathrm{vol}_{\mathbb{CP}^1}$-a.e. $y \in \mathbb{CP}^1$ the sets $u^{-1}_\infty(y)$ and $(u'_\infty)^{-1}(y)$ coincide up to $\mathscr{H}^{2m-2}$-negligible sets. We conclude that $u_\infty = u'_\infty$ $\mathcal{L}^{2m}$-a.e. on $\mathbb{B}^{2m}$ and the statement of Theorem 2.1.1 follows.

The case $n > 1$. Let $m \geqslant 3$, $n \geqslant 2$ and let $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^n)$ be weakly (J, j_n)-holomorphic and locally approximable. By the methods that we have introduced in the previous subsection, it follows that uniqueness of tangent cone holds for every $(2m-2)$-dimensional current $T^{q_1,...,q_{n-1}}_{u,\psi}$ of the form

$$\langle T^{q_1,...,q_{n-1}}_{u,\psi}, \alpha \rangle := \int_{\mathbb{B}^{2m}} (F_{q_1} \circ ... \circ F_{q_{n-1}} \circ u)^*(\psi\, \omega_{\mathbb{CP}^1}) \wedge \alpha \quad \forall\, \alpha \in \mathcal{D}^{2m-2}(\mathbb{B}^{2m}),$$

with $\psi \in C^\infty(\mathbb{CP}^n)$ and for every choice of $(q_1, ..., q_{n-1}) \in \mathbb{CP}^2 \times ... \times \mathbb{CP}^n$.

Pick any two sequences of radii $\{\rho_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\rho'_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\rho_k, \rho'_k \to 0^+$ as $k \to +\infty$ and

$$u_{\rho_k} \rightharpoonup u_\infty,$$
$$u_{\rho'_k} \rightharpoonup u'_\infty,$$

weakly in $W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^1)$. By using the technique that we have shown for the case $n = 1$, we get that

$$F_{q_1} \circ ... \circ F_{q_{n-1}} \circ u_\infty = F_{q_1} \circ ... \circ F_{q_{n-1}} \circ u'_\infty, \qquad \mathcal{L}^{2m}\text{- a.e. on } \mathbb{B}^{2m},$$

for every choice of $(q_1, ..., q_{n-1}) \in \mathbb{CP}^2 \times ... \times \mathbb{CP}^n$. By using iteratively Lemma 2.A.3, we obtain $u_\infty = u'_\infty$ $\mathcal{L}^{2m}$-a.e. on $\mathbb{B}^{2m}$ and the statement of Theorem 2.1.1 follows.

Remark 2.6.1. The advantage of the previous approach relies in the fact the we don't get only uniqueness of tangent cone for the current T_u but also for its "localizations" $T_{u,\psi}$ (see the beginning of subsection 6.3) through smooth functions $\psi \in C^\infty(\mathbb{CP}^n)$. This allows more flexibility and we would like to drag the attention of the reader on the fact we could exploit such flexibility in order to get a new proof the result in [7]. Given an integer (p,p)-cycle $\Sigma \subset \mathbb{B}^{2m}$, we could consider a weakly holomorphic and locally approximable map $u \in W^{1,2}(\mathbb{B}^{2m}, \mathbb{CP}^{m-p})$ such that $u(\Sigma) = \{y\} \in \mathbb{CP}^{m-p}$. By localizing the associated cycle T_u through a sequence $\{\psi_\varepsilon\} \subset C^\infty(\mathbb{CP}^{m-p})$ such that $\psi_\varepsilon \to \delta_y$ in $\mathcal{D}'(\mathbb{CP}^{m-p})$, we could exploit our techniques to get uniqueness of tangent cone for Σ ultimately.

Appendix to Chapter 2

2.A. Slicing through singular meromorphic maps

Let $m \in \mathbb{N}$ be such that $m \geqslant 3$ and fix any point $p \in \mathbb{CP}^{m-1}$. Let $\pi : \mathbb{C}^m \smallsetminus \{0\} \to \mathbb{CP}^{m-1}$ be the quotient map given by

$$\pi(z_1, ..., z_m) := [z_1; ...; z_m], \qquad \forall z \in \mathbb{C}^m \smallsetminus \{0\}.$$

Denote by L_p the complex line generated by p in $\mathbb{CP}^{m-1}$ and let $T_p : \mathbb{C}^m \smallsetminus L_p \to L_p^\perp \smallsetminus \{0\}$ be given by the restriction to $\mathbb{C}^m \smallsetminus L_p$ of the standard orthogonal projection from $\mathbb{C}^m$ into $L_p^\perp$. Fix a complex orthonormal basis $\{e_1^p, ..., e_{m-1}^p\}$ of $L_p^\perp$ and let $\varphi_p : L_p^\perp \to \mathbb{C}^{m-1}$ be the following linear isomorphism:

$$\varphi_p\left(\sum_{j=1}^{m-1} \alpha_j e_j^p\right) := (\alpha_1, ..., \alpha_m), \qquad \forall\, (\alpha_1, ..., \alpha_{m-1}) \in \mathbb{C}^{m-1}.$$

Let $\pi_p : L_p^\perp \smallsetminus \{0\} \to \mathbb{CP}^{m-2}$ be the smooth submersion given by $\pi_p := \tilde{\pi} \circ \varphi_p$, where

$$\tilde{\pi}(\alpha_1, ..., \alpha_{m-1}) := [\alpha_1; ...; \alpha_{m-1}], \qquad \forall\, (\alpha_1, ...\alpha_{m-1}) \in \mathbb{C}^{m-1} \smallsetminus \{0\}.$$

Eventually, notice that the map $F_p : \mathbb{CP}^{m-1} \smallsetminus \{p\} \to \mathbb{CP}^{m-2}$ given by

$$F_p([z_1; ...; z_m]) = (\pi_p \circ T_p)(z_1, ..., z_m), \qquad \forall\, [z_1, ..., z_m] \in \mathbb{CP}^{m-1} \smallsetminus \{p\},$$

is well-defined and smooth, since the map $\pi_p \circ T_p$ is constant on the fibres of π.

Lemma 2.A.1. *Let $m \in \mathbb{N}$ be such that $m \geqslant 3$. Then, for every $p \in \mathbb{CP}^{m-1}$ the following facts hold:*

(1) the map $F_p \circ \pi$ belongs to $W^{1,2m-4}(\mathbb{B}^{2m}, \mathbb{CP}^{m-2})$;

(2) $F_p \circ \pi$ is weakly (J_0, j_{m-2})-holomorphic, where j_{m-2} is the standard complex structure on $\mathbb{CP}^{m-2}$;

(3) $F_p \circ \pi$ is such that $d\big((F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big) = 0$, distributionally on $\mathbb{B}^{2m}$.

Proof. Fix any $p \in \mathbb{CP}^{m-1}$ and notice that the complex line L_p is indeed a real 2-plane in $\mathbb{R}^{2m}$. Thus, $\mathscr{H}^{2m-\alpha}(L_p \cap \mathbb{B}^{2m}) = 0$, for every $\alpha \in [1, 2m-2)$. Hence, $L_p \cap \mathbb{B}^{2m}$ has vanishing $(2m-4)$-capacity. Since $F_p \circ \pi \in L^\infty(\mathbb{B}^{2m}) \cap C^\infty(\mathbb{B}^{2m} \smallsetminus L_p)$ and the classical differential of $F_p \circ \pi$ on $\mathbb{B}^{2m} \smallsetminus L_p$ can be estimated by

$$|d(F_p \circ \pi)| = |d(\pi_p \circ T_p)| \leqslant |\textstyle\bigwedge_2 d\pi_p \circ T_p| \leqslant \dfrac{C}{\operatorname{dist}(\,\cdot\,, L_p)}, \tag{2.A.1}$$

we obtain that $d(F_p \circ \pi) \in L^{2m-4}(\mathbb{B}^{2m} \smallsetminus L_p)$. Point (1) immediately follows.

For what concerns (2), we know that the weak differential of $F_p \circ \pi$ coincides $\mathcal{L}^{2m}$-a.e. with its classical differential on $\mathbb{B}^{2m} \smallsetminus L_p$, where $F_p \circ \pi$ is smooth. Moreover $F_p \circ \pi = \pi_p \circ\, T_p$ on $\mathbb{B}^{2m} \smallsetminus L_p$. Since both π_p and T_p are holomorphic maps, then $F_p \circ \pi$ is holomorphic on $\mathbb{B}^{2m} \smallsetminus L_p$. Then, since

$\mathbb{B}^{2m} \setminus L_p$ has full $\mathcal{L}^{2m}$-measure in $\mathbb{B}^{2m}$, the fact that the weak differential of $F_p \circ \pi$ commutes with the complex structures J_0 and j_{m-2} for $\mathcal{L}^{2m}$-a.e. $x \in \mathbb{B}^{2m}$ follows and we have proved (2).
We are just left to prove (3). Fix any $\alpha \in \mathcal{D}^3(\mathbb{B}^{2m})$. For every any $\varepsilon > 0$ we define

$$L_p^\varepsilon := \big(L_p + B_\varepsilon(0)\big) \cap \mathbb{B}^{2m} \tag{2.A.2}$$

and we notice that

$$\begin{aligned}
\left|\int_{\mathbb{B}^{2m}} (F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}} \wedge d\alpha\right| &= \left|\int_{L_p^\varepsilon} (F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}} \wedge d\alpha\right| \\
&\leqslant \int_{L_p^\varepsilon} |d\alpha|\big|(F_p \circ \pi)^* \operatorname{vol}_{\mathbb{CP}^{m-2}}\big|\, d\mathcal{L}^{2m} \\
&\leqslant ||d\alpha||_{L^\infty} \int_{L_p^\varepsilon} |d(F_p \circ \pi)|^{2m-4} d\mathcal{L}^{2m} \\
&\leqslant ||d\alpha||_{L^\infty} ||d(F_p \circ \pi)||_{L^{2m-3}}^{2m-4} \mathcal{L}^{2m}(L_p^\varepsilon)^{1/q'} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$, where $q := (2m-3)/(2m-4)$ and $q' = 2m-3$ is the conjugate exponent of q. By arbitrariness of $\alpha \in \mathcal{D}^3(\mathbb{B}^{2m})$, point (3) follows. □

Lemma 2.A.2. *For every $m \geqslant 3$, there exists a constant $B_m > 0$ such that*

$$\omega_{\mathbb{CP}^{m-1}} = B_m \int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}}\, dp.$$

Proof. Throughout this proof, given any $m \geqslant 1$ and a unitary matrix $A \in U(m)$, we will denote by $\bar{A} : \mathbb{CP}^{m-1} \to \mathbb{CP}^{m-1}$ the map $[z] \mapsto [Az]$.
It is well known that, up to rescalings by constant factors, the Fubini-Study metric is the only $U(m)$-invariant symplectic form on $\mathbb{CP}^{m-1}$, for every $m \geqslant 2$. Thus, it is enough to show that

$$\bar{A}^*\bigg(\int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}}\, dp\bigg) = \int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}}\, dp, \qquad \forall\, A \in U(m).$$

Fix any $A \in U(m)$.Given $p \in \mathbb{CP}^{m-1}$, define

$$B_p := \varphi_p \circ T_p \circ A \circ S_p \circ \varphi_p^{-1} : \mathbb{C}^{m-1} \to \mathbb{C}^{m-1},$$

where $S_p : L_p^\perp \to \mathbb{C}^m$ is the left inverse of the orthogonal projection map $T_p : \mathbb{C}^m \to L_p^\perp$. As composition of linear and unitary maps, $B_p \in U(m-1)$. Moreover, by construction it holds that $F_p \circ \bar{A} = \bar{B}_p \circ F_p$.
Hence, by linearity of the integral and the definition of F_p, we have

$$\begin{aligned}
\bar{A}^*\bigg(\int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}}\, dp\bigg) &= \int_{\mathbb{CP}^{m-1}} \bar{A}^* F_p^* \omega_{\mathbb{CP}^{m-2}}\, dp \\
&= \int_{\mathbb{CP}^{m-1}} (F_p \circ \bar{A})^* \omega_{\mathbb{CP}^{m-2}}\, dp \\
&= \int_{\mathbb{CP}^{m-1}} (\bar{B} \circ F_p)^* \omega_{\mathbb{CP}^{m-2}}\, dp \\
&= \int_{\mathbb{CP}^{m-1}} F_p^* \bar{B}^* \omega_{\mathbb{CP}^{m-2}}\, dp
\end{aligned}$$

$$= \int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}} \, dp$$

and the statement follows by arbitrariness of $A \in U(m)$. $\square$

Lemma 2.A.3. *Let $m \geqslant 3$ and pick any two points $x, y \in \mathbb{CP}^{m-1}$. For every $j = 1, ..., m$, let $\tilde{e}_j := \pi(e_j) \in \mathbb{CP}^{m-1}$, where $\{e_1, ..., e_m\}$ denotes the standard complex euclidean basis of $\mathbb{C}^m$. Assume that*

$$F_{\tilde{e}_j}(x) = F_{\tilde{e}_j}(y), \qquad \forall\, j = 1, ..., m. \tag{2.A.3}$$

Then, $x = y$.

Proof. Let $x = [x_1; ...; x_m]$ and $y = [y_1; ...; y_m]$. Fix any $j = 1, ..., m$. By definition of $F_{\tilde{e}_j}$, the condition $F_{\tilde{e}_j}(x) = F_{\tilde{e}_j}(y)$ implies that there exists $\lambda_j \in \mathbb{C} \setminus \{0\}$ such that

$$\lambda_j x_i = y_i, \qquad \forall\, i = 1, ..., m \text{ with } i \neq j.$$

Hence, by enforcing (2.A.3) we get that there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$\lambda x_i = y_i, \qquad \forall\, i = 1, ..., m.$$

This implies $x = y$ in $\mathbb{CP}^{m-1}$ and the statement follows. $\square$

3. Vector fields with integer valued fluxes

3.1. Introduction

3.1.1. Motivation and statement of the results

Given a smooth map $u : X \to Y$ between closed, oriented and connected $(n-1)$-dimensional manifolds, the *degree* of u is a measure of how many times X wraps around Y under the action of u. It can be defined as follows[1]:

$$\deg(u) = \int_X u^*\omega,$$

with $\omega \in \Omega^n(Y)$ being a renormalized volume form on Y, i.e. ω is nowhere vanishing and

$$\int_Y \omega = 1.$$

Let $D \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and consider a map $u \in W^{1,p}(D, Y)$ for some $p \geqslant 1$ being *smooth up to finitely many point singularities*, which simply means that $u \in C^\infty(D \smallsetminus S_u, Y)$ for some finite set $S_u \subset D$. In this case we write $u \in R^{1,p}(D, Y)$. We define the *degree of u at some singular point $x \in S_u$* as

$$\deg(u, x) := \deg(u|_{\partial D'}) = \int_{\partial D'} u^*\omega \in \mathbb{Z}, \tag{3.1.1}$$

where $D' \subset\subset D$ is any open, piecewise smooth domain in D such that $\overline{D'} \cap S_u = \{x\}$. Notice that Definition (3.1.1) is independent from the choice of the set D'.
If $\deg(u, x) \neq 0$ for some $x \in S_u$, then we say that x is a *topological singularity* of u and we refer to the subset of S_u made of the topological singularities of u as the *topological singular set of u*, which we denote by S_u^{top}.
Notice that if $u \in R^{1,n-1}(D)$ then $u^*\omega \in \Omega_1^{n-1}(D)$, moreover by (3.1.1) we see that the $u^*\omega$ "detects" the topological singularities of u, in the sense that

$$\int_{\partial D'} u^*\omega = \sum_{x \in S_u^{top} \cap D'} \deg(u, x) \tag{3.1.2}$$

for every open, piecewise smooth domain $D' \subset\subset D$ such that $\partial D' \cap S_u = \emptyset$. From (3.1.2) one can deduce that

$$*d(u^*\omega) = \sum_{x \in S_u^{top}} \deg(u, x)\delta_x \quad \text{in } \mathcal{D}'(D).$$

[1] An alternative definition can be given in terms of the orientations of the preimages of regular points of u, see for instance [10, Chapter 7]

Remark 3.1.1. Sobolev maps that are smooth up to a finite set of topological singularities arise frequently as solutions of variational problems in critical or supercritical dimension. For example, this is the best regularity which is possible to guarantee for energy minimizing harmonic maps in $W^{1,2}(B^3, \mathbb{S}^2)$ (see [79, Theorem II]). Again, quite recently the F. Gaia and T. Rivière considered the following "weak $\mathbb{S}^1$-harmonic map equation"

$$\operatorname{div}(u \wedge \nabla u) = 0 \qquad \text{in } \mathcal{D}'(B^2)$$

and gave a completely variational characterization of the solutions in $R^{1,p}(B^2, \mathbb{S}^1)$ with finite "renormalized Dirichlet energy" for $p > 1$ (see [37, Theorem I.3] for further details).
We also remark that the presence of topological singularities is deeply linked to fundamental questions concerning the strong $W^{1,p}$-approximability through smooth maps of elements in $W^{1,p}(D, Y)$ (see [12], [43]).

The previous discussion motivates the following general definition.

Definition 3.1.1. Let $p \in [1, \infty]$. Let $F \in \Omega^{n-1}_p(D)$. We say that F has *finitely many integer singularities* if there exists a finite set of points $S \subset D$ such that $F \in \Omega^{n-1}(D \smallsetminus S)$ and

$$*dF = \sum_{x \in S} a_x \delta_x,$$

where $a_x \in \mathbb{Z}$ for every $x \in S$. The class of L^p integrable $(n-1)$-forms on D having finitely many integer singularities will be denoted by $\Omega^{n-1}_{p,R}(D)$.

As we have seen above, $u^*\omega \in \Omega^{n-1}_{1,R}(D)$ for every $u \in R^{1,n-1}(D, Y)$, for any closed, oriented and connected $n-1$-manifold Y. Other simple examples of elements of $\Omega^{n-1}_{p,R}(D)$ can be constructed as follows. Let $\sigma : D \to \mathbb{R}$ denote the fundamental solution of the Laplace equation, i.e.

$$\sigma(x) = \begin{cases} -|x| & \text{if } n = 1, \\ -\frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & \text{if } n \geqslant 3, \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Then $*d\sigma \in \Omega^{n-1}_{p,R}(D)$ for any $p \in [1, \frac{n}{n-1})$. In fact $*d(*d\sigma) = \Delta\sigma = \delta_0$.
Clearly any finite linear combination with integer coefficients of translations of $*d\sigma$ also belongs to $\Omega^{n-1}_{p,R}(D)$. In fact one can show that any element F of $\Omega^{n-1}_{p,R}(D)$ can be decomposed as such a linear combination plus some $\tilde{F} \in \Omega^{n-1}_p(D)$ with $*d\tilde{F} = 0$. In particular, if $p \geqslant \frac{n}{n-1}$ then $*dF = 0$. Thus the class $\Omega^{n-1}_{p,R}(D)$ is relatively simple from an analytical point of view and so it is natural to ask which forms in $\Omega^{n-1}_p(D)$ can be approximated by elements in $\Omega^{n-1}_{p,R}(D)$. The main purpose of the present chapter consists in giving a description of the strong and weak closure of the class $\Omega^{n-1}_{p,R}(D)$ for any open domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the open unit n-cube $Q_1(0) \subset \mathbb{R}^n$.
First, we will address the strong closure in the case of the open, unit n-dimensional cube $Q_1(0) \subset \mathbb{R}^n$. To this end we introduce the class of $(n-1)$-forms with integer-valued fluxes.
For any $F \in \Omega^{n-1}_{L^p_{loc}}$, for any $x_0 \in Q_1(0)$ let $\tilde{R}_{F,x_0} \subset (0, r_0)$ be the set of radii $\rho \in (0, r_0)$ such that

1. the hypersurface $\partial Q_\rho(x_0)$ consists $\mathscr{H}^{n-1}$-a.e. of Lebesgue points of F,

2. there holds $|F| \in L^p\big(\partial Q_\rho(x_0), \mathscr{H}^{n-1}\big)$.

One can check that $\mathcal{L}^1\big((0, r_{x_0}) \smallsetminus \tilde{R}_{F,x_0}\big) = 0$.

Definition 3.1.2. Let $p \in [1,\infty]$, let $F \in \Omega^{n-1}_{L^p_{loc}} \cap \Omega^{n-1}_p(Q_1(0),\mu)$ for some Radon measure μ, for any $x_0 \in Q_1(0)$ let $\tilde{R}_{F,x_0}$ be defined as above. We say that F has *integer-valued fluxes* if for any $x_0 \in Q_1(0)$, for $\mathcal{L}^1-$a.e. $\rho \in \tilde{R}_{F,x_0}$ there holds[2]

$$\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}. \tag{3.1.3}$$

The space of $L^p(\mu)$-vector fields with integer valued fluxes will be denoted by $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0),\mu)$. The set of radii $\rho \in \tilde{R}_{F,x_0}$ for which (3.1.3) holds will be denoted by R_{F,x_0}

We will always write $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ for $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0),\mathcal{L}^n)$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

First of all we observe that (3.1.2) implies that $\Omega^{n-1}_{p,R}(Q_1(0)) \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. More general examples of forms in $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ can be constructed as follows. Let again Y be a smooth, closed, oriented and connected $(n-1)$-dimensional manifold. Let $u \in W^{1,n-1}(Q_1(0),Y)$. Then for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2\,\mathrm{dist}_\infty(x_0,\partial Q_1(0)))$, $u\big|_{\partial Q_\rho(0)} \in W^{1,n-1}(\partial Q_\rho(0),Y)$. Therefore for any such ρ

$$\int_{\partial Q_\rho(0)} i^*_{\partial Q_\rho(x_0)}(u^*\omega) = \deg\left(u\big|_{\partial Q_\rho(x_0)}\right) \in \mathbb{Z} \tag{3.1.4}$$

Notice that $\deg\left(u\big|_{\partial Q_\rho(x_0)}\right)$ is well defined (by means of approximation by functions in $W^{1,\infty}(\partial Q_\rho(x_0),Y)$, see [21, Section I.3]).

We will show that in fact the closure of $\Omega^{n-1}_{p,R}(Q_1(0))$ in $\Omega^{n-1}_p(Q_1(0))$ is exactly $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. More precisely, we have

Theorem 3.1.1. *Let $n \in \mathbb{N}$ and let $p \in [1,\infty)$. Let $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0),\mu)$. Then we have*

1. *if $q \in [0,1]$ and $p \in \left[1, \frac{n}{n-1}\right)$, then there exists a sequence $\{F_k\}_{k\in\mathbb{N}}$ in $\Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \to F$ in $\Omega^{n-1}_p(Q_1(0))$ as $k \to \infty$.*
2. *if $q \in (-\infty,0]$ and $p \in \left[\frac{n}{n-1}, +\infty\right)$, then $*dF = 0$.*

The reason why we have introduced the weighted measures $\mu = f\,\mathcal{L}^n$ for $q \neq 0$ is that forms belonging to $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0),\mu)$ appear naturally in the proof of Corollary 3.2.2. Nevertheless, we advise the reader to assume $q = 0$ (i.e. $\mu = \mathcal{L}^n$) throughout Section 3.2 at a first reading of the present chapter. This allows to skip many technicalities without losing formality, since all the results of this chapter are independent on Corollary 3.2.2.

With the help of Theorem 3.1.1 we will get another characterization of the L^p-closure of $\Omega^{n-1}_{p,R}(Q_1(0))$. For this we recall the following definition (compare with [19, Section II]):

[2] Notice that for the associated vector field $V = (*F)^\flat$ condition (3.1.3) reads

$$\int_{\partial Q_\rho(x_0)} V \cdot \nu_{\partial Q_\rho(x_0)} d\mathscr{H}^{n-1} \in \mathbb{Z}.$$

Definition 3.1.3 (Connection and minimal connection)**.** Let $M \subset \mathbb{R}^n$ be any embedded Lipschitz m-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\overline{M}$ is compact as a subset of $\mathbb{R}^n$.

A 1-dimensional current $I \in \mathcal{R}_1(M)$ is said to be a *connection* for (the singular set of) F if $\mathbb{M}(I) < +\infty$ and $\partial I = *dF$ in $(W_0^{1,\infty}(M))^*$.

A 1-dimensional current $L \in \mathcal{R}_1(M)$ is said to be a *minimal connection* for (the singular set of) F if it is a connection for F and

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{D}_1(M) \\ \partial T = *dF}} \mathbb{M}(T).$$

We will see in Corollary 3.2.1 that F admits a connection if and only if it admits a minimal connection.

Here is the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of minimal connections:

Theorem 3.1.2. *Let $n \in \mathbb{N}_{>0}$, let $p \in [1, +\infty)$. Let $F \in \Omega_p^{n-1}(Q_1^n(0))$. Then, the following are equivalent:*

1. *there exists $L \in \mathcal{R}_1(Q_1(0))$ such that $\partial L = *dF$ in $(W_0^{1,\infty}(Q_1(0)))^*$.*
2. *for every Lipschitz function $f : \overline{Q_1(0)} \to [a, b] \subset \mathbb{R}$ such that $f|_{\partial Q_1(0)} \equiv b$, we have*

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b];$$

3. *$F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.*

In other words, $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ if and only if F admits a (minimal) connection. This characterization allows to generalize the definition of the class $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ to general Lipschitz domains:

Definition 3.1.4. Let $M \subset \mathbb{R}^n$ be any embedded Lipschitz m-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary). We define

$$\Omega^{n-1}_{p,\mathbb{Z}}(M) := \{F \in \Omega_p^{n-1}(M) \text{ s.t. } \exists\, L \in \mathcal{R}_1(M) \text{ connection for } F\}.$$

Notice that if $M = Q_1(0)$, Definition 3.1.2 and Definition 3.1.4 coincide by Theorem 3.1.2.

We will deduce from the previous results that the approximation result can be extended to any open domain which is bi-Lipschitz equivalent to $Q_1(0)$ of $\partial Q_1(0)$ (see Theorem 3.2.3).

We mention here two other corollaries of Theorem 3.1.1.

Corollary 3.1.1. *Let $n \in \mathbb{N}$. Let $I \in \mathcal{R}_1(Q_1^n(0))$ be an integer rectifiable 1-current. Then there exists a 1-form $\omega \in \Omega^{n-1}_{1,\mathbb{Z}}(Q_1^n(0))$ such that $*d\omega = \partial I$ and ∂I can be approximated in $(W_0^{1,\infty}(Q_1^n(0)))^*$ by finite sums of Dirac-deltas with integer coefficients. More precisely, there exist sequences $(P_i)_{i \in \mathbb{N}}$ and $(N_i)_{i \in \mathbb{N}}$ of points in $Q_1^n(0)$ such that*

$$\partial I = \sum_{i \in \mathbb{N}} (\delta_{P_i} - \delta_{N_i}) \ \text{ in } (W_0^{1,\infty}(Q_1^n(0)))^* \text{ and } \sum_{i \in \mathbb{N}} |P_i - N_i| < \infty.$$

Moreover if I is supported on a Lipschitz submanifold M of $\mathbb{R}^n$ compactly contained in $Q_1(0)$, the points in the sequences $(P_i)_{i \in \mathbb{N}}$ and $(N_i)_{i \in \mathbb{N}}$ can be chosen to belong to M.

The next corollary was obtained first by R. Schoen and K. Uhlenbeck ([78, Section 4]) and F. Bethuel and X. Zheng ([13, Theorem 4]).

Corollary 3.1.2. *Let $Q_1(0) \subset \mathbb{R}^2$ be the unit cube in $\mathbb{R}^2$. Let $u \in W^{1,p}(Q_1(0), \mathbb{S}^1)$ for some $p \in (1, \infty)$.*
If $p \geqslant 2$, then u can be approximated in $W^{1,p}$ be a sequence of functions in $C^\infty(Q_1(0), \mathbb{S}^1)$.
If $p < 2$, then u can be approximated in $W^{1,p}$ by a sequence of functions in

$$\mathcal{R} := \left\{ v \in W^{1,p}(Q_1(0), \mathbb{S}^1); v \in C^\infty(Q_1(0) \smallsetminus A, \mathbb{S}^1), \text{ where } A \text{ is some finite set} \right\}.$$

In the second part of the chapter we turn our attention to the weak closure of the space $\Omega^{n-1}_{p,R}(D)$ for a domain $D \subset \mathbb{R}^n$ which is bi-Lipschitz equivalent to $Q_1(0)$ (or equivalently of $\Omega^{n-1}_{p,\mathbb{Z}}(D)$). We will show the following.

Theorem 3.1.3 (Weak closure)**.** *Let $n \in \mathbb{N}_{\geqslant 2}$, $p \in (1, +\infty)$ and $D \subset \mathbb{R}^n$ be any open and bounded domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the n-dimensional unit cube $Q^n_1(0)$. Then, the space $\Omega^{n-1}_{p,\mathbb{Z}}(D)$ is weakly sequentially closed in $\Omega^{n-1}_p(D)$.*

Notice that, by Theorem 3.1.1 (or more generally by Theorem 3.2.3), the statement of Theorem 3.1.3 is trivial for $p \in [n/(n-1), +\infty)$. Thus, we just need to provide a proof in case $p \in (1, n/(n-1))$.

We first treat the case of the open unit n-cube $Q_1(0) \subset \mathbb{R}^n$ by exploiting the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ given by Theorem 3.1.2 and a suitable slice distance à la Ambrosio-Kirchheim (see [4] and [42]). We then address the general case by standard arguments (see Remark 3.3.8).

We remark that the case $n = 1$ is different. In fact for any interval $I \subset \mathbb{R}$ there holds $\overline{\Omega^0_{p,R}(I)}^{\Omega^0_p(I)} = \Omega^0_p(I)$ (see Lemma 3.3.3).

Our main motivation to look at forms (instead of maps) with finitely many integer topological singularities is the need of developing geometric measure theory for principal bundles in order to face the still deeply open questions arising in the study of p–Yang-Mills lagrangians. As it is described in the introduction to Chapter 4 and in [48], [65], the reason why we aim to extend the set of the (by now classical) Sobolev connections is purely analytic and justified by issues arising in the application of the direct method of calculus of variations to p–Yang-Mills lagrangians. On the other hand, the need to extend the notion of bundles in order to allow singularities to appear has already been faced in many geometric applications, which brought to the introduction of coherent and reflexive sheaves in the analysis of Yang-Mills fields over complex manifolds (see [51],[52]). However, such tools are insufficient to deal with the phenomenon of accumulation of singularities that can't be excluded a priori in the real framework. Thus, new ways to describe the behaviour of very singular Yang-Mills connections need to be investigated.

Notice that all results mentioned above can be formulated in terms of vector fields: for any $F \in \Omega^{n-1}_p(D)$ we can consider the associated vector field $V_F := (*F)^\flat$. In fact for the proof of some of the results we preferred to work with vector fields instead of $(n-1)$-forms.

3.1.2. Related literature and open problems

Theorem 3.1.1 was firstly announced to hold for a 3-dimensional domain in [48] and a full proof of the 3-dimensional case was eventually given by the author in [22]. Some form of the 2-dimensional case was treated in [61], where M. Petrache proved that a strong approximability result holds for 1-forms admitting a connection both on the 2-dimensional disk and on the 2-sphere. In both cases, the proof that we give here is more general and simple. The 3-dimensional version of Theorem 3.1.3 was already treated by M. Petrache and T. Rivière in [63]. Nevertheless, here we took the opportunity to present the arguments in a more detailed and complete way. Both Theorem 3.1.1 and Theorem 3.1.3 in dimension $n \neq 2, 3$ are instead completely new.

The first open problems that relate directly to our results are linked to the celebrated Yang-Mills Plateau problem. Indeed, the weak sequential closedness of the class $\Omega^2_{p,\mathbb{Z}}(\mathbb{B}^3)$ implies that such forms behave well-enough to be considered as suitable "very weak" curvatures for the resolution of the p–Yang-Mills Plateau problem for $U(1)$-bundles on $\mathbb{B}^3$ (see the introduction of [63] for further details). The question to address would be if and how we can exploit the same kind of techniques in order to face the existence and regularity issues linked to the so called "non abelian case" (i.e. the case of bundles having a non abelian structure group) in supercritical dimension. An interesting proposal in this sense is due to M. Petrache and T. Rivière and can be found in [62], where a suitable class of weak connections in the supercritical dimension 5 is introduced. In Chapter 4 we will describe how these methods can be used in order to establish ε-regularity for stationary Yang-Mills connections belonging to such class of weak objects in dimension 5.

One could also hope that the technique presented in this chapter could be adapted to show L^p-closeness (weak and strong) of classes of differential forms exhibiting "integer fluxes" properties similar to the one described in Definition 3.1.2. As an example we define here the class $\Omega^n_{p,H}(Q^{2n}_1(0))$ of differential forms with "Hopf singularities".
Recall that for any $n \in \mathbb{N}_{\geqslant 1}$, for any smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ the *Hopf invariant* of f is defined as follows: let ω be the standard volume form on $\mathbb{S}^n$. Let $\alpha \in \Omega^{n-1}(\mathbb{S}^{2n-1})$ be such that $f^*\omega = d\alpha$. Then the Hopf invariant of f is given by

$$H(f) := \int_{\mathbb{S}^{2n-1}} \alpha \wedge d\alpha.$$

One can show that $H(f) \in \mathbb{Z}$ and that it is independent of the choice of α (see [17], Proposition 17.22). In the spirit of Definition 3.1.2 we say that a form $F \in \Omega^n_p(Q^{2n}_1(0))$ belongs to $\Omega^n_{p,H}(Q^{2n}_1(0))$ for some $p \geqslant 2 - \dfrac{1}{n}$ if there exists $A \in \Omega^{n-1}_{W^{1,p}}(Q^{2n}_1(0))$ such that $dA = F$ and if for every $x_0 \in Q^{2n}_1(0)$ there exists a set $R_{F,x_0} \subset (0, r_{x_0})$, with $r_{x_0} := 2\,\mathrm{dist}_\infty(x_0, \partial Q^{2n}_1(0))$ such that:

1. $\mathcal{L}^1\big((0, r_{x_0}) \smallsetminus R_{F,x_0}\big) = 0$;
2. for every $\rho \in R_{F,x_0}$, the hypersurface $\partial Q^{2n}_\rho(x_0)$ consists $\mathscr{H}^{2n-1}$-a.e. of Lebesgue points of F, A and ∇A (the matrix of all the partial derivatives of the components of A);
3. for every $\rho \in R_{F,x_0}$ we have $|F|, |A|, |\nabla A| \in L^p\big(\partial Q^{2n}_\rho(x_0), \mathscr{H}^{n-1}\big)$;
4. for every $\rho \in R_{F,x_0}$ it holds that

$$\int_{\partial Q^{2n}_\rho(x_0)} i^*_{\partial Q^{2n}_\rho(x_0)}(A \wedge F) \in \mathbb{Z}.$$

Notice that if $u \in W^{1,2n-1}(Q^{2n}_1(0), \mathbb{S}^n)$, then $u^*\omega \in \Omega^n_{p,H}(Q^{2n}_1(0))$.

3.1.3. Organization of the Chapter

The chapter is organized as follows. Section 3.2 is dedicated to the strong L^p-closure of the space $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0),\mu)$. First we present some preliminary and rather technical lemmata (Sections 3.2.1–3.2.3), then we give a proof of Theorem 3.1.1 (Section 3.2.4). In Section 3.2.5 we show the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of (minimal) connections (Theorem 3.1.2). In Section 3.2.6 we exploit this result to extend the approximation result to other Lipschitz manifolds, and in particular to $\partial Q^n_1(0)$. Lastly, in Section 3.2.7 we prove Corollary 3.1.1 and Corollary 3.1.2.
In Section 3.3 we discuss the weak L^p closure of $\Omega^{n-1}_{p,R}(Q_1(0))$. First, we will introduce a slice distance à la Ambrosio-Kirchheim, first on spheres (Section 3.3.1) and then on cubes (Section 3.3.2). In Section 3.3.3 we discuss some of the properties of the slice distance and in Section 3.3.4 we use it to obtain a proof of Theorem 3.1.3. We will also discuss briefly the special case $n = 1$.

3.1.4. Notation

Let $M^m \subset \mathbb{R}^n$ be any m-dimensional, embedded Lipschitz submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\overline{M}$ is compact as a subset of $\mathbb{R}^n$.

We always assume that M is endowed with the L^∞-Riemannian metric given by $g_M := \iota^*_M g_e$, where g_e denotes the standard euclidean metric on $\mathbb{R}^n$.

For every $p \in [1,+\infty]$, we denote by $\Omega^k_p(M)$ and $\Omega^k_{W^{1,p}}(M)$ the completions of $\Omega^k(M)$ with respect to the usual L^p-norm and $W^{1,p}$-norm respectively. We call $\Omega^k_p(M)$ the space of L^p *k-forms* on M and $\Omega^k_{W^{1,p}}(M)$ the space of $W^{1,p}$ *k-forms* on M.

By the symbol "$*$", we denote the Hodge star operator associated with the metric g_M on M. By "$\flat$" and "$\sharp$" we denote the usual musical isomorphisms associated with the metric g_M. Recall that, under this notation, the map

$$\Omega^{n-1}_p(M) \ni \omega \mapsto (*\omega)^\sharp \in L^p(M,\mathbb{R}^n) \tag{3.1.5}$$

gives an isomorphism onto its image. Exploiting this fact, we frequently identify $(n-1)$-forms with vector fields on M.

Given any $F \in \Omega^k_1(M)$ we define the $(m-k)$-*current associated to* F by

$$\langle T_F, \omega\rangle := \int_M F \wedge \omega, \qquad \forall \omega \in \mathcal{D}^{m-k}(M).$$

3.2. The strong L^p-approximation Theorem

In this section we provide a proof of Theorem 3.1.1. In an attempt to make the proof more accessible, we reformulate the Theorem in terms of vector fields. For any Radon measure $\mu := f\mathcal{L}^n$ with $f = \left(\frac{1}{2} - \|\cdot\|_\infty\right)^q$, with $q \in (-\infty, 1]$ let

$$L^p_R(Q_1(0),\mu) := \{V \in L^p(Q_1(0),\mu) \text{ vector field s.t. } *V^\flat \in \Omega^{n-1}_{p,R}(Q_1(0))\}$$

and let

$$L^p_{\mathbb{Z}}(Q_1(0),\mu) := \{V \in L^p(Q_1(0),\mu) \text{ vector field s.t. } *V^\flat \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))\}.$$

We will sometime write $L^p_{\mathbb{Z}}(Q_1(0))$ for $L^p_{\mathbb{Z}}(Q_1(0),\mathcal{L}^n)$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Theorem 3.2.1. *Let $V \in L^p_{\mathbb{Z}}(Q^n_1(0),\mu)$. The following facts hold:*

1. *if $q \in [0,1]$ and $p \in \big[1, n/(n-1)\big)$, then there exists a sequence $\{V_k\}_{k\in\mathbb{N}} \subset L^p_R(Q_1(0),\mu)$ such that $V_k \to V$ strongly in $L^p(Q^n_1(0),\mu)$;*
2. *if $q \in (-\infty, 0]$ and $p \in \big[n/(n-1), +\infty\big)$, then* $\operatorname{div}(V) = 0$ *distributionally on $Q^n_1(0)$.*

The case $n = 1$ is particularly easy and is treated in Lemma 3.2.6. For the proof in the case $n \geqslant 2$ we follow the ideas of [61] and [22]. We present here a plan of the proof, reducing to the case $q = 0$ for simplicity and without losing generality.

First of all we show that for any $\varepsilon > 0$ it is possible to decompose $Q_1(0)$ into cubes Q of size ε (plus a negligible rest) so that

$$\int_{\partial Q} V \cdot \nu_{\partial Q} \, d\mathscr{H}^{n-1} \in \mathbb{Z}$$

and so that the number of cubes where the integral is different from zero is controlled (Section 3.2.1). We will then show that V can be approximated on the boundaries of the small cubes Q by smooth vector fields $(V_\varepsilon)_{\varepsilon>0}$ with similar properties (Section 3.2.2). In Section 3.2.3 we show that the vector fields V_ε can be extended inside the cubes Q in such a way that the extension $\tilde{V}_\varepsilon$ has a finite number of singularities in Q (more precisely $\tilde{V}_\varepsilon|_Q \in L^p_R(Q)$) and is close to V in $L^p(Q)$. In Section 3.2.4 we will combine the previous elements to show that the approximating fields constructed above (up to some shifting and smoothing) satisfy the claim of the Theorem.

3.2.1. Choice of a suitable cubic decomposition

Fix any $\varepsilon \in (0, 1/4)$ and $a \in Q_\varepsilon(0)$. Let

$$\begin{aligned} q_\varepsilon &:= \max\{q \in \mathbb{N} \text{ s.t. } \varepsilon q \leqslant 1-\varepsilon\}, \\ C_\varepsilon &:= \left\{ \left(j + \frac{1}{2}\right)\varepsilon - \frac{1}{2}, \text{ with } j = 1, ..., q_\varepsilon - 1 \right\}^n, \\ \mathscr{C}_{\varepsilon,a} &:= \big\{ Q_\varepsilon(x) + a, \text{ with } x \in C_\varepsilon \big\}. \end{aligned}$$

We say that $\mathscr{C}_{\varepsilon,a}$ is the *cubic decomposition* of $Q_1(0)$ with origin in a and mesh thickness ε.

Let

$$\begin{aligned} \mathscr{F}_{\varepsilon,a} &:= \big\{ F \,|\, F \text{ is an } (n-1)\text{-dimensional face of } \partial Q, \text{ for some open cube } Q \in \mathscr{C}_{\varepsilon,a} \big\}, \\ S_{\varepsilon,a} &:= \bigcup_{F \in \mathscr{F}_{\varepsilon,a}} F. \end{aligned}$$

We say that $S_{\varepsilon,a}$ is the $(n-1)$-*skeleton* of the cubic decomposition $\mathscr{C}_{\varepsilon,a}$.

Lemma 3.2.1 (Choice of the cubic decomposition)**.** *Let $n \in \mathbb{N}_{>0}$. Let $V \in L^p_{\mathbb{Z}}(Q_1(0),\mu)$ where $\mu := f\mathcal{L}^n$ with*

$$f := \left(\frac{1}{2} - \|\cdot\|_\infty\right)^q$$

for some $q \in (-\infty, 1]$. Then, there exists a subset $E_V \subset (0,1)$ satisfying the following properties:

1. *$\mathcal{L}^1\big((0,1) \setminus E_V\big) = 0$;*

2. for every $\varepsilon \in E_V$, there exists $a_\varepsilon \in Q_\varepsilon(0)$ such that $\varepsilon \in R_{V,c_Q}$ for every $Q \in \mathscr{C}_{\varepsilon,a_\varepsilon}$ and

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \varepsilon \Big(\sum_{Q \in \mathscr{C}_{\varepsilon,a_\varepsilon}} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \Big) = 0, \tag{3.2.1}$$

where $(V)_Q = \fint_Q V d\mathcal{L}^n$.

Proof. For any $x \in Q_1(0)$ let $R_{V,x} := R_{V^\flat,x}$ (see Definition 3.1.2). By assumption

$$\int_0^1 \int_{Q_{1-\rho}(0)} \mathbb{1}_{\rho \in R_{V,x}} \, d\mathcal{L}^n \, d\rho = \int_{Q_1(0)} \mathcal{L}^1(R_{V,x}) \, d\mathcal{L}^n = \int_{Q_1(0)} 2 \operatorname{dist}(x, \partial Q_1(0)) \, d\mathcal{L}^n \tag{3.2.2}$$
$$= \int_0^1 \mathcal{L}^n(Q_{1-\rho}(0)) \, d\rho.$$

For any $\rho \in (0,1)$ let

$$X_\rho = \{x \in Q_{1-\rho}(0) : \rho \notin R_{V,x}\},$$

then by (3.2.2) $\mathcal{L}^n(X_\rho) = 0$ for a.e. $\rho \in (0,1)$. Now notice that for any $\rho \in (0,1)$

$$\mathcal{L}^n(X_\rho) \geqslant \sum_{c \in C_\rho} \int_{Q_\rho(0)} \mathbb{1}_{X_\rho}(x+c) \, d\mathcal{L}^n = \int_{Q_\rho(0)} \sum_{c \in C_\rho} \mathbb{1}_{X_\rho}(x+c) \, d\mathcal{L}^n,$$

thus for a.e. $\rho \in (0,1)$ we have that for a.e. $a_\rho \in Q_\rho(0)$ there holds $\rho \in R_{V,c_Q}$ for any $Q \in \mathscr{C}_{\rho,a_\rho}$. Let E_V be the set of all such $\rho \in (0,1)$.
Now let $\varepsilon \in E_V$. We claim that

$$I_\varepsilon := \int_{Q_\varepsilon(0)} \sum_{Q \in \mathscr{C}_{\varepsilon,a}} \int_{\partial Q} |V(x) - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1}(x) \, d\mathcal{L}^n(a) = o\big(\varepsilon^{n-1}\big), \tag{3.2.3}$$

as $\varepsilon \to 0^+$ in E_V. Indeed, let $\mathscr{F}$ be the set of the faces of the cube $Q_1(0) \subset \mathbb{R}^n$ and notice that

$$\begin{aligned} I_\varepsilon &= \int_{Q_\varepsilon(0)} \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \Big| V(x+c+a) - \fint_{Q_\varepsilon(c+a)} V \Big|^p f(c+a) \, d\mathscr{H}^{n-1}(x) \, d\mathcal{L}^n(a) \\ &= \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(0)} \Big| V(x+c+a) - \fint_{Q_\varepsilon(c+a)} V \Big|^p f(c+a) \, d\mathcal{L}^n(a) \, d\mathscr{H}^{n-1}(x) \\ &= \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(c)} \Big| V(x+y) - \fint_{Q_\varepsilon(y)} V \Big|^p f(y) \, d\mathcal{L}^n(y) \, d\mathscr{H}^{n-1}(x). \end{aligned}$$

Observe that for any $c \in C_\varepsilon$, $x \in \partial Q_\varepsilon(0)$

$$\begin{aligned} &\int_{Q_\varepsilon(c)} \Big| V(x+y) - \fint_{Q_\varepsilon(y)} V(z) \Big|^p f(y) d\mathcal{L}^n(y) \\ &\leqslant \fint_{Q_\varepsilon(0)} \int_{Q_\varepsilon(c)} |V(x+y) - V(z+y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z). \end{aligned}$$

Thus for any $F_0 \in \mathscr{F}$

$$\begin{aligned}
&\sum_{c\in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(c)} \left| V(x+y) - \fint_{Q_\varepsilon(y)} V \right|^p f(y)\, d\mathcal{L}^n(y)\, d\mathscr{H}^{n-1}(x) \\
\leqslant &\int_{\varepsilon F_0} \fint_{Q_\varepsilon(0)} \int_{Q_{1-2\varepsilon}(0)} |V(x+y) - V(z+y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z) d\mathscr{H}^{n-1}(x) \\
\leqslant &\, 2^{p-1} \int_{\varepsilon F_0} \int_{Q_{1-2\varepsilon}(0)} |V(x+y) - V(y)|^p f(y) d\mathcal{L}^n(y) d\mathscr{H}^{n-1}(x) \\
&+ 2^{p-1} \varepsilon^{n-1} \fint_{Q_\varepsilon(0)} \int_{Q_{1-2\varepsilon}(0)} |V(z+y) - V(y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z) \\
\leqslant &\, 2^p \varepsilon^{n-1} \sup_{\alpha\in Q_\varepsilon(0)} \|V - V(\cdot - \alpha)\|^p_{L^p(Q_{1-2\varepsilon}, \mu)}.
\end{aligned}$$

Since $C^0_c(Q_1(0))$ is dense in $L^p(Q_1(0), \mu)$ (see [56, Theorem 4.3]), given any $\delta > 0$ we can find $\tilde{V} \in C^0_c(Q_1(0))$ such that

$$\int_{Q_1(0)} |V - \tilde{V}|^p \, d\mu \leqslant \delta.$$

Notice that by Taylor's Theorem

$$\begin{aligned}
\left| \frac{f(x+\alpha) - f(x)}{f(x)} \right| &= \left| \frac{(\frac{1}{2} - \|x+\alpha\|_\infty)^q - (\frac{1}{2} - \|x\|_\infty)^q}{(\frac{1}{2} - \|x\|)^q} \right| \leqslant q\|\alpha\|_\infty \frac{(\frac{1}{2} - \|x\|_\infty - \frac{\varepsilon}{2})^{q-1}}{(\frac{1}{2} - \|x\|_\infty)^q} \\
&\leqslant q\varepsilon 2^{1-q} \left(\frac{1}{2} - \|x\|_\infty \right)^{-1} \leqslant C
\end{aligned} \tag{3.2.4}$$

for every $x \in Q_{1-\varepsilon}(0)$, $\alpha \in Q_\varepsilon(0)$ and for some constant $C > 0$ depending only on q. Thus

$$\begin{aligned}
\|V - V(\cdot - \alpha)\|^p_{L^p(Q_{1-\varepsilon}(0),\mu)} &\leqslant 4^{p-1} \bigg(\int_{Q_{1-\varepsilon}(0)} |V - \tilde{V}|^p \, d\mu + \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - \tilde{V}(\cdot - \alpha)|^p \, d\mu \\
&\quad + \int_{Q_{1-\varepsilon}(0)} |\tilde{V}(\cdot - \alpha) - V(\cdot - \alpha)|^p \, d\mu \bigg) \\
&= 4^{p-1} \bigg(2 \int_{Q_1(0)} |V - \tilde{V}|^p \, d\mu + \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - \tilde{V}(\cdot - \alpha)|^p \, d\mu \\
&\quad + \int_{Q_{1-\varepsilon}(-\alpha)} |\tilde{V} - V|^p \, \frac{f(\cdot + \alpha) - f}{f} \, d\mu \bigg) \\
&\leqslant 4^{p-1}(2 + C)\delta + 4^{p-1} \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - \tilde{V}(\cdot - \alpha)|^p \, d\mu.
\end{aligned}$$

As $\tilde{V} \in C^0_c(Q_1(0))$

$$\sup_{\alpha\in Q_\varepsilon(0)} \|\tilde{V} - \tilde{V}(\cdot - \alpha)\|^p_{L^p(Q_{1-\varepsilon}(0),\mu)} \to 0 \text{ as } \varepsilon \to 0^+,$$

we have

$$\limsup_{\varepsilon\to 0^+} \sup_{\alpha\in Q_\varepsilon(0)} \|V - V(\cdot - \alpha)\|^p_{L^p(Q_{1-\varepsilon}(0),\mu)} \leqslant 4^{p-1}(2 + C)\delta.$$

By letting $\delta \to 0^+$ in the previous inequality we get

$$\sup_{\alpha \in Q_\varepsilon(0)} \|V - V(\cdot - \alpha)\|^p_{L^p(Q_{1-\varepsilon}(0),\mu)} \to 0 \text{ as } \varepsilon \to 0^+, \tag{3.2.5}$$

and claim (3.2.3) follows.

By Fubini's theorem, for every fixed $\varepsilon \in E_V$ there exists some non-negligible subset $T_\varepsilon \subset Q_\varepsilon(0)$ such that for any $a \in T_\varepsilon$ $\varepsilon \in R_{V,c_Q}$ for any $Q \in \mathscr{C}_{\varepsilon,a}$ and

$$\begin{aligned}\sum_{Q \in \mathscr{C}_{\varepsilon,a}} \int_{\partial Q} |V - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} &\leqslant \frac{1}{\varepsilon^n} \int_{Q_\varepsilon(0)} \sum_{Q \in \mathscr{C}_{\varepsilon,a}} \int_{\partial Q} |V - (V)_Q|^p\, f(c_Q) d\mathscr{H}^{n-1}\, d\mathcal{L}^n(a) \\ &= \frac{1}{\varepsilon^n} I_\varepsilon.\end{aligned}$$

By (3.2.3), for every $a \in T_\varepsilon$ we have

$$\sum_{Q \in \mathscr{C}_{\varepsilon,a}} \int_{\partial Q} |V - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} = o(\varepsilon^{-1})$$

as $\varepsilon \to 0^+$ in E_V. The statement follows. □

Fix any $V \in L^p_{\mathbb{Z}}(Q_1(0))$ and $\varepsilon \in E_V$. From now on, we will denote simply by $\mathscr{C}_\varepsilon$ the cubic decomposition $\mathscr{C}_{\varepsilon,a_\varepsilon}$ provided by Lemma 3.2.1. Accordingly, the subscript "a_ε" will be omitted in any writing referring to such a cubic decomposition.

Given any $Q \in \mathscr{C}_\varepsilon$, we say that Q is a *good cube* if

$$\int_{\partial Q} V \cdot \nu_{\partial Q} = 0$$

and that Q is a *bad cube* otherwise. We denote by $\mathscr{C}^g_\varepsilon$ the subfamily of $\mathscr{C}_\varepsilon$ made of all the good cubes and by $\mathscr{C}^b_\varepsilon$ the one made of all the bad cubes. Moreover, we let

$$\begin{aligned}\Omega_\varepsilon &:= \bigcup_{Q \in \mathscr{C}_\varepsilon} Q, \quad \Omega^g_\varepsilon := \bigcup_{Q \in \mathscr{C}^g_\varepsilon} Q, \quad \Omega^b_\varepsilon := \bigcup_{Q \in \mathscr{C}^b_\varepsilon} Q, \\ S_\varepsilon &:= \bigcup_{Q \in \mathscr{C}_\varepsilon} \partial Q, \quad S^g_\varepsilon := \bigcup_{Q \in \mathscr{C}^g_\varepsilon} \partial Q, \quad S^b_\varepsilon := \bigcup_{Q \in \mathscr{C}^b_\varepsilon} \partial Q.\end{aligned}$$

Lemma 3.2.2. *Assume that $n \geqslant 2$. Then, we have*

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \varepsilon^n \sum_{Q \in \mathscr{C}_\varepsilon} f(c_Q) = 0.$$

In particular if $q = 0$ (and thus $\mu = \mathcal{L}^n$) we have

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \mathcal{L}^n(\Omega^b_\varepsilon) = 0.$$

Proof. Notice that by estimate (3.2.4)

$$\frac{|f - f(c_Q)|}{f} \leqslant C \qquad \text{on } Q \text{ for every } Q \in \mathscr{C}_\varepsilon \tag{3.2.6}$$

for some universal constant $C > 0$. For every bad cube $Q \in \mathscr{C}_\varepsilon^b$, it holds that

$$1 \leqslant \left| \int_{\partial Q} V \cdot \nu_{\partial Q} \, d\mathscr{H}^{n-1} \right| \leqslant \int_{\partial Q} |V| \, d\mathscr{H}^{n-1}.$$

By multiplying the previous inequality by $f(c_Q)$ and summing over all the bad cubes, we get

$$\begin{aligned} \sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) &\leqslant \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V| f(c_Q) \, d\mathscr{H}^{n-1} \\ &\leqslant \Big(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{p}} \Big(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} f(c_Q) \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{p'}} \\ &= (2n)^{\frac{1}{p'}} \varepsilon^{\frac{n-1}{p'}} \Big(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{p}} \Big(\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \Big)^{\frac{1}{p'}}, \end{aligned}$$

which is equivalent to

$$\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \leqslant (2n)^{p-1} \varepsilon^{(p-1)(n-1)} \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1}.$$

Hence, by the triangle inequality, we get

$$\begin{aligned} \sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) &\leqslant (4n)^{p-1} \varepsilon^{(p-1)(n-1)-1} \Big(\varepsilon \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \\ &\quad + 2n \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_Q |V|^p f(c_Q) \, d\mathcal{L}^n \Big) \\ &\leqslant (4n)^{p-1} \varepsilon^{(p-1)(n-1)-1} \Big(\varepsilon \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \\ &\quad + 2n \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_Q |V|^p f \, d\mathcal{L}^n + 2n \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_Q |V|^p \frac{f(c_Q) - f}{f} f \, d\mathcal{L}^n \Big) \\ &\leqslant (4n)^{p-1} \varepsilon^{(p-1)(n-1)-1} \Big(\varepsilon \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \\ &\quad + 2n(1 + C) \int_{\Omega_\varepsilon^b} |V|^p f \, d\mathcal{L}^n \Big). \end{aligned} \tag{3.2.7}$$

Therefore

$$\begin{aligned} \varepsilon^n \sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \leqslant (4n)^{p-1} \varepsilon^{p(n-1)} \Big(&\varepsilon \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \\ &+ 2n(1 + C) \int_{Q_1(0)} |V|^p f \, d\mathcal{L}^n \Big) \end{aligned}$$

and the statement follows from (3.2.1) (here we need the assumption $n > 1$). □

Remark 3.2.1. Assume that $p \in \big[n/(n-1), +\infty\big)$. In this case $\varepsilon^{(p-1)(n-1)-1}$ remains bounded as $\varepsilon \to 0^+$. Now by Lemma 3.2.1

$$\varepsilon \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} |V - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

Moreover, by Lemma 3.2.2, we have

$$\int_{\Omega_\varepsilon^b} f\, d\mathcal{L}^n = (1+C)\varepsilon^n \sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

This implies $\mathcal{L}^n(\Omega_\varepsilon^b) \to 0$ as $\varepsilon \to 0^+$ in E_V, therefore

$$\int_{\Omega_\varepsilon^b} |V|^p f\, d\mathcal{L}^n \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V$$

by absolute continuity of the integral. Thus it follows from (3.2.7) that

$$\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \to 0^+ \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

Let N_ε^b be the number of bad cubes in $\mathscr{C}_\varepsilon$. Notice that for $q \leqslant 0$, we have $f \geqslant 2^{-q}$ on $Q_1(0)$. This implies

$$N_\varepsilon^b \leqslant 2^q \sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \to 0^+ \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

Since $N_\varepsilon^b \in \mathbb{Z}$ for any $\varepsilon \in E_V$, $N_\varepsilon^b = 0$ for every $\varepsilon \in E_V$ small enough. Hence, whenever $p \in \big[n/(n-1), +\infty\big)$ and $q \leqslant 0$ we will assume, without losing generality, that there are no bad cubes in our chosen decomposition.

3.2.2. Smoothing on the $(n-1)$-skeleton of the cubic decomposition

Lemma 3.2.3. *Let $Q \subset \mathbb{R}^n$ be an n-dimensional cube with side length R. Let $V \in L^p(Q, \mathbb{R}^m)$. Let $\varepsilon > 0$. There exists $V_\varepsilon \in C_c^\infty(Q, \mathbb{R}^m)$ such that*

$$\int_Q V_\varepsilon\, d\mathcal{L}^n = \int_Q V\, d\mathcal{L}^n$$

and

$$\|V_\varepsilon - V\|_{L^p(Q)} < \varepsilon.$$

Proof. Without loss of generality, we will assume that Q is centered in the origin of $\mathbb{R}^n$. Let $\psi \in C_c^\infty(\frac{1}{2}Q)$ and $r_0 \in (1/2, 1)$ such that

$$\int_Q \psi\, d\mathcal{L}^n = 1 \quad \text{and } R^n(1 - r_0^n) < \frac{\varepsilon}{\|\psi\|_{L^p(Q)}}.$$

Let $r \in (r_0, 1)$ be such that

$$\|V\|_{L^p(Q\smallsetminus rQ)} \leqslant \min\left\{\left(\frac{\varepsilon}{\|\psi\|_{L^p(Q)}}\right)^{\frac{1}{p}}, \varepsilon\right\}.$$

Set

$$s := \int_{Q\smallsetminus rQ} V\, d\mathcal{L}^n, \quad \tilde{V} := \chi_{rQ}V + s\psi \in L^p(Q, \mathbb{R}^m).$$

Then

$$\int_Q \tilde{V}\, d\mathcal{L}^n = \int_{rQ} V\, d\mathcal{L}^n + \left(\int_{Q\smallsetminus rQ} V\, d\mathcal{L}^n\right)\int_Q \psi\, d\mathcal{L}^n = \int_Q V\, d\mathcal{L}^n.$$

Moreover

$$\begin{aligned}|s| &= \left|\int_{Q\smallsetminus rQ} V\, d\mathcal{L}^n\right| \leqslant |Q\smallsetminus rQ|^{\frac{1}{p'}}\|V\|_{L^p(Q\smallsetminus rQ)} \\ &\leqslant (R^n(1-r^n))^{\frac{1}{p'}}\left(\frac{\varepsilon}{\|\psi\|_{L^p(Q)}}\right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{\|\psi\|_{L^p(Q)}}.\end{aligned}$$

Therefore

$$\|s\psi\|_{L^p(Q)} = \|\psi\|_{L^p(Q)}|s| \leqslant \varepsilon$$

and, by choice of $\tilde{V}$,

$$\|V-\tilde{V}\|_{L^p(Q)} \leqslant \|s\psi\|_{L^p(Q)} + \|V\|_{L^p(Q\smallsetminus rQ)} \leqslant 2\varepsilon.$$

Notice that $\tilde{V}\big|_{Q\smallsetminus rQ} \equiv 0$.
Let $\eta \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \eta\, d\mathcal{L}^n = 1$. For any $\delta > 0$ let

$$\eta_\delta(x) := \frac{1}{\delta^n}\eta\left(\frac{x}{\delta}\right) \quad \forall\, x \in \mathbb{R}^n.$$

Choose $\delta_0 > 0$ such that

$$2\delta_0 < \operatorname{dist}\big(\partial Q, \partial(rQ)\big) \quad \text{and} \quad \|\tilde{V} - \tilde{V} * \eta_{\delta_0}\|_{L^p(Q)} \leqslant \varepsilon.$$

Set $V_\varepsilon := \tilde{V} * \eta_{\delta_0}$. Then $V_\varepsilon \in C_c^\infty(Q)$,

$$\int_Q V_\varepsilon\, d\mathcal{L}^n = \int_{\mathbb{R}^n} \eta_{\delta_0}\, d\mathcal{L}^n \int_Q \tilde{V}\, d\mathcal{L}^n = \int_Q \tilde{V}\, d\mathcal{L}^n = \int_Q V\, d\mathcal{L}^n$$

and

$$\|V - V_\varepsilon\|_{L^p(Q)} \leqslant \|V - \tilde{V}\|_{L^p(Q)} + \|\tilde{V} - V_\varepsilon\|_{L^p(Q)} \leqslant 3\varepsilon.$$

□

3.2.3. Extensions on good and bad cubes

Lemma 3.2.4 (Extension on the good cubes)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and $p \in [1,\infty)$. Let $f \in L^p(\partial\Omega)$ with*

$$\int_{\partial\Omega} f \, d\mathscr{H}^{n-1} = 0 \tag{3.2.8}$$

There exists a vector field $V \in L^p(\Omega)$ such that

$$\int_{\Omega} V \cdot \nabla\varphi \, d\mathcal{L}^n = \int_{\partial\Omega} f\varphi \, d\mathscr{H}^{n-1} \qquad \forall\, \varphi \in C^\infty(\mathbb{R}^n) \tag{3.2.9}$$

and

$$\int_{\Omega} |V|^p \, d\mathcal{L}^n \leqslant C(p,\Omega) \int_{\partial\Omega} |f|^p \, d\mathscr{H}^{n-1} \tag{3.2.10}$$

for some constant $C(p,\Omega)$ depending only on p and Ω.
Moreover if $p=1$, then $V \in L^q(\Omega)$ for any $q \in \left[1, \frac{n}{n-1}\right)$.

Remark 3.2.2. Observe that (3.2.9) implies that V is a distributional solution of the following Neumann problem

$$\begin{cases} \operatorname{div}(V) = 0 & \text{in } \Omega \\ V \cdot \nu_{\partial Q} = f & \text{on } \partial\Omega. \end{cases}$$

Proof.
Step 1: First we consider the case $p \in (1,\infty)$.
Let $p' := \dfrac{p}{p-1}$. For any $u \in W^{1,p'}(\Omega)$ let

$$E_p(u) = \frac{1}{p'} \int_{\Omega} |\nabla u|^{p'} \, d\mathcal{L}^n - \int_{\partial\Omega} fu \, d\mathscr{H}^{n-1}.$$

Recall that any function $u \in W^{1,p'}(\Omega)$ has a trace in $L^{p'}(\partial\Omega)$, and that the trace operator is continuous. Thus for any $v \in W^{1,p'}(\Omega)$ with $\displaystyle\int_{\Omega} v = 0$ by Poincaré Lemma there holds

$$\left| \int_{\partial\Omega} fv \, d\mathscr{H}^{n-1} \right| \leqslant \|f\|_{L^p(\partial\Omega)} \|v\|_{L^{p'}(\partial\Omega)} \leqslant C(p,\Omega) \|f\|_{L^p(\partial\Omega)} \|\nabla v\|_{L^{p'}(\Omega)}$$

for some constant $C(p,\Omega)$ depending only on p and Ω. In particular the energy E_p is well defined on $W^{1,p'}(\Omega)$.
Let

$$\dot{W}^{1,p'}(\Omega) := \left\{ v \in W^{1,p'}(\Omega), \int_{\Omega} v \, d\mathcal{L}^n = 0 \right\}$$

and observe that E_p is strictly convex on $\dot{W}^{1,p'}(\Omega)$. Let u be the unique minimizer of E_p in $\dot{W}^{1,p'}(\Omega)$. Then[3]

$$\int_{\Omega} |\nabla u|^{p'-2} \nabla u \cdot \nabla\varphi \, d\mathcal{L}^n = \int_{\partial\Omega} f\varphi \, d\mathscr{H}^{n-1}, \qquad \forall \varphi \in C^\infty(\mathbb{R}^n). \tag{3.2.11}$$

[3]The argument above shows that (3.2.11) holds for any $\varphi \in C^\infty(\mathbb{R}^n)$ with $\int_\Omega \varphi d\mathcal{L}^n = 0$, but assumption (3.2.8) implies that (3.2.11) remains valid for any $\varphi \in C^\infty(\mathbb{R}^n)$.

Moreover, as u is a minimizer of E_p, $E_p(u) \leqslant E_p(0) = 0$. It follows that

$$\frac{1}{p'}\int_\Omega |\nabla u|^{p'}\, d\mathcal{L}^n \leqslant \int_{\partial\Omega} fu\, d\mathscr{H}^{n-1} \leqslant \|f\|_{L^p(\partial\Omega)}\|u\|_{L^{p'}(\partial\Omega)} \leqslant C(p,\Omega)\|f\|_{L^p(\partial\Omega)}\|\nabla u\|_{L^{p'}(\Omega)}.$$

Thus

$$\int_\Omega |\nabla u|^{p'}\, d\mathcal{L}^n \leqslant \big(p'C(p,\Omega)\big)^p \int_{\partial\Omega} |f|^p\, d\mathscr{H}^{n-1}.$$

Set $V := |\nabla u|^{p'-2}\nabla u \quad \text{in } \Omega$. Then by (3.2.11)

$$\int_\Omega V\cdot\nabla\varphi\, d\mathcal{L}^n = \int_{\partial\Omega} f\varphi\, d\mathscr{H}^{n-1} \quad \forall\varphi\in C^\infty(\mathbb{R}^n)$$

and

$$\int_\Omega |V|^p\, d\mathcal{L}^n = \int_\Omega |\nabla u|^{p'}\, d\mathcal{L}^n \leqslant \big(p'C(p,\Omega)\big)^p \int_{\partial\Omega} |f|^p\, d\mathscr{H}^{n-1}.$$

Step 2: Next we consider the case $p = 1$.
Let $s > n$. For any $u \in W^{1,s}(\Omega)$ let

$$E_s(u) = \frac{1}{s}\int_\Omega |\nabla u|^s\, d\mathcal{L}^n - \int_{\partial\Omega} fu\, d\mathscr{H}^{n-1}.$$

Notice that E_s is well defined and strictly convex in $\dot{W}^{1,s}(\Omega)$.
Recall the Sobolev embedding

$$W^{1,s}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$$

for $\alpha = 1-\frac{n}{s}$. Then for any $u \in W^{1,s}(\Omega)$ the trace of u on $\partial\Omega$ lies in $C^{0,\alpha}(\partial\Omega)$ and if $\int_\Omega u\, d\mathcal{L}^n = 0$. Poincaré inequality implies

$$\|u\|_{L^\infty(\partial\Omega)} \leqslant C(s,\Omega)\|\nabla u\|_{L^s(\Omega)}$$

for some constant $C(s,\Omega)$ depending only on s and Ω.
Let u be the unique minimizer of E_s in $\dot{W}^{1,s}(\Omega)$. Then since $E_s(u) \leqslant E_s(0) = 0$ there holds

$$\begin{aligned}\frac{1}{s}\int_\Omega |\nabla u|^s\, d\mathcal{L}^n &\leqslant \int_{\partial\Omega} fu\, d\mathscr{H}^{n-1}\\ &\leqslant \|f\|_{L^1(\partial\Omega)}\|u\|_{L^\infty(\partial\Omega)} \leqslant C(s,\Omega)\|f\|_{L^1(\partial\Omega)}\|\nabla u\|_{L^s(\Omega)}.\end{aligned}$$

Therefore

$$\left(\int_\Omega |\nabla u|^s\, d\mathcal{L}^n\right)^{\frac{s-1}{s}} \leqslant sC(s,\Omega)\int_{\partial\Omega} |f|\, d\mathscr{H}^{n-1}.$$

Moreover, since u is a minimizer of E_s,

$$\int_\Omega |\nabla u|^{s-2}\nabla u\cdot\nabla\varphi\, d\mathcal{L}^n = \int_{\partial\Omega} f\varphi\, d\mathscr{H}^{n-1} \quad \forall\varphi\in C^\infty(\mathbb{R}^n).$$

Similarly as before set $V := |\nabla u|^{s-2}\nabla u \quad \text{in } \Omega$. Then

$$\int_\Omega |V|\, d\mathcal{L}^n = \int_\Omega |\nabla u|^{s-1}\, d\mathcal{L}^n \leqslant \mathcal{L}^n(\Omega)^{\frac{1}{s}} \left(\int_\Omega |\nabla u|^s\, d\mathcal{L}^n\right)^{\frac{s-1}{s}}$$
$$\leqslant sC(s,\Omega)\mathcal{L}^n(\Omega)^{\frac{1}{s}} \int_{\partial\Omega} |f|\, d\mathscr{H}^{n-1}.$$

Moreover

$$\int_\Omega |V|^{\frac{s}{s-1}}\, d\mathcal{L}^n = \int_\Omega |\nabla u|^s\, d\mathcal{L}^n < \infty$$

□

Remark 3.2.3. Let $Q \subset \mathbb{R}^n$ be the unit cube and let $C(p,Q)$ be the corresponding constant in (3.2.10). By an easy scaling argument one sees that for any $\varepsilon > 0$ one can choose $C(p,\varepsilon Q) = \varepsilon C(p,Q)$.

Lemma 3.2.5 (Extension on the bad cubes). *Let $Q := Q_\varepsilon(c_Q) \subset \mathbb{R}^n$ and $r(x) := \|x - c_Q\|_\infty$, for every $x \in \mathbb{R}^n$.*
Consider any vector field $V : Q \to \mathbb{R}^n$ having the form

$$V(x) := \frac{1}{2^{n-1}} f\left(\frac{\varepsilon}{2}\frac{x-c_Q}{r(x)} + c_Q\right)\frac{x-c_Q}{r(x)^n} \qquad \forall\, x \in Q,$$

for some $f \in L^\infty(\partial Q)$. Then, the following facts hold:

1. *$V \in L^p(Q)$ for every $p \in \big[1, n/(n-1)\big)$;*
2. *for some constant $C(n,p) > 0$ depending only on n and p we have*

$$\int_Q |V|^p\, d\mathcal{L}^n \leqslant \varepsilon C(n,p) \int_{\partial Q} |f|^p\, d\mathscr{H}^{n-1}, \tag{3.2.12}$$

3. *for every $\varphi \in C^\infty(\mathbb{R}^n)$ we have*

$$\int_Q V\cdot\nabla\varphi\, d\mathcal{L}^n = \int_{\partial Q} f\varphi\, d\mathscr{H}^{n-1} - \left(\int_{\partial Q} f\, d\mathscr{H}^{n-1}\right)\varphi(c_Q). \tag{3.2.13}$$

Proof. Without losing generality, we assume that $\varepsilon = 1$ and $c_Q = 0$. First, notice that $r : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz map such that $|\nabla r(x)| = 1$, for a.e. $x \in \mathbb{R}^n$. Moreover, since all the norms are equivalent on $\mathbb{R}^n$ there exists a constant $\tilde{C}(n) > 0$ depending only on n such that $|x| \leqslant \tilde{C} r(x)$, for a.e. $x \in \mathbb{R}^n$. Now choose any $p \in \big[1, n/(n-1)\big)$. By coarea formula we have

$$\int_Q |V|^p\, d\mathcal{L}^n \leqslant \frac{\tilde{C}^p}{2^{(n-1)p}} \int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)p}} \int_{\partial Q_{2\rho}(0)} \left| f\left(\frac{x}{2\rho}\right)\right|^p d\mathscr{H}^{n-1}(x)\, d\rho$$
$$= \frac{\tilde{C}^p}{2^{(n-1)(p-1)}} \left(\int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)(p-1)}}\, d\rho\right)\left(\int_{\partial Q} |f(y)|^p\, d\mathscr{H}^{n-1}(y)\right)$$
$$= C\int_{\partial Q} |f|^p\, d\mathscr{H}^{n-1},$$

with

$$C = C(n,p) := \frac{\tilde{C}^p}{2^{(n-1)(p-1)}} \int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)(p-1)}}\, d\rho < +\infty.$$

Hence, 1. and 2. follow in once. We remark that the condition $p \in \big[1, n/(n-1)\big)$ is needed in order to guarantee the convergence of the integral in ρ.
We still need to prove 3. Pick any $\varphi \in C^\infty(\mathbb{R}^n)$. By the coarea formula we have

$$\begin{aligned}
\int_Q V \cdot \nabla\varphi \, d\mathcal{L}^n &= \frac{1}{2^{n-1}} \int_0^{\frac{1}{2}} \frac{1}{\rho^n} \int_{\partial Q_{2\rho}(0)} f\left(\frac{x}{2\rho}\right) \big(x \cdot \nabla\varphi(x)\big)\, d\mathscr{H}^{n-1}(x)\, d\rho \\
&= 2 \int_0^{\frac{1}{2}} \int_{\partial Q} f(y) \big(y \cdot \nabla\varphi(2\rho y)\big)\, d\mathscr{H}^{n-1}(y)\, d\rho \\
&= \int_{\partial Q} f(y) \int_0^{\frac{1}{2}} \frac{d}{d\rho}\big(\varphi(2\rho y)\big)\, d\rho\, d\mathscr{H}^{n-1}(y) \\
&= \int_{\partial Q} f\varphi\, d\mathscr{H}^{n-1} - \left(\int_{\partial Q} f\, d\mathscr{H}^{n-1}\right)\varphi(0)
\end{aligned}$$

and 3. follows. □

3.2.4. Proof of Theorem 3.2.1

We are finally ready to prove Theorem 3.2.1.

Proof. Let $V \in L^p_{\mathbb{Z}}(Q_1(0))$ and let $\varepsilon \in E_V$ (constructed in Lemma 3.2.1). First, we notice that by using Lemma 3.2.3 separately on every face $F \in \mathscr{F}_\varepsilon$ we can build a vector field $V_\varepsilon \in C^\infty(S_\varepsilon)$ such that

$$\int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} = \int_{\partial Q} V \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} \qquad \forall\, Q \in \mathscr{C}_\varepsilon$$

and

$$\sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} |V_\varepsilon - V|^p f(c_Q) d\mathscr{H}^{n-1} < \varepsilon.$$

Let $\tilde{V}_\varepsilon$ be the vector field defined $\mathcal{L}^n$-a.e. on Ω_ε as follows:

1. if $Q \in \mathscr{C}_\varepsilon$ is a good cube, then we let $\tilde{V}_\varepsilon := W_\varepsilon + (V)_Q$ on Q, where W_ε is the extension of the datum $f := \big(V_\varepsilon - (V)_Q\big) \cdot \nu_{\partial Q}$ given by Lemma 3.2.4 (notice that for any good cube condition (3.2.8) is satisfied by our choice of f);
2. if $Q \in \mathscr{C}_\varepsilon$ is a bad cube, then we let

$$\tilde{V}_\varepsilon := \frac{1}{2^{n-1}} f\left(\frac{\varepsilon}{2} \frac{x - c_Q}{\|x - c_Q\|_\infty} + c_Q\right) \frac{x - c_Q}{\|x - c_Q\|_\infty^n}, \qquad \forall\, x \in Q,$$

with $f := V_\varepsilon\big|_{\partial Q} \cdot \nu_{\partial Q} \in L^\infty(\partial Q)$.

We recall that no bad cubes will appear in the cubic decomposition in case $p \in \big[n/(n-1), +\infty\big)$ (see Remark 3.2.1).

Claim 1. We claim that

$$\operatorname{div}(\tilde{V}_\varepsilon) = \sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q \delta_{c_Q} \qquad \text{distributionally on } \Omega_\varepsilon,$$

where

$$d_Q := \int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q} \, d\mathscr{H}^{n-1} = \int_{\partial Q} V \cdot \nu_{\partial Q} \, d\mathscr{H}^{n-1} \in \mathbb{Z} \smallsetminus \{0\}, \qquad \forall \, Q \in \mathscr{C}_\varepsilon^b.$$

Indeed, pick any $\varphi \in C_c^\infty(\Omega_\varepsilon)$. Let $Q \in \mathscr{C}_\varepsilon$ be a good cube. By the properties of the extension given by Lemma 3.2.4 and the divergence theorem we have

$$\begin{aligned}\int_Q \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n &= \int_{\partial Q} \big(V_\varepsilon - (V)_Q\big) \cdot \nu_{\partial Q} \varphi \, d\mathscr{H}^{n-1} + \int_{\partial Q} \big((V)_Q \cdot \nu_{\partial Q}\big) \varphi \, d\mathscr{H}^{n-1} \\ &= \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, d\mathscr{H}^{n-1}.\end{aligned}$$

On the other hand, let $Q \in \mathscr{C}_\varepsilon$ be a bad cube. By (3.2.13), we have

$$\int_Q \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, d\mathscr{H}^{n-1} - d_Q \langle \delta_{c_Q}, \varphi \rangle.$$

Hence we conclude that

$$\begin{aligned}\int_{\Omega_\varepsilon} \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n &= \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, d\mathscr{H}^{n-1} - \sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q \langle \delta_{c_Q}, \varphi \rangle \\ &= - \sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q \langle \delta_{c_Q}, \varphi \rangle.\end{aligned}$$

The claim follows.

Claim 2. We claim that $\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_\varepsilon, \mu)} \to 0$ as $\varepsilon \to 0^+$ in E_V.
Recall estimate (3.2.6) and notice that

$$\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_\varepsilon, \mu)}^p \leqslant (1 + C)(A_\varepsilon + B_\varepsilon),$$

with

$$\begin{aligned}A_\varepsilon &:= \sum_{Q \in \mathscr{C}_\varepsilon^g} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q) \, d\mathcal{L}^n, \\ B_\varepsilon &:= \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q) \, d\mathcal{L}^n.\end{aligned}$$

By triangle inequality and by the estimate in Lemma 3.2.4, we have that

$$A_\varepsilon \leqslant 2^{p-1} \left(\sum_{Q \in \mathscr{C}_\varepsilon^g} \int_Q |\tilde{V}_\varepsilon - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n + \sum_{Q \in \mathscr{C}_\varepsilon^g} \int_Q |V - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n \right)$$

$$\leqslant 2^{p-1}\left(\sum_{Q\in\mathscr{C}_\varepsilon^g}\int_Q |W_\varepsilon|^p f(c_Q)\,d\mathcal{L}^n + \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_Q |V-(V)_Q|^p f(c_Q)\,d\mathcal{L}^n\right)$$

$$\leqslant 2^{p-1}\left(\varepsilon C_p \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_{\partial Q} |V_\varepsilon-(V)_Q|^p\, f(c_Q) d\mathscr{H}^{n-1} + \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_Q |V-(V)_Q|^p f(c_Q)\,d\mathcal{L}^n\right),$$

where $C_p := C(p,Q)$ (see Remark 3.2.3). Again by triangle inequality and because of our choice of V_ε, we have

$$\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_{\partial Q} |V_\varepsilon-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1} \leqslant 2^{p-1}\Bigg(\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_{\partial Q} |V_\varepsilon-V|^p f(c_Q)\,d\mathscr{H}^{n-1}$$
$$+\,\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_{\partial Q} |V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\Bigg)$$
$$\leqslant C\left(\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g}\int_{\partial Q} |V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\right).$$

Thus by Lemma 3.2.1 it follows that

$$\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon}\int_{\partial Q} |V_\varepsilon-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1} \to 0 \qquad \text{as } \varepsilon\to 0^+ \text{ in } E_V.$$

Moreover, by (3.2.6) we have

$$\sum_{Q\in\mathscr{C}_\varepsilon^g}\int_Q |V-(V)_Q|^p f(c_Q)\,d\mathcal{L}^n \leqslant 2^p(1+C)\sum_{Q\in\mathscr{C}_\varepsilon^g} \fint_{Q_\varepsilon(0)}\int_Q |V(x+y)-V(y)|^p f(y)\,d\mathcal{L}^n$$
$$\leqslant 2^p(1+C)\fint_{Q_\varepsilon(0)} \|V(x+\cdot\,)-V\|^p_{L^p(\Omega_\varepsilon,\mu)} \to 0$$

as $\varepsilon\to 0^+$ in E_V. Hence, $A_\varepsilon \to 0$ as $\varepsilon\to 0^+$ in E_V.
On the other hand, by (3.2.12) we have

$$B_\varepsilon \leqslant 2^{p-1}\left(\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |\tilde V_\varepsilon|^p f(c_Q)\,d\mathcal{L}^n + \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |V|^p f(c_Q)\,d\mathcal{L}^n\right)$$
$$\leqslant 2^{p-1}\left(C\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q} |V_\varepsilon|^p f(c_Q)\,d\mathscr{H}^{n-1} + \int_{\Omega_\varepsilon^b} |V|^p f(c_Q)\,d\mathcal{L}^n\right).$$

We notice that

$$\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q} |V_\varepsilon|^p f(c_Q)\,d\mathscr{H}^{n-1} \leqslant 2^{p-1}\Bigg(\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q} |V_\varepsilon-V|^p f(c_Q)\,d\mathscr{H}^{n-1}$$
$$+\,\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q} |V|^p f(c_Q)\,d\mathscr{H}^{n-1}\Bigg)$$
$$\leqslant 4^{p-1}\Bigg(2n\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q} |V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}$$

$$+\,\varepsilon \sum_{Q\in\mathscr{C}^b_\varepsilon} \int_{\partial Q} |(V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1}\Bigg).$$

Moreover, by (3.2.6) we have

$$\begin{aligned}
\varepsilon \sum_{Q\in\mathscr{C}^b_\varepsilon} \int_{\partial Q} |(V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} &\leqslant \varepsilon \sum_{Q\in\mathscr{C}^b_\varepsilon} \int_{\partial Q} \left(\fint_Q |V|^p f(c_Q)\, d\mathcal{L}^n \right) d\mathscr{H}^{n-1} \\
&\leqslant \sum_{Q\in\mathscr{C}^b_\varepsilon} \varepsilon \left(\fint_Q |V|^p f(c_Q)\, d\mathcal{L}^n \right) \int_{\partial Q} d\mathscr{H}^{n-1} \\
&\leqslant 2n \sum_{Q\in\mathscr{C}^b_\varepsilon} \int_Q |V|^p f(c_Q)\, d\mathcal{L}^n = 2n \int_{\Omega^b_\varepsilon} |V|^p f(c_Q)\, d\mathcal{L}^n \\
&\leqslant (1+C) 2n \int_{\Omega^b_\varepsilon} |V|^p f\, d\mathcal{L}^n.
\end{aligned}$$

Thus, we have obtained

$$B_\varepsilon \leqslant C\left(\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}^b_\varepsilon} \int_{\partial Q} |V-(V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} + \int_{\Omega^b_\varepsilon} |V|^p f\, d\mathcal{L}^n \right),$$

for some constant $C > 0$ which does not depend on $\varepsilon \in E_V$. By Lemma 3.2.1 and Remark 3.2.1 we get that $B_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in E_V. Hence, the claim follows.

Next we show that by rescaling V_ε we obtain a vector field with similar properties defined on the whole $Q_1(0)$. Let

$$\alpha_\varepsilon := \sup\{\alpha \in [1/2, 1) \text{ s.t. } Q_1(0) \subset \alpha^{-1}\Omega_\varepsilon\}, \qquad \forall\, \varepsilon \in E_V.$$

Notice that $\alpha_\varepsilon \to 1^-$ as $\varepsilon \to 0^+$ in E_V. Define the vector field $\overline{V}_\varepsilon := \alpha_\varepsilon^{n-1}\tilde{V}_\varepsilon(\alpha_\varepsilon\,\cdot) : Q_1(0) \to \mathbb{R}^n$. It's straightforward that $\overline{V}_\varepsilon \in L^p(Q_1(0), \mu)$ in case $p > 1$ and $\overline{V}_\varepsilon \in L^s(Q_1(0), \mu)$ for some $s > 1$ in case $p = 1$, for every given $\varepsilon \in E_V$. A direct computation also shows that the distributional divergence of $\overline{V}_\varepsilon$ on $Q_1(0)$ is given by

$$\operatorname{div}(\overline{V}_\varepsilon) = \sum_{Q\in\mathscr{C}^b_\varepsilon} \overline{d}_Q \delta_{\alpha_\varepsilon^{-1} c_Q},$$

with

$$\overline{d}_{Q'} = \begin{cases} d_{Q'} & \text{if } \alpha_\varepsilon^{-1} c_{Q'} \in Q_1(0), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\overline{V}_\varepsilon \to V$ in $L^p(Q_1(0), \mu)$. Indeed, we have

$$\begin{aligned}
\int_{Q_1(0)} |\overline{V}_\varepsilon - V|^p f\, d\mathcal{L}^n &= \int_{Q_1(0)} |\alpha_\varepsilon^{n-1}\tilde{V}_\varepsilon(\alpha_\varepsilon x) - V(x)|^p\, f(x) d\mathcal{L}^n(x) \\
&= \alpha_\varepsilon^{p(n-1)-n} \int_{\alpha_\varepsilon Q_1(0)} |\tilde{V}_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(\alpha_\varepsilon^{-1} y)\, d\mathcal{L}^n(y) \\
&= \alpha_\varepsilon^{p(n-1)-n} \Bigg(\int_{\Omega_\varepsilon} |\tilde{V}_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(y)\, d\mathcal{L}^n(y)
\end{aligned}$$

$$+\int_{\Omega_\varepsilon}|\tilde{V}_\varepsilon(y)-\alpha_\varepsilon^{-(n-1)}V(\alpha_\varepsilon^{-1}y)|^p\frac{f(\alpha_\varepsilon^{-1}y)-f(y)}{f(y)}f(y)\,d\mathcal{L}^n(y)\Bigg)$$
$$\leqslant C_{n,p}\bigg(\int_{\Omega_\varepsilon}|\tilde{V}_\varepsilon-V|^p f\,d\mathcal{L}^n+\int_{\Omega_\varepsilon}|V-P_{\alpha_\varepsilon^{-1}}V|^p f\,d\mathcal{L}^n\bigg),$$

(see Lemma 3.C.2 for the definition of $P_{\alpha_\varepsilon^{-1}}V$). By Lemma 3.C.2 and since $\tilde{V}_\varepsilon\to V$ in $L^p(\Omega_\varepsilon,\mu)$, our claim follows.

Thus, we have built a vector field $\overline{V}_\varepsilon$ such that:

1. $\overline{V}_\varepsilon\in L^p(Q_1(0),\mu)$ and $\overline{V}_\varepsilon\in L^s(Q_1(0),\mathcal{L}^n)$ for $s=p$ if $p>1$ and $s>1$ if $p=1$[4];
2. the distributional divergence of $\overline{V}_\varepsilon$ on $Q_1(0)$ is given by a finite sum of delta distributions supported on a finite set $X_\varepsilon\subset Q_1(0)$ with integer weights $\{d_x \text{ s.t. } x\in X_\varepsilon\}$;
3. $\|\overline{V}_\varepsilon-V\|_{L^p(Q_1(0),\mu)}\to 0$ as $\varepsilon\to 0^+$ in E_V.

Now we are ready to reach the conclusions 1 and 2 of Theorem 3.2.1.

1. If $q\in[0,1]$ and $p\in\big[1,\frac{n}{n-1}\big)$ we possibly have $X_\varepsilon\neq\emptyset$, since bad cubes can appear in the cubic decompositions. Since $\overline{V}_\varepsilon$ always belongs to $L^s(Q_1(0))$ for some $s>1$ (with $s=p$ if p itself is already greater than 1), we can Hodge-decompose $\overline{V}_\varepsilon^\flat$ as $\overline{V}_\varepsilon^\flat=d\varphi+d^*A$ for some $A\in\Omega^2_{W^{1,s}}(Q_1(0))$ and some $\varphi\in W^{1,s}(Q_1(0))$. Applying d^* to the previous decomposition we obtain
$$\Delta\varphi=d^*(\overline{V}_\varepsilon^\flat)=\operatorname{div}(\overline{V}_\varepsilon)=\sum_{x\in X_\varepsilon}d_x\delta_x.$$
By standard elliptic regularity, $\varphi\in C^\infty(Q_1(0)\smallsetminus X_\varepsilon)$. Choose $A_\varepsilon\in\Omega^2(Q_1(0))$ such that $\|A_\varepsilon-A\|_{\Omega^2_{W^{1,s}}(Q_1(0))}\leqslant\varepsilon$. Then $\|d^*A_\varepsilon-d^*A\|_{\Omega^1_{L^p(\mu)}(Q_1(0))}\leqslant\varepsilon$. Let $v_\varepsilon:=d\varphi+d^*(A_\varepsilon)$ and let $U_\varepsilon:=v_\varepsilon^\#$. Then $U_\varepsilon\in L^p_R(Q_1(0),\mu)$ for every $\varepsilon\in E_V$ and $U_\varepsilon\to V$ in $L^p(Q_1(0),\mu)$.
2. If $q\in(-\infty,0]$ and $p\in\big[\frac{n}{n-1},+\infty\big)$ no bad cubes are allowed in the cubic decomposition, thus $(\overline{V}_\varepsilon)_{\varepsilon\in E_V}$ is a sequence of divergence-free vector fields converging to V in $L^p(Q_1(0),\mu)$ as $\varepsilon\to 0^+$ in E_V. Hence V itself is divergence-free.

□

Remark 3.2.4. Notice that if $p=1$, $V_k\in L^s(Q_1^n(0))$ for any $k\in\mathbb{N}$ and for any $s\in\big[1,\frac{n}{n-1}\big)$.

Remark 3.2.5. Observe that this proof can be used to show that the analogous approximation result holds if we assume that V satisfies the first three conditions of Definition 3.1.2 and in addition we require that for every $\rho\in R_{F,x_0}$ we have that
$$\int_{\partial Q_\rho(x_0)}i^*_{\partial Q_\rho(x_0)}F\in S$$
for a set $S\subset\mathbb{R}$ such that $0\in S$ and 0 is an isolated point in S. In this case the vector field V can be approximated in L^p by a sequence of vector fields $(V_n)_{n\in\mathbb{N}}$ smooth outside a finite set of points and such that for any $n\in\mathbb{N}$, $\operatorname{div}(V_n)$ is a finite sum of deltas with coefficients in S.

[4] In fact even when μ is different from $\mathcal{L}^n$, $\tilde{V}_\varepsilon$ is constructed through Lemmata 3.2.4 and 3.2.5 as extension of a smooth boundary datum, thus $\tilde{V}_\varepsilon$ lies in $L^r(\Omega_\varepsilon)$ for any $r\in[1,\frac{n}{n-1})$ if $p<n$ and in $L^r(\Omega_\varepsilon)$ for any $r\in[1,\infty)$ if $p\geqslant\frac{n}{n-1}$. It follows that $\overline{V}_\varepsilon\in L^r(\Omega_\varepsilon)$ for any $r\in[1,\frac{n}{n-1})$ if $p<n$ and $\overline{V}_\varepsilon\in L^r(\Omega_\varepsilon)$ for any $r\in[1,\infty)$ if $p\geqslant\frac{n}{n-1}$.

Lemma 3.2.6. *Let $I \subset \mathbb{R}$ be a connected interval and $p \in [1, +\infty)$. Then*

$$\overline{L^p_R(I)}^{L^p} = L^p(I, \mathbb{Z}) + \mathbb{R} = L^p_{\mathbb{Z}}(I).$$

Proof. We start by showing the first equality. Notice that

$$L^p_R(I) = \left\{ V = c + \sum_{j \in J} a_j \chi_{I_j} \,:\, (I_j)_{j \in J} \text{ is a finite partition of } I, a_j \in \mathbb{Z} \, \forall \, j \in J,\, c \in [0,1) \right\}.$$

In other words, $L^p_R(I)$ consists of all integer-valued step functions and their translations by a constant.
First we show the inclusion " $\supset$ ": let $f = g + a$ with $g \in L^p(I, \mathbb{Z})$ and $a \in [0,1)$ and let $\varepsilon > 0$. For any $k \in \mathbb{N}$ set $g_k := \mathbb{1}_{|g| \leqslant k} g$. Then there exists $K \in \mathbb{N}$ such that $\|g_K - g\|_{L^p(I)} < \frac{\varepsilon}{2}$.
Now for any $j \in \{-K, ..., K\}$ $g_K^{-1}(j) = g^{-1}(j)$ is a measurable set, therefore there exists a finite collection of disjoint intervals $(I^j_i)_{i \in J^j}$ such that $\mathcal{L}\left(\bigcup_{i \in J^j} I^j_i \triangle g^{-1}(j)\right) \leqslant \frac{\varepsilon}{2(2K+1)^2}$.
For any $j \in \{-K, ..., K\}$ set $A_j := \bigcup_{i \in J^j} I^j_i \smallsetminus \left(\bigcup_{j' \neq j} \bigcup_{i \in J^{j'}} I^{j'}_i\right)$. Then A_j is a finite union of intervals and

$$\mathcal{L}(A_j \triangle g^{-1}(j)) \leqslant \mathcal{L}(A_j \smallsetminus g^{-1}(j)) + \sum_{j' \neq j} \mathcal{L}\left(\bigcup_{i \in J^j} I^{j'}_i \cap g^{-1}(j)\right) \leqslant \frac{\varepsilon}{2(2K+1)}.$$

Set

$$\tilde{g}_K = \begin{cases} j & \text{if } x \in A_j \\ 0 & \text{otherwise} \end{cases}$$

Then by construction $\tilde{g}_K \in L^p_R(I)$ and $\|\tilde{g}_K - g\|_{L^p(I)} \leqslant \varepsilon$. We conclude that any $g \in L^p(I, \mathbb{Z})$ lies in the closure of $L^p_R(I)$. This shows " $\supset$ ". As $L^p(I, \mathbb{Z}) + \mathbb{R} = L^p(I, \mathbb{Z}) + [0,1)$ is closed in $L^p(I)$, the inclusion "$\subset$" holds as well.
Next we show the second equality. Let's start with "$\subset$". Let $g \in L^p(I, \mathbb{Z})$, $a \in \mathbb{R}$ and $f = g + a$. Let $x_0 \in I$, then for a.e. $r \in (0, \text{dist}(x_0, \partial I))$ we have $f(x_0 + r) - f(x_0 - r) = g(x_0 + r) - g(x_0 - r) \in \mathbb{Z}$. Let $\tilde{R}_{F, x_0}$ denote the set of all such r. Set R_{R, x_0} to be the intersection of $\tilde{R}_{F, x_0}$ with the set of Lebesgue points of f. Then f satisfies Definition 3.1.2 and thus $f \in L^p_{\mathbb{Z}}(I)$.
To show "$\supset$" let $f \in L^p_{\mathbb{Z}}(I)$. Set $F : I \to \mathbb{S}^1$, $x \mapsto e^{i 2\pi f(x)}$. Then F is a measurable bounded function. Notice that for any $x_0 \in I$, for a.e. $r \in (0, \text{dist}(x_0, \partial I))$ there holds $F(x_0 - r) = F(x_0 + r)$. This implies that F is constant: this can be seen for instance approximating F by smooth functions with the same symmetry properties away from ∂I (convolving with a symmetric mollifier with small support), which then have to be constant. Choose $a \in \mathbb{R}$ such that $F \equiv e^{i 2\pi a}$, then $f - a \in L^p(I, \mathbb{Z})$. This completes the proof. □

Remark 3.2.6. From Theorem 3.2.1 it follows immediately that

$$\overline{L^p_R(Q_1(0))}^{L^p} = L^p_{\mathbb{Z}}(Q_1(0)).$$

To see this it is enough to check that $L^p_{\mathbb{Z}}(Q_1(0))$ is closed in L^p, which can be shown by simple application of the coarea formula.

Notice that in case $p \in [n/(n-1), +\infty)$ we can approximate V strongly in L^p with smooth and divergence free vector fields. This is a straightforward consequence of Hodge decomposition.

3.2.5. A characterization of $\Omega^{n-1}_{p,\mathbb{Z}}$

First of all we apply the Strong approximation Theorem to obtain a useful characterization of the class $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.

Theorem 3.2.2. *Let $n \in \mathbb{N}_{>0}$, $p \in [1, +\infty)$. Let $F \in \Omega^{n-1}_p(Q^n_1(0))$. Then, the following are equivalent:*

1. *there exists $L \in \mathcal{R}_1(Q_1(0))$ such that $\partial L = *dF$ in $(W^{1,\infty}_0(Q_1(0)))^*$ and*

$$\mathbb{M}(L) = \sup_{\substack{\varphi\in\mathcal{D}(Q_1(0)),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_{Q_1(0)} F \wedge d\varphi.$$

2. *for every Lipschitz function $f : \overline{Q_1(0)} \to [a, b] \subset \mathbb{R}$ such that $f|_{\partial Q_1(0)} \equiv b$, we have*

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b];$$

3. *$F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.*

Proof. We just need to show that $1 \Rightarrow 2$, $2 \Rightarrow 3$ and $3 \Rightarrow 1$. We prove these implications separately.

$1 \Rightarrow 2$. Assume 1. Let $L \in \mathcal{R}_1(Q_1(0))$ be given by

$$\langle L, \omega\rangle = \int_\Gamma \theta\langle\omega, \vec{L}\rangle\, d\mathscr{H}^1, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)),$$

where $\Gamma \subset Q_1(0)$ is a locally 1-rectifiable set, $\vec{L}$ is a Borel measurable unitary vector field on Γ and $\theta \in L^1(\Gamma, \mathscr{H}^1)$ is a $\mathbb{Z}$-valued function. Pick any Lipschitz function $f : Q_1(0) \to \mathbb{R}$ and let $\varphi \in C^\infty_c((-\infty, b))$ be such that $\int_\mathbb{R} \varphi\, d\mathcal{L}^1 = 0$. By the coarea formula we have

$$\int_{Q_1(0)} F \wedge f^*(\varphi \operatorname{vol}_\mathbb{R}) = \int_{-\infty}^{+\infty} \varphi(t)\bigg(\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F\bigg)\, dt.$$

At the same time, by the coarea formula for countably 1-rectifiable sets, we have

$$\langle L, f^*(\varphi \operatorname{vol}_\mathbb{R})\rangle = \int_\Gamma \theta\langle f^*(\varphi \operatorname{vol}_\mathbb{R}), \vec{L}\rangle\, d\mathscr{H}^1 = \int_{-\infty}^{+\infty} \varphi(t)\bigg(\int_{\Gamma\cap f^{-1}(t)} \tilde{\theta}\bigg)\, dt,$$

where $\tilde{\theta} : \Gamma \to \mathbb{Z}$ is given by $\tilde{\theta} := \operatorname{sgn}(\langle f^* \operatorname{vol}_\mathbb{R}, \vec{L}\rangle)\theta$. Let $\Phi \in C^\infty_c((-\infty, b))$ satisfy $d\Phi = \varphi \operatorname{vol}_\mathbb{R}$. Notice that since f is proper $f^*\Phi \in W^{1,\infty}_0(Q_1(0))$. Then, by hypothesis, we have

$$\begin{aligned}\int_{Q_1(0)} F \wedge f^*(\varphi \operatorname{vol}_\mathbb{R}) &= \int_{Q_1(0)} F \wedge df^*\Phi = \langle *dF, f^*\Phi\rangle = \langle \partial L, f^*\Phi\rangle \\ &= \langle L, d(f^*\Phi)\rangle = \langle L, f^*(d\Phi)\rangle = \langle L, f^*(\varphi \operatorname{vol}_\mathbb{R})\rangle.\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \varphi(t)\bigg(\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F - \int_{\Gamma\cap f^{-1}(t)} \tilde{\theta}\bigg)\, dt = 0, \qquad \forall\, \varphi \in C^\infty_c((-\infty, b)) \text{ s.t. } \int_\mathbb{R} \varphi = 0.$$

We conclude that

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F - \int_{\Gamma\cap f^{-1}(t)} \tilde{\theta} = c, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a,b],$$

for some constant $c \in \mathbb{R}$. We claim that $c = 0$. In fact let $m \in \mathbb{N} \smallsetminus \{0\}$. Integrating both sides on $(-m, m)$ we get

$$\int_{\{|f|<m\}} F \wedge f^* \operatorname{vol}_{\mathbb{R}} - \int_{\Gamma\cap\{|f|<m\}} \theta\langle f^* \operatorname{vol}_{\mathbb{R}}, \vec{L}\rangle = 2mc. \tag{3.2.14}$$

Since $f^* \operatorname{vol}_{\mathbb{R}} = df$, we have

$$\int_{Q_1(0)} F \wedge f^* \operatorname{vol}_{\mathbb{R}} - \int_{\Gamma} \theta\langle f^* \operatorname{vol}_{\mathbb{R}}, \vec{L}\rangle = \int_{Q_1(0)} F \wedge df - \langle L, df\rangle = \langle *dF - \partial L, f\rangle = 0.$$

Thus, by letting $m \to +\infty$ in (3.2.14), we get that the left-hand-side converges to 0 whilst the right-hand-side diverges to $+\infty$, unless $c = 0$. Hence we conclude that $c = 0$, i.e.

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F = \int_{\Gamma\cap f^{-1}(t)} \tilde{\theta} \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a,b],$$

since $\Gamma \cap f^{-1}(t)$ consists of finitely many points for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.
$2 \Rightarrow 3$. Assume 2. Given $x_0 \in Q_1(0)$, let $f_{x_0} := \min\left\{\|\cdot - x_0\|_\infty, \frac{r_{x_0}}{2}\right\}$. We claim that we can find R_{F,x_0} as in Definition 3.1.2. Indeed, let $L \subset Q_1(0)$ be the set of the Lebesgue points of F. Let $r_{x_0} := 2\operatorname{dist}_\infty(x_0, \partial Q_1(0))$. Then, by the coarea formula, we have

$$0 = \mathcal{L}^n\big(Q_{r_{x_0}}(x_0) \smallsetminus L\big) = \frac{1}{2^n}\int_0^{r_{x_0}} \mathscr{H}^{n-1}\big((Q_1(0) \smallsetminus L) \cap \partial Q_\rho(x_0)\big)\, d\rho,$$

which implies that there exists a set $E_{x_0} \subset (0, r_{x_0})$ such that

1. $\mathcal{L}^1\big((0, r_{x_0}) \smallsetminus E_{x_0}\big) = 0$;
2. for every $\rho \in E_{x_0}$, $\mathscr{H}^{n-1}$-a.e. $x \in \partial Q_\rho(x_0)$ is a Lebesgue point for F.

Hence, for every $\rho \in E_{x_0}$ it makes sense to consider the pointwise restriction of F to $\partial Q_\rho(x_0)$. Notice that, by the coarea formula, we have

$$\int_{E_{x_0}} \left(\int_{\partial Q_\rho(x_0)} \big|F\big|^p\, d\mathscr{H}^{n-1}\right) d\rho = 2^n \int_{Q_{r_{x_0}}(x_0)} |F|^p\, d\mathcal{L}^n < +\infty,$$

which implies

$$\int_{\partial Q_\rho(x_0)} \big|F\big|^p\, d\mathscr{H}^{n-1} < +\infty, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).$$

Moreover, by Statement 2., we have

$$\int_{f_{x_0}^{-1}(\rho)} i^*_{f_{x_0}^{-1}(\rho)} F = \int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).$$

Our claim follows immediately.

$3 \Rightarrow 1$. Assume 3. By Theorem 3.1.1, we can find a sequence $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \to F$ strongly in L^p. Since the estimate

$$|\langle T_{F_k} - T_F, \omega\rangle| \leqslant C||F_k - F||_{L^p}$$

holds for every $\omega \in \mathcal{D}^1(Q_1(0))$ such that $||\omega||_{L^\infty} \leqslant 1$ and for every $k \in \mathbb{N}$, we conclude that

$$\sup_{\substack{\varphi\in W^{1,\infty}_0(Q_1(0)),\\ \|d\varphi\|_{L^\infty}\leqslant 1}} \langle *dF_k - *dF, \varphi\rangle \leqslant C||F_k - F||_{L^p} \to 0 \quad \text{as } k \to \infty. \tag{3.2.15}$$

Fix any $\varepsilon \in (0,1)$. By (3.2.15) we can find a subsequence $\left\{F_{k_j(\varepsilon)}\right\}_{j\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that

$$\|*dF_{k_j(\varepsilon)} - *dF_{k_{j+1}(\varepsilon)}\|_{(W^{1,\infty}_0(Q_1(0)))^*} \leqslant \frac{\varepsilon}{2^j}, \qquad \text{for every } j \in \mathbb{N}.$$

For every $h \in \mathbb{N}$, let L^ε_h be a minimal connection for the singular set of $F_{k_h(\varepsilon)}$ (the existence of such a minimal connection is proved in Proposition 3.A.1. Analogously, for every $j \in \mathbb{N}$, let $L^\varepsilon_{j,j+1}$ be the minimal connection for the singular set of $F_{k_j(\varepsilon)} - F_{k_{j+1}(\varepsilon)}$. Define the following sequence of integer 1-currents on $Q_1(0)$:

$$\tilde{L}^\varepsilon_h := \begin{cases} L^\varepsilon_0 & \text{if } h = 0, \\ L^\varepsilon_0 - \displaystyle\sum_{j=0}^{h-1} L^\varepsilon_{j,j+1} & \text{if } h > 0, \end{cases} \qquad \text{for every } h \in \mathbb{N}.$$

Clearly

$$\partial\tilde{L}^\varepsilon_h = \partial L^\varepsilon_0 - \sum_{j=0}^{h-1} \partial L^\varepsilon_{j,j+1} = \partial L^\varepsilon_0 - \sum_{j=0}^{h-1}(\partial L^\varepsilon_j - \partial L^\varepsilon_{j+1}) = \partial L^\varepsilon_h = *dF_{k_h(\varepsilon)}.$$

Moreover, since $L^\varepsilon_{j,j+1}$ is a minimal connection, it holds that

$$\mathbb{M}(L^\varepsilon_{j,j+1}) = \|*dF_{k_j(\varepsilon)} - *dF_{k_{j+1}(\varepsilon)}\|_{(W^{1,\infty}_0(Q_1(0)))^*} \leqslant \frac{\varepsilon}{2^j}, \qquad \text{for every } j \in \mathbb{N}.$$

Thus,

$$\mathbb{M}(\tilde{L}^\varepsilon_{h+1} - \tilde{L}^\varepsilon_h) = \mathbb{M}(L^\varepsilon_{h,h+1}) \leqslant \frac{\varepsilon}{2^h}, \qquad \text{for every } h \in \mathbb{N},$$

which amounts to saying that the sequence $\{\tilde{L}^\varepsilon_h\}_{h\in\mathbb{N}}$ is a Cauchy sequence in mass. Hence, by the closure of integer currents under mass convergence (see Lemma 3.C.1), there exists an integer 1-current $\tilde{L}^\varepsilon \in \mathcal{R}_1(Q_1(0))$ such that

$$\mathbb{M}(\tilde{L}^\varepsilon_h - \tilde{L}^\varepsilon) \to 0 \quad \text{as } h \to \infty,$$

Notice that

$$\partial\tilde{L}^\varepsilon = \lim_{h\to\infty} \partial\tilde{L}^\varepsilon_h = \lim_{h\to\infty} *dF_{k_h(\varepsilon)} = *dF \quad \text{in } (W^{1,\infty}_0(Q_1(0)))^*.$$

The family of integer 1-cycles $\{\tilde{L}^\varepsilon - \tilde{L}^{1/2}\}_{0<\varepsilon<1} \subset \mathcal{R}_1(Q_1(0))$ is uniformly bounded in mass. Indeed, first we notice that by (2.10) it holds that

$$\mathbb{M}(L^\varepsilon_h) = \|*dF_{k_h(\varepsilon)}\|_{(W^{1,\infty}_0(Q_1(0)))^*} \leqslant C, \quad \forall\, h \in \mathbb{N}, \forall\, \varepsilon \in (0,1),$$

where $C > 0$ is a constant independent on h and ε. Thus, we have

$$\mathbb{M}(\tilde{L}_h^\varepsilon) \leqslant \mathbb{M}(L_0^\varepsilon) + \sum_{j=0}^{h-1} \mathbb{M}(L_{j,j+1}^\varepsilon) \leqslant C + \sum_{j=0}^{+\infty} \frac{\varepsilon}{2^j} \leqslant C + \varepsilon \leqslant C + 1 \quad \forall\, h \in \mathbb{N}, \forall\, \varepsilon \in (0,1).$$

Since $\tilde{L}_h^\varepsilon \to \tilde{L}^\varepsilon$ in mass as $h \to +\infty$ for every $\varepsilon \in (0,1)$, we have

$$\mathbb{M}(\tilde{L}^\varepsilon - \tilde{L}^{1/2}) \leqslant \mathbb{M}(\tilde{L}^\varepsilon) + \mathbb{M}(\tilde{L}^{1/2}) \leqslant 2(C+1).$$

Hence, by standard compactness arguments for currents (see for instance [53], Theorem 7.5.2), we can find a sequence $\varepsilon_k \to 0$ and an integer 1-cycle $\tilde{L} \in \mathcal{R}_1(Q_1(0))$ with finite mass such that $\tilde{L}^{\varepsilon_k} - \tilde{L}^{1/2} \to \tilde{L}$ weakly in $\mathcal{D}_1(Q_1(0))$ as $k \to +\infty$. If we let $L := \tilde{L}^{1/2} + \tilde{L}$ then we get $\tilde{L}^{\varepsilon_k} \to L$ weakly in $\mathcal{D}_1(Q_1(0))$. By construction, L is again an integer 1-current with finite mass such that $\partial L = \partial \tilde{L}^{1/2} = *dF$ in $(W^{1,\infty}(Q_1(0)))^*$. We claim that

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T).$$

By contradiction, assume that we can find $T \in \mathcal{M}_1(Q_1(0))$ such that $\partial T = *dF$ and

$$\mathbb{M}(T) < \mathbb{M}(L) \leqslant \liminf_{k \to \infty} \mathbb{M}(\tilde{L}^{\varepsilon_k}),$$

where the last inequality follows by weak convergence and lower semicontinuity of the mass. Then, we can find some $h \in \mathbb{N}$ such that

$$\mathbb{M}(T) < \mathbb{M}(\tilde{L}^{\varepsilon_h}) - 2\varepsilon_h.$$

Moreover, since $\mathbb{M}(L_0^\varepsilon - \tilde{L}^\varepsilon) \leqslant 2\varepsilon$ for every $0 < \varepsilon < 1$, it holds that

$$\mathbb{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) \leqslant 2\varepsilon_h.$$

We define $\tilde{T} := T + L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}$ and we notice that $\partial \tilde{T} = *dF_{k_0(\varepsilon_h)}$. Moreover, by the minimality of $L_0^{\varepsilon_h}$, we conclude that

$$\mathbb{M}(L_0^{\varepsilon_h}) \leqslant \mathbb{M}(\tilde{T}) \leqslant \mathbb{M}(T) + \mathbb{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) < \mathbb{M}(L_0^{\varepsilon_h}),$$

which is a contradiction. Thus, our claim follows.
Since $L \in \mathcal{R}_1(Q)$, we get that

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{R}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T)$$

and, by Lemma 3.A.2, we have

$$\inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in \mathcal{D}(Q_1(0)),\\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi.$$

Hence, 1. follows. $\square$

Remark 3.2.7. We notice that in the proof of Theorem 3.2.2 we have never used the minimality property of L while showing that $1 \Rightarrow 2$. Hence whenever $F \in \Omega_p^{n-1}(Q_1^n(0))$ admits a connection we have $F \in \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0))$ in the sense of Definition 3.1.2 and the conclusions of Theorem 3.1.1 hold for F.

By the previous remark we can deduce the following result from the proof of Theorem 3.2.2.

Corollary 3.2.1. *Let $F \in \Omega_p^{n-1}(Q_1^n(0))$ and assume that there exists an integer 1-current of finite mass $I \in \mathcal{R}_1(Q_1(0))$ such that $\partial I = *dF$. Then there exists an integer 1-current $L \in \mathcal{R}_1(Q_1(0))$ of finite mass such that $\partial L = *dF$ and*

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{R}_1(Q_1(0)) \\ \partial T = *dF}} \mathbb{M}(T).$$

In other words, whenever there exists a connection for F, then there exists a minimal connection for F.

3.2.6. The case of $\partial Q_1^{n+1}(0)$

In order to extend the previous result to more general manifolds we introduce the following definition.

Definition 3.2.1. Let M be a Lipschitz m-manifold embedded in $\mathbb{R}^n$. Let $p \in [1, \infty)$. Set

$$\Omega^1_{p,R,\infty}(M) := \left\{ \alpha \in \Omega^1_p(M) \cap \Omega^1_{L^\infty_{loc}}(M \setminus S) : *d\alpha = \sum_{i \in I} d_i \delta_{p_i} \right\},$$

where I is a finite index set, $d_i \in \mathbb{Z}$, $p_i \in S$ for any $i \in I$ and $F := \{p_i\}_{i \in I}$.

The previous definition is motivated by the following observation: let M be a Lipschitz m-manifold, N a smooth m-manifold, $\varphi : M \to N$ a bi-Lipschitz map. Let $F \in \Omega^1_{p,R}(N)$. Then $\varphi^* F \in \Omega^\infty_{p,R,\infty}$ (see Lemma 3.2.7).

Corollary 3.2.2. *Let $n \in \mathbb{N}_{\geqslant 2}$. Assume that $F \in \Omega_{p,\mathbb{Z}}^{n-2}(\partial Q_1^n(0))$ admits a connection. Then, the following facts hold:*

1. *if $p \in \big[1, (n-1)/(n-2)\big)$, then there exists a sequence $\{F_k\}_{k \in \mathbb{N}} \subset \Omega^{n-2}_{p,R,\infty}(\partial Q_1^n(0))$ such that $F_k \to F$ strongly in L^p;*
2. *if $p \in \big[(n-1)/(n-2), +\infty\big)$, then $*dF = 0$ distributionally on ∂Q^n.*

Proof. Let $N = (0, ..., 0, \frac{1}{2})$ be the north pole in $\partial Q_1^n(0) \subset \mathbb{R}^m$ and let

$$U := \big\{(x_1, ..., x_{n-1}, x_n) \in \partial Q_1^n(0) \text{ s.t. } x_n = \frac{1}{2}\big\}$$

be the upper face of $\partial Q_1^n(0)$. Let $q := (n-1) - (n-2)p$. For every $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we let $x' := (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$. Define $\Phi : \partial Q_1^n(0) \setminus N \subset \mathbb{R}^n \to Q_1^{n-1}(0)$ by

$$\Phi(x) := \begin{cases} \left(\dfrac{1}{2} - \dfrac{\sqrt{2}}{4} \|x'\|_\infty^{\frac{1}{2}}\right) \dfrac{x'}{\|x'\|_\infty} & \text{on } U \setminus N, \\ g(x) & \text{on } \partial Q_1^n(0) \setminus U, \end{cases}$$

where the map $g : \overline{\partial Q_1^n(0) \setminus U} \to \overline{Q_{1/2}^{n-1}(0)}$ is any bi-Lipschitz homeomorphism such that $g \equiv \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\|x'\|_\infty^{\frac{1}{2}}\right)\frac{x'}{\|x'\|_\infty}$ on ∂U. Notice that Φ is an homeomorphism from $\partial Q_1^n(0) \setminus N$ to $Q_1^{n-1}(0)$. We denote its inverse map by Ψ.
We have that Ψ is Lipschitz on $Q_1^{n-1}(0)$ and Φ is Lipschitz on every compact set $K \subset \partial Q_1^n(0) \setminus N$, since there exists $C > 0$ such that

$$
\begin{aligned}
|d\Phi(x)| &\leqslant \frac{C}{\|x-N\|_\infty^{\frac{1}{2}}}, \qquad \forall\, x \in \partial Q_1^n(0) \setminus N,\\
|d\Psi(y)| &\leqslant C\left(\frac{1}{2} - \|y\|_\infty\right), \qquad \forall\, y \in Q_1^{n-1}(0).
\end{aligned}
$$

Define $\tilde{F} := \Psi^* F$ and fix any $\varepsilon \in (0, 1/4)$. Notice that

$$
\begin{aligned}
\int_{Q_1^{n-1}(0)} \left(\frac{1}{2} - \|\cdot\|_\infty\right)^q |\tilde{F}|^p \, d\mathscr{H}^{n-1} &\leqslant C \int_{Q_1^{n-1}(0)} \left(\frac{1}{2} - \|\cdot\|_\infty\right)^{n-1} |F \circ \Psi|^p \, d\mathscr{H}^{n-1}\\
&\leqslant C \int_{\partial Q_1^n(0)} |F|^p \, d\mathscr{H}^{n-1} < +\infty.
\end{aligned}
$$

Moreover if $I \in \mathcal{R}_1(\partial Q_1^n(0))$ and $*dF = \partial I$, then $\Phi_* I \in \mathcal{R}_1(\partial Q_1^{n-1}(0))$ and $*d\tilde{F} = \partial \Phi_* I$ (see Lemma 3.2.7). This implies that $\tilde{F} \in \Omega_{p,\mathbb{Z}}^{n-2}(Q_1^{n-1}(0), \mu)$ with $\mu := (\frac{1}{2} - \|\cdot\|_\infty)^q \, \mathcal{L}^{n-1}$ in the sense of Definition 3.1.2.
Let's consider first the case $p \in \big[1, (n-1)/(n-2)\big)$. Notice that in this case $q > 0$.
By Theorem 3.2.1 with $f := (\frac{1}{2} - \|\cdot\|_\infty)^q$ there exists a $(n-2)$-form $\tilde{F}_\varepsilon \in \Omega_{p,R}^{n-2}(\Omega_{\varepsilon,a_\varepsilon})$ such that $\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(S_{\varepsilon,a_\varepsilon})} \leqslant \varepsilon$ and

$$
\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(\Omega_{\varepsilon,a_\varepsilon},\mu)}^p \to 0 \tag{3.2.16}
$$

as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$.
Define $F_\varepsilon := \Phi_{a_\varepsilon}^* \tilde{F}_\varepsilon$ on $\partial Q_1^n(0) \setminus U_\varepsilon$, with $\Phi_{a_\varepsilon} := \Phi + a_\varepsilon$ and $U_\varepsilon := \Phi_{a_\varepsilon}^{-1}(Q_1^{n-1}(0) \setminus \Omega_{\varepsilon,a_\varepsilon})$.
Notice that

$$
\begin{aligned}
\|F_\varepsilon - F\|_{L^p(\partial Q_1^n(0)\setminus U_\varepsilon)}^p &\leqslant C \int_{\partial Q_1^n(0)\setminus U_\varepsilon} \frac{1}{\|\cdot - N\|_\infty^{(n-2)p}} |\tilde{F}_\varepsilon \circ \Phi_{a_\varepsilon} - \tilde{F} \circ \Phi_{a_\varepsilon}|^p \, d\mathscr{H}^{n-1}\\
&\quad + C \int_{\partial Q_1^n(0)\setminus U_\varepsilon} \frac{1}{\|\cdot - N\|_\infty^{(n-2)p}} |\tilde{F} \circ \Phi_{a_\varepsilon} - \tilde{F} \circ \Phi|^p \, d\mathscr{H}^{n-1}\\
&\leqslant C\bigg(\int_{\Omega_{\varepsilon,a_\varepsilon}} |\tilde{F}_\varepsilon - \tilde{F}|^p f \, d\mathscr{H}^{n-1}\\
&\quad + \int_{Q_{1-\varepsilon}^{n-1}(0)} |\tilde{F}(\cdot - a_\varepsilon) - \tilde{F}|^p f \, d\mathscr{H}^{n-1}\bigg).
\end{aligned}
$$

The first term tends to zero as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$ by (3.2.16), while the second tends to zero as $\varepsilon \to 0^+$ by (3.2.5). Therefore we have

$$
\|F_\varepsilon - F\|_{L^p(\partial Q_1^n(0)\setminus U_\varepsilon)}^p \to 0
$$

as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$.

Now we notice that

$$\begin{aligned}\int_{\partial U_\varepsilon} i^*_{\partial U_\varepsilon} F_\varepsilon &= -\int_{\partial(\partial Q_1^n(0)\smallsetminus U_\varepsilon)} i^*_{\partial(\partial Q_1^n(0)\smallsetminus U_\varepsilon)} F_\varepsilon \\ &= \int_{\partial \Omega_{\varepsilon,a_\varepsilon}} i^*_{\partial\Omega_{\varepsilon,a_\varepsilon}} \tilde{F}_\varepsilon = \int_{\partial \Omega_{\varepsilon,a_\varepsilon}} i^*_{\partial\Omega_{\varepsilon,a_\varepsilon}} \tilde{F} =: b_\varepsilon \in \mathbb{Z}.\end{aligned}$$

If $b_\varepsilon = 0$, then we use Lemma 3.2.4 to extend F_ε inside U_ε. If $b_\varepsilon \neq 0$, then we use Lemma 3.2.5 to extend F_ε inside U_ε (notice that U_ε is an $(n-1)$-cube of side-length $8\varepsilon^2$ contained in U and centered at N, for ε sufficiently small). In both cases, the following estimate holds:

$$\begin{aligned}\int_{U_\varepsilon} |F_\varepsilon|^p \, d\mathscr{H}^{n-1} &\leqslant C\varepsilon^2 \int_{\partial U_\varepsilon} |F_\varepsilon|^p \, d\mathscr{H}^{n-2} \\ &\leqslant C\varepsilon^2 \left(\int_{\partial U_\varepsilon} |F - F_\varepsilon|^p d\mathscr{H}^{n-2} + \int_{\partial U_\varepsilon} |F|^p d\mathscr{H}^{n-2} \right).\end{aligned}$$

Notice that the first term on the right hand side tends to zero as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$, since $\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(S_{\varepsilon,a_\varepsilon})} \leqslant \varepsilon$ for any $\varepsilon \in E_{\tilde{F}}$. In order to control the second term, pick any $\delta > 0$ sufficiently small and notice that, by coarea formula, we have

$$\begin{aligned}\fint_0^\delta \varepsilon^2 \int_{\partial U_\varepsilon} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) &\leqslant \int_0^\delta \varepsilon \int_{Q^{n-1}_{8\varepsilon^2}(N)} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) \\ &\leqslant \frac{1}{16} \int_0^{8\delta^2} \int_{\partial Q^{n-1}_\zeta(N)} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\zeta) \\ &\leqslant \frac{2^{n-1}}{16} \int_{Q^{n-1}_{8\delta^2}(N)} |F|^p \, d\mathscr{H}^{n-1} \to 0^+\end{aligned}$$

as $\delta \to 0^+$. This implies that we can pick a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset E_{\tilde{F}}$ such that $\varepsilon_j \to 0^+$ and

$$\varepsilon_j^2 \int_{\partial U_{\varepsilon_j}} |F|^p \, d\mathscr{H}^{m-1} \to 0^+$$

as $j \to +\infty$. Thus

$$\|F_{\varepsilon_j} - F\|^p_{L^p(U_{\varepsilon_j})} \leqslant 2^{p-1} \Big(\|F_{\varepsilon_j}\|^p_{L^p(U_{\varepsilon_j})} + \|F\|^p_{L^p(U_{\varepsilon_j})} \Big) \to 0$$

as $j \to +\infty$, since $\mathscr{H}^{n-1}\big(U_{\varepsilon_j}\big) \to 0^+$ as $\varepsilon_j \to 0^+$. Hence, we conclude that

$$\|F_{\varepsilon_j} - F\|^p_{L^p(\partial Q_1^n(0))} \to 0$$

as $j \to +\infty$. Moreover, by construction we have that $*dF_{\varepsilon_j}$ is a finite sum of Dirac-deltas with integer coefficients, for any $j \in \mathbb{N}$. Thus arguing as in the final step of the proof of Theorem 3.2.1 (i.e. by Hodge decomposition), for any $j \in \mathbb{N}$ we can find $\hat{F}_{\varepsilon_j} \in \Omega^1_{p,R,\infty}(\partial Q_1^n(0))$ such that $\|F_{\varepsilon_j} - \hat{F}_{\varepsilon_j}\|_{L^p(\partial Q_1^n(0))} < \varepsilon_j$. The sequence $\{\hat{F}_j\}_{j\in\mathbb{N}}$ then has the desired properties. This concludes the proof in the case $p \in \big[1, (n-1)/(n-2)\big)$.

If $p \in \big[(n-1)/(n-2), +\infty\big)$ notice that $q \leqslant 0$. Therefore by Remark 3.2.1 we may assume, up to passing to a subsequence, that no bad cube appears in the construction of $\tilde{F}_\varepsilon$. Hence repeating the first part of the proof as in the previous case, we get $b_{\varepsilon_j} = 0$ and $*dF_{\varepsilon_j} = 0$ on $\partial Q_1^n(0)$ for any $j \in \mathbb{N}$. Thus we obtain that F can be approximated in $\Omega_p^{n-1}(\partial Q_1^n(0))$ by a sequence $(F_{\varepsilon_j})_{j\in\mathbb{N}}$ such that for any $j \in \mathbb{N}$ there holds $*dF_{\varepsilon_j} = 0$, and this property passes to the limit. □

Remark 3.2.8. Notice that if $p = 1$, for any $k \in \mathbb{N}$ we have that $F_k \in \Omega_q^{n-1}(\partial Q_1^n(0))$ for some $q > 1$ which does not depend on k (compare with Remark 3.2.4).

Lemma 3.2.7. *Let $M, N \subset \mathbb{R}^n$ be Lipschitz m-manifolds in $\mathbb{R}^n$. Let $\varphi : M \to N$ be a bi-Lipschitz map.*
Let $F \in \Omega_p^{m-1}(M)$ and assume that there exists a 1-rectifiable current $I \in \mathcal{R}_1(M)$ of finite mass such that $*dF = \partial I$ in $(W_0^{1,\infty}(M))^*$. Then $\varphi_* F \in \Omega_p^{m-1}(N)$ and $*d(\varphi_* F) = \partial \varphi_* I$.
If $(F_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega_{R,p,\infty}^{m-1}(M)$ and $F_k \to F$ in $\Omega_p^{m-1}(M)$ as $k \to \infty$, then $(\varphi_* F_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega_{p,R,\infty}^{m-1}(N)$ and $\varphi_* F_k \to \varphi_* F$ in $\Omega_p^{m-1}(N)$ as $k \to \infty$.
If in addition we assume that N is smooth and closed or a bounded simply connected Lipschitz domain and for any $k \in \mathbb{N}$ we have $F_k \in \Omega_q^{m-1}(N)$ for some $q > 1$ (possibly dependent on k), then $\varphi_* F$ can be approximated in $\Omega_p^{m-1}(N)$ by $(m-1)$-forms in $\Omega_{p,R}^{m-1}(N)$.

Proof. Assume that $*dF = \partial I$ holds in $(W_0^{1,\infty}(M))^*$. Then for any $f \in W_0^{1,\infty}(N)$ $\varphi^* f \in W_0^{1,\infty}(M)$ and thus

$$\langle *d(\varphi_* F), f\rangle = \int_N \varphi_* F \wedge df = \int_M F \wedge d\varphi^* f = \langle \partial I, \varphi^* f\rangle = \langle I, \varphi^* df\rangle = \langle \partial \varphi_* I, f\rangle,$$

therefore $*d(\varphi_* F) = \partial \varphi_* I$ in $(W_0^{1,\infty}(N))^*$.
Now assume that $(F_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega_R^{m-1}(M)$ such that $F_k \to F$ in $\Omega_p^{m-1}(M)$ as $k \to \infty$. Then for any $k \in \mathbb{N}$ there exists $I_k \in \mathcal{R}_1(M)$ of finite mass and so that ∂I_k supported in a finite subset of M such that $*dF_k = \partial I_k$. As we saw above, $*d(\varphi_* F_k) = \partial \varphi_* I_k$, therefore $\varphi_* F_k \in \Omega_{p,R,\infty}^{m-1}(N)$. Moreover we have $\varphi_* F_k \to \varphi_* F$ in $\Omega_p^{m-1}(N)$ as $k \to \infty$.
Finally if N is smooth and closed or a bounded Lipschitz domain and for any $k \in \mathbb{N}$ we have that $F_k \in \Omega_q^{m-1}(N)$ for some $q > 1$, we can improve the approximating sequence $(\varphi^* F_k)_{k\in\mathbb{N}}$ as follows: for any $k \in \mathbb{N}$, let $\alpha_k \in \Omega_{W^{1,q}}^{m-2}(N)$, $\beta_k \in \Omega_{W^{1,q}}^{m}(N)$ and $h_k \in \Omega_h^{m-1}(N)$ (the space of harmonic $(m-1)$-forms on N) such that $\varphi_* F_k = d\alpha_k + d^*\beta_k + h_k$. Then $*\Delta\beta = *d\varphi_* F_k = *\varphi_* dF_k$. Since $\varphi_* dF_k = \partial \varphi_* I_k$, $\varphi_* dF_k$ is supported in a finite set of points, thus β is smooth in N outside of a finite number of points. Now let $\tilde{\alpha}_k \in \Omega^{m-2}(N)$ such that $\|\alpha_k - \tilde{\alpha}_k\|_{W^{1,p}} \leqslant \frac{1}{2^k}$ and set $\tilde{F}_k := d\tilde{\alpha}_k + d\beta_k + h_k$. Then by construction $\tilde{F}_k \in \Omega_R^{m-1}(N)$ and $\tilde{F}_k \to \varphi_* F$ in $\Omega_p^{m-1}(N)$ as $k \to \infty$. □

Theorem 3.1.1, Corollary 3.2.2, Lemma 3.2.7 and Remark 3.2.7 can be combined to obtain the following general statement.

Theorem 3.2.3. *Let $M \subset \mathbb{R}^n$ be any embedded m-dimensional Lipschitz submanifold of $\mathbb{R}^n$ which is bi-Lipschitz equivalent either to $Q_1^m(0)$ or $\partial Q_1^{m+1}(0)$. Then:*

$$\overline{\Omega_{p,R,\infty}^{m-1}(M)}^{L^p} = \begin{cases} \Omega_{p,\mathbb{Z}}^{m-1}(M) & \text{if } p \in [1, m/(m-1)), \\ \{F \in \Omega_p^{m-1}(M) \text{ s.t. } *dF = 0\} & \text{if } p \in [m/(m-1), +\infty). \end{cases}$$

Moreover, if M is smooth we have $\overline{\Omega_{p,R}^{m-1}(M)}^{L^p} = \overline{\Omega_{p,R,\infty}^{m-1}(M)}^{L^p}$.

3.2.7. Corollaries of Theorem 3.1.1

Finally we present a couple of Corollaries of Theorem 3.1.1.
First we show that the boundary of any $I \in \mathcal{R}_1(D)$ having finite mass can be approximated strongly in $(W_0^{1,\infty}(D))^*$ by finite sums of deltas with integer coefficients.

Corollary 3.2.3. *Let $D \subset \mathbb{R}^n$ be any open and bounded domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to $Q_1^n(0)$. Let $I \in \mathcal{R}_1(D)$ with finite mass. Then, there exists a vector field $V \in L^1(D)$ such that*

$$\operatorname{div}(V) = \partial I \quad \text{in } (W_0^{1,\infty}(D))^*.$$

Thus ∂I can be approximated strongly in $(W_0^{1,\infty}(D))^*$ by finite sums of deltas with integer coefficients. More precisely there exist sequences of points $(P_i)_{i\in\mathbb{N}}$ and $(N_i)_{i\in\mathbb{N}}$ in D such that

$$\partial I = \sum_{i\in\mathbb{N}} (\delta_{P_i} - \delta_{N_i}) \text{ in } (W_0^{1,\infty}(D))^* \text{ and } \sum_{i\in\mathbb{N}} |P_i - N_i| < \infty. \tag{3.2.17}$$

Proof. By Lemma 3.2.7, it is enough to consider the case $D = Q_1(0)$. Let I be as above. By [2, Theorem 5.6] there exists a map $u \in W^{1,n-1}(Q_1(0), \mathbb{S}^{n-1})$ such that

$$*d\left(\frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}u_i \bigwedge_{j\neq i} du_j\right) = \alpha_{n-1}\partial I,$$

where α_{n-1} denotes the volume of the $(n-1)$-dimensional ball.
Set

$$\omega := \frac{1}{n\alpha_{n-1}}\sum_{i=1}^{n}(-1)^{i-1}u_i \bigwedge_{j\neq i} du_j$$

Notice that $\omega \in \Omega_1^{n-1}(Q_1(0))$ and $*d\omega = \partial I$. Now let $V := (*\omega)^\sharp$. Then $V \in L^1(Q_1(0))$ and

$$\operatorname{div}(V) = \partial I.$$

By Theorem 3.1.1, there exists a sequence $(V_k)_{k\in\mathbb{N}}$ in $L^1_R(Q_1(0))$ such that $V_k \to V$ in L^1. Then

$$\operatorname{div}(V_k) \to \operatorname{div}(V) = \partial I \quad \text{in } (W_0^{1,\infty}(Q_1(0)))^*.$$

As for any $k \in \mathbb{N}$ we have that $\operatorname{div}(V_k)$ is a finite sum of deltas with integer coefficients, by [68, Proposition A.1][5] (3.2.17) holds. □

Corollary 3.2.4. *Let M be a complete Lipschitz m-manifold, with or without boundary, compactly contained in the open cube $Q_2(0)$. Let $I \in \mathcal{R}_1(Q_2(0))$ be a rectifiable current of finite mass supported on M. Then there exist two sequences of points $(p_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ in M such that*

$$\partial I = \sum_{i\in\mathbb{N}} (\delta_{p_i} - \delta_{n_i}) \text{ in } (W^{1,\infty}(Q_2(0)))^* \text{ and } \sum_{i\in\mathbb{N}} |p_i - n_i| < \infty.$$

[5] Here Proposition A.1 in [68] is applied to the following metric space: for any $x, y \in Q_1(0)$ let $\overline{d}(x,y) = \min\{d(x,y), \operatorname{dist}(x, \partial Q_1(0)) + \operatorname{dist}(y, \partial Q_1(0)\}$, where d denotes the Euclidean distance in $Q_1(0)$. Let $(\widetilde{Q}_1(0), \overline{d})$ denote the completion of $Q_1(0)$ with respect to the distance $\overline{d}$. Then Lipschitz functions on $(\widetilde{Q}_1(0), \overline{d})$ corresponds to functions in $W_0^{1,\infty}(Q_1(0))$ (with same Lipschitz constant) modulo additive constants.

Proof. Let $I \in \mathcal{R}_1(M)$ be a rectifiable 1-current of finite mass supported in $M \subset Q_2(0)$. By Corollary 3.2.3 there exists a vector field $V \in L^1(Q_2(0))$ such that $\mathrm{div}(V) = \partial I$. Thus we can apply the arguments of the proof of Theorem 3.2.1 to V. For any $\varepsilon \in E_V$ this yields a vector field $\tilde{V}_\varepsilon \in L^1(\Omega_{a_\varepsilon,\varepsilon})$ with the following properties: all bad cubes $Q \in \mathscr{C}_{a_\varepsilon,\varepsilon}$ are such that $\overline{Q} \cap M \neq \emptyset$, therefore the topological singularities of $\tilde{V}_\varepsilon$ lie at a distance of at most $\sqrt{n}\varepsilon$ from M. Moreover notice that if ε is sufficiently small, $\int_{Q_2(0)} \mathrm{div}(\tilde{V}_\varepsilon)\, d\mathcal{L}^n = 0$ (one can see this by testing $\mathrm{div}(\tilde{V}_\varepsilon)$ against a function $\varphi \in C_c^\infty(Q_2(0))$ such that $\varphi \equiv 1$ in a neighbourhood of M). Thus $\mathrm{div}(\tilde{V}_\varepsilon)$ can represented by

$$\mathrm{div}(V_\varepsilon) = \sum_{i=1}^{Q^\varepsilon} (\delta_{p_i^\varepsilon} - \delta_{n_i^\varepsilon})$$

for some $Q^\varepsilon \in \mathbb{N}$ and points p_i^ε and n_i^ε (possibly repeated) in a $\sqrt{n}\varepsilon$-neighbourhood of M. By the argument of Lemma 3.2.2 (with $Q_2(0)$ in place of $Q_1(0)$) we have $\varepsilon Q^\varepsilon \to 0$ as $\varepsilon \to 0^+$ in E_V. Now for any $i \in \{1, ..., Q^\varepsilon\}$ let $\tilde{p}_i^\varepsilon$ and $\tilde{n}_i^\varepsilon$ in M such that $|p_i^\varepsilon - \tilde{p}_i^\varepsilon| < 2\sqrt{n}\varepsilon$ and $|n_i^\varepsilon - \tilde{n}_i^\varepsilon| < 2\sqrt{n}\varepsilon$. Let $I_{p_i^\varepsilon} \in \mathcal{R}_1(Q_2(0))$ be the rectifiable current given by integration on the segment joining p_i^ε and $\tilde{p}_i^\varepsilon$ oriented from p_i^ε to $\tilde{p}_i^\varepsilon$ and let $I_{n_i^\varepsilon} \in \mathcal{R}_1(Q_2(0))$ be the rectifiable current given by integration on the segment joining n_i^ε and $\tilde{n}_i^\varepsilon$ oriented from $\tilde{n}_i^\varepsilon$ to n_i^ε. Let $I_\varepsilon \in \mathcal{R}_1(Q_2(0))$ be a rectifiable 1-current of finite mass such that $\mathrm{div}(\tilde{V}_\varepsilon) = \partial I_\varepsilon$. Set $\tilde{I}_\varepsilon = I_\varepsilon + \sum_{i=1}^{Q^\varepsilon}(I_{p_i^\varepsilon} + I_{n_i^\varepsilon})$. Then

$$\partial \tilde{I}_\varepsilon = \sum_{i=1}^{Q^\varepsilon} (\delta_{\tilde{p}_i^\varepsilon} - \delta_{\tilde{n}_i^\varepsilon})$$

is supported in M. Moreover we have

$$\|\partial I - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(M))^*} \leqslant \|\partial I - \partial I_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*} + \|\partial I_\varepsilon - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*}$$

(here M is endowed with the euclidean distance in $Q_2(0)$; notice that we are making use of the fact that any Lipschitz function on M can be extended to a Lipschitz function on $Q_2(0)$ with same Lipschitz constant). Now since $\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_{a_\varepsilon,\varepsilon})} \to 0$ as $\varepsilon \to 0$ in E_V and ∂I, ∂I_ε are supported in a compact subset of $Q_2(0)$, the first term on the right hand side tends to zero as $\varepsilon \to 0^+$ in E_V. Moreover the second term is bounded by $4\sqrt{n}\varepsilon Q^\varepsilon$ (see for instance Lemma 2 in [18]) and thus tends to 0 as $\varepsilon \to 0^+$ in E_V. This shows that ∂I belongs to the (strong) $(W^{1,\infty}(M))^*$ closure of the class of 0-currents T on M such that

$$T = \sum_{j \in J} (\delta_{p_j} - \delta_{n_j}) \text{ in } (W^{1,\infty}(M))^* \text{ and } \sum_{j \in J} |p_j - n_j| < \infty \tag{3.2.18}$$

for a countable set J and points p_j, n_j in M. By [68, Proposition A.1] applied to the complete metric space (M, d) (where d denotes the Euclidean distance in $Q_2(0)$) this space is closed in $(W^{1,\infty}(M))^*$, therefore ∂I is also of this form. Since any Lipschitz function $\varphi \in W^{1,\infty}(Q_2(0))$ has a Lipschitz trace $\varphi\big|_M$ on M and $\langle \partial I, \varphi \rangle_{Q_2(0)} = \langle \partial I, \varphi\big|_M \rangle_M$, we conclude that ∂I can be represented as in 3.2.18 also as an element of $(W^{1,\infty}(Q_2(0)))^*$. □

Theorem 3.1.1 could also be useful to obtain approximation results for Sobolev maps with values into manifolds. For instance we can use it to recover the following result, due to R. Schoen and K. Uhlenbeck (for $p = 2$, see [78, Section 4]) and F. Bethuel and X. Zheng (for $p > 2$, see [13, Theorem 4]).

Corollary 3.2.5. *Let $u \in W^{1,p}(Q_1(0), \mathbb{S}^1)$ for some $p \in (1, \infty)$.*
If $p \geqslant 2$, then

$$\operatorname{div}(u \wedge \nabla^\perp u) = 0 \quad \text{in } \mathcal{D}'(Q_1(0))$$

and u can be approximated in $W^{1,p}$ by a sequence of functions in $C^\infty(Q_1(0), \mathbb{S}^1)$.
If $p < 2$, then

$$\frac{1}{2\pi} \operatorname{div}(u \wedge \nabla^\perp u) = \partial I, \tag{3.2.19}$$

where $I \in \mathcal{R}_1(Q_1(0))$ is a 1-rectifiable current of finite mass, and u can be approximated in $W^{1,p}$ by a sequence of functions in

$$\mathcal{R} := \left\{ v \in W^{1,p}(Q_1(0), \mathbb{S}^1); v \in C^\infty(Q_1(0) \setminus A, \mathbb{S}^1), \text{ where } A \text{ is some finite set} \right\}.$$

Proof. First we claim that the vector field $u \wedge \nabla^\perp u$ belongs to $L^p_{\mathbb{Z}}(Q_1(0))$. In fact notice that for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2 \operatorname{dist}_\infty(x_0, \partial Q_1(0)))$ $\partial Q_\rho(x_0)$ consists $\mathscr{H}^{n-1}$-a.e. of Lebesgue points of $u \wedge \nabla^\perp u$. Moreover for almost any such ρ we have

$$\frac{1}{2\pi} \int_{\partial Q_\rho(x_0)} (u \wedge \nabla^\perp u) \cdot \nu_{\partial Q_\rho(x_0)} = \deg \left(u\big|_{\partial Q_\rho(x_0)} \right) \in \mathbb{Z}.$$

Hence the vector field $\frac{1}{2\pi} u \wedge \nabla^\perp u$ belongs to $L^p_{\mathbb{Z}}(Q_1(0))$.
Thus if $p \geqslant 2$ by Theorem 3.1.1 there holds $\operatorname{div}(u \wedge \nabla^\perp u) = 0$ in $\mathcal{D}'(Q_1(0))$, while if $p < 2$ there exists a sequence of vector fields $(V_n)_{n \in \mathbb{N}}$ in $L^p_R(Q_1(0))$ such that

$$V_n \to \frac{1}{2\pi} u \wedge \nabla^\perp u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

For any $n \in \mathbb{N}$ by Hodge decomposition there exist $a_n \in W^{1,p}(Q_1(0))$, $b_n \in W^{1,p}_0(Q_1(0))$ such that

$$2\pi V_n = \nabla^\perp a_n + \nabla b_n.$$

For any $n \in \mathbb{N}$ let $\tilde{a}_n \in C^\infty(Q_1(0))$ be such that $\|\tilde{a}_n - a_n\|_{L^p} \leqslant \frac{1}{n}$. Moreover notice that there exists $d_n \in W^{1,p}(Q_1(0), \mathbb{S}^1) \cap C^\infty(Q_1(0) \setminus A)$, where A is a finite set, such that

$$\nabla b_n = d_n \wedge \nabla^\perp d_n.$$

In fact

$$\Delta b_n = 2\pi \operatorname{div}(V_n) = 2\pi \sum_{i=1}^{Q^n} d_i^n \delta_{p_i^n}$$

for some $Q^n \in \mathbb{N}$, $p_i^n \in Q_1(0)$ and $d_i^n \in \mathbb{Z}$, thus $b_n = -\sum_{i=1}^{Q^n} \log|x - p_i^n|^{d_i^n} + h_n$ for an harmonic function h_n. Then d_n can be chosen to be

$$d_n(x) = e^{-i\tilde{h}_n} \prod_{i=1}^{Q^n} \left(\frac{x - p_i^n}{|x - p_i^n|} \right)^{d_n},$$

where $\tilde{h}_n$ is the harmonic conjugate of h_n (the product has to be understood as complex multiplication in $\mathbb{C} \simeq \mathbb{R}^2$).
For any $n \in \mathbb{N}$ set $u_n := e^{i\tilde{a}_n} d_n$. Then by construction $u_n \in \mathcal{R}$ and

$$u_n \wedge \nabla^\perp u_n \to u \wedge \nabla^\perp u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

Therefore there is $c \in [0, 2\pi)$ so that up to a subsequence

$$e^{ic} u_n \to u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

□

Remark 3.2.9. Equation (3.2.19) was obtained in [20, Theorem 3'] with the help of the approximation result of F. Bethuel and X. Zheng.

3.3. The weak L^p-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0))$

In the present section we follow the ideas presented in [63] in order to prove that the space $\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0))$ is weakly sequentially closed for every $n \geqslant 2$ and $p \in (1, +\infty)$. The main reason why such techniques couldn't be used before in this context for $n \neq 3$ was the lack of a strong approximation theorem like Theorem 3.1.1 for general dimension n. Such result is needed in order to define a suitable notion of distance between the cubical slices of a form $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q^n(0))$, given by $(x \mapsto x_0 + \rho x)^* i^*_{\partial Q_\rho(x_0)} F$ for $\mathcal{L}^1$-a.e. ρ (see Section 3.3.1 and 3.3.2 for the precise definition). Once we have turned the space of the cubical slices of F into a metric space, we will show that the "slice function" associated to F, given by $\rho \mapsto (x \mapsto x_0 + \rho x)^* i^*_{\partial Q_\rho(x_0)} F$, is locally $\frac{1}{p'}$−Hölder continuous (see Section 3.3.3). Moreover we will see that if $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ converges weakly in L^p, then the sequence of the slice functions associated to each F_k is locally uniformly $\frac{1}{p'}$−Hölder continuous. Lastly, we will use the previous facts together with some technical lemmata to conclude the proof of Theorem 3.1.3 for $D = Q^n_1(0)$. Notice that by Theorem 3.1.1 the result is clear if $p \in \big[n/(n-1), \infty\big)$, here we will focus on the case $p \in \big(1, n/(n-1)\big)$.

3.3.1. Slice distance on $\mathbb{S}^{n-1}$

Throughout the following section, we will assume that $p \in \big(1, n/(n-1)\big)$. Moreover, we will denote by "$*$" the Hodge star operator associated with the standard round metric on $\mathbb{S}^{n-1}$. We will denote by Z the linear subspace of $\Omega^{n-1}_p(\mathbb{S}^{n-1})$ given by

$$Z := \left\{ h \in \Omega^{n-1}_p(\mathbb{S}^{n-1}) \text{ s.t. } \int_{\mathbb{S}^{n-1}} h \in \mathbb{Z} \right\}.$$

Remark 3.3.1. It's clear that Z is weakly (and thus strongly) L^p closed in $\Omega^{n-1}_p(\mathbb{S}^{n-1})$. Indeed, let $\{h_k\}_{k\in\mathbb{N}} \subset Z$ be any sequence such that $h_k \rightharpoonup h$ weakly in $\Omega^{n-1}_p(\mathbb{S}^{n-1})$, i.e.

$$\int_{\mathbb{S}^{n-1}} \varphi h_k \to \int_{\mathbb{S}^{n-1}} \varphi h, \qquad \forall\, \varphi \in L^{p'}(\mathbb{S}^{n-1}).$$

Then, the statement follows by picking $\varphi \equiv 1$ and noticing that a convergent sequence of integer numbers is definitively constant.

Fix any arbitrary point $q \in \mathbb{S}^{n-1}$. We define the functions $d, \tilde{d} : Z \times Z \to [0, +\infty]$ by

$$d(h_1, h_2) := \inf \left\{ ||\alpha||_{L^p} \text{ s.t. } *(h_1 - h_2) = d^*\alpha + \partial I + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1}), I \in \mathcal{R}_1(\mathbb{S}^{n-1})$, and

$$\tilde{d}(h_1, h_2) := \inf \left\{ ||\alpha||_{L^p} \text{ s.t. } *(h_1 - h_2) = d^*\alpha + \partial I + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$, $I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$.

Remark 3.3.2 (d and $\tilde{d}$ are always finite on Z). We claim that $d, \tilde{d} < +\infty$. Since obviously $d(h_1, h_2) \leqslant \tilde{d}(h_1, h_2)$, it is enough to show that $\tilde{d}(h_1, h_2) < +\infty$, for every $h_1, h_2 \in Z$. This just amounts to saying that given any $h_1, h_2 \in Z$ we can always find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$, $I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ satisfying

$$*(h_1 - h_2) = d^*\alpha + \partial I + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q.$$

Indeed, let

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2 \in \mathbb{Z}.$$

Consider the following first order differential system on $\mathbb{S}^{n-1}$:

$$\begin{cases} d^*\omega = *(h_1 - h_2) - a\delta_q =: F, \\ d\omega = 0. \end{cases}$$

Since $p \in \big(1, n/(n-1)\big)$, $F \in \mathcal{L}(W^{1,p'}(\mathbb{S}^{n-1}))$. Moreover, $\langle F, 1 \rangle = 0$. Hence, by Lemma 3.B.2, we know that the previous differential system has a solution $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ and the statement follows.

Remark 3.3.3. As observed above, it is clear that $d(h_1, h_2) \leqslant \tilde{d}(h_1, h_2)$, for every $h_1, h_2 \in Z$. We claim that actually $d(h_1, h_2) = \tilde{d}(h_1, h_2)$, for every $h_1, h_2 \in Z$. In order to prove the remaining inequality, fix any $h_1, h_2 \in Z$ and let $\{\alpha_k\}_{k \in \mathbb{N}} \subset \Omega^1_p(\mathbb{S}^{n-1})$, $\{I_k\}_{k \in \mathbb{N}} \subset \mathcal{R}_1(\mathbb{S}^{n-1})$ be such that

$$\begin{cases} *(h_1 - h_2) = d^*\alpha_k + \partial I_k + a\delta_q, \quad \forall\, k \in \mathbb{N}, \\ ||\alpha_k||_{L^p} \to d(h_1, h_2) \text{ as } k \to \infty, \end{cases}$$

with

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2.$$

By Corollary 3.B.1, the linear differential equation

$$\Delta u = *(h_1 - h_2) - a\delta_q$$

has a weak solution $\psi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$. Let $\omega_k := d\psi - \alpha_k$, for every $k \in \mathbb{N}$. Notice that

$$d^*\omega_k = \partial I_k, \qquad \forall\, k \in \mathbb{N}.$$

By Theorem 3.2.3, for every $k \in \mathbb{N}$ there exists a sequence $\{\omega_k^j\}_{j\in\mathbb{N}} \subset \Omega^1_{p,R}(\mathbb{S}^{n-1})$ such that $r_k^j := \omega_k - \omega_k^j \to 0$ strongly in L^p as $j \to \infty$. By construction, it follows that

$$*(h_1 - h_2) = d^*(\alpha_k + r_k^j) + d^*\omega_k^j + a\delta_q, \qquad \forall\, k, j \in \mathbb{N}.$$

We observe that by Proposition 3.A.1 for every $k, j \in \mathbb{N}$ there exist $I_k^j \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ such that $d^*\omega_k^j = \partial I_k^j$. This implies that

$$\tilde{d}(h_1, h_2) \leqslant ||\alpha_k + r_k^j||_{L^p} \leqslant ||\alpha_k||_{L^p} + ||r_k^j||_{L^p}, \qquad \forall\, k, j \in \mathbb{N}.$$

By letting first $j \to \infty$ and then $k \to \infty$ in the previous inequality, our claim follows.

Proposition 3.3.1. (Z, d) *is a metric space.*

Proof. We need to check symmetry, triangular inequality and non-degeneracy.
Symmetry. This is clear since both the L^p-norm and the space $\mathcal{R}_1(\mathbb{S}^{n-1})$ are invariant under sign change.
Triangular inequality. Let $h_1, h_2, h_3 \in Z$. By definition of infimum, for every $\varepsilon > 0$ we can write

$$\begin{cases} *(h_1 - h_2) = d^*\alpha_\varepsilon + \partial I_\varepsilon + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2\right)\delta_q, \\ *(h_2 - h_3) = d^*\alpha'_\varepsilon + \partial I'_\varepsilon + \left(\int_{\mathbb{S}^{n-1}} h_2 - h_3\right)\delta_q, \end{cases}$$

with $\alpha_\varepsilon, \alpha'_\varepsilon \in \Omega^1_p(\mathbb{S}^{n-1})$ and $I_\varepsilon, I'_\varepsilon \in \mathcal{R}_1(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} ||\alpha_\varepsilon||_{L^p} \leqslant d(h_1, h_2) + \varepsilon \\ ||\alpha'_\varepsilon||_{L^p} \leqslant d(h_2, h_3) + \varepsilon. \end{cases}$$

We notice that

$$*(h_1 - h_3) = d^*(\alpha_\varepsilon + \alpha'_\varepsilon) + \partial(I_\varepsilon + I'_\varepsilon) + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_3\right)\delta_q, \qquad \forall\, \varepsilon > 0.$$

Then, by definition of d, we have

$$d(h_1, h_3) \leqslant ||\alpha_\varepsilon + \alpha'_\varepsilon||_{L^p} \leqslant ||\alpha_\varepsilon||_{L^p} + ||\alpha'_\varepsilon||_{L^p} \leqslant d(h_1, h_2) + d(h_2, h_3) + 2\varepsilon, \qquad \forall\, \varepsilon > 0.$$

By letting $\varepsilon \to 0^+$ in the previous inequality, we get our claim.
Non-degeneracy. Assume that $d(h_1, h_2) = 0$, for some $h_1, h_2 \in Z$. Let

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2 \in \mathbb{Z}.$$

Then, since $d = \tilde{d}$ (see Remark 3.3.3) and by definition of $\tilde{d}$, there exist $\{\alpha_k\}_{k\in\mathbb{N}} \subset \Omega^1_p(\mathbb{S}^{n-1})$ and $\{I_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ such that

$$*(h_1 - h_2) = d^*\alpha_k + \partial I_k + a\delta_q$$

and $\alpha_k \to 0$ strongly in L^p as $k \to \infty$. Observe that

$$\partial I_k \to *(h_1 - h_2) - a\delta_q \quad \text{ in } (W^{1,\infty}(\mathbb{S}^{n-1}))^*.$$

Now for any $k \in \mathbb{N}$, ∂I_k can be represented as

$$\partial I_k = \sum_{j=1}^{J_k} (\delta_{p_j^k} - \delta_{n_j^k})$$

for some $J_k \in \mathbb{N}$ and points p_j^k, n_j^k in $\mathbb{S}^{n-1}$. But the space of distributions of the form

$$\sum_{j \in J} (\delta_{p_j} - \delta_{n_j}) \text{ such that } \sum_{j \in J} |p_j - n_j| < \infty \tag{3.3.1}$$

(for a countable set J and points p_j, n_j in $\mathbb{S}^{n-1}$) is closed with respect to the (strong) topology of $(W^{1,\infty}(\mathbb{S}^{n-1}))^*$ (see Proposition A.1 in [68]), thus there exists a distribution T as in (3.3.1) such that

$$*(h_1 - h_2) = T - a\delta_q. \tag{3.3.2}$$

But this implies that $*(h_1 - h_2) = 0$, since the left-hand-side in (3.3.2) is in L^p whilst the right-hand-side is in L^p if and only if it is equal to zero. □

Remark 3.3.4. Notice that the proof above relies on the fact that $d = \tilde{d}$, which was proved using Corollary 3.2.2. As the proof of the Corollary 3.2.2 was rather cumbersome, we remark here that there is a way to skip that passage. Indeed, in the proof of the non-degeneracy of d it is not necessary to assume that $\{I_k\}_{k \in \mathbb{N}}$ lies in $\mathcal{N}_1(\mathbb{S}^{n-1})$. In fact, it follows from Corollary 3.2.4 that ∂I_k is of the form (3.3.1), and thus the limit of $\{I_k\}_{k \in \mathbb{N}}$ in $(W^{1,\infty}(\mathbb{S}^{n-1}))^*$ will also be of that form.

Proposition 3.3.2. *Let $\{h_k\}_{k \in \mathbb{N}} \subset Z$ and $h \in Z$. Then the following are equivalent:*

1. *$\{h_k\}_{k \in \mathbb{N}} \subset Z$ is uniformly bounded w.r.t. the L^p-norm and $d(h_k, h) \to 0$ as $k \to \infty$;*
2. *$h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$.*

Proof. We prove separately the two implications.

$2 \Rightarrow 1$. Pick any subsequence of $\{h_k\}_{k \in \mathbb{N}}$ (not relabelled). For every $k \in \mathbb{N}$, let

$$a_k := \langle h_k - h, 1 \rangle = \int_{\mathbb{S}^{n-1}} h_k - h.$$

Since $h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$, it follows that $a_k \to 0$ as as $k \to \infty$. Since $\{a_k\}_{k \in \mathbb{N}} \subset \mathbb{Z}$, there exists $K \in \mathbb{N}$ such that $a_k = 0$ for every $k \geqslant K$. Fix any $k \geqslant K$. By Lemma 3.B.2, the linear differential system

$$\begin{cases} d^*\omega = *(h_k - h) \\ d\omega = 0, \end{cases}$$

respectively (if $n = 2$)

$$\begin{cases} d^*\omega = *(h_k - h) \\ d\omega = 0 \\ \displaystyle\int_{\mathbb{S}^1} \omega = 0, \end{cases}$$

has a unique weak solution $\alpha_k \in \Omega^1_p(\mathbb{S}^{n-1})$. By Remark 3.B.3 we have

$$||\alpha_k||_{W^{1,p}} \leqslant C\big(||d\alpha_k||_{L^p} + ||d^*\alpha_k||_{L^p}\big) = C||h_k - h||_{L^p}.$$

Since $\{h_k\}_{k\in\mathbb{N}}$ is weakly convergent, we know that it is also uniformly bounded w.r.t the L^p-norm. Then $\{\alpha_k\}_{k\geqslant K}$ is uniformly bounded w.r.t the $W^{1,p}$-norm. Hence, by weak compactness in $W^{1,p}$, there exists a subsequence $\{\alpha_{k_l}\}_{l\in\mathbb{N}} \subset \{\alpha_k\}_{k\geqslant K}$ and a one-form $\alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ such that $\alpha_{k_l} \rightharpoonup \alpha$ weakly in $W^{1,p}$. By Rellich-Kondrakov theorem, it follows that $\alpha_{k_l} \to \alpha$ strongly in L^p. We claim that $\alpha = 0$. Indeed,

$$\begin{aligned}
\langle \alpha, \omega\rangle_{L^p-L^{p'}} &= \lim_{l\to\infty} \langle \alpha_{k_l}, d\varphi + d^*\beta\rangle_{L^p-L^{p'}} \\
&= \lim_{l\to\infty} -\langle d^*\alpha_{k_l}, \varphi\rangle_{L^p-L^{p'}} \\
&= \lim_{l\to\infty} -\int_{\mathbb{S}^{n-1}} (h_{k_l} - h)\wedge\varphi = 0, \qquad \forall\, \omega = d\varphi + d^*\beta \in \Omega^1(\mathbb{S}^{n-1}),
\end{aligned}$$

respectively (if $n = 2$)

$$\begin{aligned}
\langle \alpha, \omega\rangle_{L^p-L^{p'}} &= \lim_{l\to\infty} \langle \alpha_{k_l}, d\varphi + d^*\beta + \eta\rangle_{L^p-L^{p'}} \\
&= \lim_{l\to\infty} \langle \alpha_{k_l}, d\varphi + d^*\beta\rangle_{L^p-L^{p'}} \\
&= \lim_{l\to\infty} -\langle d^*\alpha_{k_l}, \varphi\rangle_{L^p-L^{p'}} \\
&= \lim_{l\to\infty} -\int_{\mathbb{S}^{n-1}} (h_{k_l} - h)\wedge\varphi = 0, \qquad \forall\, \omega = d\varphi + d^*\beta + \eta \in \Omega^1(\mathbb{S}^1),
\end{aligned}$$

where $\eta \in \Omega^1(\mathbb{S}^1)$ is a harmonic 1-form on $\mathbb{S}^1$ (hence a constant 1-form) and the second equality follows because α_{k_l} is distributionally closed.

Hence, we have shown that $\alpha_{k_l} \to 0$ strongly in L^p as $l \to \infty$. As $*(h_{k_l} - h) = d^*\alpha_{k_l}$, for every $l \in \mathbb{N}$, we have

$$d(h_{k_l}, h) \leqslant ||\alpha_{k_l}||_{L^p} \to 0, \qquad \text{as } l \to \infty.$$

We have just proved that any subsequence of $\{h_k\}_{k\in\mathbb{N}}$ has a further subsequence converging to h with respect to d, therefore 1. follows.

$1 \Rightarrow 2$. Pick any subsequence of $\{h_k\}_{k\in\mathbb{N}}$ (not relabelled). Since $\{h_k\}_{k\in\mathbb{N}} \subset Z$ is uniformly bounded w.r.t. the L^p-norm, by weak L^p-compactness there exists a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ of $\{h_k\}_{k\in\mathbb{N}}$ and a $h_w \in Z$ such that $h_{k_l} \rightharpoonup h_w$ weakly in L^p. Since we have just shown that $2 \Rightarrow 1$, we know that $d(h_{k_l}, h_w) \to 0$ as $l \to \infty$. By uniqueness of the limit, we get $h_w = h$. We have just proved that any subsequence of $\{h_k\}_{k\in\mathbb{N}} \subset Z$ has a further subsequence converging to h weakly in L^p, hence, 2. follows. □

3.3.2. Slice distance on $\partial Q^n_1(0)$

Let $Q_1(0) \subset \mathbb{R}^n$ be the unit cube in $\mathbb{R}^n$ centered at the origin and let $\Psi : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be a bi-Lipschitz homeomorphism. We let Y be the linear subspace of $\Omega^{n-1}_p(\partial Q_1(0))$ given by

$$Y := \left\{ h \in \Omega^{n-1}_p(\partial Q_1(0)) \text{ s.t. } \int_{\partial Q_1(0)} h \in \mathbb{Z} \right\}.$$

Remark 3.3.5. Notice that $h \in Y$ if and only if $\Psi^* h \in Z$. Indeed, given any $h \in Z$ we have

$$C_\Psi^{-1} \int_{\partial Q_1(0)} |h|^p \, d\mathscr{H}^{n-1} \leqslant \int_{\mathbb{S}^{n-1}} |\Psi^* h|^p \, d\mathscr{H}^{n-1} \leqslant C_\Psi \int_{\partial Q_1(0)} |h|^p \, d\mathscr{H}^{n-1},$$

with $C_\Psi := \big(\max\{\|d\Psi\|_{L^\infty}, \|d\Psi^{-1}\|_{L^\infty}\}\big)^{(n-1)(p-1)}$, and

$$\int_{\mathbb{S}^{n-1}} \Psi^* h = \int_{\partial Q_1(0)} h.$$

Thus, the functions $d_\Psi, \tilde{d}_\Psi : Y \times Y \to [0, +\infty)$ given by

$$\begin{aligned} d_\Psi(h_1, h_2) &:= d(\Psi^* h_1, \Psi^* h_2) \qquad \forall\, h_1, h_2 \in Y, \\ \tilde{d}_\Psi(h_1, h_2) &:= \tilde{d}(\Psi^* h_1, \Psi^* h_2) \qquad \forall\, h_1, h_2 \in Y, \end{aligned}$$

are well-defined and coincide on $Y \times Y$ by Remarks 3.3.2 and 3.3.3. Moreover, (Y, d_Ψ) is a metric space as a direct consequence of Proposition 3.3.1 and the following statement is a corollary of Proposition 3.3.2.

Corollary 3.3.1. *Let $\{h_k\}_{k \in \mathbb{N}} \subset Y$ and $h \in Y$. Then, the following are equivalent:*

1. *$\{h_k\}_{k \in \mathbb{N}} \subset Y$ is uniformly bounded w.r.t. the L^p-norm and $d_\Psi(h_k, h) \to 0$ as $k \to \infty$;*
2. *$h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$.*

Remark 3.3.6. Let $\Psi_1, \Psi_2 : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be bi-Lipschitz homeomorphisms. We claim that the distances d_{Ψ_1} and d_{Ψ_2} induced on Y by Ψ_1 and Ψ_2 respectively are equivalent. Indeed notice that given any bi-Lipschitz map $\Lambda : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ we have

$$d(\Lambda^* h_1, \Lambda^* h_2) \leqslant \|d\Lambda\|_{L^\infty}^{n-2} \|d\Lambda^{-1}\|_{L^\infty}^{\frac{n-1}{p}} d(h_1, h_2), \qquad \forall\, h_1, h_2 \in Z. \tag{3.3.3}$$

To see this notice that if $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ is a competitor in the definition of $d(h_1, h_2)$ then the form given by $(-1)^{n-2} * \Lambda^*(*\alpha) \in \Omega^1_p(\mathbb{S}^{n-1})$ is a competitor in the definition of $d(\Lambda^* h_1, \Lambda^* h_2)$. Hence

$$d(\Lambda^* h_1, \Lambda^* h_2) \leqslant \|{*\Lambda^*(*\alpha)}\|_{L^p} \leqslant \|d\Lambda\|_{L^\infty}^{n-2} \|d\Lambda^{-1}\|_{L^\infty}^{\frac{n-1}{p}} \|\alpha\|_{L^p},$$

for every competitor α in the definition of $d(h_1, h_2)$. By taking the infimum on all the competitors in the previous inequality, (3.3.3) follows. By applying (3.3.3) we obtain

$$\begin{aligned} d_{\Psi_2}(h_1, h_2) &= d(\Psi_2^* h_1, \Psi_2^* h_2) = d\big((\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_1, (\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_2\big) \\ &\leqslant C_{\Psi_1 \Psi_2} d(\Psi_1^* h_1, \Psi_1^* h_2) = C_{\Psi_1 \Psi_2} d_{\Psi_1}(h_1, h_2) \qquad \forall\, h_1, h_2 \in Y, \end{aligned}$$

Analogously, we get

$$d_{\Psi_1}(h_1, h_2) \leqslant C_{\Psi_1 \Psi_2} d_{\Psi_2}(h_1, h_2) \qquad \forall\, h_1, h_2 \in Y,$$

with $C_{\Psi_1 \Psi_2} := \max\big\{\|d(\Psi_2^{-1} \circ \Psi_1)\|_{L^\infty}, \|d(\Psi_1^{-1} \circ \Psi_2)\|_{L^\infty}\big\}^{n-2+\frac{n-1}{p}}$.

3.3.3. Slice functions and their properties

Definition 3.3.1 (Slice functions)**.** Let $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. Given any arbitrary $x_0 \in Q_1(0)$, we let $\rho_0 := 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0))$.
We call the *slice function of F at x_0* the map $s : \mathrm{Dom}(s) \subset (0, \rho_0) \to Y$ given by

$$s(\rho) := (x \mapsto \rho x + x_0)^* i^*_{\partial Q_\rho(x_0)} F, \qquad \forall\, \rho \in \mathrm{Dom}(s),$$

where $\mathrm{Dom}(s)$ is the subset of $(0, \rho_0)$ defined as follows: $\rho \in \mathrm{Dom}(s)$ if and only if the following conditions hold:

1. $\mathscr{H}^{n-1}$-a.e. point in $\partial Q_\rho(x_0)$ is a Lebesgue point for F,
2. $|F| \in L^p(\partial Q_\rho(x_0), \mathscr{H}^{n-1})$,
3. ρ is a Lebesgue point for the L^p-function
$$(0, \rho_0) \ni \rho \mapsto \int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F,$$
4. $(x \mapsto \rho x + x_0)^* i^*_{\partial Q_\rho(x_0)} F \in Y$.

Remark 3.3.7. Notice that $\mathrm{Dom}(s)$ has $\mathcal{L}^1$ full measure in $(0, \rho_0)$ and $s \in L^p\big((0, \rho_0); Y\big)$, in the following sense: letting $j_\rho : x \mapsto \rho x + x_0$, we have

$$\|s(\rho)\|^p_{L^p} = \int_{\partial Q_1(0)} |j^*_\rho F|^p \, d\mathscr{H}^{n-1} = \int_{\partial Q_\rho(x_0)} |F|^p \rho^{(n-1)(p-1)} \, d\mathscr{H}^{n-1}$$

and thus

$$\int_0^{\rho_0} ||s(\rho)||^p_{L^p} \, d\rho \leqslant \int_0^{\rho_0} \int_{\partial Q_\rho} |F|^p \rho^{(n-1)(p-1)} \, d\mathscr{H}^{n-1} d\rho \leqslant 2^n \int_{Q_{\rho_0}(x_0)} |F|^p d\mathscr{H}^n$$

Proposition 3.3.3. *Let $x_0 \in Q_1(0)$ and set $\rho_0 := 2\,\mathrm{dist}(x_0, \partial Q_1(0))$. Fix any $(n-1)$-form $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ and let $s \in L^p\big((0, \rho_0), Y\big)$ be the slice function of F at x_0. Let $K \subset (0, \rho_0)$ be compact. Then, there exists a subset $E \subset K$ such that $\mathcal{L}^1(K \smallsetminus E) = 0$ and a representative $\tilde{s}$ of s defined pointwise on E such that*

$$d_\Psi\big(\tilde{s}(\rho_1), \tilde{s}(\rho_2)\big) \leqslant C_{p,K,\Psi} ||F||_{L^p} |\rho_1 - \rho_2|^{\frac{1}{p'}}, \qquad \forall\, \rho_1, \rho_2 \in E, \tag{3.3.4}$$

with

$$C_{p,K,\Psi} := C_{p,\Psi} \max_{\rho \in K} \rho^{1-n}$$

Proof. Denote by $T_F \in \mathcal{D}_1(Q_1(0))$ the 1-current on $Q_1(0)$ given by

$$\langle T_F, \omega \rangle = \int_{Q_1(0)} F \wedge \omega, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)).$$

Since $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, by Theorem 3.2.2 there exists $I \in \mathcal{R}_1(Q_1(0))$ such that $\mathbb{M}(I) < +\infty$ and $*dF = \partial I$. By definition of integral 1-current, there exist a locally 1-rectifiable set $\Gamma \subset Q_1(0)$, a

Borel measurable unitary vector field $\vec{I}$ on Γ and a positive $\mathbb{Z}$-valued $\mathscr{H}^1 \llcorner \Gamma$-integrable function $\theta \in L^1(\Gamma, \mathscr{H}^1)$ such that

$$\langle I, \omega \rangle = \int_\Gamma \theta \langle \omega, \vec{I} \rangle \, d\mathscr{H}^1, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)).$$

By the coarea formula, there exists $G \subset K$ such that $\mathcal{L}^1(K \smallsetminus G) = 0$ and such that $\Gamma \cap \partial Q_\rho(x_0)$ is a finite set for every $\rho \in G$.

Let $\Psi : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be the bi-Lipschitz map given by

$$\Psi(x) := \frac{x}{2\|x\|_\infty}, \qquad \forall\, x \in \mathbb{S}^{n-1}.$$

Consider the map $\Phi : \mathbb{S}^{n-1} \times [0, \rho_0] \to \operatorname{Im}(\Phi) = \overline{Q_{\rho_0}(x_0)} \subset \overline{Q_1(0)}$ given by

$$\Phi(y,t) := x_0 + t\Psi(y), \qquad \forall\, (y,t) \in \mathbb{S}^{n-1} \times [0, \rho_0].$$

Notice that $\Phi\big|_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}$ is a bi-Lipschitz homeomorphism onto its image for every $\rho_1, \rho_2 \in (0,1)$. We claim that estimate (3.3.4) holds on a full-measure subset of G. Indeed, fix any $\rho_1, \rho_2 \in G$. Without loss of generality, assume that $\rho_2 > \rho_1$. Let $\hat{\Phi} := \Phi\big|_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}$. Define $\pi := \operatorname{pr}_1 \circ \Phi^{-1} : \overline{Q_{\rho_0}(x_0)} \to \mathbb{S}^{n-1}$, where $\operatorname{pr}_1 : \mathbb{S}^{n-1} \times [0, \rho_0] \to \mathbb{S}^{n-1}$ is the canonical projection on the first factor, and notice that π is a Lipschitz and proper map. Then, $\pi_*\big(T_F \llcorner \operatorname{Im}(\hat{\Phi})\big) \in \mathcal{D}_1(\mathbb{S}^{n-1})$ can be expressed as follows: for any $\omega \in \Omega^1(\mathbb{S}^{n-1})$

$$\begin{aligned}
\langle \pi_*\big(T_F \llcorner \operatorname{Im}(\hat{\Phi})\big), \omega \rangle &= \langle T_F \llcorner \operatorname{Im}(\hat{\Phi}), \pi^*\omega \rangle = \int_{\operatorname{Im}(\hat{\Phi})} F \wedge \pi^*\omega \\
&= \int_{\operatorname{Im}(\hat{\Phi})} (\Phi^{-1})^*(\Phi^* F \wedge \Phi^*\pi^*\omega) \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \Phi^* F \wedge \operatorname{pr}_1^*\omega \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \operatorname{pr}_1^*\omega \wedge *(*\Phi^* F) \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \langle \operatorname{pr}_1^*\omega, *\Phi^* F \rangle \, d\operatorname{vol}_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}(y,t) \\
&= \int_{\rho_1}^{\rho_2} \left(\int_{\mathbb{S}^{n-1}} \langle \operatorname{pr}_1^*\omega, *\Phi^* F \rangle \, d\operatorname{vol}_{\mathbb{S}^{n-1}}(y) \right) dt \\
&= \int_{\rho_1}^{\rho_2} \left(\int_{\mathbb{S}^{n-1}} \langle \omega, i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F \rangle \, d\operatorname{vol}_{\mathbb{S}^{n-1}}(y) \right) dt \\
&= \int_{\rho_1}^{\rho_2} \left(\int_{\mathbb{S}^{n-1}} \omega \wedge \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F \big) \right) dt \\
&= \int_{\mathbb{S}^{n-1}} \omega \wedge \left(\int_{\rho_1}^{\rho_2} \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F \big) \, dt \right) \\
&= (-1)^{n-1} \int_{\mathbb{S}^{n-1}} \omega \wedge \alpha,
\end{aligned}$$

where

$$\alpha := (-1)^{n-1} \int_{\rho_1}^{\rho_2} \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F \big) \, dt \in \Omega^{n-2}_p(\mathbb{S}^{n-1}).$$

In particular,

$$\begin{aligned}\langle \partial\pi_*\big(T_F \llcorner \mathrm{Im}(\hat\Phi)\big), \varphi\rangle &= \langle \pi_*\big(T_F \llcorner \mathrm{Im}(\hat\Phi)\big), d\varphi\rangle \\ &= (-1)^{n-1}\int_{\mathbb{S}^{n-1}} d\varphi\wedge\alpha = \langle d^*(*\alpha), \varphi\rangle, \qquad \forall\,\varphi\in C^\infty(\mathbb{S}^{n-1}).\end{aligned}$$

Recall that the restriction of an integral current to a measurable set is still an integral current. Moreover, the push-forward of an integral current through a Lipschitz and proper map remains an integral current (see [53, Chapter 7, §7.5]). Then, $\tilde I := -\pi_*\big(I \llcorner \mathrm{Im}(\hat\Phi)\big) \in \mathcal{R}_1(\mathbb{S}^{n-1})$.
So far, we have shown that

$$\partial\pi_*\big((T_F - I)\llcorner \mathrm{Im}(\hat\Phi)\big) = \partial\pi_*\big(T_F\llcorner \mathrm{Im}(\hat\Phi)\big) - \partial\pi_*\big(I\llcorner \mathrm{Im}(\hat\Phi)\big) = d^*(*\alpha) + \partial\tilde I.$$

Let $\zeta\in C_c^\infty((-1,1))$ such that $\displaystyle\int_{\mathbb{R}}\zeta = 1$. For any $\varepsilon\in(0,\min\{\rho_1,\rho_0-\rho_2\})$ set $\zeta_\varepsilon = \dfrac{1}{\varepsilon}\zeta\left(\dfrac{\cdot}{\varepsilon}\right)$ and let χ_ε be the unique solution of

$$\begin{cases}\chi_\varepsilon'(x) = \zeta_\varepsilon(x-\rho_1) - \zeta_\varepsilon(x-\rho_2)\\ \chi_\varepsilon(0) = 0.\end{cases}$$

Let $\psi\in C^\infty(\mathbb{S}^{n-1}\times[0,\rho_0])$ and let $\mathrm{pr}_2 : \mathbb{S}^{n-1}\times[0,\rho_0]\to[0,\rho_0]$ be the projection on the second factor. We compute

$$\begin{aligned}\langle(\Phi^{-1})_*I, \psi\, d(\chi_\varepsilon\circ\mathrm{pr}_2)\rangle &= \int_{\Phi^{-1}(\Gamma)} \theta_{(\Phi^{-1})_*I}\psi(\chi_\varepsilon'\circ\mathrm{pr}_2)\langle d\mathrm{pr}_2, \vec I_{(\Phi^{-1})_*I}\rangle\, d\mathscr{H}^1\\ &= \int_{\rho_1-\varepsilon}^{\rho_1+\varepsilon}\zeta_\varepsilon(t)\bigg(\int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{t\})}\psi\tilde\theta\, d\mathscr{H}^0\bigg)\, d\mathcal{L}^1(t)\\ &\quad - \int_{\rho_2-\varepsilon}^{\rho_2+\varepsilon}\zeta_\varepsilon(t)\bigg(\int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{t\})}\psi\tilde\theta\, d\mathscr{H}^0\bigg)\, d\mathcal{L}^1(t)\end{aligned}$$

with $\tilde\theta = \theta_{(\Phi^{-1})_*I}\mathrm{sgn}(\langle d\mathrm{pr}_2, \vec I_{(\Phi_{-1})_*I}\rangle)\in L^1(\Phi^{-1}(\Gamma),\mathbb{Z})$. Moreover

$$\begin{aligned}\langle(\Phi^{-1})_*T_F, \psi\, d(\chi_\varepsilon\circ\mathrm{pr}_2)\rangle &= \int_{\mathbb{S}^{n-1}\times[0,\rho_0]}\psi(\chi_\varepsilon'\circ\mathrm{pr}_2)\Phi^*F\wedge d\mathrm{pr}_2\\ &= \int_{\rho_1-\varepsilon}^{\rho_1+\varepsilon}\zeta_\varepsilon(t)\left(\int_{\mathbb{S}^{n-1}\times\{t\}}\psi(\Phi|_{\mathbb{S}^{n-1}\times\{t\}})^*F\right)d\mathcal{L}^1(t)\\ &\quad - \int_{\rho_2-\varepsilon}^{\rho_2+\varepsilon}\zeta_\varepsilon(t)\left(\int_{\mathbb{S}^{n-1}\times\{t\}}\psi(\Phi|_{\mathbb{S}^{n-1}\times\{t\}})^*F\right)d\mathcal{L}^1(t).\end{aligned}$$

Now observe that

$$\langle(\Phi^{-1})_*(T_F - I), (\chi_\varepsilon\circ\mathrm{pr}_2)\, d\psi\rangle \to \langle((\Phi^{-1})_*(T_F-I))\llcorner(\mathbb{S}^{n-1}\times[\rho_1,\rho_2]), d\psi\rangle$$

as $\varepsilon\to0^+$, by dominated convergence. On the other hand, since $\partial(T_F - I) = 0$, we have

$$\langle(\Phi^{-1})_*(T_F - I), (\chi_\varepsilon\circ\mathrm{pr}_2)\, d\psi\rangle = \langle(\Phi^{-1})_*I, \psi\, d(\chi_\varepsilon\circ\mathrm{pr}_2)\rangle - \langle(\Phi^{-1})_*T_F, \psi\, d(\chi_\varepsilon\circ\mathrm{pr}_2)\rangle.$$

Therefore for almost every $\rho_1, \rho_2 \in (0, \rho_0)$ (depending on ψ) we have

$$\begin{aligned}\langle \partial((\Phi^{-1})_*(T_F - I) \llcorner (\mathbb{S}^{n-1} \times [\rho_1, \rho_2])), \psi \rangle = &\int_{\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_1\})} \psi \tilde{\theta} \, d\mathscr{H}^0 \\ &- \int_{\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_2\})} \psi \tilde{\theta} \, d\mathscr{H}^0 \\ &- \int_{\mathbb{S}^{n-1} \times \{\rho_1\}} \psi (\Phi|_{\mathbb{S}^{n-1} \times \{\rho_1\}})^* F \\ &+ \int_{\mathbb{S}^{n-1} \times \{\rho_2\}} \psi (\Phi|_{\mathbb{S}^{n-1} \times \{\rho_2\}})^* F. \end{aligned} \tag{3.3.5}$$

Now let $\{\psi_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{S}^{n-1} \times [0, \rho_0])$ be a countable sequence dense in $C^1(\mathbb{S}^{n-1} \times [0, \rho_0])$. For every $k \in \mathbb{N}$, let $E_k \subset G$ be the set such that (3.3.5) holds with $\psi = \psi_k$ (i.e. the set of the $\rho \in G$ wich are "ζ_ε-Lebesgue points" of the integrands in (3.3.5), with $\psi = \psi_k$) and define

$$E := \bigcap_{k \in \mathbb{N}} E_k.$$

Then $\mathcal{L}^1(E) = \mathcal{L}^1(K)$ and for every $\rho_1, \rho_2 \in E$ estimate (3.3.5) holds with $\psi = \psi_k$ for every $k \in \mathbb{N}$. By density of $\{\psi_k\}_{k \in \mathbb{N}}$ in $C^1(\mathbb{S}^{n-1} \times [0, \rho_0])$, we can pass to the limit in (3.3.5) and get that for any given couple of parameters $\rho_1, \rho_2 \in \tilde{E}$ such estimate holds for every $\psi \in C^\infty(\mathbb{S}^{n-1} \times [0, \rho_0])$. In particular, for every $\rho_1, \rho_2 \in E$, $\varphi \in C^\infty(\mathbb{S}^{n-1})$ we have

$$\begin{aligned}\langle \partial \pi_*((T_F - I) \llcorner \mathrm{Im}(\hat{\Phi})), \varphi \rangle = &\sum_{x \in \Gamma_{\rho_1}} \tilde{\theta}(x, \rho_1) \varphi(x) - \sum_{x \in \Gamma_{\rho_2}} \tilde{\theta}(x, \rho_2) \varphi(x) \\ &- \int_{\mathbb{S}^{n-1}} \varphi \, \Psi^* s(\rho_1) + \int_{\mathbb{S}^{n-1}} \varphi \, \Psi^* s(\rho_2),\end{aligned}$$

where

$$\begin{aligned}\Gamma_{\rho_1} &:= \mathrm{pr}_1(\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_1\})) \subset \mathbb{S}^{n-1}, \\ \Gamma_{\rho_2} &:= \mathrm{pr}_1(\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_2\})) \subset \mathbb{S}^{n-1}\end{aligned}$$

are finite set for any ρ_1, $\rho_2 \in G$.

Gathering together what we have proved so far, we have

$$*\big(\Psi^* s(\rho_2) - \Psi^* s(\rho_1)\big) = d^*(*\alpha) + \partial I' + \left(\int_{\mathbb{S}^{n-1}} \Psi^* s(\rho_2) - \Psi^* s(\rho_1) \right) \delta_q,$$

where $I' \in \mathcal{R}_1(\mathbb{S}^{n-1})$ is any rectifiable one-current of finite mass such that

$$\partial I' = \sum_{x \in \Gamma_{\rho_2}} \tilde{\theta}(x, \rho_2) \delta_x - \sum_{x \in \Gamma_{\rho_1}} \tilde{\theta}(x, \rho_1) \delta_x + \partial \tilde{I} + \left(\sum_{x \in \Gamma_{\rho_1}} \tilde{\theta}(x, \rho_1) - \sum_{x \in \Gamma_{\rho_2}} \tilde{\theta}(x, \rho_2) \right) \delta_q,$$

i.e. $*\alpha$ is a competitor in the definition of $d\big(\Psi^* s(\rho_2), \Psi^* s(\rho_1)\big)$. Hence, in order to estimate $d\big(\Psi^* s(\rho_2), \Psi^* s(\rho_1)\big)$ we just need to find an upper bound for $||*\alpha||_{L^p}$.

Notice that $|d\Phi| \leqslant t|d\Psi| + \frac{\sqrt{n}}{2}$. Moreover since

$$\Phi^{-1}(x) = \left(\Psi^{-1} \left(\frac{x - x_0}{2\|x - x_0\|_\infty} \right), 2\|x - x_0\|_\infty \right)$$

we have $|J\Phi^{-1}(x)| \leqslant 2^{-(n-2)}\left(\frac{\|d\Psi^{-1}\|_{L^\infty}}{\|x-x_0\|_\infty}\right)^{n-1}$. Therefore

$$\begin{aligned}\|{*\alpha}\|^p_{L^p} &\leqslant \int_{\mathbb{S}^{n-1}}\left|\int_{\rho_1}^{\rho_2}|\Phi^*F|dt\right|^p d\mathscr{H}^{n-1} \leqslant |\rho_1-\rho_2|^{\frac{p}{p'}}\int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}|\Phi^*F|^p d\mathscr{H}^{n-1}dt\\ &\leqslant \left(2^{-(n-2)}\left(\|d\Psi\|_{L^\infty}+\frac{\sqrt{n}}{2}\right)^{p(n-1)}\frac{\|d\Psi^{-1}\|^{n-1}_{L^\infty}}{\rho_1^{n-1}}\right)|\rho_1-\rho_2|^{\frac{p}{p'}}\|F\|^p_{L^p(Q_1(0))}\end{aligned}$$

and our claim follows.

□

3.3.4. Proof of Theorem 3.1.3 for $Q_1^n(0)$

For the proof of Theorem 3.1.3 we need two technical Lemmas.

Lemma 3.3.1. *Let $\{f_k\}_{k\in\mathbb{N}} \subset L^1(0,1)$ be such that $||f_k||_{L^1} \leqslant C$ for any $k \in \mathbb{N}$. Then there exist a sequence of compact subsets $\{W_h\}_{h\in\mathbb{N}_{\geqslant 2}}$ of $(0,1)$ such that for every $h \in \mathbb{N}_{\geqslant 2}$ the following properties hold:*

1. *$L^1(W_h) = 1 - \dfrac{C+2}{h}$;*
2. *$W_h \subset (1/h, 1)$;*
3. *for almost every $\rho \in W_h$ and every $k \in \mathbb{N}$ there exists $k' > k$ such that $|f_{k'}(\rho)| \leqslant h$.*

Proof. Let $h \in \mathbb{N}_{\geqslant 2}$. For any $l \in \mathbb{N}$ let

$$A_l^h := \bigcap_{k=l}^{\infty} f_k^{-1}([-h,h]^c).$$

Notice that for any $l \in \mathbb{N}$ $A_l^h \subset A_{l+1}^h$ and set

$$I_h = \bigcup_{l=1}^{\infty} A_l^h.$$

Let $m \in \mathbb{N}$ and let $k \geqslant m$. Notice that

$$C \geqslant \int_0^1 |f_k(\rho)|\,d\rho \geqslant \int_{A_m^h} |f_k(\rho)|\,d\rho > h\mathcal{L}^1(A_m^h).$$

By letting $m \to \infty$ in the previous inequality, be obtain

$$L^1(I_h) \leqslant \frac{C}{h}.$$

Then, by defining $E_h := I_h^c \cap (1/h, 1)$, we clearly get

$$\mathcal{L}^1(E_h) = \mathcal{L}^1((I_h \cup (0,1/h])^c) = 1 - \mathcal{L}^1(I_h \cup (0,1/h]) \geqslant 1 - \frac{C+1}{h}.$$

Moreover, E_h and any of its subsets satisfy the properties 2. and 3.. Finally, since E_h is measurable, we can find a compact set $W_h \subset E_h$ such that

$$\mathcal{L}^1(W_h) = 1 - \frac{C+2}{h}.$$

By construction, W_h satisfies 1, 2 and 3. □

Since we do not know if the space (Y, d) is complete, we will also need the following lemma.

Lemma 3.3.2. *Let $K \subset [0,1]$ be compact and let $S \subset K$ be dense and countable. Let $\{f_k\}_{k\in\mathbb{N}} \subset C^0(K, Y)$ be such that*

1. *$\{f_k\}_{k\in\mathbb{N}}$ is uniformly Cauchy from K to (Y, d) (i.e. it is a Cauchy sequence w.r.t. uniform convergence);*
2. *for some $C > 0$,*

$$\sup_{\rho\in S}\sup_{k\in\mathbb{N}} ||f_k(\rho)||_{L^p} \leqslant C;$$

3. *for some $A > 0$ and some $\alpha \in (0,1]$, we have*

$$d\big(f_k(\rho), f_k(\rho')\big) \leqslant A|\rho - \rho'|^\alpha, \qquad \forall\, \rho, \rho' \in S.$$

Then, there exists $f \in C^0(K, Y)$ such that $f_k \to f$ uniformly.

Proof. Fix any $\rho \in S$. By hypothesis 2, $\{f_k(\rho)\}_{k\in\mathbb{N}}$ is bounded in L^p and therefore it has a subsequence converging weakly in L^p to a limit $f(\rho) \in Y$ (recall that Y is closed with respect to the weak L^p convergence). By Corollary 3.3.1, such a subsequence converges in (Y, d) to the same limit $f(\rho)$. Since $\{f_k\}_{k\in\mathbb{N}}$ is uniformly Cauchy from K to (Y, d), we have that $\{f_k(\rho)\}_{k\in\mathbb{N}}$ is Cauchy in (Y, d), therefore $f_k(\rho) \xrightarrow{d} f(\rho)$ and, by Corollary 3.3.1, $f_k(\rho) \rightharpoonup (\rho)$ weakly in L^p.
Fix any $\rho \in K$ and let $\{\rho_i\}_{i\in\mathbb{N}} \subset S$ be such that $\rho_i \to \rho$. We claim that there exists $f_\rho \in Y$ such that $f(\rho_i) \to f_\rho$ w.r.t. d and f_ρ doesn't depend on the choice of the sequence $\{\rho_i\}_{i\in\mathbb{N}}$. Indeed by lower semicontinuity of the L^p norm w.r.t. the weak L^p convergence, we have

$$||f(\rho_i)||_{L^p} \leqslant \liminf_{k\to\infty} ||f_k(\rho_i)||_{L^p} \leqslant C,$$

for every $i \in \mathbb{N}$. Then there exists a subsequence $\{\rho_{i_j}\}_{j\in\mathbb{N}}$ and a $f_\rho \in L^p$ such that $f(\rho_{i_j}) \rightharpoonup f_\rho$ weakly in L^p. Since Y is weakly closed in L^p we have $f_\rho \in Y$. By Corollary 3.3.1 we also have $f(\rho_{i_j}) \to f_\rho$ w.r.t. d.
Relabel the subsequence as $\{\rho_i\}_{i\in\mathbb{N}}$. To see that f_ρ doesn't depend on the subsequence (and thus it doesn't depend on $\{\rho_i\}_{i\in\mathbb{N}}$), assume that $\{\tilde\rho_i\}_{i\in\mathbb{N}}$ is another sequence in S with $\tilde\rho_i \to \rho$ and $f(\tilde\rho_i) \to \tilde f_\rho$ w.r.t. d. To see that $f_\rho = \tilde f_\rho$, first notice that by hypothesis 3. and by triangle inequality we have

$$\begin{aligned} d\big(f(\rho_i), f(\tilde\rho_i)\big) &\leqslant d\big(f(\rho_i), f_k(\rho_i)\big) + d\big(f_k(\rho_i), f_k(\tilde\rho_i)\big) + d\big(f_k(\tilde\rho_i), f(\tilde\rho_i)\big) \\ &\leqslant d\big(f(\rho_i), f_k(\rho_i)\big) + A|\rho_i - \tilde\rho_i|^\alpha + d\big(f_k(\tilde\rho_i), f(\tilde\rho_i)\big). \end{aligned}$$

for every $i \in \mathbb{N}$. Hence, passing to the limit as $k \to \infty$ in the previous inequality, we get

$$d\big(f(\rho_i), f(\tilde\rho_i)\big) \leqslant A|\rho_i - \tilde\rho_i|^\alpha.$$

Thus we finally obtain

$$\begin{aligned} d(f_\rho, \tilde f_\rho) &\leqslant d\big(f_\rho, f(\rho_i)\big) + d\big(f(\rho_i), f(\tilde\rho_i)\big) + d\big(f(\tilde\rho_i), \tilde f_\rho\big) \\ &\leqslant d\big(f_\rho, f(\rho_i)\big) + A|\rho_i - \tilde\rho_i|^\alpha + d\big(f(\tilde\rho_i), \tilde f_\rho\big) \end{aligned}$$

and, passing to the limit as $i \to \infty$, we get $d(f_\rho, \tilde{f}_\rho) = 0$, i.e. $f_\rho = \tilde{f}_\rho$.
For any $\rho \in K$ let

$$f(\rho) := \lim_{i\to\infty} f(\rho_i),$$

where $\{\rho_i\}_{i\in\mathbb{N}} \subset S$ is any sequence such that $\rho_i \to \rho$ and the limit is understood w.r.t. d. Then f is a well-defined function on K.
To see that $f_k \to f$ uniformly, let $\varepsilon > 0$ and let $K \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}_{\geqslant K}$

$$\sup_{\rho\in S} d(f_m(\rho), f_n(\rho)) < \varepsilon.$$

Let $\rho \in K$ and let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence in S such that $\rho_i \to \rho$. Then for any $k \in \mathbb{N}_{\geqslant K}$ we have

$$d(f_k(\rho), f(\rho)) = \lim_{i\to\infty} d(f_k(\rho_i), f(\rho_i)) = \lim_{i\to\infty} \lim_{m\to\infty} d(f_k(\rho_i), f_m(\rho_i)) \leqslant \varepsilon.$$

This shows that f is the uniform limit of $\{f_k\}_{k\in\mathbb{N}}$ in K, with respect to d. As $f_k \in C^0(K, Y)$ for any $k \in \mathbb{N}$, we have that $f \in C^0(K, Y)$ (this also follows directly from the construction of f). □

Theorem 3.3.1 (Weak closure for $Q_1^n(0)$). *Fix any $n \in \mathbb{N}_{\geqslant 2}$ and assume that $p \in \big(1, n/(n-1)\big)$. Then $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1^n(0))$ is weakly sequentially closed.*

Proof. Assume that $F \in \Omega_p^{n-1}(Q_1(0))$ belongs to the weak L^p-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, i.e. there exists $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \xrightarrow{L^p} F$. What we need to show is that $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, which amounts to saying that

$$\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}, \tag{3.3.6}$$

for every $x_0 \in Q_1(0)$ and for a.e. $\rho \in \big(0, 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0))\big)$. Without losing generality, we will just show (3.3.6) for $x_0 = 0$.

Step 1. For any $k \in \mathbb{N}$ let s_k be the slice function of F_k at 0. Fix any $h \in \mathbb{N}_{\geqslant 2}$ and let $W_h \subset (1/h, 1)$ be the compact set given by applying Lemma 3.3.1 with $f_k = \|s_k\|_{L^p}$ and $C = 2^{\frac{n}{p}} \sup_{k\in\mathbb{N}} \|F_k\|_{L^p}$ (see Remark 3.3.7). Let $E_h^k \subset W_h$ denote the subset associated to W_h and s_k by Proposition 3.3.3, let $\tilde{E}_h = \bigcap_{k\in\mathbb{N}} E_h^k$ and let s_k denote its $\frac{1}{p'}$-Hölder representative on $\tilde{E}_h$, for any $k \in \mathbb{N}$. By property 3. in Lemma 3.3.1, for almost every $\rho \in \tilde{E}_h$ we can find a subsequence $\{s_{k_\rho}(\rho)\}_{k_\rho\in\mathbb{N}} \subset \{s_k(\rho)\}_{k\in\mathbb{N}}$ such that $s_{k_\rho}(\rho)$ is uniformly bounded in L^p by h. Denote by E_h the set of all such ρ and observe that $\mathcal{L}^1(E_h) = \mathcal{L}^1(W_h) = 1 - \frac{C+2}{h}$. Then for any $\rho \in E_h$, $\{s_{k_\rho}(\rho)\}_{k_\rho\in\mathbb{N}}$ has a subsequence that converges weakly in L^p or, equivalently, with respect to d_Ψ (see Corollary 3.3.1).
Let $S_h \subset E_h$ be a countable dense subset. By a diagonal extraction argument, we find a subsequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ such that $\{s_{k_l}(\rho)\}_{l\in\mathbb{N}}$ is convergent with respect to d_Ψ and weakly in L^p, and is uniformly bounded in L^p by h, for every $\rho \in S_h$.

Step 2. Next we claim that for any $l \in \mathbb{N}$, s_{k_l} can be extended to a $\frac{1}{p'}$-Hölder continuous function on $\overline{E_h}$ (with the same Hölder constant, which is bounded uniformly in l).
In fact let $f \in \{s_{k_l}\}_{l\in\mathbb{N}}$, let $\rho \in \overline{E_h} \smallsetminus E_h$, let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence in S_h such that $\rho_i \to \rho$ as $i \to \infty$ (observe that such a sequence exists, since S_h is dense in E_h, which in turn is dense in

$\overline{E_h}$). Since $\{f(\rho_i)\}_{i\in\mathbb{N}} \subset Y$ is uniformly bounded in L^p, by weak L^p-compactness there exists a subsequence $\{f(\rho_{i_j})\}_{j\in\mathbb{N}}$ of $\{f(\rho_i)\}_{i\in\mathbb{N}}$ such that $f(\rho_{i_j}) \rightharpoonup f_\rho$ weakly in L^p for some $f_\rho \in Y$. By Corollary 3.3.1, we know that $d_\Psi\big(f(\rho_{i_j}), f_\rho\big) \to 0$. Since f is $\frac{1}{p'}$-Hölder continuous on $\overline{E_h}$, f_ρ does not depend on the sequence $\{\rho_i\}_{i\in\mathbb{N}}$. Hence, the function

$$\tilde{f}(\rho) := \begin{cases} f(\rho) & \text{if } \rho \in E_h \\ f_\rho & \text{if } \rho \in \overline{E_h} \setminus E_h \end{cases}$$

is well-defined on $\overline{E_h}$ and satisfies (3.3.4) on $\overline{E_h}$.
In the following, in order to simplify the notation, we will denote again by s_{k_l} the $\frac{1}{p'}$-Hölder extension of s_{k_l} to $\overline{E_h}$, for any $l \in \mathbb{N}$.

Step 3. We show that $\{s_{k_l}\}_{l\in\mathbb{N}}$ converges uniformly on $\overline{E_h}$ to some $s \in C^0(\overline{E_h}, Y)$.
Fix any $\varepsilon > 0$. By Step 2. we know that the sequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ is equicontinuous from $\overline{E_h}$ to (Y, d_Ψ). Therefore we can choose $\delta > 0$ such that

$$d_\Psi\big(s_{k_l}(\rho), s_{k_l}(\rho')\big) < \varepsilon, \qquad \forall\, \rho, \rho' \in \overline{E_h} \text{ s.t. } |\rho - \rho'| < \delta \text{ and } \forall\, l \in \mathbb{N}.$$

Notice that $\{(\rho - \delta, \rho + \delta)\}_{\rho\in S_h}$ is an open cover of $\overline{E_h}$. Since $\overline{E_h}$ is compact, we can find a finite set $\{\rho_1, ..., \rho_m\} \subset S_h$ such that $\{(\rho_j - \delta, \rho_j + \delta)\}_{j=1,...,m}$ is a finite open cover of $\overline{E_h}$.
Now let $\rho \in \overline{E_h}$. Observe that there exists a point $\rho_j \in \{\rho_1, ..., \rho_m\}$ such that $\rho \in (\rho_j - \delta, \rho_j + \delta)$, i.e. $|\rho - \rho_j| < \delta$. By our choice of δ, this implies $d\big(s_{k_l}(\rho), s_{k_l}(\rho_j)\big) < \varepsilon$, for every $l \in \mathbb{N}$. By triangle inequality, we have

$$\begin{aligned} d_\Psi\big(s_{k_l}(\rho), s_{k_m}(\rho)\big) &\leqslant d_\Psi\big(s_{k_l}(\rho), s_{k_l}(\rho_j)\big) + d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big) + d_\Psi\big(s_{k_m}(\rho_j), s_{k_m}(\rho)\big) \\ &< 2\varepsilon + d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big). \end{aligned}$$

But since $\rho_j \in S_h$, we know that there exists $L_j > 0$ such that

$$d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big) < \varepsilon, \qquad \forall\, l, m \geqslant L_j.$$

Hence, by letting $L := \max_{j=1,...,m} L_j$, we have that

$$d_\Psi\big(s_{k_l}(\rho), s_{k_m}(\rho)\big) < 3\varepsilon, \qquad \forall\, l, m \geqslant L, \forall\, \rho \in \overline{E_h}.$$

Here we have just proved that the sequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ is uniformly Cauchy on $\overline{E_h}$. Since $\{s_{k_l}\}_{l\in\mathbb{N}}$ satisfies all the hypotheses of Lemma 3.3.2, we get that there exists $s \in C^0(\overline{E_h}, Y)$ such that $s_{k_l} \to s$ uniformly on $\overline{E_h}$ w.r.t. d_Ψ.
Notice that since $\|s_{k_l}(\rho)\|_{L^p} \leqslant h$ for any $l \in \mathbb{N}$, for any $\rho \in E_h$, and since $s_{k_l}(\rho) \to s(\rho)$ w.r.t. d_Ψ for any $\rho \in \overline{E_h}$, by Corollary 3.3.1 there holds $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p for any $\rho \in E_h$.

Step 4. Let $s_0 : E_h \to Y$ be the restriction to E_h of the slice function of F at 0. We claim that $s = s_0$ a.e. in E_h. To show this we will prove that

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s_0(\rho)\big)\, d\rho \to 0, \qquad \text{as } l \to \infty,$$

for every $\varphi \in C_c^\infty((0,1))$ and for every $\psi \in \mathrm{Lip}(\partial Q_1(0))$. Indeed, an explicit computation gives

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s_0(\rho)\big)\, d\rho$$

$$= \int_{E_h} \varphi(\rho) \Big(\int_{\partial Q_\rho(0)} \psi\Big(\frac{\cdot}{\rho}\Big) i^*_{\partial Q_\rho(0)} F_{k_l} - \int_{\partial Q_\rho} \psi\Big(\frac{\cdot}{\rho}\Big) i^*_{\partial Q_\rho(0)} F \Big)\, d\rho$$
$$= 2^n \int_{Q_1(0)} \mathbb{1}_{E_h}(2r)\varphi(2|\cdot|)\psi\Big(\frac{\cdot}{|\cdot|}\Big) dr \wedge (F_{k_l} - F) \to 0 \text{ as } l \to \infty,$$

since

$$\mathbb{1}_{E_h}(2r)\varphi(2|\cdot|)\psi\Big(\frac{\cdot}{|\cdot|}\Big) dr \in \Omega^1_{p'}(Q_1(0))$$

and $F_{k_l} \xrightarrow{L^p} F$. On the other hand, since $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p for any $\rho \in E_h$,

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s(\rho)\big)\, d\rho \to 0 \qquad \text{as } l \to \infty$$

for every $\varphi \in C^\infty_c((0,1))$ and for every $\psi \in \mathrm{Lip}(\partial Q_1(0))$, we obtain

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s(\rho) - s_0(\rho)\big)\, d\rho = 0, \qquad \forall\, \varphi \in C^\infty_c((0,1)), \forall\, \psi \in \mathrm{Lip}(\partial Q_1(0)).$$

This means that $s_0(\rho) = s(\rho) \in Y$ for a.e. $\rho \in E_h$.

Step 5. Finally we show that (3.3.6) holds for almost any $\rho \in (0,1)$.
In fact for any $\rho \in E_h$ such that $s_0(\rho) = s(\rho)$ we have

$$\int_{\partial Q_\rho(0)} i^*_{\partial Q_\rho} F = \int_{\partial Q_1(0)} s_0(\rho) = \int_{\partial Q_1(0)} s(\rho) = \lim_{k_l \to \infty} \int_{\partial Q_1(0)} s_{k_l}(\rho) \in \mathbb{Z},$$

since $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p. Thus (3.3.6) holds for $\mathcal{L}^1$-a.e. $\rho \in E_h$. Since the previous step can be repeated for any $h \in \mathbb{N}_{\geqslant 2}$, and since

$$\lim_{h \to +\infty} \mathcal{L}^1(E_h) = \lim_{h \to +\infty} 1 - \frac{C+1}{h} = 1,$$

we conclude that (3.3.6) holds for $\mathcal{L}^1$-a.e. $\rho \in (0,1)$. □

Remark 3.3.8. Let $D \subset \mathbb{R}^n$ be any open and bounded domain which is bi-Lipschitz equivalent to $Q_1(0)$. From Theorem 3.3.1 follows that $\Omega^{n-1}_{p,\mathbb{Z}}(D)$ (see Definition 3.1.4) is a weakly sequentially closed subspace of $\Omega^{n-1}_p(D)$. Indeed, let $\varphi : Q_1(0) \to D$ be any bi-Lipschitz homeomorphism and let $\{F_k\}_{k \in \mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(D)$ be such that $F_k \xrightarrow{L^p} F$ on D. Then, by Lemma 3.2.7 we have $\{\varphi^* F_k\}_{k \in \mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ and as φ is bi-Lipschitz we have $\varphi^* F_k \xrightarrow{L^p} \varphi^* F$ on $Q_1(0)$. By the Weak Closure Theorem (Theorem 3.3.1), $\varphi^* F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. Thus $F \in \Omega^{n-1}_{p,\mathbb{Z}}(D)$ (again by Lemma 3.2.7). This shows that Theorem 3.1.3 holds true.

Observe that Theorem 3.1.3 does not hold if $n = 1$. In fact in this case the following holds.

Lemma 3.3.3. *Let I be a bounded connected interval in $\mathbb{R}$. Let $p \in [1, \infty)$.*
The weak closure of $L^p_{\mathbb{Z}}(I)$ in $L^p(I)$ is $L^p(I)$.

Proof. Since $C^0(\overline{I})$ is dense in $L^p(I)$, it is enough to show that any function $f \in C^0(\overline{I})$ can be approximated weakly in $L^p(I)$ by functions in $L^p_{\mathbb{Z}}(I)$. Without loss of generality we can assume that $I = [0,1)$. For any $n \in \mathbb{N}$ let's define $f_n : I \to \mathbb{R}$ as follows:
for any $k \in \{1, ..., 2^n\}$ let $I_k^n := [\frac{k-1}{2^n}, \frac{k}{2^n})$, for any $k \in \{1, ..., 2^n\}$ let $c_k := \fint_{I_k^n} f(x)dx$ and for any $x \in I_k^n$ set

$$f_n(x) := \begin{cases} \lceil c_k \rceil & \text{if } x - \frac{k-1}{2^n} \leqslant \frac{c_k}{2^n \lceil c_k \rceil} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in L^p(I, \mathbb{Z})$ and $\int_{I_k^n} f_n(x)dx = \int_{I_k^n} f(x)dx$ for any $k \in \{1, ..., 2^n\}$, for any $n \in \mathbb{N}$.
Moreover notice that since f is bounded, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(I)$. Therefore if $p > 1$ $(f_n)_{n\in\mathbb{N}}$ converges weakly in $L^p(I)$, up to a subsequence, to a function $\tilde{f} \in L^p(I)$. Testing against continuous functions on $\overline{I}$ it is easy to check that $f = \tilde{f}$.
If $p = 1$ we have to check that, up to a subsequence,

$$\lim_{n\to\infty} \int_I f_n g = \int_I fg \quad \forall\, g \in L^\infty(I). \tag{3.3.7}$$

Since $L^\infty(I) \subset L^q(I)$ for any $q > 1$, (3.3.7) follows from the case $p > 1$ (with $p = q'$). □

Remark 3.3.9. Let $n \geqslant 2$ and let $D \subset \mathbb{R}^n$ be any open, bounded and Lipschitz domain in $\mathbb{R}^n$. It is still unknown if the space $L^1_{\mathbb{Z}}(D)$ is weakly sequentially closed. Surely it is not weakly-$*$ closed, a proof this fact can be easily achieved by generalising the arguments in [63, Section 8].

Appendix to Chapter 3

3.A. Minimal connections for forms with finitely many integer singularities

Throughout this appendix, we will denote by $M^m \subset \mathbb{R}^n$ some arbitrary embedded Lipschitz and connected m-dimensional submanifold of $\mathbb{R}^n$. Let $p \in [1, \infty]$. We will denote by $\Omega^1_{p,R,\infty}(M)$ the space introduced in Definition 3.2.1.

Lemma 3.A.1. *Let $F \in \Omega^{m-1}_{p,R,\infty}(M)$. Then, there exists a connection for F.*

Proof. Throughout the following proof, given any couple of points $x, y \in M$ we will denote by (x, y) an arbitrarily chosen oriented Lipschitz curve with finite length joining x and y. By assumption, it holds that

$$*dF = \sum_{j=1}^{N} d_j \delta_{x_j}, \qquad \text{for some } d_1, ..., d_N \in \mathbb{Z} \smallsetminus \{0\} \text{ and } x_1, ..., x_N \in M.$$

We define

$$\begin{aligned} \{i_1, ..., i_p\} &:= \big\{ j \in \{1, ..., N\} \text{ s.t. } d_j > 0 \big\}, \\ \{j_1, ..., j_q\} &:= \big\{ j \in \{1, ..., N\} \text{ s.t. } d_j < 0 \big\}, \\ d &:= \sum_{j=1}^{N} d_j \in \mathbb{Z}. \end{aligned}$$

We build a family $\mathscr{F} = \{I_\alpha\}_{\alpha \in A}$ of oriented Lipschitz curves in M as follows. If there is no point x_j such that $d_j < 0$, then we set $\mathscr{F} = \emptyset$. Else, we start from x_{i_1} and we add to the family $\mathscr{F}$ the curves $\big(x_{j_1}, x_{i_1}\big), ..., \big(x_{j_{k_1}}, x_{i_1}\big)$, until we reach the condition $k_1 = q$ or the condition

$$r_1 := d_{i_1} + \sum_{l=1}^{k_1} d_{j_l} \leqslant 0.$$

If $k_1 = q$, then we stop. Else, we move to the point x_{i_2}.
If $r_1 = 0$, then we add to $\mathscr{F}$ the segments $\big(x_{j_{k_1+1}}, x_{i_2}\big), ..., \big(x_{j_{k_2}}, x_{i_2}\big)$, where $k_2 \in \{1, ..., q\}$ is the smallest value such that

$$r_2 := d_{i_2} + \sum_{l=k_1+1}^{k_2} d_{j_l} \leqslant 0.$$

If there is no $k \in \{1, ..., q\}$ such that

$$d_{i_2} + \sum_{l=k_1+1}^{k} d_{j_l} \leqslant 0,$$

then we add to $\mathscr{F}$ the segments $\big(x_{j_{k_1+1}}, x_{i_2}\big), ..., \big(x_{j_q}, x_{i_2}\big)$ and we set $k_2 = q$.

If $r_1 < 0$, then we add to $\mathscr{F}$ the segments $\left(x_{j_{k_1}}, x_{i_2}\right)$ and $\left(x_{j_{k_1+1}}, x_{i_2}\right), ..., \left(x_{j_{k_2}}, x_{i_2}\right)$, where $k_2 \in \{1, ..., q\}$ is the smallest value such that

$$r_2 := d_{i_2} + r_1 + \sum_{l=k_1+1}^{k_2} d_{j_l} \leqslant 0.$$

If there is no $k \in \{1, ..., q\}$ such that

$$d_{i_2} + r_1 + \sum_{l=k_1+1}^{k} d_{j_l} \leqslant 0,$$

then we add to $\mathscr{F}$ the segments $\left(x_{j_{k_1}}, x_{i_2}\right)$ and $\left(x_{j_{k_1+1}}, x_{i_2}\right), ..., \left(x_{j_q}, x_{i_2}\right)$ and we set $k_2 = q$. We proceed iteratively in this way, moving on to the subsequent points x_{i_s} until $k_s = q$ or $s = p$. Then, the construction of the family $\mathscr{F}$ is complete. We let x_{i_h} be the last node that is visited before the iteration stops and, for every $I_\alpha = (x_j, x_i) \in \mathscr{F}$, we define its multiplicity m_α as

$$m_\alpha := \begin{cases} |d_j| - |r_l| & \text{if } i = i_l \text{ and } j = i_{k_l}, \\ \min\{|d_i|, |r_{l-1}|\} & \text{if } i = i_l \text{ and } j = i_{k_{l-1}}, \\ \min\{|d_j|, |d_i|\} & \text{else.} \end{cases}$$

Finally, we divide three cases:

1. *Case* $d = 0$. Notice that this is always the case if M has no boundary. We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

2. *Case* $d > 0$. We fix a point $x_0 \in \partial M$ and we let $I_s^p := (x_0, x_{i_s})$, for every $s = h, ..., p$. We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + r_h \int_{I_h^b} \omega + \sum_{s=h+1}^{p} d_{i_s} \int_{I_s^b} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

3. *Case* $d < 0$. We fix a point $x_0 \in \partial M$ and we let $I_s^n := (x_{j_s}, x_0)$, for every $s = k_h, ..., q$ We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + |r_h| \int_{I_h^b} \omega + \sum_{s=k_h+1}^{q} |d_{i_s}| \int_{I_s^b} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

By direct computation, we verify that I has the desired properties and the statement follows. □

Lemma 3.A.2. *Let* $F \in \Omega_{p,\mathbb{Z}}^{m-1}(M)$ *(see Definition 3.1.4). Then,*

$$\inf_{\substack{T \in \mathcal{D}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{M}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W_0^{1,\infty}(M), \\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi < +\infty, \tag{3.A.1}$$

where $\mathcal{M}_1(M)$ *denotes the set of all the* 1*-currents with finite mass on* M*. Moreover, the infimum on the left-hand-side of the previous chain of equalities is achieved.*

Proof. By definition, there exists an integer 1-current $I \in \mathcal{R}_1(M)$ with finite mass such that $\partial I = *dF$. Hence

$$\inf_{\substack{T\in\mathcal{D}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) \leqslant \mathbb{M}(I) < +\infty.$$

The first equality in (3.A.1) is clear.
Notice that for every $T \in \mathcal{M}_1(M)$ such that $\partial T = *dF$ it holds that

$$\int_M F \wedge d\varphi = \langle *dF, \varphi\rangle = \langle \partial T, \varphi\rangle = \langle T, d\varphi\rangle \leqslant \mathbb{M}(T)||d\varphi||_{L^\infty(M)}, \qquad \forall\, \varphi \in W_0^{1,\infty}(M).$$

Hence,

$$\inf_{\substack{T\in\mathcal{M}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) \geqslant \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi. \tag{3.A.2}$$

To prove that the former inequality is actually an equality, it suffices to show that the supremum on its right-hand-side is greater than the mass of some 1-current with finite mass T on M such that $\partial T = *dF$. Define the vector subspace $X \subset \Omega^1_\infty(M)$ given by

$$X := \{\omega \in \Omega^1_\infty(M) \text{ s.t. } \omega = d\varphi, \text{ for some } \varphi \in W_0^{1,\infty}(M)\}.$$

Consider the linear functional $\phi : X \subset (\Omega^1(M), \|\cdot\|_{L^\infty}) \to \mathbb{R}$ given by

$$\langle \phi, \omega\rangle = \int_M F \wedge \omega, \qquad \forall\, \omega \in X.$$

By (3.A.2) we get that ϕ is continuous on X, i.e.

$$||\phi||_{\mathcal{L}(X)} = \sup_{\substack{\omega\in X,\\ ||\omega||_{L^\infty}\leqslant 1}} \int_M F\wedge\omega = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi \leqslant \inf_{\substack{T\in\mathcal{M}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) < +\infty.$$

By Hahn-Banach theorem, we can extend ϕ to a linear functional $T : \Omega^1(M) \to \mathbb{R}$ such that

$$||T||_{\mathcal{L}(\Omega^1_\infty(M))} = ||\phi||_{\mathcal{L}(X)} = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi$$

But then, T is a 1-current on M having finite mass and such that

$$\mathbb{M}(T) \leqslant ||T||_{\mathcal{L}(\Omega^1_\infty(M))} = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi.$$

Moreover,

$$\langle \partial T, \varphi\rangle = \langle T, d\varphi\rangle = \langle \phi, d\varphi\rangle = \int_M F \wedge d\varphi = \langle *dF, \varphi\rangle, \qquad \forall\, \varphi \in W_0^{1,\infty}(M).$$

Hence,

$$\mathbb{M}(T) = \inf_{\substack{\tilde{T}\in\mathcal{D}_1(M),\\ \partial\tilde{T} = *dF}} \mathbb{M}(\tilde{T}) = \inf_{\substack{\tilde{T}\in\mathcal{M}_1(M),\\ \partial\tilde{T} = *dF}} \mathbb{M}(\tilde{T}) = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi < +\infty$$

and the statement follows. □

Proposition 3.A.1. *Let $F \in \Omega^{m-1}_{p,R}(M)$. Then, there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and*

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{D}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W^{1,\infty}_0(M),\\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi.$$

In particular,

$$\mathbb{M}(L) \leqslant C||F||_{L^p}.$$

Proof. Notice that by [38, Chapter 1, Section 3.4, Theorem 8], we have

$$\inf_{\substack{T \in \mathcal{R}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{D}_1(M),\\ \partial T = *dF}} \mathbb{M}(T).$$

Since the mass $\mathbb{M}(\,\cdot\,)$ is lower semicontinuous with respect to the weak convergence in $\mathcal{D}_1(M)$ and since $\mathbb{M}$-bounded subsets of the competition class $\mathcal{R}_1(M) \cap \{T \in \mathcal{D}_1(M) \text{ s.t. } \partial T = *dF\}$ are weakly sequentially compact (for a reference, see e.g. [53, Equation (7.5), Theorem 7.5.2]), by the direct method of calculus of variations we conclude that there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{R}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{D}_1(M),\\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W^{1,\infty}_0(M),\\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi,$$

where the last equality follows from Lemma 3.A.2. The statement follows. □

3.B. Laplace equation on spheres

Let $n \in \mathbb{N}$ be such that $n \geqslant 2$ and fix any $p \in (1, +\infty)$. We let

$$\dot{W}^{1,p}(\mathbb{S}^{n-1}) := \left\{ u \in W^{1,p}(\mathbb{S}^{n-1}) \text{ s.t. } \bar{u} := \int_{\mathbb{S}^{n-1}} u \, d\,\mathrm{vol}_{\mathbb{S}^{n-1}} = 0 \right\}.$$

We can endow the space $\dot{W}^{1,p}(\mathbb{S}^{n-1})$ with the usual $W^{1,p}$-norm induced by $W^{1,p}(\mathbb{S}^{n-1})$, given by

$$||u||_{W^{1,p}} := ||u||_{L^p} + ||du||_{L^p}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Lemma 3.B.1 (Poincaré inequality on $\dot{W}^{1,p}$)**.** *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{S}^{n-1}} |u|^p \, d\,\mathrm{vol}_{\mathbb{S}^{n-1}} \leqslant C \int_{\mathbb{S}^{n-1}} |du|^p \, d\,\mathrm{vol}_{\mathbb{S}^{n-1}}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Proof. By contradiction, assume that for every $k > 0$ there exists $u_k \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $||u_k||_{L^p} = 1$ and

$$1 > k \int_{\mathbb{S}^{n-1}} |du_k|^p \, d\,\mathrm{vol}_{\mathbb{S}^{n-1}}.$$

This implies immediately that $||du_k||_{L^p} \to 0$ as $k \to \infty$. In particular, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded with respect to the $W^{1,p}$-norm. Hence, by weak compactness of $W^{1,p}(\mathbb{S}^{n-1})$, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_j} \rightharpoonup u$ in $W^{1,p}(\mathbb{S}^{n-1})$. Moreover, by Rellich-Kondrakov theorem, we have $u_{k_j} \to u$ strongly in $L^p(\mathbb{S}^{n-1})$. Since $du_{k_j} \to 0$ strongly in L^p we get $du = 0$. Then, u is constant on $\mathbb{S}^{n-1}$. Since $u_{k_j} \rightharpoonup u$ in $L^p(\mathbb{S}^{n-1})$, it follows that

$$0 = \lim_{j\to\infty} \int_{\mathbb{S}^{n-1}} u_{k_j}\, d\,\mathrm{vol}_{\mathbb{S}^{n-1}} = \int_{\mathbb{S}^{n-1}} u\, d\,\mathrm{vol}_{\mathbb{S}^{n-1}}$$

and this leads to $u = 0$. But this is absurd, since by strong L^p-convergence of $\{u_{k_j}\}_{j\in\mathbb{N}}$ to u we obtain $||u||_{L^p} = 1$. □

Remark 3.B.1. By Lemma 3.B.1, we conclude that we can endow $\dot{W}^{1,p}$ with the following much more convenient norm:

$$||u||_{\dot{W}^{1,p}} := ||du||_{L^p}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Moreover, such a norm is equivalent to $W^{1,p}$-norm.

Remark 3.B.2. Notice that a linear functional on $W^{1,p}(\mathbb{S}^{n-1})$ restricts to an element of $(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*$ if and only if it is $W^{1,p}$-continuous and $\langle F, 1\rangle = 0$.

Lemma 3.B.2. *Let $F \in (W^{1,p'}(\mathbb{S}^{n-1}))^*$ be such that $\langle F, 1\rangle = 0$. Then, the following facts hold.*

1. *If $n \geqslant 3$ the linear differential system*

$$\begin{cases} d^*\omega = F \\ d\omega = 0 \end{cases}$$

has a unique weak solution $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$.

2. *If $n = 2$ the linear differential system*

$$\begin{cases} d^*\omega = F \\ d\omega = 0 \\ \displaystyle\int_{\mathbb{S}^1} \omega = 0 \end{cases}$$

has a unique weak solution $\alpha \in \Omega^1_p(\mathbb{S}^1)$.

In both cases, α satisfies the following estimate:

$$||\alpha||_{L^p} \leqslant C||F||_{\mathcal{L}(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))},$$

for some constant $C > 0$ depending only n.

Proof. Observe that, by Remark 3.B.2, F restricts to an element of $(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*$. Consider the linear functional $\phi : \Omega^1(\mathbb{S}^{n-1}) \to \mathbb{R}$ given by

$$\langle \phi, \omega\rangle = \langle F, u\rangle, \qquad \forall\, \omega = du + d^*\beta + \eta \in \Omega^1(\mathbb{S}^{n-1}),$$

where η is a harmonic 1-form on $\mathbb{S}^{n-1}$. Since ϕ is continuous and linear on $\Omega^1(\mathbb{S}^{n-1})$ w.r.t. the $L^{p'}$-norm, by Hahn-Banach theorem there exists a unique (recall that L^p-spaces are strictly convex) extension $\Phi \in (\Omega^1_{p'}(\mathbb{S}^{n-1}))^*$ of ϕ such that

$$||\Phi||_{(\Omega^1_{p'}(\mathbb{S}^{n-1}))^*} = ||\phi||_{(\Omega^1(\mathbb{S}^{n-1}))^*} \leqslant C||F||_{\mathcal{L}(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))}.$$

By Riesz representation theorem, there exists a unique $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ such that

$$\langle \alpha, \omega \rangle_{L^p - L^{p'}} := \int_{\mathbb{S}^{n-1}} \alpha \wedge *\omega = \langle \Phi, \omega \rangle, \qquad \forall\, \omega \in \Omega^1_{p'}(\mathbb{S}^{n-1}) \tag{3.B.1}$$

and

$$||\alpha||_{L^p} = ||\Phi||_{(\Omega^1_{p'}(\mathbb{S}^{n-1}))^*} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Finally, by applying equation (3.B.1) with get

$$\langle \alpha, du \rangle_{L^p - L^{p'}} = \langle \Phi, du \rangle = \langle \phi, du \rangle = \langle F, u \rangle, \qquad \forall\, u \in C^\infty(\mathbb{S}^{n-1}),$$

and

$$\langle \alpha, d^*\beta \rangle_{L^p - L^{p'}} = \langle \Phi, d^*\beta \rangle = \langle \phi, d^*\beta \rangle = 0, \qquad \forall\, \beta \in \Omega^2(\mathbb{S}^{n-1}).$$

The two previous equations are exactly the weak forms of the equations $d^*\alpha = F$ and $d\alpha = 0$ respectively. Moreover, in case $n = 2$, we have

$$\int_{\mathbb{S}^1} \alpha = \int_{\mathbb{S}^1} \alpha \wedge *1 = \langle \alpha, *1 \rangle_{L^p - L^{p'}} = \langle \Phi, *1 \rangle = \langle F, 1 \rangle = 0.$$

This concludes about the existence of a solution to the differential systems given in points 1 and 2. For what concerns uniqueness, assume that α and α' are two solutions of the differential system given in point 1 (resp. 2) and define $\beta = \alpha - \alpha'$. Then, we distinguish the two cases:

Case $n \geqslant 3$. In this case, β satisfies

$$\begin{cases} d^*\beta = 0 \\ d\beta = 0. \end{cases}$$

Hence, β is a harmonic 1-forms on $\mathbb{S}^{n-1}$ for $n \geqslant 1$, which implies $\beta = 0$.

Case $n = 2$. In this case, β satisfies

$$\begin{cases} d^*\beta = 0 \\ d\beta = 0 \\ \displaystyle\int_{\mathbb{S}^1} \beta = 0. \end{cases}$$

Hence, β is a harmonic 1-forms on $\mathbb{S}^1$, which implies $\beta = c\,\mathrm{vol}_{\mathbb{S}^1}$ for some $c \in \mathbb{R}$. But since β has vanishing integral on $\mathbb{S}^1$, we get $\beta = 0$. ☐

Definition 3.B.1 (Sobolev spaces of differential forms)**.** Fix any $k \in \mathbb{N} \setminus \{0\}$. We define the *Sobolev space of $W^{1,p}$-regular differential k-forms on* $\mathbb{S}^{n-1}$ by

$$\Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}) := \big\{\omega \in \Omega^k_p(\mathbb{S}^{n-1}) \text{ s.t. } d\omega, d^*\omega \in L^p\big\}.$$

We endow such space with the norm

$$||\omega||_{W^{1,p}} := ||\omega||_{L^p} + ||d\omega||_{L^p} + ||d^*\omega||_{L^p}, \qquad \forall\, \omega \in \Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}).$$

Remark 3.B.3. It can be shown (see [80, §3 and §4]) that such Sobolev spaces are completely equivalent to the usual ones, namely the space of k-forms having local coefficients in $W^{1,p}$. Moreover, in case $n \geqslant 3$ there exists $C > 0$ such that

$$||\omega||_{W^{1,p}} \leqslant C\big(||d\omega||_{L^p} + ||d^*\omega||_{L^p}\big), \qquad \forall\, \omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}). \tag{3.B.2}$$

Indeed, let

$$\begin{aligned} X &:= \{d\alpha \text{ s.t. } \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})\}, \\ Y &:= \{d^*\beta \text{ s.t. } \beta \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})\}. \end{aligned}$$

By [80, Proposition 7.1], both X and Y are closed linear subspaces respectively of $\Omega^2_p(\mathbb{S}^{n-1})$ and $\dot{W}^{1,p}(\mathbb{S}^{n-1})$. Then, $X \oplus Y$ is a Banach space with respect to the standard norm on the direct sum of two Banach spaces. We claim that the liner operator $T : \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \to X \oplus Y$ given by $T\omega = (d\omega, d^*\omega)$, for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ is a continuous linear bijection between Banach spaces. Indeed, the fact that T is injective follows form the fact that there no non-zero harmonic forms on $\mathbb{S}^{n-1}$ for $n \geqslant 3$. Hence, we just need to show that T is surjective. Pick any $(d\alpha, d^*\beta) \in X \oplus Y$. By Lemma 3.B.2, the linear differential system

$$\begin{cases} d^*\omega = d^*(\beta - \alpha) \\ d\omega = 0 \end{cases}$$

has a unique weak solution $\tilde{\omega} \in \Omega^1_p(\mathbb{S}^{n-1})$. Since by construction we have $d\tilde{\omega}, d^*\tilde{\omega} \in L^p$, we conclude that $\tilde{\omega} \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$. Then, by letting $\omega := \tilde{\omega} + \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ we see that

$$T\omega = (d\omega, d^*\omega) = (d\tilde{\omega} + d\alpha, d^*\tilde{\omega} + d^*\alpha) = (d\alpha, d^*\beta)$$

and we have proved our claim. This proves that T has a continuous inverse and the statement follows with $C = ||T^{-1}||_{\mathcal{L}(X\oplus Y, \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}))}$.

In case $n = 2$, the estimate (3.B.2) still holds for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^1)$ such that

$$\int_{\mathbb{S}^1} \omega = 0.$$

The proof is completely analogous.

Remark 3.B.4 (L^p-Hodge decomposition). Let

$$\begin{aligned} X &:= \{d\varphi \text{ s.t. } \varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})\}, \\ Y &:= \{d^*\beta \text{ s.t. } \beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})\}, \end{aligned}$$

$$Z := \{\eta \in \Omega^1(\mathbb{S}^{n-1}) \text{ s.t. } \Delta\eta = (dd^* + d^*d)\eta = 0\}.$$

Then, as a particular consequence of the L^p-Hodge decomposition theorem (see e.g. [80, Proposition 6.5]), the operator $T : X \oplus Y \oplus Z \to \Omega^1_p(\mathbb{S}^{n-1})$ given by

$$T(d\varphi, d^*\beta, \eta) := d\varphi + d^*\beta + \eta$$

is a continuous and linear isomorphism between Banach spaces. Hence, T has a continuous inverse. We let

$$C_H := ||T^{-1}||_{\mathcal{L}(\Omega^1_p(\mathbb{S}^{n-1}), X\oplus Y)}.$$

We conclude that for every $\omega \in \Omega^1_p(\mathbb{S}^{n-1})$ there exist $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$, $\beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})$ and $\eta \in Z$ such that $\omega = d\varphi + d^*\beta + \eta$ and

$$||d\varphi||_{L^p} + ||d^*\beta||_{L^p} + ||\eta||_{L^p} \leqslant C_H||\omega||_{L^p}. \tag{3.B.3}$$

Lemma 3.B.3 (A weak version of Poincarè lemma). *Let $n \geqslant 3$ and let $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ be such that $d\alpha = 0$ weakly on $\mathbb{S}^{n-1}$. Then, there exists a Sobolev function $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $d\varphi = \alpha$ weakly on $\mathbb{S}^{n-1}$.*

Proof. We follow the notation of Remark 3.B.4 and we notice that, since $n \geqslant 3$ we have $Z = \{0\}$. Hence, we write $\alpha = d\varphi + d^*\beta$, for $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ and $\beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})$. We observe that

$$\begin{aligned}
\langle d^*\beta, \omega\rangle_{L^p - L^{p'}} &= \langle d^*\beta, d\psi + d^*\gamma\rangle_{L^p - L^{p'}} \\
&= \langle d^*\beta, d^*\gamma\rangle_{L^p - L^{p'}} \\
&= \langle \alpha - d\varphi, d^*\gamma\rangle_{L^p - L^{p'}} \\
&= \langle \alpha, d^*\gamma\rangle_{L^p - L^{p'}} = 0, \qquad \forall\, \omega = d\psi + d^*\gamma \in \Omega^1(\mathbb{S}^{n-1}).
\end{aligned}$$

This implies $d^*\beta = 0$ and the statement follows. □

Corollary 3.B.1 (Laplace equation on spheres). *Let $F \in \mathcal{L}(W^{1,p'}(\mathbb{S}^{n-1}))$ such that $\langle F, 1\rangle = 0$. Then, the linear differential equation*

$$\Delta u = F$$

has a unique weak solution $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ satisfying

$$||\varphi||_{W^{1,p}} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Proof. First, we face the case $n \geqslant 3$. By Lemma 3.B.2 we can find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} d^*\alpha = F \\ d\alpha = 0 \end{cases}$$

and

$$||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Since $d\alpha = 0$, by Lemma 3.B.3 there exists $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $\alpha = d\varphi$. Hence, we get

$$\Delta\varphi = d^* d\varphi = d^*\alpha = F.$$

Moreover, by Lemma 3.B.1, we have

$$||\varphi||_{W^{1,p}} \leqslant C||d\varphi||_{L^p} = C||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

This concludes the proof in case $n \geqslant 3$.

If $n = 2$, then by Lemma 3.B.2 we can find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} d^*\alpha = F \\ d\alpha = 0 \\ \displaystyle\int_{\mathbb{S}^1} \alpha = 0 \end{cases}$$

and

$$||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

By setting $\varphi := *\alpha$, the statement follows. □

3.C. Some technical lemmata

In this section we will make use of the following notation: let T be an m-rectifiable current in $\mathbb{R}^n$, then T can be represented as follows:

$$\langle T, \omega\rangle = \int_{\mathbb{R}^n} \theta\langle\omega, \xi\rangle d\mathscr{H}^m\big|_{\Sigma} \quad \forall\omega \in \Omega^m(\mathbb{R}^n),$$

where Σ is a locally m-rectifiable set,

$$\theta : \Sigma \to \mathbb{Z}$$

is a locally $\mathscr{H}^m$-integrable, non-negative function and

$$\xi : \Sigma \to \Lambda_m\mathbb{R}^n$$

is an $\mathscr{H}^m$-measurable function such that for $\mathscr{H}^m$-almost every point $x \in \Sigma$, $\xi(x)$ is a simple unit m-vector in $T_x\Sigma$.

In this case we write

$$T = \tau(\Sigma, \theta, \xi).$$

Lemma 3.C.1. *For any $k \in \mathbb{N}$ let*

$$T_k = \tau(\Sigma_k, \theta_k, \xi_k)$$

be an m-rectifiable current on $\mathbb{R}^n$ of finite mass. Assume that $(T_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the convergence in mass.

Then there exists an m-rectifiable current

$$T = \tau(\Sigma, \theta, \xi)$$

such that

$$T_k \to T \quad (k \to \infty) \quad \textit{in mass}.$$

Proof. Replacing the original sequence by a subsequence if necessary, we may assume that for any $k \in \mathbb{N}$

$$\mathbb{M}(T_k - T_{k+1}) \leqslant 2^{-k}.$$

Now for any $k \in \mathbb{N}$ let

$$\tilde{T}_k := \begin{cases} T_1 & \text{if } k = 1 \\ T_k - T_{k-1} & \text{if } k > 1. \end{cases}$$

Then for any $k \in \mathbb{N}$ we have

$$T_k = \sum_{i=1}^{k} \tilde{T}_i.$$

For any $k \in \mathbb{N}$ write

$$\tilde{T}_k = \langle \tilde{\Sigma}_k, \tilde{\theta}_k, \tilde{\xi}_k \rangle.$$

Notice that

$$\sum_{i=1}^{k} \tilde{\theta}_i \tilde{\xi}_i = \theta_k \xi_k \qquad \mathscr{H}^m\text{- a.e. on } \Sigma,$$

for every $k \in \mathbb{N}$. Set

$$\Sigma := \bigcup_{k \in \mathbb{N}} \big(\tilde{\Sigma}_k \setminus \tilde{\theta}_k^{-1}(0) \big).$$

Then Σ is m-rectifiable as countable union of m-rectifiable sets. Moreover $\mathscr{H}^m(\Sigma) < \infty$. In fact

$$\mathscr{H}^m(\Sigma) \leqslant \sum_{k \in \mathbb{N}} \mathscr{H}^m \big(\tilde{\Sigma}_k \setminus \tilde{\theta}_k^{-1}(0) \big) \leqslant \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty.$$

Next let

$$\theta = \sum_{k \in \mathbb{N}} \tilde{\theta}_k,$$

where $\tilde{\theta}_k$ is extended by zero on $\Sigma \setminus \tilde{\Sigma}_k$ for any $k \in \mathbb{N}$.
By Beppo-Levi Theorem

$$\begin{aligned} \int_{\mathbb{R}^n} |\theta| \, d\mathscr{H}^m \llcorner \Sigma &= \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \Sigma \\ &= \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty. \end{aligned} \tag{3.C.1}$$

Therefore θ is finite $\mathscr{H}^m$-a.e. in Σ, i.e. for $\mathscr{H}^m$-a.e. $x \in \Sigma$ there are only finitely many $k \in \mathbb{N}$ so that $\theta_k(x) \neq 0$. In particular the sum

$$\sum_{k \in \mathbb{N}} \tilde{\theta}_k(x) \xi_k(x)$$

is well defined and finite for $\mathscr{H}^m$-a.e. $x \in \Sigma$ (again ξ_k is extended by zero on $\Sigma \smallsetminus \tilde{\Sigma}_k$ for any $k \in \mathbb{N}$) and we can write

$$\theta(x)\xi(x) = \sum_{k \in \mathbb{N}} \tilde{\theta}_k(x)\xi_k(x)$$

for some $\theta(x) \in \mathbb{Z}_{\geqslant 0}$ and for some simple unit m-vector $\xi(x)$ in $T_x\Sigma$, for $\mathscr{H}^m$-a.e. $x \in \Sigma$.
Observe that θ is an $\mathscr{H}^m$-measurable function on Σ as the absolute value of the a.e.-limit of $\mathscr{H}^m$-measurable functions. Analogously, ξ is an $\mathscr{H}^m$-measurable map on Σ as the a.e.-limit of $\mathscr{H}^m$-measurable maps. We set

$$T := \tau(\Sigma, \theta, \xi)$$

and we claim that

$$T_k \to T \quad (k \to \infty) \quad \text{in mass.}$$

In fact we know that since the space of m-currents is complete under the convergence in mass (as a dual space), there exists an m-current T' such that

$$T_k \to T' \quad (k \to \infty) \quad \text{in mass.}$$

To see that $T = T'$ observe that for any $\omega \in \mathcal{D}^m(\mathbb{R}^n)$

$$\theta_k\langle \omega, \xi_k\rangle = \sum_{i=1}^{k} \tilde{\theta}_i \langle \omega, \tilde{\xi}_i\rangle \to \theta\langle \omega, \xi\rangle \quad \mathscr{H}^m\text{-a.e. in } \Sigma,$$

thus by (3.C.1) and Dominated Convergence Theorem we conclude that

$$\langle T_k, \omega\rangle \to \langle T, \omega\rangle \quad (k \to \infty).$$

In particular $T = T'$. □

Lemma 3.C.2. *Let $\alpha \in (1, +\infty)$, $q \in (-\infty, 1]$, $\varepsilon \in (0, 1)$ and let $\Omega \subset Q^n_{1-\varepsilon}(0)$ be open, Lipschitz and bounded. For any $p \in [1, +\infty)$ and $\mu := f\mathcal{L}^n$ with $f = (\frac{1}{2} - \|\cdot\|_\infty)^q$, consider the continuous linear operator $P_\alpha : L^p(\Omega, \mu; \mathbb{R}^n) \to L^p(\Omega, \mu; \mathbb{R}^n)$ given by*

$$(P_\alpha V)(x) := \begin{cases} \alpha^{n-1}V(\alpha x) & \text{if } x \in \alpha^{-1}\Omega, \\ 0 & \text{on } \Omega \smallsetminus \alpha^{-1}\Omega. \end{cases}$$

Then:

1. *For every $\alpha \in (1, +\infty)$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$ holds that*

$$\|P_\alpha V\|_{L^p(\mu)} \leqslant C\alpha^{n-1-\frac{n}{p}}\|V\|_{L^p(\mu)},$$

for some constant $C > 0$ depending only on q and p.

2. *For every $V \in L^p(\Omega, \mu; \mathbb{R}^n)$ we have that $P_\alpha V \to V$ in $L^p(\Omega, \mu; \mathbb{R}^n)$ as $\alpha \to 1^+$.*

Proof. Fist we prove 1. Fix any $V \in L^p(\Omega, \mu; \mathbb{R}^n)$ and compute

$$\int_\Omega |P_\alpha V|^p \, d\mu = \alpha^{p(n-1)} \int_{\alpha^{-1}\Omega} |V(\alpha x)|^p \, d\mu(x)$$
$$\leqslant \alpha^{p(n-1)-n} \bigg(\int_\Omega |V(y)|^p \, d\mu(y) + \int_\Omega |V(y)|^p \frac{f(\alpha^{-1}y) - f(y)}{f(y)} \, d\mu(y) \bigg).$$

As in (3.2.4) we can estimate

$$\left| \frac{f(\alpha^{-1}y) - f(y)}{f(y)} \right| \leqslant q \left(\frac{1}{2} - \|y\|_\infty \right)^{-1} \|y\|_\infty (1 - \alpha^{-1}) \leqslant C$$

for any $y \in Q_{1-\varepsilon}(0)$ and any $\alpha \geqslant 1$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$, for some constant C depending only on q. Therefore

$$\int_\Omega |P_\alpha V|^p \, d\mu \leqslant (C+1)\alpha^{p(n-1)-n} \int_\Omega |V(y)|^p \, d\mu(y).$$

Hence 1. follows. We are left to prove 2.. Fix any $\delta > 0$ and let $V_\delta \in C_c^0(\Omega; \mathbb{R}^n)$ be such that

$$\|V_\delta - V\|_{L^p(\mu)} \leqslant \delta.$$

By 1, we have

$$\|P_\alpha V - V\|_{L^p(\mu)} \leqslant \|P_\alpha(V - V_\delta)\|_{L^p(\mu)} + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} + \|V_\delta - V\|_{L^p(\mu)}$$
$$\leqslant (C\alpha^{n-1-\frac{n}{p}} + 1)\delta + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)},$$

for every $\alpha \in (1, +\infty)$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$. Since V_δ is continuous and compactly supported, it follows from dominated convergence that $\|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} \to 0$ as $\alpha \to 1^+$. Hence, by letting $\alpha \to 1^+$ in the previous inequality we get

$$\limsup_{\alpha \to 1^+} \|P_\alpha V - V\|_{L^p(\mu)} \leqslant (C+1)\delta.$$

As $\delta > 0$ was arbitrary, 2 follows. ◻

4. Coulomb gauges and regularity for Yang–Mills fields in supercritical dimension

4.1. Introduction

The study of the variations of the Yang–Mills Lagrangian has known a spectacular development since the very first analytical works by K. Uhlenbeck, which have been central in producing new invariants of differential structures on topological 4-manifolds. In particular, S. Donaldson proved his celebrated result on the existence of non-smoothable topological 4-manifolds by studying properties of the the moduli space of the anti-self-dual instantons[1] over such manifolds (see [30], [31]). Examples of these manifolds had already been constructed by Freedman in [36]. Moreover, Taubes proved the existence of uncountably many fake $\mathcal{R}^4$'s by means of gauge theoretic methods in [87]. For a complete discussion of these topics, we refer the reader to [32] or [35].

Due to the successful use of the Yang–Mills energy in dimension 4, it is natural to explore its behavior in higher dimensions, specifically in the *supercritical* regime. In fact, S. Donaldson and R. Thomas outlined a research program in this direction in [33], [88]. However, analyzing the Yang–Mills Lagrangian becomes increasingly challenging in dimensions higher than 4, where we are led to consider*singular* solutions, which naturally emerge in this more complicated context.

Locally, the Yang–Mills Lagrangian is defined as follows. Let A be a 1-form on the flat n-dimensional unit ball $\mathbb{B}^n$ taking values into the Lie Algebra $\mathfrak{g}$ of a compact matrix Lie group G. The *Yang–Mills energy* of this "connection 1-form" is given by

$$\mathrm{YM}(A) := \int_{\mathbb{B}^n} |dA + A \wedge A|^2 \, d\mathcal{L}^n, \tag{4.1.1}$$

where $\mathcal{L}^n$ denotes the *Lebesgue measure* on $\mathbb{R}^n$, $A \wedge A$ is the $\mathfrak{g}$-valued 2-form given by

$$A \wedge A(X, Y) := [A(X), A(Y)] \qquad \forall\, X, Y \in \mathbb{R}^n \tag{4.1.2}$$

is the Lie Algebra bracket on $\mathfrak{g}$. The curvature $F_A := dA + A \wedge A$ is the sum of a *linear operation* on A (i.e. dA) and a *bilinear* one (i.e. $A \wedge A$). One of the main difficulties in the analysis of the Yang–Mills functional is to understand which one of the two is "taking over" the other along sequences with uniformly bounded Yang–Mills energy. As we will see shortly, the general answer to this question strongly depends on the dimension n of the base manifold.

The Yang–Mills Lagrangian is conformally invariant in dimension 4. This makes the 4-dimensional case critical from a purely analytic perspective, as we are explaining in the forthcoming

[1] Instantons in dimension 4 are very special critical points of the Yang–Mills functional solving a first order PDE, in a similar fashion as holomorphic functions are very special critical points of the Dirichlet energy in dimension 2.

paragraphs. The Yang–Mills Lagrangian is invariant under the following *gauge change operation*:

$$\mathrm{YM}(A^g) = \mathrm{YM}(A), \qquad \text{where} \qquad A^g := g^{-1}\, dg + g^{-1}\, A\, g \tag{4.1.3}$$

for any choice of map g from $\mathbb{B}^n$ into G. These maps are called *local gauge transformations* or simply *local gauges* and they realize the so called *local gauge group*. The equality between the Yang–Mills energy of A and the Yang–Mills energy of A^g is a direct consequence of the following identity[2]

$$F_{A^g} = g^{-1} F_A g \quad \Rightarrow \quad |F_{A^g}| = |F_A|. \tag{4.1.4}$$

This huge invariance group is both a source of difficulties and a big advantage in studying this Lagrangian.

Our starting point consists in considering the space of connection forms modulo this gauge group action. One of the main challenges in the field consists in proving the existence of a gauge g in which the connection form A^g is "optimally" controlled[3] by its Yang–Mills energy $\mathrm{YM}(A^g) = \mathrm{YM}(A)$. For instance, in order to make the functional YM as much coercive as possible, a reasonable quest suggested by the abelian case ($G = U(1)$) in electromagnetism consists in looking for the *Coulomb condition* to be fulfilled. This amounts to finding a local gauge g such that

$$\begin{cases} d^* A^g = 0 & \text{in } \mathbb{B}^n \\ \langle A^g, x\rangle = 0 & \text{on } \partial\mathbb{B}^n. \end{cases} \tag{4.1.5}$$

This condition is equivalent to the following non-linear elliptic PDE

$$\begin{cases} -\mathrm{div}\left(g^{-1}\nabla g\right) = \mathrm{div}(g^{-1} A g) & \text{in } \mathbb{B}^n \\ -g^{-1}\partial_\nu g = g^{-1}\langle A, x\rangle\, g & \text{on } \partial\mathbb{B}^n. \end{cases} \tag{4.1.6}$$

We shall come back to the difficulty of solving (4.1.6) later in this introduction but, assuming such a g has been obtained, we control

$$C^{-1}\|A^g\|^2_{W^{1,2}(\mathbb{B}^n)} \leqslant \int_{\mathbb{B}^4} |dA^g|^2 + |d^* A^g|^2\, d\mathcal{L}^n \leqslant \mathrm{YM}(A) + \int_{\mathbb{B}^n} |A^g \wedge A^g|^2\, d\mathcal{L}^n. \tag{4.1.7}$$

From the Sobolev embedding theorem, for $n > 2$ we have[4]

$$W^{1,2}(\mathbb{B}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{B}^n). \tag{4.1.8}$$

In dimension $n = 3, 4$, Hölder inequality implies that $L^{\frac{2n}{n-2}}(\mathbb{B}^n) \hookrightarrow L^4(\mathbb{B}^n)$. Hence, for $n \leqslant 4$ we obtain the bound

$$\int_{\mathbb{B}^n} |A^g \wedge A^g|^2\, d\mathcal{L}^n \leqslant \|A^g\|^4_{L^4(\mathbb{B}^n)} \leqslant C\|A^g\|^4_{W^{1,2}(\mathbb{B}^n)}, \tag{4.1.9}$$

[2] The action of the local gauge group on a given connection — i.e. an equivariant horizontal plane distribution in the principal G-bundle which is represented in a local trivialization by a $\mathfrak{g}$ valued 1-forms A in the base — is converted into the adjoint action of G on $\mathfrak{g}$ at the level of the curvature which is itself measuring the lack of integrability of this plane distribution (recall that the curvature is the vertical projection of the bracket of horizontal lifts of vector fields in the base).

[3] The adjective "optimally" refers to the smallest possible classical function space that A^g belongs to.

[4] For $n = 2$, we have

$$W^{1,2}(\mathbb{B}^n) \hookrightarrow \bigcap_{p=1}^{+\infty} L^p(\mathbb{B}^n).$$

and finally

$$\|A^g\|^2_{W^{1,2}(\mathbb{B}^n)} \leqslant C\,\mathrm{YM}(A) + \; C\|A^g\|^4_{W^{1,2}(\mathbb{B}^n)}. \tag{4.1.10}$$

Assuming now that a smallness condition of the form $C\|A^g\|^2_{W^{1,2}(\mathbb{B}^n)} \leqslant 2^{-1}$ is known, one gets in return the "a priori" estimate

$$\|A^g\|^2_{W^{1,2}(\mathbb{B}^n)} \leqslant C\,\mathrm{YM}(A). \tag{4.1.11}$$

One of the main achievements of [90] is to convert the a priori estimate (4.1.11) into an existence result for (4.1.6) such that (4.1.11) eventually holds, provided $\mathrm{YM}(A)$ is small enough.

This achievement is not straightforward at all in dimension $n \geqslant 4$. A first attempt would be to use the variational nature of the problem. Indeed, equation (4.1.6) happens to be the Euler–Lagrange equation of

$$\|A^g\|^2_{L^2(\mathbb{B}^n)} = \int_{\mathbb{B}^4} |dg + Ag|^2 \; d\mathcal{L}^n. \tag{4.1.12}$$

Hence, independently of dimension, a solution to (4.1.6) can be obtained by a direct minimization of (4.1.12) and by applying the fundamental principles of the calculus of variations. Nevertheless, this variational strategy is hitting a serious regularity issue in the sense that, for a generic $A \in L^2(\mathbb{B}^n)$, a minimizer g of (4.1.12) is a priori only in $W^{1,2}$ and in general (4.1.11) is not satisfied by any of these minimizers[5]

Here, another dichotomy appears within the low dimensions $n \leqslant 4$. For $\underline{n < 4}$, because of the Sobolev embedding

$$W^{2,2}(\mathbb{B}^n) \hookrightarrow C^0(\mathbb{B}^n), \tag{4.1.13}$$

the group multiplication

$$\begin{aligned} \mathcal{M} : W^{2,2}(\mathbb{B}^n, G)^2 &\to W^{2,2}(\mathbb{B}^n, G) \\ (g,h) &\mapsto gh \end{aligned} \tag{4.1.14}$$

is smooth between the two Banach manifolds $W^{2,2}(\mathbb{B}^n, G)^2$ and $W^{2,2}(\mathbb{B}^n, G)$. This allows to implement an argument based on the local inversion theorem in order to prove that any $A \in W^{1,2}(\wedge^1\mathbb{B}^n \otimes \mathfrak{g})$ satisfying $\mathrm{YM}(A) < \varepsilon$ admits a local gauge $g \in W^{2,2}(\mathbb{B}^n, G)$ solving (4.1.6) and (4.1.11).

Coming now to the critical dimension 4, the group multiplication map $\mathcal{M}$ is still well-defined, since the group G is assumed to be compact, but $\mathcal{M}$ ceases to be continuous. Moreover, $W^{2,2}(\mathbb{B}^n, G)$ looses its natural Banach manifold structure. These facts prevent implementing the strategy involving the direct use of the local inversion theorem. K. Uhlenbeck instead developed a very clever continuity argument leading to a $W^{1,2}$ controlled representative satisfying (4.1.11), under small YM-energy assumption.

To summarize, the balance between the linear part dA and the bilinear part $A \wedge A$ of the curvature form can be settled in favor of the linear and more regularizing part dA as long as $\mathrm{YM}(A)$ is small

[5] Even for a smooth data A, minimizers to the Dirichlet energy for maps from $\mathbb{B}^4$ into G are known to be at most in $W^{2,(\frac{3}{2},\infty)}(\mathbb{B}^4, G)$ (see [79]) and are certainly not automatically in $W^{2,2}(\mathbb{B}^4, G)$, which must be the case if A^g is in $W^{1,2}$.

enough and up to dimension 4. This enables to implement classical variational strategies for YM within the framework of Sobolev connections[6] on smooth principal G-bundles (see e.g. [81] for minimization procedures).

In dimension larger than 4, the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ does not hold anymore. This fundamental fact compromises Uhlenbeck's procedure to extract controlled gauges. Even worse, one can produce a sequence of smooth $\mathfrak{su}(2)$-valued 1 forms $\{A_k\}_{k\in\mathbb{N}}$ on $\mathbb{B}^5$ such that

$$\begin{cases} A_k \rightharpoonup A_\infty & \text{weakly in } L^2(\mathbb{B}^5) \\ dA_k + A_k \wedge A_k \rightharpoonup F_\infty & \text{weakly in } L^2(\mathbb{B}^5) \\ d^* A_k = 0 & \text{in } \mathbb{B}^5. \end{cases} \tag{4.1.16}$$

and

$$\operatorname{Spt}\ d\left(\operatorname{tr}(F_\infty \wedge F_\infty)\right) = \overline{\mathbb{B}^5}. \tag{4.1.17}$$

Assume that there exists $B \in W^{1,2}(\wedge^1\mathbb{B}^5 \otimes \mathfrak{su}(2))$ such that

$$F_\infty = dB + B \wedge B = F_B. \tag{4.1.18}$$

Then, for any smooth function φ compactly supported in $\mathbb{B}^5$ the coarea formula gives

$$\int_{B^5} d\varphi \wedge \operatorname{tr}(F_B \wedge F_B) = \int_{-\infty}^{+\infty} d\mathcal{L}^1(s) \int_{\varphi^{-1}(s)} \operatorname{tr}(F_B \wedge F_B). \tag{4.1.19}$$

Thanks to Sard's theorem and Fubini's theorem, for $\mathscr{H}^1$-a.e. $s \in \mathbb{R}$, $\varphi^{-1}(s)$ is a smooth closed 4-dimensional manifold and the restriction of B to this submanifold is in $W^{1,2}$. On such a 4-dimensional manifold, by a straightforward strong approximation procedure for B in $W^{1,2}(\wedge^1\varphi^{-1}(s) \otimes \mathfrak{su}(2))$, we can derive the following identity from the classical expression of the *transgression form* for the second Chern class:

$$\operatorname{tr}(F_B \wedge F_B) = d\left[\operatorname{tr}\left(B \wedge dB + \frac{1}{3}\, B \wedge [B, B]\right)\right]. \tag{4.1.20}$$

This implies that

$$\int_{\varphi^{-1}(s)} \operatorname{tr}(F_B \wedge F_B) = 0 \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in \mathbb{R}. \tag{4.1.21}$$

Combining (4.1.19) and (4.1.21), we get

$$d\left(\operatorname{tr}(F_B \wedge F_B)\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{B}^5). \tag{4.1.22}$$

This fact is contradicting (4.1.17) and we conclude that the weak limit of the smooth curvatures F_{A_k} on $\mathbb{B}^5$ cannot be, even locally, the curvature of a Sobolev $W^{1,2}$-connection. For sequences

[6] A Sobolev $W^{k,p}$-connection on a smooth principal G-bundle $\pi : P \to M^4$ is given by a collection of $W^{k,p}$ $\mathfrak{g}$-valued 1-forms A_U on each open set U over which P is trivial (i.e. $\pi^{-1}(U) \simeq U \times G$ as principal G-bundle isomorphism) and related to each other by the classical gauge equivalence relations

$$A_V := g_{UV}^{-1} dg_{UV} + g_{UV}^{-1}\, A_U\, g_{UV} \quad \text{on } U \cap V \tag{4.1.15}$$

where g_{UV} are the G-valued transition functions defining the smooth bundle P (see for instance [35]).

of smooth Yang–Mills fields (smooth critical points of YM) with uniformly bounded energy, the possibility for their weak limits[7] not to satisfy (4.1.22) is not excluded at all. In fact, we believe that it is possible to produce such sequences where (4.1.22) is violated.

These facts left the variational geometric analysis community in some perplexity and the following question arose naturally:

What is the space of weak limits of curvatures of smooth Yang–Mills connections in supercritical dimension?

The above considerations are excluding the space of the curvatures of Sobolev connections, which has been the unique framework adopted so far to approach the variational issues related to the Yang–Mills lagrangian in subcritical and critical dimension.

The work in [66] was motivated by this question and brought an answer to it in the abelian case. In this framework, a 2-form F on a closed oriented connected surface Σ is the curvature of some complex line bundle E over Σ if and only if

$$\int_\Sigma F \in 2\pi\mathbb{Z} \qquad \text{and then} \qquad c_1(E) = \int_\Sigma F. \tag{4.1.23}$$

The authors introduced on $\mathbb{B}^3$ the space of *weak L^p-curvatures*

$$\mathcal{F}^p_{\mathbb{Z}} := \left\{ F \in L^p(\mathbb{B}^3) \ : \ \forall\, \varphi \in \mathrm{Lip}_c(\mathbb{B}^3) \ \text{ and for } \mathscr{H}^1\text{-a.e. } s \in \mathbb{R}, \ \int_{\varphi^{-1}(s)} F \in 2\pi\mathbb{Z} \right\}. \tag{4.1.24}$$

The main result in [66] establishes that for every $p > 1$ the space $\mathcal{F}^p_{\mathbb{Z}}$ is sequentially closed for the weak convergence in L^p. This statement and some complementary results were made more precise and extended to higher dimensions in [22] and [23].

Inspired by the abelian case, M. Petrache and the second author introduced the space of *weak connections* on a n-dimensional manifold N^n. The first main idea consists first, for the dimensions $n \leqslant 4$, in "wrapping together" the space of principal G-connections for any possible principal G-bundles in a single definition. We define

$$\mathbb{A}_G(N^n) := \left\{ \begin{array}{c} \nabla := (U_i, A_i)_{i\in I} \ ; \ (U_i)_{i\in I} \text{ realizes an open cover of } N^n \\ A_i \in W^{1,2}(\wedge^1 U_i \otimes \mathfrak{g}), \ \forall\, i \neq j \ \ \exists\, g_{ij} \in W^{2,2}(U_i \cap U_j, G) \ \text{ s. t.} \\ A_j = A_i^{g_{ij}} := g_{ij}^{-1} dg_{ij} + g_{ij}^{-1}\, A_i\, g_{ij} \quad \text{ on } U_i \cap U_j \\ \forall\, i \neq j \neq k \quad g_{ij}\, g_{jk}\, g_{ki} \equiv \mathrm{id}_G \quad \text{ on } U_i \cap U_j \cap U_k \\ |F_\nabla| \in L^2(N^n) \end{array} \right\}. \tag{4.1.25}$$

This is the definition of Sobolev connections considered in the classical analytical works on gauge theory since the early eighties ([32], [35]). In order to implement analysis arguments, we need to find some natural generalization of Sobolev connections to higher dimension that enjoys a good closure property. Aiming to do so, it is tempting to work with a single form to represent the connection. The problem of constructing a global gauge to any Sobolev connection in dimension at most 4 was considered for the first time in [64].

[7]These objects are called *admissible Yang–Mills connections* in [89].

Theorem 4.1.1 ([64]). *Let $\nabla \in \mathbb{A}_{\mathrm{SU}(2)}(N^n)$ be a Sobolev connection of some principal* SU(2)*-bundle over a closed oriented Riemannian manifold N^n of dimension $n \leqslant 4$. Then there exists $A \in L^{4,\infty}(\wedge^1 N^n \otimes \mathfrak{su}(2))$ such that, about every point, there exists locally a $W^{1,(4,\infty)}$ trivialization in which*

$$\nabla = d + A, \tag{4.1.26}$$

where $L^{4,\infty}$ is the weak Marcinkiewicz space

$$L^{4,\infty}(N^n) : \left\{ f \text{ is measurbale and } \sup_{\lambda>0} \lambda \, |\{x \in N^n \ ; \ |f(x)| > \lambda\}|^{\frac{1}{4}} < +\infty \right\} \tag{4.1.27}$$

with $|\cdot|$ being the measure induced by the volume form of N^n.

Representing a smooth connection on a non trivial bundle by a global 1-form is obviously impossible if one does not give up regularity. In fact, for instance, if the bundle is a non trivial SU(2)-bundle over the 4-sphere, one can prove that no smooth connection ∇ has a global representative in L^4. Hence, the regularity $L^{4,\infty}$ is optimal in that sense[8].

Theorem 4.1.1 is leading naturally to the following definition

$$A_G(N^n) := \left\{ A \in L^2(\wedge^1 N^n \otimes \mathfrak{g}) \, : \, F_A \in L^2(N^n) \text{ and } \exists \text{ locally } g \in W^{1,2} \text{ s.t. } A^g \in W^{1,2} \right\}. \tag{4.1.28}$$

Thanks to this theorem, for $n \leqslant 4$ we have

$$A_{\mathrm{SU}(2)}(N^n) = \mathbb{A}_{\mathrm{SU}(2)}(N^n) \tag{4.1.29}$$

Finally, again for $n \leqslant 4$, one can consider the following apparently weaker definition

$$\mathfrak{a}_G(N^n) := \left\{ A \in L^2(\wedge^1 N^n \otimes \mathfrak{g}) \, : \, F_A \in L^2(N^n) \text{ and } \exists \text{ locally } g \in W^{1,2} \text{ s.t. } A^g \in L^4 \right\}. \tag{4.1.30}$$

This definition more flexible than the definition of A_G, because it extends to spaces X^n which are bi-Lipschitz homeomorphic to a smooth Riemannian oriented closed manifold N^n and $n \leqslant 4$. Moreover, we prove in the Appendix A of the present paper that for any Riemannian manifold of dimension less or equal than 4 and for any compact Lie group G

$$A_G(N^n) = \mathfrak{a}_G(N^n). \tag{4.1.31}$$

In [74], a proof of the sequential weak closure of $\mathfrak{a}_G(M^4)$ under Yang–Mills energy control is given for every closed oriented 4-dimensional Riemannian manifold N^4. The proof is based on the analysis mostly developed by K. Uhlenbeck during the 80s in [90], [91] and [92], combined with some more recent arguments involving the use of interpolation spaces introduced in this context by the second author in [72].

Coming now to the supercritical dimensions, as underlined earlier one would also wish to produce a class of "objects" containing smooth connections and which is weakly sequentially closed under Yang–Mills energy control exclusively. In order to do so, the main idea in [66] is to propose

[8] In [64], the authors were asking the question whether a global gauge in the optimal space $L^{4,\infty}(\wedge^1 N^n \otimes \mathfrak{su}(2))$ and satisfying simultaneously the Coulomb condition $d^* A = 0$. A partial answer to this question which is still open as such is given in [93].

an inductive definition by mean of generic slicing. Let $n > 4$ and denote by $\mathcal{L}(N^n)$ the space of Lipschitz functions whose level sets are almost always bi-Lipschitz equivalent to a smooth manifold. We introduce the following space:

$$\mathfrak{a}_G(N^n) := \left\{ \begin{array}{c} A \in L^2(\wedge^1 N^n \otimes \mathfrak{g}) \, : \, F_A \in L^2(N^n) \\ \text{s.t. } \forall \varphi \in \mathcal{L}(N^n) \text{ and for } \mathscr{H}^1\text{-a.e. } s \in \mathbb{R} \\ \iota^*_{\varphi^{-1}(s)} A \in \mathfrak{a}_G(\varphi^{-1}(s)) \end{array} \right\}, \tag{4.1.32}$$

where $\iota_{\varphi^{-1}(s)}$ is the canonical inclusion of $\varphi^{-1}(s)$ in M^m.

In [65], a proof of the sequential weak closure of $\mathfrak{a}_G(M^5)$ under Yang–Mills energy control is proposed[9]. More precisely, the authors show that

$$\begin{cases} \forall \, \{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{a}_G(N^5) \text{ s.t. } \limsup\limits_{k\to+\infty} \mathrm{YM}(A_k) < +\infty \\ \exists \, \{A_{k_n}\}_{n\in\mathbb{N}}, \; A \in \mathfrak{a}_G(N^5) \text{ and } \{g_n\}_{n\in\mathbb{N}} \subset W^{1,2}(N^5, G) \\ \text{s.t. } A_{k_n}^{g_n} \rightharpoonup A \text{ in } L^2(N^5) \text{ and } \mathrm{YM}(A) \leqslant \liminf\limits_{k\to+\infty} \mathrm{YM}(A_{k_n}). \end{cases} \tag{4.1.33}$$

The proof of (4.1.33) uses a strong approximation property of elements in $\mathfrak{a}_G(\mathbb{B}^5)$ by connection forms which are smooth away from finitely many points, modulo gauge transformations. More specifically, in [65] the authors show that for any $A \in \mathfrak{a}_G(\mathbb{B}^5)$ there exists a sequence of $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{a}_G(\mathbb{B}^5)$ such that each A_k is gauge equivalent to a connection form in $\mathbb{B}^5$ which is smooth away from finitely many points and

$$\begin{cases} A_k \to A & \text{strongly in } L^2(\mathbb{B}^5), \\ F_{A_k} \rightharpoonup F_A & \text{weakly in } L^2(\mathbb{B}^5), \\ \mathrm{tr}\,(F_{A_k} \wedge F_{A_k}) \rightharpoonup \mathrm{tr}\,(F_A \wedge F_A) & \text{weakly in } \mathcal{D}'(\mathbb{B}^5). \end{cases} \tag{4.1.34}$$

The space $\mathcal{F}_G(\mathbb{B}^5)$ of smooth connections away from isolated points is the smallest space such that the strong approximation property (4.1.34) holds true. This makes $\mathcal{F}_G(\mathbb{B}^5)$ a natural subspace in $\mathfrak{a}_G(\mathbb{B}^5)$, in the same way as the space $R^\infty(\mathbb{B}^3, \mathbb{S}^2)$ of maps in the Sobolev space $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ that are smooth away from finitely many isolated topological singularities is the smallest subspace in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ being sequentially dense with respect to the $W^{1,2}$-norm (see [13, Theorem 4]).

Because of the sequential weak closure property (4.1.33), $\mathfrak{a}_G(N^5)$ is a space in which variational problems related to the Yang–Mills lagrangian on N^5 are well-posed. We can then define the notion of weak Yang–Mills connections.

Definition 4.1.1 (Weak Yang–Mills connections)**.** Let G be a compact matrix Lie group. We

[9] The proof of the sequential weak closure in [65] is based on the Proposition 2.1 in the same paper. As discussed in Remark 4.2.1, [65, Proposition 2.1] has the missing term $\|A\|^2_{L^2}$ on the right-hand-side of the inequality (2.2). This was first noticed by S. Sil. Our Proposition 4.2.1 below is a suitable replacement of [65, Proposition 2.1] as explained in section II. We give a complete and detailed proof of the sequential weak closure of $\mathfrak{a}_G(M^5)$ under controlled Yang–Mills energy in [24]

say that $A \in \mathfrak{a}_G(\mathbb{B}^5)$ is a *weak Yang–Mills connection* on $\mathbb{B}^5$ if

$$d_A^* F_A = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{5} \partial_{x_i}(F_A)_{ij} + [A_i, (F_A)_{ij}] \quad \forall\, j = 1 \cdots 5 \quad \text{distributionally on } \mathbb{B}^5, \tag{4.1.35}$$

i.e.

$$\int_{\mathbb{B}^5} \langle F_A, d_A\varphi \rangle = 0 \qquad \forall\, \varphi \in C_c^\infty(\wedge^1 \mathbb{B}^5 \otimes \mathfrak{g}), \tag{4.1.36}$$

where we have

$$d_A\varphi := d\varphi + [A \wedge \varphi] = d\varphi + A \wedge \varphi + \varphi \wedge A, \qquad \forall \varphi \in C_c^\infty(\wedge^1 \mathbb{B}^5 \otimes \mathfrak{g}).$$

and $\langle \cdot\, , \cdot \rangle$ denotes the scalar product on $\wedge^2 \mathbb{B}^5 \otimes \mathfrak{g}$.

Among weak Yang–Mills connections, we shall be particularly interested with the ones that satisfy the following *stationarity condition* (which is automatically satisfied by smooth solutions to (4.1.35) or by YM-energy minimizers for instance).

Definition 4.1.2 (Stationary weak Yang–Mills connections)**.** Let G be a compact matrix Lie group. We say that a weak Yang–Mills connection $A \in \mathfrak{a}_G(\mathbb{B}^5)$ on $\mathbb{B}^5$ is stationary if

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{YM}(\Phi_t^* A) = 0, \tag{4.1.37}$$

for every smooth 1-parameter group of diffeomorphisms Φ_t of $\mathbb{B}^5$ with compact support.

If A is a stationary Yang–Mills connection, by standard methods it can be shown that the following *monotonicity property* holds true: for every given $x \in \mathbb{B}^5$, the function

$$\big(0, \operatorname{dist}(x, \partial\mathbb{B}^5)\big) \ni \rho \to \frac{e^{c\Lambda\rho}}{\rho} \int_{B_\rho(x)} |F_A|^2 \, d\mathcal{L}^5 \tag{4.1.38}$$

is non-decreasing, where $c > 0$ is a universal constant and Λ depends on $B_1(x)$. In particular, we have

$$\sup_{\substack{x \in B_{\frac{1}{2}}(0),\\ 0<\rho<\frac{1}{4}}} \frac{1}{\rho} \int_{B_\rho(x)} |F_A|^2 \, d\mathcal{L}^5 \leqslant C \int_{B_1^5(0)} |F_A|^2 \, d\mathcal{L}^5,$$

for some constant $C > 0$ independent on A. We then naturally introduce the following spaces which are known as Morrey-Sobolev space for any domain $\Omega \subset \mathcal{R}^n$

$$M^0_{p,q}(\Omega) := \left\{ f \in L^p(\Omega) \,:\, |f|^p_{M^0_{p,q}(\Omega)} := \sup_{\substack{x\in\Omega,\\ \rho>0}} \frac{1}{\rho^{n-pq}} \int_{B_\rho(x)\cap\Omega} |f|^p \, d\mathcal{L}^n \right\}. \tag{4.1.39}$$

It is strongly motivated by the analysis of weak stationary Yang–Mills Fields to ask whether Uhlenbeck's Coulomb gauge extraction extends in the higher dimension 5 to weak connections having a curvature with small Morrey $M^0_{2,2}$-norm. The following theorem answers positively to this question and is one of the main results of the present work.

Theorem 4.1.2. *Let G be a compact matrix Lie group. There exists $\varepsilon_G > 0$ such that for every weak connection $A \in \mathfrak{a}_G(\mathbb{B}^5)$ satisfying*

$$|F_A|^2_{M^0_{2,2}(\mathbb{B}^5)} = \sup_{\substack{x\in\mathbb{B}^5,\\ \rho>0}} \frac{1}{\rho}\int_{B_\rho(x)\cap\mathbb{B}^5} |F_A|^2 \, d\mathcal{L}^5 < \varepsilon_G \tag{4.1.40}$$

there exists $g \in W^{1,2}(\mathbb{B}^5, G)$ such that

$$d^* A^g = 0, \tag{4.1.41}$$

and

$$|\nabla A^g|^2_{M^0_{2,2}(\mathbb{B}^5)} = \sup_{\substack{x\in\mathbb{B}^5,\\ \rho>0}} \frac{1}{\rho}\int_{B_\rho(x)\cap\mathbb{B}^5} \sum_{i=1}^{5} |\partial_{x_i} A^g|^2 \, d\mathcal{L}^5 \leqslant C_G |F_A|^2_{M^0_{2,2}(\mathbb{B}^5)}, \tag{4.1.42}$$

where $C_G > 0$ is a constant depending only on G.

This result has been conjectured to hold in [65]. Such a Coulomb gauge extraction theorem has been first established in [89] for smooth connections and in [57] under the assumption that the connection can be approximated strongly by smooth connections with curvatures having small Morrey norm (4.1.40). Later on, the same statement was proved in a particular case assuming that the connection is a weak limit of smooth Yang–Mills fields (see [86]). Such weak limits are smooth away from a closed codimension 4 rectifiable set and and referred to as *admissible Yang–Mills connections*. We also remark that energy identities and bubbling analysis for Yang–Mills fields in supercritical dimension are due to the subsequent works of the second author and A. Naber–D. Valtorta, in [73] and [60] respectively.

In [86], the authors prove a strong approximability property of admissible Yang–Mills connections by smooth connections with small Morrey norm (see [86, Proposition 4.4]). In [83, Theorem 34], the author shows the existence of Morrey norm controlled local Coulomb gauges in supercritical dimension by exploiting an approximation procedure, in the same spirit as in [57]. However, approximating connections in the (stronger) Morrey norm requires additional assumptions[10] which are not available in our context.

The main achievement of the present chapter is to prove that **any weak connection satisfying** (4.1.40) (which includes all the previous cases) can be approximated by smooth connections with small Morrey norms.

Combining Theorem 4.1.2 and the main result in [57] we can derive the following ε-regularity statement by using the same arguments presented in [57, Section 4].

Theorem 4.1.3 (ε-regularity)**.** *Let G be a compact matrix Lie group. There exists $\varepsilon_G \in (0,1)$ such that for every stationary weak Yang–Mills $A \in \mathfrak{a}_G(\mathbb{B}^5)$ satisfying*

$$\mathrm{YM}(A) = \int_{\mathbb{B}^5} |F_A|^2 \, d\mathcal{L}^5 < \varepsilon_G$$

there exist $g \in W^{1,2}(B_{\frac{1}{2}}(0), G)$ such that $A^g \in C^\infty(B_{\frac{1}{2}}(0))$.

[10] In particular, in [83, Theorem 34] the author exploits a “vanishing Morrey norm” condition.

Finally, standard covering arguments give the following bound on the singular set of stationary weak Yang–Mills connections, which is the main result of the present chapter.

Theorem 4.1.4. *Let G be a compact matrix Lie group and let $A \in \mathfrak{a}_G(\mathbb{B}^5)$ be a stationary weak Yang–Mills connection on $\mathbb{B}^5$. Then*

$$\mathscr{H}^1(\mathrm{Sing}(A)) = 0,$$

where $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^5$ and $\mathrm{Sing}(A) \subset \mathbb{B}^5$ is the singular set of A, given by $\mathrm{Sing}(A) := \mathbb{B}^5 \setminus \mathrm{Reg}(A)$ where

$$\mathrm{Reg}(A) := \left\{ x \in \mathbb{B}^5 \text{ s.t. } \exists\, \rho > 0,\ g \in W^{1,2}(B_\rho(x), G) \text{ s.t. } A^g \in C^\infty(B_\rho(x)) \right\}.$$

Organization of the chapter

As mentioned above, the proof of the main Theorem 4.1.2 consists in approximating strongly in L^2 every weak connection A (i.e. every $A \in \mathfrak{a}_G(\mathbb{B}^5)$) whose curvature has a small Morrey $M^0_{2,2}$-norm by a sequence of smooth $\mathfrak{g}$-valued 1-forms A_j having small Morrey $M^0_{2,2}$-norms.

Section 4.2 is devoted to the construction of the building blocks. It contains two main results, explaining how to extend in an "optimal way" inside a cube a given connection at its boundary. We propose two ways of extending this connection. First, we assume some smallness condition on the L^2-norms both of the curvature and of the connection at the boundary (Corollary 4.2.1). Then, we assume smallness of the L^2-norm of the curvature only (Corollary 4.2.2).

In Section 4.3 we introduce some terminology. We say that a cube is *good* if on its boundary both the L^2-norms of the curvature and of the connection are small. We call a cube *bad* if on its boundary just the L^2-norm of the curvature is small.[11] Then, in a second step, by the mean of the coarea formula and the mean value theorem we prove the existence of a so called *admissible covers* by small cubes of comparable sizes, so that the L^2-norm of the curvature is small on the boundary of each of the cubes.

Section 4.4 is devoted to what is called the "first smoothification". In the first smoothification, we replace the initial connection in every cube of a chosen admissible cover by mean of the extensions introduced in the previous section. When the cube is good, we use Corollary 4.2.1 whilst, if the cube is bad, we exploit by Corollary 4.2.2. The main result in Section IV is Theorem 4.4.1. The consequence of the "first smoothification" is that the newly obtained connection forms $A_{i,\Lambda}$ converging strongly in the L^2-norm to A as $i \to +\infty$ and $\Lambda \to +\infty$, still having a small Morrey norm of the curvature, enjoys the following property: every trace on a generic cube of size comparable to the size of the admissible covering satisfies the small L^2-condition.

In Section 4.5 we shall proceed to the "second smoothification", that is, the L^2 strong approximation of the initial weak connection A by a sequence A_i of smooth connections whose curvature has small Morrey $M^0_{2,2}$-norm (Theorem 4.5.1). The proof of the second smoothification goes as

[11] This terminology in the dichotomy between good and bad cubes is reminiscent of the one introduced in the framework of the strong approximation of weak connections in $\mathbb{B}^5$ ([65] and [24]). Nevertheless, the meaning that we associate to it is different, in the sense that assuming the smallness condition of the Morrey norm allows us to decompose the domain into cubes at the boundary of which the L^2-norm of the curvature is always small. The dichotomy between good and bad cubes is made on the base of the smallness of L^2-norm of the connection only.

follows. Starting from the approximating sequences $A_{i,\Lambda}$ given by the first smoothification, we take a grid of size comparable to the size of the admissible cover associated to $A_{i,\Lambda}$ and we apply the replacement results of Section II (the building blocks Corollary 4.2.1 and Corollary 4.2.2) on each of the disjoint cubes of the grid iteratively in such a way that, in the procedure, each cube to be replaced is facing at least one cube which has not been replaced yet.

In Section 4.6, we combine the approximation given by the second smoothification with the main result in [57] in order to prove our main theorem (Theorem 4.1.2).

4.2. The building blocks for the approximation theorems

4.2.1. Extension of weak connections

In this subsection we build the fundamental statements that will be used in Section 4.4 in order to prove the strong L^2-approximation theorems for weak connections (Theorem 4.4.1 and Theorem 4.5.1) under Morrey norm control. In particular, we will need Corollaries 4.2.1 and 4.2.2 to extend weak connections from the boundary of 5-cubes to their interior. If we can assume the L^2-smallness of both the connection and its curvature on the boundary of the cube, then we will use Corollary 4.2.1. In case we can only assume the L^2-smallness of the curvature of the connection on the boundary, we will exploit Corollary 4.2.2.

Extension under L^2-smallness of the connection and its curvature

To ease the reading, throughout this subsection we will denote by "$d_{\mathbb{R}^5}$" and "$d_{\mathbb{S}^4}$" the standard differential of k-forms respectively on $\mathbb{R}^5$ and on the round sphere $\mathbb{S}^4$. More precisely we have

$$d_{\mathbb{S}^4} := \iota_{\mathbb{S}^4}^* d_{\mathbb{R}^5},$$

where $\iota_{\mathbb{S}^4}$ is the canonical embedding of $\mathbb{S}^4$ into $\mathbb{R}^5$. The following proposition is one of the building blocks of our approximation procedure.

Proposition 4.2.1 (Harmonic extension under smallness condition on F_A and A)**.** *Let G be a compact matrix Lie group and let $f : \mathbb{R}^5 \to \mathbb{R}^5$ be a bi-Lipschitz homeomorphism such that $f \in W^{1,\infty}_{loc}(\mathbb{R}^5, \mathbb{R}^5)$ and*

$$\langle d_{\mathbb{S}^4} f_i, d_{\mathbb{S}^4} f_j \rangle_{L^2(\mathbb{S}^4)} = 0, \qquad \forall\, i,j = 1,...,5 \;\; s.t. \;\; i \neq j. \tag{4.2.1}$$

Let

$$C_f := \sqrt{\sum_{i=1}^{5} \frac{1}{\|d_{\mathbb{S}^4} f_i\|^2_{L^2(\mathbb{S}^4)}}}. \tag{4.2.2}$$

There are constants $\varepsilon(G,f) \in (0,1)$ and $C(G,f)$ depending only on G, C_f and on $\|d_{\mathbb{R}^5} f\|_{L^\infty(\mathbb{B}^5)}$ such that for any $A \in \mathfrak{a}_G(\mathbb{S}^4)$ satisfying

$$\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} < \varepsilon(G,f) \tag{4.2.3}$$

the following facts hold.

(i) There exist $g \in W^{1,2}(\mathbb{S}^4, G)$ *and a* $\mathfrak{g}$*-valued 1-form* $\tilde{A} \in L^5(\wedge^1 \mathbb{B}^5 \otimes \mathfrak{g})$ *such that*

$$\iota_{\mathbb{S}^4}^* \tilde{A} = A^g$$

and

$$\|A^g\|_{L^4(\mathbb{S}^4)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\right),$$

$$\|d_{\mathbb{S}^4} g\|_{L^2(\mathbb{S}^4)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^2\right), \tag{4.2.4}$$

$$\|F_{\tilde{A}}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \left\|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\right\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^3\right), \tag{4.2.5}$$

for every constant $\mathfrak{g}$*-valued* 1*-form* $\bar{A}$ *on* $\mathbb{R}^5$.

(ii) There exists $\tilde{g} \in W^{1,2}(\mathbb{B}^5, G)$ *satisfying*

$$\|\tilde{g} - \mathrm{id}_G\|_{L^4(\mathbb{B}^5)} + \|d\tilde{g}\|_{L^2(\mathbb{B}^5)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^2\right), \tag{4.2.6}$$

for every constant $\mathfrak{g}$*-valued* 1*-form* $\bar{A}$ *on* $\mathbb{R}^5$*, such that the* $\mathfrak{g}$*-valued* 1*-form* $\hat{A} := \tilde{A}^{\tilde{g}^{-1}} \in L^2(\mathbb{B}^5)$ *satisfies the following properties.*

(a) $F_{\hat{A}} \in L^2(\mathbb{B}^5)$.

(b) $\iota_{\mathbb{S}^4}^* \hat{A} = A \in L^2(\mathbb{S}^4)$.

(c) Let $\Omega \subset \mathbb{R}^5$ *be an open set such that* $\Omega \cap \mathbb{B}^5$ *has a* 4*-dimensional compact Lipschitz boundary which can be included in a union of* N *submanifolds of* $\overline{\mathbb{B}^5}$ *of class* C^2*. Then we have*

$$\|F_{\hat{A}}\|_{L^2(\partial\Omega\cap\mathbb{B}^5))} \leqslant K_G \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \left\|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\right\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^3\right) \tag{4.2.7}$$

and

$$\|\tilde{A}\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} \leqslant K_G \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \left\|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\right\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\right), \tag{4.2.8}$$

for every constant $\mathfrak{g}$*-valued* 1*-form* $\bar{A}$ *on* $\mathbb{R}^5$*, where* $K_G = K_G(\Omega \cap \mathbb{B}^5) > 0$ *depends only on* G *and on* $\Omega \cap \mathbb{B}^5$ *(that is, on the number* N *of submanifolds containing* $\partial(\Omega \cap \mathbb{B}^5)$ *as well as their* C^2 *norms), on* C_f *and on* $\|d_{\mathbb{R}^5} f\|_{L^\infty(\mathbb{B}^5)}$.

Moreover,

$$\left\|\tilde{A} - f^* \bar{A}\right\|_{L^5(\mathbb{B}^5)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^2\right), \tag{4.2.9}$$

$$\left\|d\tilde{A}\right\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^2\right), \tag{4.2.10}$$

$$\left\|\hat{A} - f^* \bar{A}\right\|_{L^2(\mathbb{B}^5)} \leqslant C(G,f) \, \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^* \bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}^2\right), \tag{4.2.11}$$

for every constant $\mathfrak{g}$*-valued* 1*-form* $\bar{A}$ *on* $\mathbb{R}^5$.

Proof of Proposition 4.2.1. Notice that since $f = (f_1, ..., f_5)$ is a bi-Lipschitz homeomorphism, we have that $\|d_{\mathbb{S}^4} f_i\|_{L^2(\mathbb{S}^4)} \neq 0$ for every $i = 1, ..., 5$. Let ξ be the constant $\mathfrak{g}$-valued 1-form on $\mathbb{R}^5$ given by

$$\xi := \sum_{i=1}^{5} \frac{1}{\|d_{\mathbb{S}^4} f_i\|^2_{L^2(\mathbb{S}^4)}} \langle A, d_{\mathbb{S}^4} f_i \rangle_{L^2(\mathbb{S}^4)} d_{\mathbb{R}^5} x_i, \tag{4.2.12}$$

where $\{x_i\}_{i=1,...,5}$ represent the standard euclidean coordinates on $\mathbb{R}^5$. We have

$$|\xi| = \sqrt{\sum_{i=1}^{5} |\xi_i|^2} \leqslant \sqrt{\sum_{i=1}^{5} \frac{1}{\|d_{\mathbb{S}^4} f_i\|^2_{L^2(\mathbb{S}^4)}}} \, \|A\|_{L^2(\mathbb{S}^4)} = C_f \|A\|_{L^2(\mathbb{S}^4)}, \tag{4.2.13}$$

with

$$C_f := \sqrt{\sum_{i=1}^{5} \frac{1}{\|d_{\mathbb{S}^4} f_i\|^2_{L^2(\mathbb{S}^4)}}}. \tag{4.2.14}$$

Let $\eta \in L^2(\mathbb{S}^4)$ be given by

$$\eta := A - \iota^*_{\mathbb{S}^4} f^* \xi = A - \sum_{i=1}^{5} \left\langle A, \frac{d_{\mathbb{S}^4} f_i}{\|d_{\mathbb{S}^4} f_i\|_{L^2(\mathbb{S}^4)}} \right\rangle_{L^2(\mathbb{S}^4)} \frac{d_{\mathbb{S}^4} f_i}{\|d_{\mathbb{S}^4} f_i\|_{L^2(\mathbb{S}^4)}}. \tag{4.2.15}$$

Notice that, since η is the L^2-orthogonal projection of A on the linear subspace $\{d_{\mathbb{S}^4} f_i\}^{\perp}_{i=1,...,5} \subset L^2(\mathbb{S}^4)$, we have

$$\|\eta\|_{L^2(\mathbb{S}^4)} \leqslant \|A - \iota^*_{\mathbb{S}^4} f^* \bar{A}\|_{L^2(\mathbb{S}^4)}, \tag{4.2.16}$$

for every constant $\mathfrak{g}$-valued 1-form $\bar{A}$ on $\mathbb{R}^5$. Moreover,

$$d_{\mathbb{S}^4} \eta = d_{\mathbb{S}^4} A - d_{\mathbb{S}^4}(\iota^*_{\mathbb{S}^4} f^* \xi) = d_{\mathbb{S}^4} A - \iota^*_{\mathbb{S}^4} f^*(d_{\mathbb{R}^5} \xi) = d_{\mathbb{S}^4} A.$$

For future reference, we record the simple bound

$$\|f^*\xi\|_{L^\infty(\mathbb{B}^5)} \leqslant C \|d_{\mathbb{R}^5} f\|_{L^\infty(\mathbb{B}^5)} \, |\xi| \leqslant C \|d_{\mathbb{R}^5} f\|_{L^\infty(\mathbb{B}^5)} \, C_f \|A\|_{L^2(\mathbb{S}^4)} = C \tilde{C}_f \|A\|_{L^2(\mathbb{S}^4)}, \tag{4.2.17}$$

where $C_f > 0$ is given by (4.2.14) and we let $\tilde{C}_f := \|d_{\mathbb{R}^5} f\|_{L^\infty(\mathbb{B}^5)} \, C_f$.

Since $d_{\mathbb{R}^5}(f^*\xi \wedge f^*\xi) = 0$ and $f^*\xi \in L^\infty(\mathbb{B}^5)$, for every exponent $p \in [1, +\infty)$ there exists a unique[12] $\alpha \in W^{1,p}(\wedge^1 \mathbb{B}^5 \otimes \mathfrak{g})$ such that

$$\begin{cases} d_{\mathbb{R}^5} \alpha = f^*\xi \wedge f^*\xi & \text{in } \mathbb{B}^5, \\ d^*_{\mathbb{R}^5} \alpha = 0 & \text{in } \mathbb{B}^5, \\ \alpha(\partial_r) = 0 & \text{in } \mathbb{B}^5. \end{cases}$$

[12] The form α can be obtained variationally by minimizing

$$\int_{\mathbb{B}^5} |\alpha|^2 \, d\mathcal{L}^5$$

among any form satisfying $d_{\mathbb{R}^5} \alpha = f^*\xi \wedge f^*\xi$.

Moreover, from [47], for any $p \in [1,+\infty)$ we have

$$\|\alpha\|_{W^{1,p}(\mathbb{B}^5)} \leqslant C_p \|f^*\xi\|^2_{L^{2p}(\mathbb{B}^5)} \leqslant C_p \|f^*\xi\|^2_{L^\infty(\mathbb{B}^5)} \leqslant C_p\tilde{C}_f^2 \, \|A\|^2_{L^2(\mathbb{S}^4)}, \tag{4.2.18}$$

where $C_p > 0$ depends only on $p \in [1,+\infty)$. In particular, we deduce

$$\|\alpha\|_{L^\infty(\mathbb{B}^5)} \leqslant C\tilde{C}_f^2 \|A\|^2_{L^2(\mathbb{S}^4)}. \tag{4.2.19}$$

We define

$$\omega := \eta + \iota^*_{\mathbb{S}^4}\alpha = A - \iota^*_{\mathbb{S}^4} f^*\xi + \iota^*_{\mathbb{S}^4}\alpha \in L^2(\mathbb{S}^4).$$

Observe that

$$\begin{aligned} F_\omega = F_{\eta+\iota^*_{\mathbb{S}^4}\alpha} &= d_{\mathbb{S}^4}\eta + \iota^*_{\mathbb{S}^4}(f^*\xi\wedge f^*\xi) + \eta\wedge\eta + \eta\wedge\iota^*_{\mathbb{S}^4}\alpha \wedge + \iota^*_{\mathbb{S}^4}\alpha\wedge\eta + \iota^*_{\mathbb{S}^4}(\alpha\wedge\alpha) \\ &= \big(F_\eta + \iota^*_{\mathbb{S}^4}(f^*\xi\wedge f^*\xi)\big) + \eta\wedge\iota^*_{\mathbb{S}^4}\alpha\wedge + \iota^*_{\mathbb{S}^4}\alpha\wedge\eta + \iota^*_{\mathbb{S}^4}(\alpha\wedge\alpha). \end{aligned} \tag{4.2.20}$$

Since $d(f^*\xi) = 0$, we also have

$$\begin{aligned} F_A = F_{\eta+\iota^*_{\mathbb{S}^4}f^*\xi} &= F_\eta + \eta\wedge\iota^*_{\mathbb{S}^4}f^*\xi + \iota^*_{\mathbb{S}^4}f^*\xi\wedge\eta + \iota^*_{\mathbb{S}^4}(f^*\xi\wedge f^*\xi) \\ &= \big(F_\eta + \iota^*_{\mathbb{S}^4}(f^*\xi\wedge f^*\xi)\big) + \eta\wedge\iota^*_{\mathbb{S}^4}f^*\xi + \iota^*_{\mathbb{S}^4}f^*\xi\wedge\eta. \end{aligned} \tag{4.2.21}$$

Combining (4.2.20) and (4.2.21) we obtain

$$F_\omega = F_A - \eta\wedge\iota^*_{\mathbb{S}^4}f^*\xi - \iota^*_{\mathbb{S}^4}f^*\xi\wedge\eta + \eta\wedge\iota^*_{\mathbb{S}^4}\alpha\wedge + \iota^*_{\mathbb{S}^4}\alpha\wedge\eta + \iota^*_{\mathbb{S}^4}(\alpha\wedge\alpha). \tag{4.2.22}$$

Thus,

$$\begin{aligned} \|F_\omega\|_{L^2(\mathbb{S}^4)} &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|d_{\mathbb{R}^5}f\|_{L^\infty(\mathbb{B}^5)}|\xi|\|\eta\|_{L^2(\mathbb{S}^4)} + \|\alpha\|_{L^\infty(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)} + \|\alpha\|^2_{L^\infty(\mathbb{S}^4)}\right) \\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^2\|A\|^2_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^4\|A\|^4_{L^2(\mathbb{S}^4)}\right) \\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G)\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^4\|A\|^4_{L^2(\mathbb{S}^4)}\right) \\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G + \tilde{C}_f^3\varepsilon_G^2)\|A\|^2_{L^2(\mathbb{S}^4)}\right) \\ &\leqslant C\left(\varepsilon_G + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G + \tilde{C}_f^3\varepsilon_G)\varepsilon_G^2\right), \end{aligned} \tag{4.2.23}$$

for some universal constant $C > 0$. By assumption, since $A \in \mathfrak{a}_G(\mathbb{S}^4)$ and since the curvature of A is small enough in L^2-norm, there exists $\hat{g} \in W^{1,2}(\mathbb{S}^4, G)$ such that $A^{\hat{g}} \in W^{1,2}(\mathbb{S}^4)$. We observe that

$$\begin{aligned} \omega^{\hat{g}} &= \hat{g}^{-1}d_{\mathbb{S}^4}\hat{g} + \hat{g}^{-1}(\eta + \iota^*_{\mathbb{S}^4}\alpha)\hat{g} \\ &= \hat{g}^{-1}d_{\mathbb{S}^4}\hat{g} + \hat{g}^{-1}(A - \iota^*_{\mathbb{S}^4}f^*\xi + \iota^*_{\mathbb{S}^4}\alpha)\hat{g} = A^{\hat{g}} + \hat{g}^{-1}\iota^*_{\mathbb{S}^4}(\alpha - f^*\xi)\hat{g} \in L^4(\mathbb{S}^4). \end{aligned}$$

and

$$\|F_{\omega^{\hat{g}}}\|_{L^2(\mathbb{S}^4)} = \|F_\omega\|_{L^2(\mathbb{S}^4)} \leqslant C\left(\varepsilon_G + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G + \tilde{C}_f^3\varepsilon_G^2)\varepsilon_G^2\right).$$

If ε_G in (4.2.3) is small enough, we can apply Proposition 4.A.3 to $\omega^{\hat{g}}$ and we get the existence of a gauge $h \in W^{1,4}(\mathbb{S}^4, G)$ such that, by letting $g := \hat{g}h \in W^{1,2}(\mathbb{S}^4, G)$, we have $d^*_{\mathbb{S}^4}\omega^g = 0$ and

$$\omega^g = A^g - g^{-1}\iota^*_{\mathbb{S}^4}f^*\xi\, g + g^{-1}\,\iota^*_{\mathbb{S}^4}\alpha\, g. \tag{4.2.24}$$

Moreover,

$$
\begin{aligned}
\|\omega^g\|_{W^{1,2}(\mathbb{S}^4)} &\leqslant C\|F_\omega\|_{L^2(\mathbb{S}^4)} \\
&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G)\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^4\|A\|^4_{L^2(\mathbb{S}^4)}\right),
\end{aligned} \tag{4.2.25}
$$

where $C>0$ only depends on G. We have

$$
\begin{aligned}
\|A^g\|_{L^4(\mathbb{S}^4)} &\leqslant \|\omega^g\|_{L^4(\mathbb{S}^4)} + \|\alpha - f^*\xi\|_{L^4(\mathbb{S}^4)} \\
&\leqslant C(G,f)\ \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A-\iota^*_{\mathbb{S}^4}f^*\bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\right).
\end{aligned}
$$

Observe that by substituting g with $g\,g_0$ for some $g_0\in G$ we still have $d^*_{\mathbb{S}^4}\omega^g=0$ and (4.2.25) with the constant $C>0$ being unchanged. Since we have $d_{\mathbb{S}^4}g = g\omega^g - \omega g$, recalling that $\varepsilon_G<1$ by assumption, we get

$$
\begin{aligned}
\|d_{\mathbb{S}^4}g\|_{L^2(\mathbb{S}^4)} &\leqslant C\ \left(\|\omega^g\|_{L^2(\mathbb{S}^4)} + \|\omega\|_{L^2(\mathbb{S}^4)}\right) \\
&\leqslant C\left(\|F_\omega\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \|\alpha\|_{L^2(\mathbb{S}^4)}\right) \\
&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f(1+\tilde{C}_f\varepsilon_G + \tilde{C}_f^3\varepsilon_G^2)\|A\|^2_{L^2(\mathbb{S}^4)}\right) \\
&\quad + C\|\eta\|_{L^2(\mathbb{S}^4)} + C\tilde{C}_f^2\ \|A\|^2_{L^2(\mathbb{S}^4)} \\
&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right),
\end{aligned} \tag{4.2.26}
$$

where $\hat{C}_f := \tilde{C}_f(1+2\tilde{C}_f+\tilde{C}_f^3)$. Sobolev–Poincaré inequality gives the existence of $C>0$ (independent on A) such that

$$
\|g-\bar{g}\|_{L^4(\mathbb{S}^4)} \leqslant C\|d_{\mathbb{S}^4}g\|_{L^2(\mathbb{S}^4)}, \tag{4.2.27}
$$

where $\bar{g}$ is the average of g on $\mathbb{S}^4$. Thus, we deduce the existence of $x_0\in\mathbb{S}^4$ such that

$$
|g(x_0)-\bar{g}| \leqslant C\|d_{\mathbb{S}^4}g\|_{L^2(\mathbb{S}^4)}. \tag{4.2.28}
$$

Replacing g by $gg^{-1}(x_0)$ and combining (4.2.27) and (4.2.28) we obtain

$$
\begin{aligned}
\|g-\mathrm{id}_G\,\|_{L^4(\mathbb{S}^4)} &\leqslant C\,\|d_{\mathbb{S}^4}g\|_{L^2(\mathbb{S}^4)} \\
&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right).
\end{aligned} \tag{4.2.29}
$$

We denote $\tilde{g} := g(x/|x|)\in W^{1,2}(\mathbb{B}^5)$ the radial extension of g in $\mathbb{B}^5$. A straightforward estimate gives

$$
\begin{aligned}
\|\tilde{g}-\mathrm{id}_G\,\|_{L^4(\mathbb{B}^5)} &= \left(\int_{\mathbb{B}^5}\left|g\left(\frac{x}{|x|}\right)-\mathrm{id}_G\right|^4\ d\mathcal{L}^5(x)\right)^{\frac{1}{4}} \\
&\leqslant \left(\int_0^1 r^4\,d\mathcal{L}^1(r)\int_{\mathbb{S}^4}|g-\mathrm{id}_G|^4\ d\mathscr{H}^4\right)^{\frac{1}{4}} \leqslant \|g-\mathrm{id}_G\,\|_{L^4(\mathbb{S}^4)} \\
&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right).
\end{aligned} \tag{4.2.30}
$$

Moreover, using (4.2.26), we get

$$\begin{aligned}\left(\int_{\mathbb{B}^5}|d\tilde{g}|^2\,d\mathcal{L}^5\right)^{\frac{1}{2}} &\leqslant C\left(\int_{\mathbb{B}^5}\left|dg\left(\frac{x}{|x|}\right)\right|^2\frac{1}{|x|^2}\,d\mathcal{L}^5(x)\right)^{\frac{1}{2}}\\ &\leqslant C\left(\int_0^1 r^2\,d\mathcal{L}^1(r)\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^4}|d_{\mathbb{S}^4}g^2|\,d\mathscr{H}^4\right)^{\frac{1}{2}}\\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|\eta\|_{L^2(\mathbb{S}^4)}+\hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right).\end{aligned}\tag{4.2.31}$$

Let $\tilde{\omega}$ be the unique minimizers of

$$\inf\left\{\int_{\mathbb{B}^5}\left(|d_{\mathbb{R}^5}C|^2+|d^*_{\mathbb{R}^5}C|^2\right)dx^5\,:\,\iota^*_{\mathbb{S}^4}C=\omega^g\right\}.\tag{4.2.32}$$

Classical analysis for differential forms gives that $\tilde{\omega}$ solves

$$\begin{cases} d^*_{\mathbb{R}^5}\tilde{\omega}=0 & \text{in } \mathbb{B}^5,\\ d^*_{\mathbb{R}^5}d_{\mathbb{R}^5}\tilde{\omega}=0 & \text{in } \mathbb{B}^5,\\ \iota^*_{\mathbb{S}^4}\tilde{\omega}=\omega^g=A^g+g^{-1}\iota^*_{\mathbb{S}^4}(\alpha-f^*\xi)g & \text{on } \partial\mathbb{B}^5\end{cases}\tag{4.2.33}$$

and that $\tilde{\omega}\in(W^{\frac{3}{2},2}\cap C^\infty)(\mathbb{B}^5)$. By classical elliptic regularity theory and Sobolev embedding theorem, we have the estimates

$$\begin{aligned}\|\tilde{\omega}\|_{W^{1,\frac{5}{2}}(\mathbb{B}^5)} &\leqslant C\|\tilde{\omega}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)}\leqslant C\|\omega^g\|_{W^{1,2}(\mathbb{S}^4)}\\ &\leqslant C\big(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)}+\|A\|^4_{L^2(\mathbb{S}^4)}\big),\end{aligned}\tag{4.2.34}$$

for some constant $C>0$ depending only on G and on f (notice that the last inequality follows from (4.2.25)). Recall the continuous linear embedding

$$W^{1,\frac{5}{2}}(\mathbb{B}^5)\hookrightarrow L^5(\mathbb{B}^5).\tag{4.2.35}$$

Hence, in particular

$$\begin{aligned}\|\tilde{\omega}\|_{L^5(\mathbb{B}^5)} &\leqslant C\|\tilde{\omega}\|_{W^{1,\frac{5}{2}}(\mathbb{B}^5)}\leqslant C\|\tilde{\omega}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)}\leqslant C\|\omega^g\|_{W^{1,2}(\mathbb{S}^4)}\\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)}+\|A\|^4_{L^2(\mathbb{S}^4)}\right).\end{aligned}\tag{4.2.36}$$

We let $\tilde{A}\in L^5(\wedge^1\mathbb{B}^5\otimes\mathfrak{g})$ be given by

$$\tilde{A}=\tilde{\eta}+f^*\xi=\tilde{\omega}-\alpha+\zeta+f^*\xi,\tag{4.2.37}$$

where we denote

$$\tilde{\eta}:=\tilde{\omega}-\alpha+\zeta\tag{4.2.38}$$

and ζ is constructed as follows. We apply Proposition 4.B.1 to G and we find a smooth Riemannian manifold M_G and a measurable map

$$\operatorname{Ext}(g):M_G\longrightarrow W^{1,\frac{5}{2}}(\mathbb{B}^5,G)\tag{4.2.39}$$

$$p \longmapsto g_p \tag{4.2.40}$$

such that (4.B.1), (4.B.4) and (4.B.6) hold. We then choose

$$\zeta := \alpha - f^*\xi - \fint_{M_G} g_p^{-1}(\alpha - f^*\xi) g_p \, d\,\mathrm{vol}_{M_G}(p) \tag{4.2.41}$$

$$= \fint_{M_G} \left(\mathrm{id}_G - g_p^{-1}\right)(\alpha - f^*\xi) g_p \, d\,\mathrm{vol}_{M_G}(p) + \fint_{M_G} (\alpha - f^*\xi)\left(\mathrm{id}_G - g_p\right) d\,\mathrm{vol}_{M_G}(p). \tag{4.2.42}$$

We notice that $\iota_{\mathbb{S}^4}^*\zeta = 0$ and $\iota_{\mathbb{S}^4}^*\tilde{\omega} = A^g + g^{-1}\,\iota_{\mathbb{S}^4}(\alpha - f^*\xi)\,g$, so that $\iota_{\mathbb{S}^4}^*\tilde{A} = A^g$ by construction[13].

We have also

$$\begin{aligned} d\zeta &:= f^*\xi \wedge f^*\xi - \fint_{M_G} g_p^{-1}\, f^*\xi \wedge f^*\xi\, g_p \; dp - \fint_{M_G} dg_p^{-1} \wedge (\alpha - f^*\xi)\; g_p \, d\,\mathrm{vol}_{M_G}(p) \\ &\quad + \fint_{M_G} g_p^{-1}\,(\alpha - f^*\xi) \wedge dg_p \, d\,\mathrm{vol}_{M_G}(p) \\ &= \fint_{M_G} \left(\mathrm{id}_G - g_p^{-1}\right) f^*\xi \wedge f^*\xi \; g_p \, d\,\mathrm{vol}_{M_G}(p) + \fint_{M_G} f^*\xi \wedge f^*\xi \; \left(\mathrm{id}_G - g_p\right) d\,\mathrm{vol}_{M_G}(p) \\ &\quad - \fint_{M_G} dg_p^{-1} \wedge (\alpha - f^*\xi)\; g_p \, d\,\mathrm{vol}_{M_G}(p) + \fint_{M_G} g_p^{-1}\,(\alpha - f^*\xi) \wedge dg_p \, d\,\mathrm{vol}_{M_G}(p). \end{aligned} \tag{4.2.43}$$

Using (4.B.3), we have

$$\begin{aligned} \|d\zeta\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} &\leqslant C\,\|f^*\xi\|^2_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G| \; d\,\mathrm{vol}_{M_G}(p) \right\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} \\ &\qquad + C\,\|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |d_x g_p| \; d\,\mathrm{vol}_{M_G}(p) \right\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} \\ &\leqslant C_f\,\|A\|_{L^2(\mathbb{S}^4)}\;\|dg\|_{L^2(\mathbb{S}^4)} \\ &\leqslant C\,\|A\|_{L^2(\mathbb{S}^4)}\;\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^*\bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)}\right) \end{aligned} \tag{4.2.44}$$

and, by Poincaré–Sobolev inequality,

$$\begin{aligned} \|\zeta\|_{L^5(\mathbb{B}^5)} &\leqslant \|\alpha - f^*\xi\|_\infty \left| \fint_{M_G} \int_{\mathbb{B}^5} |g_p - \mathrm{id}_G|^5 \; d\mathcal{L}^5 \, d\,\mathrm{vol}_{M_G}(p) \right|^{\frac{1}{5}} \\ &\leqslant C_{f,G}\,\|A\|_{L^2(\mathbb{S}^4)} \left| \fint_{M_G} \int_{\mathbb{B}^5} |g_p - \mathrm{id}_G|^5 \; d\mathcal{L}^5 \, d\,\mathrm{vol}_{M_G}(p) \right|^{\frac{1}{5}} \\ &\leqslant C_{f,G}\,\|A\|_{L^2(\mathbb{S}^4)} \left| \fint_{M_G} \int_{\mathbb{B}^5} |d_x g_p|^{\frac{5}{2}} \; d\mathcal{L}^5 \, d\,\mathrm{vol}_{M_G}(p) \right|^{\frac{2}{5}} \\ &\leqslant C_f\,\|A\|_{L^2(\mathbb{S}^4)}\;\|dg\|_{L^2(\mathbb{S}^4)} \\ &\leqslant C\,\|A\|_{L^2(\mathbb{S}^4)}\;\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A - \iota_{\mathbb{S}^4}^* f^*\bar{A}\|_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)}\right). \end{aligned} \tag{4.2.45}$$

[13] At this stage, in order to control the L^2-norm of $F_{\tilde{A}}$ in $\mathbb{B}^5$ it could be tempting to estimate separately $\|d\tilde{A}\|_{L^2(\mathbb{B}^5)}$ and $\|\tilde{A} \wedge \tilde{A}\|_{L^2(\mathbb{B}^5)}$. This however is not going to lead to the expected estimate because each of the terms require to estimate respectively $\|d\alpha\|_{L^2(\mathbb{B}^5)}$ and $\|f^*\xi \wedge f^*\xi\|_{L^2(\mathbb{B}^5)}$ which would give in the r.h.s of (4.2.5) a term proportional to $\|A\|^2_{L^2(\mathbb{B}^5)}$ as in [65, Proposition 2.1] but which is preventing to prove the desired approximability property 4.1.34. While in the combination $d\tilde{A} + \tilde{A} \wedge \tilde{A}$ these two contributions fortunately cancel each other which is one of the main advantage of this construction.

Hence,

$$\|d\zeta+\zeta\wedge\zeta\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}\leqslant C\,\|A\|_{L^2(\mathbb{S}^4)}\;\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\|A\|^2_{L^2(\mathbb{S}^4)}\right).\tag{4.2.46}$$

We now explicitly compute

$$\begin{aligned}F_{\tilde{A}}&=d_{\mathbb{R}^5}\tilde{A}+\tilde{A}\wedge\tilde{A}=d_{\mathbb{R}^5}(\tilde{\omega}+f^*\xi-\alpha+\zeta)+(\tilde{\omega}+f^*\xi-\alpha+\zeta)\wedge(\tilde{\omega}+f^*\xi-\alpha+\zeta)\\&=F_{\tilde{\omega}}+F_\zeta-f^*\xi\wedge f^*\xi+\tilde{\omega}\wedge f^*\xi-\tilde{\omega}\wedge\alpha+\tilde{\omega}\wedge\zeta+f^*\xi\wedge\tilde{\omega}+f^*\xi\wedge f^*\xi-f^*\xi\wedge\alpha\\&\quad+f^*\xi\wedge\zeta+-\alpha\wedge\tilde{\omega}-\alpha\wedge f^*\xi+\alpha\wedge\alpha+\zeta\wedge\tilde{\omega}+\zeta\wedge f^*\xi-\zeta\wedge\alpha\\&=F_{\tilde{\omega}}+F_\zeta+(\tilde{\omega}\wedge f^*\xi+f^*\xi\wedge\tilde{\omega})-(\tilde{\omega}\wedge\alpha+\alpha\wedge\tilde{\omega})+(\tilde{\omega}\wedge\zeta+\zeta\wedge\tilde{\omega})\\&\quad-(f^*\xi\wedge\alpha+\alpha\wedge f^*\xi)+(f^*\xi\wedge\zeta+\zeta\wedge f^*\xi)-(\alpha\wedge\zeta+\zeta\wedge\alpha)+\alpha\wedge\alpha.\end{aligned}\tag{4.2.47}$$

Hence, we obtain

$$\begin{aligned}\|F_{\tilde{A}}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}&\leqslant C\,\|d\tilde{\omega}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}+\|\tilde{\omega}\|^2_{L^5(\mathbb{B}^5)}+\|F_\zeta\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}+C\,\|\alpha\|_{L^\infty(\mathbb{B}^5)}\,\|f^*\xi\|_{L^\infty(\mathbb{B}^5)}\\&\quad+C\,\|\zeta\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}\,\left(\|f^*\xi\|_{L^\infty(\mathbb{B}^5)}+\|\alpha\|_\infty\right)+C\,\|\alpha\|^2_\infty\\&\leqslant C\,\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\,\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\|A\|^3_{L^2(\mathbb{S}^4)}\right).\end{aligned}\tag{4.2.48}$$

Observe that

$$\tilde{A}-f^*\bar{A}=\tilde{\eta}+f^*\xi-f^*\bar{A}=\tilde{\eta}+f^*(\xi-\bar{A}),\tag{4.2.49}$$

where $\tilde{\eta}:=\tilde{\omega}-\alpha+\zeta$. We have

$$\begin{aligned}\|\tilde{\eta}\|_{L^5(\mathbb{B}^5)}&\leqslant\|\tilde{\omega}\|_{L^5(\mathbb{B}^5)}+\|\alpha\|_{L^5(\mathbb{B}^5)}+\|\zeta\|_{L^5(\mathbb{B}^5)}\\&\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\|\eta\|_{L^2(\mathbb{S}^4)}+\|A\|^4_{L^2(\mathbb{S}^4)}\right)+C\tilde{C}_f^2\|A\|^2_{L^2(\mathbb{S}^4)}\\&\quad+C\tilde{C}_f\|A\|_{L^2(\mathbb{S}^4)}\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right)\\&\quad+C\tilde{C}_f^2\|A\|^2_{L^2(\mathbb{S}^4)}\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\hat{C}_f\|A\|^2_{L^2(\mathbb{S}^4)}\right).\end{aligned}\tag{4.2.50}$$

We have also

$$\begin{aligned}\|f^*(\xi-\bar{A})\|_{L^5(\mathbb{B}^5)}&\leqslant C\|d_{\mathbb{R}^5}f\|_{L^\infty(\mathbb{B}^5)}\|\xi-\bar{A}\|_{L^5(\mathbb{B}^5)}\\&\leqslant C\|d_{\mathbb{R}^5}f\|_{L^\infty(\mathbb{B}^5)}|\xi-\bar{A}|=C\|d_{\mathbb{R}^5}f\|_{L^\infty(\mathbb{B}^5)}\left(\sum_{i=1}^5|\xi_i-\bar{A}_i|^2\right)^{\frac{1}{2}},\end{aligned}\tag{4.2.51}$$

where $C>0$ is a universal constant. Notice that, by Cauchy–Schwarz inequality, we have

$$|\xi_i-\bar{A}_i|^2=\frac{|\langle A-\iota_{\mathbb{S}^4}^*f^*\bar{A},d_{\mathbb{S}^4}f_i\rangle_{L^2(\mathbb{S}^4)}|^2}{\|d_{\mathbb{S}^4}f_i\|^4_{L^2(\mathbb{S}^4)}}\leqslant\frac{\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|^2_{L^2(\mathbb{S}^4)}}{\|d_{\mathbb{S}^4}f_i\|^2_{L^2(\mathbb{S}^4)}},\qquad\forall i=1,...,5,\tag{4.2.52}$$

which by (4.2.51) implies

$$\|f^*(\xi-\bar{A})\|_{L^5(\mathbb{B}^5)}\leqslant C\tilde{C}_f\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)},\tag{4.2.53}$$

for some universal constant $C>0$. We now need an estimate on L^2-norm of $\tilde{\eta}$. Using (4.2.49), (4.2.16), (4.2.38) and (4.2.53), we have

$$\begin{aligned}\|\tilde{A}-f^*\bar{A}\|_{L^5(\mathbb{B}^5)} &\leqslant \|\tilde{\eta}\|_{L^5(\mathbb{B}^5)}+\|f^*(\xi-\bar{A})\|_{L^5(\mathbb{B}^5)}\\ &\leqslant C\left[\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}^4\right]+C\tilde{C}_f^2\|A\|_{L^2(\mathbb{S}^4)}^2\\ &\quad+C\tilde{C}_f\|A\|_{L^2(\mathbb{S}^4)}\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\hat{C}_f\|A\|_{L^2(\mathbb{S}^4)}^2\right)\\ &\quad+C\tilde{C}_f^2\|A\|_{L^2(\mathbb{S}^4)}^2\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\hat{C}_f\|A\|_{L^2(\mathbb{S}^4)}^2\right)\\ &\quad+C\tilde{C}_f\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)},\end{aligned}\tag{4.2.54}$$

where C only depends on G. Thus, we get (4.2.9) for $B:=\tilde{A}$. We have

$$\begin{aligned}\|d\tilde{A}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} &\leqslant \|d\tilde{\omega}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}+\|d\alpha\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}+\|d\zeta\|_{L^{\frac{5}{2}}(\mathbb{B}^5)}\\ &\leqslant C\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}^4\right)+C\tilde{C}_f^2\|A\|_{L^2(\mathbb{S}^4)}^2\\ &\quad+C\,\|A\|_{L^2(\mathbb{S}^4)}\left(\|F_A\|_{L^2(\mathbb{S}^4)}+\|A-\iota_{\mathbb{S}^4}^*f^*\bar{A}\|_{L^2(\mathbb{S}^4)}+\|A\|_{L^2(\mathbb{S}^4)}^2\right).\end{aligned}\tag{4.2.55}$$

Hence, we obtain (4.2.10).

We now prove (4.2.11). Notice that

$$\begin{aligned}F_{\hat{A}} &= d_{\mathbb{R}^5}\hat{A}+\hat{A}\wedge\hat{A}=d_{\mathbb{R}^5}(\tilde{A}^{\tilde{g}^{-1}})+\tilde{A}^{\tilde{g}^{-1}}\wedge\tilde{A}^{\tilde{g}^{-1}}\\ &=\tilde{g}\,F_{\tilde{A}}\,\tilde{g}^{-1}\in L^2(\mathbb{B}^5)\end{aligned}\tag{4.2.56}$$

and

$$\begin{aligned}\iota_{\mathbb{S}^4}^*\hat{A} &= \iota_{\mathbb{S}^4}^*(\tilde{g}d_{\mathbb{R}^5}(\tilde{g}^{-1})+\tilde{g}\tilde{A}\tilde{g}^{-1})=\iota_{\mathbb{S}^4}^*(-d_{\mathbb{R}^5}\tilde{g}\tilde{g}^{-1}+\tilde{g}\tilde{A}\tilde{g}^{-1})\\ &=-d_{\mathbb{S}^4}gg^{-1}+gA^gg^{-1}=-d_{\mathbb{S}^4}gg^{-1}+g(g^{-1}d_{\mathbb{S}^4}g+g^{-1}Ag)g^{-1}\\ &=A\in L^2(\mathbb{S}^4).\end{aligned}\tag{4.2.57}$$

We have

$$\begin{aligned}\|\hat{A}-f^*\bar{A}\|_{L^2(\mathbb{B}^5)} &\leqslant \|\hat{A}-\tilde{A}\|_{L^2(\mathbb{B}^5)}+\|\tilde{A}-f^*\bar{A}\|_{L^2(\mathbb{B}^5)}\\ &\leqslant \|\tilde{g}\,\tilde{A}\,\tilde{g}^{-1}-\tilde{A}\|_{L^2(\mathbb{B}^5)}+\|\tilde{g}\,d\tilde{g}^{-1}\|_{L^2(\mathbb{B}^5)}+\|\tilde{A}-f^*\bar{A}\|_{L^2(\mathbb{B}^5)}\\ &\leqslant \|\tilde{g}-\mathrm{id}_G\|_{L^4(\mathbb{B}^5)}\,\|\tilde{A}\|_{L^4(\mathbb{B}^5)}+\|d\tilde{g}\|_{L^2(\mathbb{B}^5)}+\|\tilde{A}-f^*\bar{A}\|_{L^2(\mathbb{B}^5)}.\end{aligned}\tag{4.2.58}$$

Combining (4.2.31), (4.2.54), (4.2.58) and (4.2.54), we obtain (4.2.11).

Coming now to the trace estimates

$$
\begin{aligned}
&\|\zeta\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} \leqslant 2\ \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G|\ d\,\mathrm{vol}_{M_G}(p) \right\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} \\
&\leqslant 2\ \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G|\ d\,\mathrm{vol}_{M_G}(p) \right\|^{\frac{1}{2}}_{L^2(\partial\Omega\cap\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G|\ d\,\mathrm{vol}_{M_G}(p) \right\|^{\frac{1}{2}}_{L^\infty(\partial\Omega\cap\mathbb{B}^5)} \\
&\leqslant C(G)\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G|\ d\,\mathrm{vol}_{M_G}(p) \right\|^{\frac{1}{2}}_{L^2(\partial\Omega\cap\mathbb{B}^5)} \\
&\leqslant C(G)\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left(\fint_{M_G} \|g_p - \mathrm{id}_G\|_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \right)^{\frac{1}{2}} \\
&\leqslant C(G)\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left(\fint_{M_G} \|g_p - \mathrm{id}_G\|^2_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \right)^{\frac{1}{4}} \\
&\leqslant C(G,\Omega)\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)}\, \|g - \mathrm{id}_G\|^{\frac{1}{2}}_{W^{1,2}(\mathbb{S}^4)} \\
&\leqslant C(G,\Omega) \left(\tilde{C}_f \|A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^2 \|A\|^2_{L^2(\mathbb{S}^4)} \right) \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \hat{C}_f \|A\|^2_{L^2(\mathbb{S}^4)} \right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.2.59}
$$

We have also using (4.2.43)

$$
\begin{aligned}
\|d\zeta\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} &\leqslant C\, \|f^*\xi\|^2_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |g_p - \mathrm{id}_G|\ d\,\mathrm{vol}_{M_G}(p) \right\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} \\
&\quad + C\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left\| \fint_{M_G} |dg_p|\ d\,\mathrm{vol}_{M_G}(p) \right\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} \\
&\leqslant C\, \|f^*\xi\|^2_{L^\infty(\mathbb{B}^5)} \fint_{M_G} \|g_p - \mathrm{id}_G\|_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \\
&\quad + C\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \fint_{M_G} \|dg_p\|_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \\
&\leqslant C\, \|f^*\xi\|^2_{L^\infty(\mathbb{B}^5)} \left(\fint_{M_G} \|g_p - \mathrm{id}_G\|^2_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \right)^{\frac{1}{2}} \\
&\quad + C\, \|\alpha - f^*\xi\|_{L^\infty(\mathbb{B}^5)} \left(\fint_{p\in M_G} \|dg_p\|^2_{L^2(\partial\Omega\cap\mathbb{B}^5)}\ d\,\mathrm{vol}_{M_G}(p) \right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.2.60}
$$

Using Proposition 4.B.1 together with (4.2.17) and (4.2.19), we finally get

$$
\|d\zeta\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} \leqslant C_G(\Omega) \big(\tilde{C}_f \|A\|_{L^2(\mathbb{S}^4)} + \tilde{C}_f^2 \|A\|^2_{L^2(\mathbb{S}^4)} \big) \big(\|F_A\|_{L^2(\mathbb{S}^4)} + \|\eta\|_{L^2(\mathbb{S}^4)} + \hat{C}_f \|A\|^2_{L^2(\mathbb{S}^4)} \big).
\tag{4.2.61}
$$

Moreover, thanks to the continuous embedding $W^{\frac{3}{2},2}(\mathbb{B}^5) \hookrightarrow W^{1,2}(\partial\Omega\cap\mathbb{B}^5)$, we have

$$
\begin{aligned}
\|\tilde{\omega}\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} + \|d_{\partial\Omega}\tilde{\omega}\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} &\leqslant C(\Omega)\, \|\tilde{\omega}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C(\Omega)\, \|\omega^g\|_{W^{1,2}(\mathbb{S}^4)} \\
&\leqslant C(\Omega) \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \|\eta\|_{L^2(\mathbb{S}^4)} + \|A\|^4_{L^2(\mathbb{S}^4)} \right),
\end{aligned}
\tag{4.2.62}
$$

where $C(\Omega)$ depends only on the C^2-norm of the C^2 components of $\partial(\Omega \cap \mathbb{B}^5)$, G, C_f and $\|df\|_{L^\infty(\mathbb{B}^5)}$. Combining now (4.2.17), (4.2.19), (4.2.59), (4.2.61), (4.2.62) and (4.2.47), we obtain (4.2.7).

Combining (4.2.59), (4.2.62), (4.2.19) and the definition of ξ, we obtain

$$\|\hat{A}^{\tilde{g}}\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} = \|\tilde{A}\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} \leqslant K_G \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \left\|A - \iota^*_{\mathbb{S}^4} f^* \bar{A}\right\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\right). \tag{4.2.63}$$

This concludes the proof of Proposition 4.2.1. □

From now on, we will use the following notation:

$$|x|_\infty := \sup_{i=1,\dots,5} |x_i|,$$
$$|x| := \left(\sum_{i=1}^5 x_i^2\right)^{\frac{1}{2}},$$

for every $x = (x_1, ..., x_5) \in \mathbb{R}^5$. We have obviously

$$\sqrt{5}^{-1}\, |x| \leqslant |x|_\infty \leqslant |x| \qquad \forall\, x \in \mathbb{R}^5.$$

Let $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^5, \mathbb{R}^5)$ be the bi-Lipschitz homeomorphism given by

$$\varphi(x) := \begin{cases} \dfrac{|x|}{|x|_\infty} x & \text{if } x \in \mathcal{R}^5 \smallsetminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$$

Corollary 4.2.1. *Let G be a compact matrix Lie group. Let $Q \subset \mathbb{R}^5$ be any open cube with edge-length $k\varepsilon$, where $k > 0$ is a universal constant. There are constants $\varepsilon_G \in (0, 1)$ and $C_G > 0$ depending only on G such that for any $A \in \mathfrak{a}_G(\partial Q)$ satisfying*

$$\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A\|_{L^2(\partial Q)} < \varepsilon_G \tag{4.2.64}$$

the following facts hold.

(i) There exist $g \in W^{1,2}(\partial Q, G)$ and a $\mathfrak{g}$-valued 1-form $\tilde{A} \in L^5(Q)$ such that for every 4-dimensional face F of ∂Q we have

$$\begin{cases} \|A^g\|_{W^{1,2}(F)} \leqslant C_G \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A\|_{L^2(\partial Q)}\right), \\ \iota^*_F \tilde{A} = A^g \end{cases} \tag{4.2.65}$$

and

$$\|F_{\tilde{A}}\|_{L^{\frac{5}{2}}(Q)} \leqslant C_G \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-2}\, \|A\|_{L^2(\partial Q)}\, \|A - \iota^*_{\partial Q}\bar{A}\|_{L^2(\partial Q)} + \varepsilon^{-3}\|A\|^3_{L^2(\partial Q)}\right), \tag{4.2.66}$$

$$\|dg\|_{L^2(\partial Q)} \leqslant C_G \left(\varepsilon\|F_A\|_{L^2(\partial Q)} + \|A - \iota^*_{\partial Q}\bar{A}\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A\|^2_{L^2(\partial Q)}\right), \tag{4.2.67}$$

for every constant $\mathfrak{g}$-valued 1-form $\bar{A}$ on $\mathbb{R}^5$.

(ii) There exists $\tilde{g} \in W^{1,2}(Q,G)$ such that the $\mathfrak{g}$-valued 1-form given by $\hat{A} := \tilde{A}^{\tilde{g}^{-1}} \in L^2(Q)$ satisfies the following properties.

a) $F_{\hat{A}} \in L^2(Q)$.

b) $\iota^*_{\mathbb{S}^4}\hat{A} = A \in L^2(\partial Q)$.

c) Let $Q' \subset \mathbb{R}^5$ be an open cube such that $Q' \cap Q$ is a rectangle of minimum edge-length $\alpha\varepsilon > 0$ where $\alpha > 0$ is a universal constant. Then, we have

$$\|F_{\hat{A}}\|^2_{L^2(\partial Q'\cap Q)} \leqslant C_G \left(\|F_A\|^2_{L^2(\partial Q)} + \varepsilon^{-4}\|A\|^2_{L^2(\partial Q)}\|A - \iota^*_{\partial Q}\bar{A}\|^2_{L^2(\partial Q)} + \varepsilon^{-6}\|A\|^6_{L^2(\partial Q)}\right), \tag{4.2.68}$$

$$\|\tilde{A}\|_{L^4(\partial Q'\cap Q)} \leqslant C_G \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-2}\|A\|_{L^2(\partial Q)}\left\|A - \iota^*_{\partial Q}\bar{A}\right\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A\|_{L^2(\partial Q)}\right), \tag{4.2.69}$$

for every constant $\mathfrak{g}$-valued 1-form $\bar{A}$ on $\mathbb{R}^5$.

Moreover, we have the estimates

$$\|\tilde{A} - \bar{A}\|_{L^5(Q)} \leqslant C_G \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A - \iota^*_{\partial Q}\bar{A}\|_{L^2(\partial Q)} + \varepsilon^{-2}\|A\|^2_{L^2(\partial Q)}\right), \tag{4.2.70}$$

$$\|d\tilde{A}\|_{L^{\frac{5}{2}}(Q)} \leqslant C_G \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A - \iota^*_{\partial Q}\bar{A}\|_{L^2(\partial Q)} + \varepsilon^{-2}\|A\|^2_{L^2(\partial Q)}\right), \tag{4.2.71}$$

$$\|\hat{A} - \bar{A}\|_{L^2(Q)} \leqslant C_G\sqrt{\varepsilon} \left(\|F_A\|_{L^2(\partial Q)} + \varepsilon^{-1}\|A - \iota^*_{\partial Q}\bar{A}\|_{L^2(\partial Q)} + \varepsilon^{-2}\|A\|^2_{L^2(\partial Q)}\right), \tag{4.2.72}$$

for every constant $\mathfrak{g}$-valued 1-form $\bar{A}$ on $\mathbb{R}^5$.

Proof of Corollary 4.2.1. Corollary 4.2.1 is deduced by applying Proposition 4.2.1 with $f = \varphi$ to the 1-form $\big((\varepsilon\ \cdot\ + c_Q)\circ\varphi\big)^*A$, where c_Q denotes the center of the cube Q, and then pulling the resulting data back on Q by $\big((\varepsilon\ \cdot\ + c_Q)\circ\varphi\big)^{-1}$. □

Extension under L^2-smallness of the curvature only

Proposition 4.2.2 (Harmonic extension under smallness condition on F_A only)**.** *Let G be a compact matrix Lie group. There are constants $\varepsilon_G \in (0,1)$ and $C_G > 0$ depending only on G such that for any $A \in \mathfrak{a}_G(\mathbb{S}^4)$ satisfying*

$$\|F_A\|_{L^2(\mathbb{S}^4)} < \varepsilon_G \tag{4.2.73}$$

the following facts hold.

(i) There exist $g \in W^{1,2}(\mathbb{S}^4, G)$ and a $\mathfrak{g}$-valued 1-form $\tilde{A} \in (W^{\frac{3}{2},2} \cap C^\infty)(\mathbb{B}^5)$ satisfying

$$\begin{cases} \|\tilde{A}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C_G\|F_A\|_{L^2(\mathbb{S}^4)}, \\ \|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C_G\|F_A\|_{L^2(\mathbb{S}^4)}, \\ \iota^*_{\mathbb{S}^4}\tilde{A} = A^g, \end{cases} \tag{4.2.74}$$

and

$$\|dg\|_{L^2(\mathbb{S}^4)} \leqslant C_G\big(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\big). \tag{4.2.75}$$

(ii) There exists $\tilde{g} \in W^{1,2}(\mathbb{B}^5, G)$ such that the $\mathfrak{g}$-valued 1-form $\hat{A} := \tilde{A}^{\tilde{g}^{-1}} \in L^2(\mathbb{B}^5)$ satisfies the following properties.

(1) $F_{\hat{A}} \in L^2(\mathbb{B}^5)$.

(2) $\iota_{\mathbb{S}^4}^* \hat{A} = A \in L^2(\mathbb{S}^4)$.

(3)

$$\int_{\mathbb{B}^5} |\hat{A}|^2 \, d\mathcal{L}^5 \leqslant C(G) \, \left(\|F_A\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)} \right). \tag{4.2.76}$$

(4) Let $\Omega \subset \mathbb{R}^5$ be an open set such that $\Omega \cap \mathbb{B}^5$ has a 4-dimensional compact Lipschitz boundary which can be included in a union of N submanifolds of $\overline{\mathbb{B}^5}$ of class C^2. Then we have

$$\|F_{\hat{A}}\|_{L^2(\partial\Omega\cap\mathbb{B}^5)} \leqslant K_G \|F_A\|_{L^2(\mathbb{S}^4)}, \tag{4.2.77}$$

and

$$\|\tilde{A}\|_{L^4(\partial\Omega\cap\mathbb{B}^5)} \leqslant K_G \|F_A\|_{L^2(\mathbb{S}^4)}, \tag{4.2.78}$$

where $K_G = K_G(\Omega \cap \mathbb{B}^5) > 0$ depends only on G and on $\Omega \cap \mathbb{B}^5$ (that is the number N of submanifolds containing $\partial\Omega$ as well as their C^2 norms).

Proof of Proposition 4.2.2. To ease the reading, throughout this proof we will denote by "$d_{\mathbb{R}^5}$" and "$d_{\mathbb{S}^4}$" the standard differential of k-forms on $\mathbb{R}^5$ and on the round sphere $\mathbb{S}^4$ respectively. First, notice that since $A \in \mathfrak{a}_G(\mathbb{S}^4)$, by definition there exists locally $\hat{g} \in W^{1,2}$ such that $A^{\hat{g}} \in W^{1,2}$. For ε_G in (4.2.73) small enough the existence of such a g is global and A has a representative globally on $\mathbb{S}^4$ which is in $W^{1,2}(\wedge^1\mathbb{S}^4)$. We can apply Uhlenbeck's Coulomb gauge extraction theorem (see Proposition 4.A.1) to $A^{\hat{g}}$ and we get the existence of a gauge $h \in W^{1,4}(\mathbb{S}^4, G)$ such that, letting $g := \hat{g}h \in W^{1,2}(\mathbb{S}^4, G)$, we have $d^*_{\mathbb{S}^4} A^g = 0$ and

$$\|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C(G) \|F_A\|_{L^2(\mathbb{S}^4)}, \tag{4.2.79}$$

where $C(G) > 0$ depends only on G. Observe that by changing g into $g\,g_0$ we still have a solution of (4.2.25) with the constant $C(G) > 0$ being unchanged. We have $d_{\mathbb{S}^4} g = g\,A^g - A\,g$, hence

$$\begin{aligned} \|d_{\mathbb{S}^4} g\|_{L^2(\mathbb{S}^4)} &\leqslant C(G) \left(\|A^g\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \right) \\ &\leqslant C(G) \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \right). \end{aligned} \tag{4.2.80}$$

Poincaré inequality gives the existence of $C > 0$ (independent on A) such that

$$\|g - \bar{g}\|_{L^2(\mathbb{S}^4)} \leqslant C \; \|d_{\mathbb{S}^4} g\|_{L^2(\mathbb{S}^4)}, \tag{4.2.81}$$

where $\bar{g}$ is the average of g on $\mathbb{S}^4$. Thus, we deduce the existence of $x_0 \in \mathbb{S}^4$ such that

$$|g(x_0) - \bar{g}| \leqslant C \; \|d_{\mathbb{S}^4} g\|_{L^2(\mathbb{S}^4)}. \tag{4.2.82}$$

Replacing g by $gg^{-1}(x_0)$ and combining (4.2.81) and (4.2.82) we obtain

$$\|g - \mathrm{id}_G\|_{L^2(\mathbb{S}^4)} \leqslant C \, \|dg\|_{L^2(\mathbb{S}^4)} \leqslant C(G) \left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \right). \tag{4.2.83}$$

We denote $\tilde{g} := g(x/|x|)$ the radial extension of g in $\mathbb{B}^5$ (here "$|\cdot|$" stands for the standard Euclidean norm of x). A straightforward estimate gives

$$\|\tilde{g} - \mathrm{id}_G\|_{L^2(\mathbb{B}^5)} \leqslant \|g - \mathrm{id}_G\|_{L^2(\mathbb{S}^4)} \leqslant C(G)\left(\|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}\right). \tag{4.2.84}$$

Moreover, by the coarea formula we have

$$\begin{aligned}\int_{\mathbb{B}^5} |d\tilde{g}|^2 \, d\mathcal{L}^5 &= \int_0^1 \rho^{-2} \int_{\partial B_r(0)} \left| dg\left(\frac{x}{|x|}\right)\right|^2 \, d\mathscr{H}^4(x) \, d\mathcal{L}^1(\rho) \leqslant C \, \|dg\|^2_{L^2(\mathbb{S}^4)} \\ &\leqslant C(G)\left(\|F_A\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)}\right).\end{aligned} \tag{4.2.85}$$

We now extend A^g by $\tilde{A}$ that we choose to be the unique minimizer of

$$\inf\left\{ \int_{\mathbb{B}^5} \left(|d_{\mathbb{R}^5} G|^2 + |d^*_{\mathbb{R}^5} G|^2\right) d\mathcal{L}^5 \, : \, \iota^*_{\mathbb{S}^4} G = A^g \right\}. \tag{4.2.86}$$

Classical analysis for differential forms gives that $\tilde{A} \in (W^{\frac{3}{2},2} \cap C^\infty)(\mathbb{B}^5)$ solves

$$\begin{cases} d^*_{\mathbb{R}^5} \tilde{A} = 0, \\ d^*_{\mathbb{R}^5} d_{\mathbb{R}^5} \tilde{A} = 0, \\ \iota^*_{\mathbb{S}^4} \tilde{A} = A^g. \end{cases} \tag{4.2.87}$$

Moreover, the following estimate holds:

$$\|\tilde{A}\|_{W^{1,\frac{5}{2}}(\mathbb{B}^5)} \leqslant C \, \|\tilde{A}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C \, \|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C(G) \, \|F_A\|_{L^2(\mathbb{S}^4)}. \tag{4.2.88}$$

Thus, (i) follows. Let $\hat{A} := \tilde{A}^{\tilde{g}^{-1}} \in L^2(\mathbb{B}^5)$. Notice that

$$F_{\hat{A}} = d_{\mathbb{R}^5} \hat{A} + \hat{A} \wedge \hat{A} = d_{\mathbb{R}^5}(\tilde{A}^{\tilde{g}^{-1}}) + \tilde{A}^{\tilde{g}^{-1}} \wedge \tilde{A}^{\tilde{g}^{-1}} = \tilde{g} F_{\tilde{A}} \tilde{g}^{-1} \in L^2(\mathbb{B}^5) \tag{4.2.89}$$

and

$$\iota^*_{\mathbb{S}^4} \hat{A} = \iota^*_{\mathbb{S}^4}(\tilde{g} d_{\mathbb{R}^5}(\tilde{g}^{-1}) + \tilde{g}\tilde{A}\tilde{g}^{-1}) = \iota^*_{\mathbb{S}^4}(-d_{\mathbb{R}^5}\tilde{g}\tilde{g}^{-1} + \tilde{g}\tilde{A}\tilde{g}^{-1}) \tag{4.2.90}$$

$$= -d_{\mathbb{S}^4} g g^{-1} + g A^g g^{-1} = -d_{\mathbb{S}^4} g g^{-1} + g(g^{-1} d_{\mathbb{S}^4} g + g^{-1} A g) g^{-1} = A \in L^2(\mathbb{S}^4). \tag{4.2.91}$$

We have also

$$\begin{aligned}\int_{\mathbb{B}^5} |\hat{A}|^2 \, d\mathcal{L}^5 &= \int_{\mathbb{B}^5} |\tilde{g}\tilde{A}\tilde{g}^{-1} + \tilde{g}\, d\tilde{g}^{-1}|^2 \, d\mathcal{L}^5 \leqslant 2 \int_{\mathbb{B}^5} |\tilde{A}|^2 \, d\mathcal{L}^5 + 2 \int_{\mathbb{B}^5} |d\tilde{g}|^2 \, d\mathcal{L}^5 \\ &\leqslant C(G)\left(\|F_A\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)}\right).\end{aligned} \tag{4.2.92}$$

Lastly, fix any open set $\Omega \subset \mathbb{R}^5$ such that $\Omega \cap \mathbb{B}^5$ has Lipschitz boundary which can be included in a union of N submanifolds of $\overline{\mathbb{B}^5}$ of class C^2. Recalling that $\varepsilon_G < 1$ by assumption, we have

$$\begin{aligned}\|F_{\hat{A}}\|_{L^2(X)} = \|F_{\tilde{A}}\|_{L^2(X)} &\leqslant C\left(\|\nabla \tilde{A}\|_{L^2(X))} + \|\tilde{A}\|^2_{L^2(X)}\right) \\ &\leqslant C(\Omega)\|\tilde{A}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C(\Omega, G) \, \|F_A\|_{L^2(\mathbb{S}^4)},\end{aligned} \tag{4.2.93}$$

where $C(\Omega, G) > 0$ depends only on G and on $\Omega \cap \mathbb{B}^5$. This concludes the proof of Proposition 4.2.2.

□

Completely analogously to the way we proved Corollary 4.2.1, by exploiting Proposition 4.2.2 one derives from proposition 4.2.2 the following statement.

Corollary 4.2.2. *Let G be a compact matrix Lie group. Let $Q \subset \mathbb{R}^5$ be any open cube with edge-length $k\varepsilon$, where $k > 0$ is a universal constant. There are constants $\varepsilon_G \in (0,1)$ and $C_G > 0$ depending only on G such that for any $A \in \mathfrak{a}_G(\partial Q)$ satisfying*

$$\|F_A\|_{L^2(\partial Q)} < \varepsilon_G \tag{4.2.94}$$

the following facts hold.

(i) There exist a gauge $g \in W^{1,2}(\partial Q, G)$ and a 1-form $\tilde{A} \in W^{1,\frac{5}{2}}(Q)$ such that for every 4-dimensional face F of ∂Q we have

$$\begin{cases} \|d\tilde{A}\|_{L^{\frac{5}{2}}(Q)} + \|\tilde{A}\|_{L^5(Q)} \leqslant C_G \, \|F_A\|_{L^2(\partial Q)}, \\ \|A^g\|_{W^{1,2}(F)} \leqslant C_G \|F_A\|_{L^2(\partial Q)}, \\ \iota_F^* \tilde{A} = A^g \end{cases} \tag{4.2.95}$$

and

$$\|F_{\tilde{A}}\|_{L^{\frac{5}{2}}(Q)} \leqslant C_G \, \|F_A\|_{L^2(\partial Q)}, \tag{4.2.96}$$

$$\|dg\|_{L^2(\partial Q)} \leqslant C_G \big(\varepsilon \|F_A\|_{L^2(\partial Q)} + \|A\|_{L^2(\partial Q)}\big). \tag{4.2.97}$$

(ii) There exists $\tilde{g} \in W^{1,2}(Q, G)$ such that the $\mathfrak{g}$-valued 1-form $\hat{A} := \tilde{A}^{\tilde{g}^{-1}} \in L^2(Q)$ satisfies the following properties.

(a) $F_{\hat{A}} \in L^2(Q)$.

(b) $\iota_{\mathbb{S}^4}^* \hat{A} = A \in L^2(\partial Q)$.

(c) Let $Q' \subset \mathbb{R}^5$ be an open cube such that $Q' \cap Q$ is a rectangle of minimum edge-length $\alpha\varepsilon > 0$ where $\alpha > 0$ is a universal constant. Then, we have

$$\|F_{\hat{A}}\|_{L^2(\partial Q' \cap Q)} \leqslant K_G \|F_A\|_{L^2(\partial Q)}, \tag{4.2.98}$$

and

$$\|\tilde{A}\|_{L^4(\partial Q' \cap Q)} \leqslant K_G \|F_A\|_{L^2(\partial Q)}, \tag{4.2.99}$$

where $K_G > 0$ depends only on G.

Remark 4.2.1. [65, Proposition 2.1], has a missing term on the right-hand-side of equation (2.2). This has been noticed by S. Sil. Proposition 4.2.1 is a suitable replacement of [65, Proposition 2.1].

4.2.2. Construction of optimally regular gauges on the boundary of cubes

We start by recalling the following optimal extension theorem, whose proof can be found in [67, Theorem 2].

Proposition 4.2.3. *Let $n \in \mathbb{N}$ be such that $n \geqslant 1$ and let $N \hookrightarrow \mathbb{R}^k$ be a closed embedded submanifold of $\mathbb{R}^k$. Then, for every $u \in W^{\frac{n}{n+1},n+1}(\mathbb{S}^n, N)$ there exists an extension $\tilde{u} \in W^{1,(n+1,\infty)}(\mathbb{B}^{n+1}, N)$ such that $\tilde{u}|_{\mathbb{S}^n} = u$ in the sense of traces.*

Next, we leverage on the Proposition 4.2.3 to prove the following gluing lemma.

Lemma 4.2.1. *Let G be a compact matrix Lie group and let $n \in \mathbb{N}$ be such that $n \geqslant 2$. Let*

$$\mathbb{S}^n_+ := \{x = (x_1, ..., x_{n+1}) \in \mathbb{S}^n \, : \, x_{n+1} > 0\}. \tag{4.2.100}$$

Then, for every $g \in W^{1,n}(\mathbb{S}^n_+, G)$ there exists $\tilde{g} \in W^{1,(n+1,\infty)}(\mathbb{B}^{n+1}, G)$ such that $\tilde{g}|_{\mathbb{S}^n_+} = g$.

Proof of Lemma 4.2.1. Let

$$\mathbb{S}^n_- := \{x = (x_1, ..., x_{n+1}) \in \mathbb{S}^n \, : \, x_{n+1} < 0\}. \tag{4.2.101}$$

and define $h \in W^{1,n}(\mathbb{S}^n_-, G)$ by

$$h(x_1, ..., x_{n+1}) := g(x_1, ..., -x_{n+1}) \qquad \forall\, x = (x_1, ..., x_{n+1}) \in \mathbb{S}^n_-. \tag{4.2.102}$$

Let $\hat{g} \in W^{1,n}(\mathbb{S}^n, G)$ be given by

$$\hat{g} := \begin{cases} g & \text{on } \mathbb{S}^n_+ \\ h & \text{on } \mathbb{S}^n_- \end{cases} \tag{4.2.103}$$

Since $W^{1,n}(\mathbb{S}^n, G) \hookrightarrow W^{\frac{n}{n+1}, n+1}(\mathbb{S}^n, G)$, by Proposition 4.2.3 there exists $\tilde{g} \in W^{1,(n+1,\infty)}(\mathbb{B}^{n+1}, N)$ such that $\tilde{g}|_{\mathbb{S}^n} = \hat{g}$ in the sense of traces. Since $\hat{g} = g$ on $\mathbb{S}^n_+$, this concludes the proof of Lemma 4.2.1. □

Corollary 4.2.3. *Let G be a compact matrix Lie group. There exist constants $\varepsilon_G \in (0,1)$ and $C_G > 0$ depending only in G such that the following holds. Let $Q \subset \mathbb{R}^5$ be an open cube in $\mathbb{R}^5$ and let $\mathscr{F}$ be any non-empty proper subset of 4-dimensional faces of ∂Q. Let*

$$\Omega := \bigcup_{F \in \mathscr{F}} F. \tag{4.2.104}$$

Then, for every $g \in W^{1,4}(\Omega, G)$ there exists an extension $\tilde{g} \in W^{1,5}(Q, G)$ such that $\tilde{g}|_{\Omega} = g$ in the sense of traces.

Proof of Corollary 4.2.3. Let $\Phi : \overline{Q} \to \overline{\mathbb{B}^5}$ be a bi-Lipschitz homeomorphism such that

$$\Omega = \mathbb{S}^4_+, \qquad \partial\Omega = \Gamma, \qquad \partial Q \smallsetminus \Omega = \mathbb{S}^4_-. \tag{4.2.105}$$

By apply Lemma 4.2.1 to $(\Phi^{-1})^* g \in W^{1,4}(\mathbb{S}^4_+, G)$ and pullback the resulting extension on $\mathbb{B}^5$ by Φ we get the desired $\tilde{g}$. This concludes the proof of Corollary 4.2.3. □

4.3. The notion of admissible covers and good and bad cubes

4.3.1. The notion of admissible cubic ε_i-covers

From now on, for every $c \in \mathbb{R}^5$ and $\varepsilon > 0$ we let

$$Q_\varepsilon(c) := \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^5 + c$$

be the open cube with center c, edge-length $\varepsilon > 0$ and faces parallel to the coordinate planes. We shall be using the following definition.

Definition 4.3.1. Let $A \in \mathfrak{a}_G(\mathbb{B}^5)$, $c \in \mathbb{R}^5$ and $\rho > 0$. We say that $A \in \mathcal{A}_G(Q_\rho(c))$ if there exists an L^2 1-form that we denote $\iota^*_{\partial Q_\rho(c)} A$ such that

(i)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\rho-\varepsilon}^{\rho+\varepsilon} \int_{\partial Q_\rho(c)} \left| \iota^*_{D_r} \iota^*_{\partial Q_r(c)} A - \iota^*_{\partial Q_\rho(c)} A \right|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^1(r) = 0,$$

where $D_r(x) := \frac{\rho}{r}(x-c) + c$.

(ii)

$$\iota^*_{\partial Q_\rho(c)} A \in \mathfrak{a}_G(\partial Q_\rho(c)).$$

As a direct consequence of Fubini and Lebesgue theorems, we have the following proposition.

Proposition 4.3.1. *Let $A \in \mathfrak{a}_G(\mathbb{B}^5)$ and $c \in \mathbb{B}^5$, then for $\mathcal{L}^1$-a.e. $\rho > 0$ such that $Q_\rho(c) \subset \mathbb{B}^5$ there holds $A \in \mathcal{A}_G(\partial Q_\rho(c))$. A positive number ρ such that $A \in \mathcal{A}_G(\partial Q_\rho(c))$ is called an admissible edge-length for A.*

For the purposes of the present subsection, given any $\varepsilon \in \left(0, \frac{1}{4}\right)$ we let

$$\mathscr{C}_\varepsilon := \varepsilon \mathbb{Z}^5 \cap \overline{Q_{1-2\varepsilon}(0)}.$$

Note that

$$\#\{c' \in \mathscr{C}_\varepsilon \,:\, Q_{2\varepsilon}(c) \cap Q_{2\varepsilon}(c') \neq \emptyset\} \leqslant N \qquad \forall\, c \in \mathscr{C}_\varepsilon, \tag{4.3.1}$$

where $N \in \mathbb{N}$ is independent on ε.

Lemma 4.3.1 (Choice of an admissible cubic cover)**.** *Let $A \in \mathfrak{a}_G(Q_1^5(0))$. There exists a universal constant $K > 0$ such that for a sequence $\{\varepsilon_i\}_{i \in \mathbb{N}} \in \left(0, \frac{1}{4}\right)$ satisfying $\varepsilon_i \to 0$ for $i \to +\infty$, for i large enough we can find a family of admissible edge-lengths $\{\rho_{i,c}\}_{c \in \mathscr{C}_{\varepsilon_i}} \subset \left(\frac{3}{2}\varepsilon_i, 2\varepsilon_i\right)$ for which the following facts hold.*

(i) $A \in \mathcal{A}_G(\partial Q_{\rho_{i,c}}(c))$ for every $c \in \mathscr{C}_{\varepsilon_i}$.

(ii) For every $c \in \mathscr{C}_{\varepsilon_i}$ we have

$$\int_{\partial Q_{\rho_{i,c}}(c)} |F_A|^2 \, d\mathscr{H}^4 \leqslant \frac{K}{2\varepsilon_i} \int_{Q_{2\varepsilon_i}(c)} |F_A|^2 \, d\mathcal{L}^5 \tag{4.3.2}$$

and

$$\int_{\partial Q_{\rho_{i,c}}(c)} |A|^2 \, d\mathscr{H}^4 \leqslant \frac{K}{2\varepsilon_i} \int_{Q_{2\varepsilon_i}(c)} |A|^2 \, d\mathcal{L}^5. \tag{4.3.3}$$

(iii) It holds that

$$\lim_{i \to +\infty} \varepsilon_i \sum_{c \in \mathscr{C}_{\varepsilon_i}} \int_{\partial Q_{\rho_{i,c}}(c)} \left| A - (A)_{Q_{2\varepsilon_i}(c)} \right|^2 d\mathscr{H}^4 = 0, \tag{4.3.4}$$

$$\lim_{i\to+\infty}\sum_{c\in\mathscr{C}_{\varepsilon_i}}\int_{Q_{\rho_{i,c}}(c)}\left|A-(A)_{Q_{2\varepsilon_i}(c)}\right|^2 d\mathcal{L}^5=0 \tag{4.3.5}$$

and

$$\lim_{i\to+\infty}\varepsilon_i\sum_{c\in\mathscr{C}_{\varepsilon_i}}\int_{\partial Q_{\rho_{i,c}}(c)}\left|F_A-(F_A)_{Q_{2\varepsilon_i}(c)}\right|^2 d\mathscr{H}^4=0, \tag{4.3.6}$$

where we have used the following notation:

$$(A)_{Q_{2\varepsilon_i}(c)}:=\fint_{Q_{2\varepsilon_i}(c)}A\,d\mathcal{L}^5 \qquad \text{and} \qquad (F_A)_{Q_{2\varepsilon_i}(c)}:=\fint_{Q_{2\varepsilon_i}(c)}F_A\,d\mathcal{L}^5.$$

Proof of Lemma 4.3.1. Fix any $\varepsilon\in\left(0,\frac{1}{4}\right)$. By assumption, for every $c\in\mathscr{C}_\varepsilon$ there exists a full $\mathcal{L}^1$-measure set $R_{A,c}\subset\left(\frac{3}{2}\varepsilon,2\varepsilon\right)$ such that $\iota^*_{\partial Q_\rho(c)}A\in\mathfrak{a}_G(\partial Q_\rho(c))$ for every $\rho\in R_{A,c}$. Given a constant $K>0$ (to be fixed), for any $c\in\mathscr{C}_\varepsilon$ we define the set

$$E_{\varepsilon,K,c}:=\left\{\rho\in\left(\frac{3}{2}\varepsilon,2\varepsilon\right)\text{ s.t. }A\notin\mathfrak{a}_G(\partial Q_\rho(c))\quad\text{or}\quad\int_{\partial Q_\rho(c)}|F_A|^2\,d\mathscr{H}^4>\frac{K}{2\varepsilon}\int_{Q_{2\varepsilon}(c)}|F_A|^2\,d\mathcal{L}^5\right\}. \tag{4.3.7}$$

By integration on $E_{\varepsilon,K,c}$ and Fubini together with mean value theorem we get

$$\begin{aligned}\frac{K}{2\varepsilon}\left(\int_{Q_{2\varepsilon}(c)}|F_A|^2\,d\mathcal{L}^5\right)\mathcal{L}^1(E_{\varepsilon,K,c})&<\int_{E_{\varepsilon,K,c}}\int_{\partial Q_\rho(c)}|F_A|^2\,d\mathscr{H}^4\,d\mathcal{L}^1(\rho) &(4.3.8)\\ &\leqslant\int_0^{2\varepsilon}\int_{\partial Q_\rho(c)}|F_A|^2\,d\mathscr{H}^4\,d\mathcal{L}^1(\rho) &(4.3.9)\\ &=\int_{Q_{2\varepsilon}(c)}|F_A|^2\,d\mathcal{L}^5. &(4.3.10)\end{aligned}$$

This implies that

$$\mathcal{L}^1(E_{\varepsilon,K,c})<\frac{2\varepsilon}{K},\qquad\forall\,c\in\mathscr{C}_\varepsilon. \tag{4.3.11}$$

Fix $i\in\mathbb{N}$ and let $A_i\in C_c^\infty(\wedge^1Q_1^5(0)\otimes\mathfrak{g})$ be such that

$$\int_{Q_1^5(0)}|A_i-A|^2\,d\mathcal{L}^5\leqslant 2^{-i}. \tag{4.3.12}$$

Let

$$G_{\varepsilon,K,i}:=\left\{\rho\in\left(\frac{3}{2}\varepsilon,2\varepsilon\right)\text{ s.t. }\varepsilon\sum_{c\in\mathscr{C}_\varepsilon}\int_{\partial Q_\rho(c)}|A_i-A|^2\,d\mathscr{H}^4>K\int_{Q_1^5(0)}|A_i-A|^2\,d\mathcal{L}^5\right\}, \tag{4.3.13}$$

$$H_{\varepsilon,K,i}:=\left\{\rho\in\left(\frac{3}{2}\varepsilon,2\varepsilon\right)\text{ s.t. }f_i(\rho)>K\fint_{\frac{3}{2}\varepsilon}^{2\varepsilon}f_i(\rho)\,d\mathcal{L}^1(\rho)\right\}, \tag{4.3.14}$$

with

$$f_i(\rho):=\sum_{c\in\mathscr{C}_\varepsilon}\int_{\partial Q_\rho(c)}\left|A_i-(A_i)_{Q_{2\varepsilon}(c)}\right|^2 d\mathscr{H}^4. \tag{4.3.15}$$

Notice that, again by Fubini theorem together with mean value theorem and integration as above, we get

$$\begin{aligned} \mathcal{L}^1(G_{\varepsilon,K,i}) &\leqslant \frac{C\varepsilon}{K} \\ \mathcal{L}^1(H_{\varepsilon,K,i}) &\leqslant \frac{\varepsilon}{K} \end{aligned} \tag{4.3.16}$$

for every $i \in \mathbb{N}$, where $C > 0$ is a universal cover. By Fubini theorem, the mean value theorem and Poincaré inequality, we have

$$\int_{\frac{3}{2}\varepsilon}^{2\varepsilon} \sum_{c\in\mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} \big|A_i - (A_i)_{Q_{2\varepsilon}(c)}\big|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^1(\rho) \leqslant \sum_{c\in\mathscr{C}_\varepsilon} \int_{Q_{2\varepsilon}(c)} \big|A_i - (A_i)_{Q_{2\varepsilon}(c)}\big|^2 \, d\mathcal{L}^5 \tag{4.3.17}$$

$$\leqslant \sum_{c\in\mathscr{C}_\varepsilon} C_P\big(Q_{2\varepsilon}(c)\big) \int_{Q_{\frac{3\varepsilon}{4}}(c)} |\nabla A_i|^2 \, d\mathcal{L}^5 \tag{4.3.18}$$

$$\leqslant 4C_P(Q_1^5(0))\varepsilon^2 \sum_{c\in\mathscr{C}_\varepsilon} \int_{Q_{2\varepsilon}(c)} |\nabla A_i|^2 \, d\mathcal{L}^5 \tag{4.3.19}$$

$$\leqslant C\varepsilon^2 \int_{Q_1^5(0)} |\nabla A_i|^2 \, d\mathcal{L}^5, \tag{4.3.20}$$

where we denote $C_P(\Omega) > 0$ for the open and bounded domain $\Omega \subset \mathbb{R}^5$ and $C > 0$ is universal. Hence for any $\rho \in (G_{\varepsilon,K,i} \cup H_{\varepsilon,K,i})^c$ (where the superscript "c" denotes the complement of the set in $(\varepsilon, 2\varepsilon)$) one deduces

$$\sum_{c\in\mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} \big|A_i - (A_i)_{Q_{2\varepsilon}(c)}\big|^2 \, d\mathscr{H}^4 \leqslant K \fint_{\frac{3}{2}\varepsilon}^{2\varepsilon} \sum_{c\in\mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} |A_i - (A_i)_{Q_{2\varepsilon}(c)}|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^1(\rho) \tag{4.3.21}$$

$$\leqslant KC\varepsilon \int_{Q_1^5(0)} |\nabla A_i|^2 \, d\mathcal{L}^5. \tag{4.3.22}$$

Moreover, for every $\rho \in (\varepsilon, 2\varepsilon)$ we have

$$\varepsilon \sum_{c\in\mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} \big|(A_i)_{Q_{2\varepsilon}(c)} - (A)_{Q_{2\varepsilon}(c)}\big|^2 \, d\mathscr{H}^4 \leqslant \varepsilon \sum_{c\in\mathscr{C}_\varepsilon} |\partial Q_\rho(c)| \left(\fint_{Q_{2\varepsilon}(c)} |A - A_i| \, d\mathcal{L}^5 \right)^2 \tag{4.3.23}$$

$$\leqslant C \sum_{c\in\mathscr{C}_\varepsilon} \int_{Q_{2\varepsilon}(c)} |A - A_i|^2 \, d\mathcal{L}^5 \leqslant C\ 2^{-i}. \tag{4.3.24}$$

Hence, by triangle inequality, we get that for every $\rho \in (G_{\varepsilon,K,i} \cup H_{\varepsilon,K,i})^c$ we have

$$\varepsilon \sum_{c\in\mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} \big|A - (A)_{Q_{2\varepsilon}(c)}\big|^2 \, d\mathscr{H}^4 \leqslant \big(K + C\big)2^{-i} + KC\varepsilon \int_{Q_1^5(0)} |\nabla \tilde{A}_i|^2 \, d\mathcal{L}^5. \tag{4.3.25}$$

Analogously, we consider $F_i \in C_c^\infty(\wedge^2 Q_1^5(0) \otimes \mathfrak{g})$ such that

$$\int_{Q_1^5(0)} |F_i - F_A|^2 \, dx^5 \leqslant 2^{-i} \tag{4.3.26}$$

and we get that for every $\varepsilon \in \left(0, \frac{1}{4}\right)$ and for every $i \in \mathbb{N}$ there are two $\mathcal{L}^1$-measurable sets $G'_{\varepsilon,K,i}, H'_{\varepsilon,K,i} \subset \left(\frac{3}{2}\varepsilon, 2\varepsilon\right)$ satisfying

$$\begin{aligned}\mathcal{L}^1(G'_{\varepsilon,K,i}) &\leqslant \frac{C\varepsilon}{K} \\ \mathcal{L}^1(H'_{\varepsilon,K,i}) &\leqslant \frac{\varepsilon}{K}\end{aligned} \tag{4.3.27}$$

such that for every $\rho \in (G'_{\varepsilon,K,i} \cup H'_{\varepsilon,K,i})^c$ we have

$$\varepsilon \sum_{c \in \mathscr{C}_\varepsilon} \int_{\partial Q_\rho(c)} \left|F_A - (F_A)_{Q_{2\varepsilon}(c)}\right|^2 d\mathscr{H}^4 \leqslant \big(K + C\big)2^{-i} + KC\varepsilon \int_{Q_1^5(0)} |\nabla F_i|^2 \, d\mathcal{L}^5. \tag{4.3.28}$$

Now, first we let $\tilde{K} > 0$ be big enough (independent on ε and i) so that

$$\mathcal{L}^1(G_{\varepsilon,\tilde{K},i} \cup H_{\varepsilon,\tilde{K},i} \cup G'_{\varepsilon,\tilde{K},i} \cup H'_{\varepsilon,\tilde{K},i}) \leqslant \frac{\varepsilon}{2}. \tag{4.3.29}$$

Then, we notice that for every $i \in \mathbb{N}$ there exists $\varepsilon_i < 2^{-(i+1)}$ small enough so that

$$\tilde{K}C\varepsilon_i \left(\int_{Q_1^5(0)} |\nabla A_i|^2 \, d\mathcal{L}^5 + \int_{Q_1^5(0)} |\nabla F_i|^2 \, d\mathcal{L}^5 \right) \leqslant 2^{-(i+1)}. \tag{4.3.30}$$

Then, for every $i \in \mathbb{N}$ and for every $c \in \mathscr{C}_{\varepsilon_i}$ we pick $\rho_{i,c} > 0$ in the non-empty set

$$(G_{\varepsilon_i,\tilde{K},i} \cup H_{\varepsilon_i,\tilde{K},i} \cup G'_{\varepsilon_i,\tilde{K},i} \cup H'_{\varepsilon_i,\tilde{K},i})^c \cap R_{A,c} \cap E_{\varepsilon_i,\tilde{K},c}. \tag{4.3.31}$$

Notice that, for such admissible edge-lengths, we have

$$\varepsilon_i \sum_{c \in \mathscr{C}_{\varepsilon_i}} \int_{\partial Q_{\rho_{i,c}}(c)} \left|A - (A)_{Q_{2\varepsilon_i}(c)}\right|^2 d\mathscr{H}^4 \leqslant \big(K + C\big)\, 2^{-i} + 2^{-(i+1)} \tag{4.3.32}$$

and

$$\varepsilon_i \sum_{c \in \mathscr{C}_{\varepsilon_i}} \int_{\partial Q_{\rho_{i,c}}(c)} \left|F_A - (F_A)_{Q_{2\varepsilon_i}(c)}\right|^2 d\mathscr{H}^4 \leqslant \big(K + C\big)\, 2^{-i} + 2^{-(i+1)}. \tag{4.3.33}$$

This concludes the proof of Lemma 4.3.1. □

Definition 4.3.2. Let $A \in \mathfrak{a}_G(Q_1^5(0))$. Under the same notation that we have used in the previous Lemma 4.3.1, for every $\varepsilon_i \in \left(0, \frac{1}{4}\right)$ we say that ε_i is an *admissible scale* for A and that the collection of cubes $\{Q_{\rho_{i,c}}(c)\}_{c \in \mathscr{C}_{\varepsilon_i}}$ is a *admissible cubic ε_i-cover* relative to A.

Remark 4.3.1. Let $A \in \mathfrak{a}_G(Q_1^5(0))$. Let $\varepsilon \in \left(0, \frac{1}{4}\right)$ and let $\mathcal{Q}_\varepsilon$ be an admissible cubic ε-cover relative to A. For every $Q \in \mathcal{Q}_\varepsilon$, denote by $\bar{A}_Q$ the constant 2-form given by $(A)_{Q_{2\varepsilon}(c_Q)}$, where c_Q denotes the center of Q. Notice that, by Jensen inequality, we have

$$|\bar{A}_Q|^2 \leqslant \fint_{Q_{2\varepsilon}(c_Q)} |A|^2 \, d\mathcal{L}^5, \qquad \forall\, Q \in \mathcal{Q}_\varepsilon. \tag{4.3.34}$$

We have

$$\sum_{Q \in \mathcal{Q}_\varepsilon} \int_{\partial Q} |A|^2 \, d\mathscr{H}^4 \leqslant 2 \sum_{Q \in \mathcal{Q}_\varepsilon} \left(\int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 + \int_{\partial Q} |\bar{A}_Q|^2 \, d\mathscr{H}^4 \right) \tag{4.3.35}$$

$$\leqslant C \sum_{Q\in\mathcal{Q}_\varepsilon} \left(\int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 + \varepsilon^4 |\bar{A}_Q|^2 \right) \tag{4.3.36}$$

$$\leqslant C \sum_{Q\in\mathcal{Q}_\varepsilon} \left(\int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 + \varepsilon^4 \fint_{Q_{2\varepsilon}(c_Q)} |A|^2 \, d\mathcal{L}^5 \right) \tag{4.3.37}$$

$$\leqslant C \Big(\sum_{Q\in\mathcal{Q}_\varepsilon} \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 + \frac{1}{\varepsilon} \int_{Q_1^5(0)} |A|^2 \, d\mathcal{L}^5 \Big), \tag{4.3.38}$$

where $C > 0$ is independent on ε. By (4.3.4), for ε small enough we have

$$\varepsilon \sum_{Q\in\mathcal{Q}_\varepsilon} \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 \leqslant \int_{Q_1^5(0)} |A|^2 \, d\mathcal{L}^5. \tag{4.3.39}$$

This implies that for such small values of the parameter ε it holds that

$$\varepsilon \sum_{Q\in\mathcal{Q}_\varepsilon} \int_{\partial Q} |A|^2 \, d\mathscr{H}^4 \leqslant C \int_{Q_1^5(0)} |A|^2 \, d\mathcal{L}^5, \tag{4.3.40}$$

where $C > 0$ is independent on ε. Exactly by the same procedure, we get that

$$\varepsilon \sum_{Q\in\mathcal{Q}_\varepsilon} \int_{\partial Q} |F_A|^2 \, d\mathscr{H}^4 \leqslant C \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5 \tag{4.3.41}$$

5 for $\varepsilon > 0$ sufficiently small.

4.3.2. The notion of good and bad cubes

Definition 4.3.3 (Good and bad cubes). Let $A \in \mathfrak{a}_G(Q_1^5(0))$. Let $\varepsilon \in \left(0, \frac{1}{4}\right)$ be an admissible scale for A and let $\mathcal{Q}_\varepsilon$ be an admissible cubic ε-cover relative to A. Given any $\Lambda > 0$, we say that $Q \in \mathcal{Q}_\varepsilon$ is a Λ-*good* cube if the following conditions hold:

(1) $\displaystyle \frac{1}{\varepsilon^3} \int_{\partial Q} |F_A|^2 \, d\mathscr{H}^4 \leqslant \varepsilon^{\frac{1}{2}} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5$,

(2) $\displaystyle \frac{1}{\varepsilon^3} \int_{\partial Q} |A|^2 \, d\mathscr{H}^4 \leqslant \varepsilon^{\frac{1}{2}} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5$,

(3) $\displaystyle \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 \leqslant \frac{1}{\varepsilon} \int_{Q_{2\varepsilon}(c_Q)} |A|^2 \, d\mathcal{L}^5$,

(4) $\displaystyle \frac{1}{\varepsilon^4} \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 \leqslant \Lambda^{-1} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5$,

(5) $\displaystyle \fint_{Q_{2\varepsilon}(c_Q)} |A|^2 \, d\mathcal{L}^5 \leqslant \Lambda$.

Otherwise, we say that Q is a Λ-*bad* cube. We denote by $\mathcal{Q}^g_{\varepsilon,\Lambda}$ the set of all the Λ-good cubes and $\mathcal{Q}^b_{\varepsilon,\Lambda} := \mathcal{Q}_\varepsilon \smallsetminus \mathcal{Q}^g_{\varepsilon,\Lambda}$.

The following technical lemma is important for our argument. Given a sequence of admissible scales $\{\varepsilon_i\}_{i\in\mathbb{N}}$, the lemma states that as $i \in \mathbb{N}$ increases, asymptotically, the L^2-norm of A on the Λ-bad cubes is controlled by the L^2 norm of A on the union of cubes where property (5) in Definition 4.3.3 fails.

Lemma 4.3.2. *Let $A \in \mathfrak{a}_G(Q_1^5(0))$ and let $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ be a sequence of admissible scales for A. Let $\Lambda > 0$ and let $\mathcal{Q}_{\varepsilon_i}$ be an admissible cubic ε_i-cover relative to A and Λ, for every $i \in \mathbb{N}$. Then, for some universal constant $C > 0$ (depending on the intersection property of the cubic cover) there holds*

$$\lim_{i\to+\infty} \sum_{Q\in\mathcal{Q}^b_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2 \, d\mathcal{L}^5 \leqslant C \lim_{i\to+\infty} \int_{\Omega_{i,\Lambda,A}} |A|^2 \, d\mathcal{L}^5, \tag{4.3.42}$$

where $\Omega_{i,\Lambda,A} \subset Q_1^5(0)$ is given by

$$\Omega_{i,\Lambda,A} := \bigcup \left\{ Q \in \mathcal{Q}_{\varepsilon_i} \text{ s.t. } \fint_{Q_{2\varepsilon_i}(c_Q)} |A|^2 \, d\mathcal{L}^5 > \Lambda \right\}. \tag{4.3.43}$$

Proof of Lemma 4.3.2. Observe the following.

- If $Q \in \mathcal{Q}_{\varepsilon_i}$ is such that (1) fails, we have

$$\varepsilon_i \int_{\partial Q} |F_A|^2 \, d\mathscr{H}^4 > \varepsilon_i^{\frac{9}{2}} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5. \tag{4.3.44}$$

- If $Q \in \mathcal{Q}_{\varepsilon_i}$ is such that (2) fails, we have

$$\varepsilon_i \int_{\partial Q} |A|^2 \, d\mathscr{H}^4 > \varepsilon_i^{\frac{9}{2}} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5, \tag{4.3.45}$$

- If $Q \in \mathcal{Q}_{\varepsilon_i}$ is such that (3) fails, we have

$$\varepsilon_i \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 > \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2 \, d\mathcal{L}^5. \tag{4.3.46}$$

- If $Q \in \mathcal{Q}_{\varepsilon_i}$ is such that (4) fails, we have

$$\varepsilon_i \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 > \varepsilon_i^5 \Lambda^{-1} \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5. \tag{4.3.47}$$

Summing up over the appropriate cubes and by Remark 4.3.1, for $\varepsilon_i > 0$ small enough we get the following set of estimates:

$$\operatorname{card}(\{Q \in \mathcal{Q}_{\varepsilon_i} \text{ s.t. (1) fails}\}) < C\varepsilon_i^{-\frac{9}{2}}, \tag{4.3.48}$$

$$\operatorname{card}(\{Q \in \mathcal{Q}_{\varepsilon_i} \text{ s.t. (2) fails}\}) < C\left(\int_{Q_1^5(0)} |A|^2 \, d\mathcal{L}^5 \right) \left(\int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5 \right)^{-1} \varepsilon_i^{-\frac{9}{2}}, \tag{4.3.49}$$

$$\sum_{\substack{Q\in\mathcal{Q}_{\varepsilon_i} \\ \text{s.t. (3) fails}}} \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2 \, d\mathcal{L}^5 < \varepsilon_i \sum_{Q\in\mathcal{Q}_{\varepsilon_i}} \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4, \tag{4.3.50}$$

$$\operatorname{card}(\{Q \in \mathcal{Q}_{\varepsilon_i} \text{ s.t. (4) fails}\}) < \Lambda \left(\int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5 \right)^{-1} \varepsilon_i^{-5} \left(\varepsilon_i \sum_{Q\in\mathcal{Q}_{\varepsilon_i}} \int_{\partial Q} |A - \bar{A}_Q|^2 \, d\mathscr{H}^4 \right). \tag{4.3.51}$$

In particular, we note that the $\mathcal{L}^5$-measure of the union of the cubes in $\mathcal{Q}_{\varepsilon_i}$ such that (1),(2),(4) fail vanishes at the limit $i \to +\infty$. Thus,

$$\lim_{i\to+\infty} \sum_{\substack{Q\in\mathcal{Q}_{\varepsilon_i} \text{ s.t.}\\ (1),(2),(3),(4)\\ \text{fail}}} \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2\, d\mathcal{L}^5 = 0. \tag{4.3.52}$$

Finally, we get

$$\lim_{i\to+\infty} \sum_{Q\in\mathcal{Q}^b_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2\, d\mathcal{L}^5 \leqslant \lim_{i\to+\infty} \sum_{\substack{Q\in\mathcal{Q}_{\varepsilon_i}\\ \text{s.t. (5) fails}}} \int_{Q_{2\varepsilon_i}(c_Q)} |A|^2\, d\mathcal{L}^5 \tag{4.3.53}$$

and the statement follows by the intersection properties of the cubic cover. This concludes the proof of Lemma 4.3.2. □

Remark 4.3.2. Let $A \in \mathfrak{a}_G(Q_1^5(0))$. Let $\varepsilon \in \left(0, \frac{1}{4}\right)$ be a good scale for A and let $\mathcal{Q}_\varepsilon$ be an admissible cubic ε-cover relative to A. Notice that the property (3) in Definition 4.3.3 immediately implies that

$$\int_{\partial Q} |A|^2\, d\mathscr{H}^4 \leqslant \int_{\partial Q} |A - \bar{A}_Q|^2\, d\mathscr{H}^4 + \int_{\partial Q} |\bar{A}_Q|^2\, d\mathscr{H}^4 \tag{4.3.54}$$

$$\leqslant \frac{1}{\varepsilon} \int_{Q_{2\varepsilon}(c_Q)} |A|^2\, d\mathcal{L}^5 + C\varepsilon^4 \frac{1}{\varepsilon^5} \int_{Q_{2\varepsilon}(c_Q)} |A|^2\, d\mathcal{L}^5 \leqslant \frac{C}{\varepsilon} \int_{Q_{2\varepsilon}(c_Q)} |A|^2\, d\mathcal{L}^5 \tag{4.3.55}$$

for every good Λ-good cube $Q \in \mathcal{Q}_\varepsilon$, where $C > 0$ depends only on the dimension.

4.4. The first smoothification

The goal of this section is to approximate in L^2-norm any weak connection A whose curvature is small in Morrey norm by forms $A_{i,\Lambda}$ whose curvature is suitably controlled in L^2-norm on a set of good slices at the same scale ρ.

Lemma 4.4.1. *Let $Q = Q_\rho(c) \subset \mathbb{R}^5$ be an open cube in $\mathbb{R}^5$. Fix any $\mathfrak{g}$-valued 1-form $A \in L^2(\partial Q)$ with $F_A \in L^2(\partial Q)$. Let $\{Q_i\}_{i=0,\dots,k-1}$ be a collection of open cubes such that ∂Q is transversal to ∂Q_i for all i and assume that $\partial Q \subset \bigcup_{i=0}^{k-1} Q_i$, $A \in \mathfrak{a}_G(\partial Q \cap Q_i))$ for every $i = 0, \dots, k-1$. Then, there exists $\varepsilon_G \in (0,1)$ depending only on G such that if*

$$\int_{\partial Q} |F_A|^2\, d\mathscr{H}^4 < \varepsilon_G \tag{4.4.1}$$

we can find a global gauge $g \in W^{1,2}(\partial Q, G)$ such that $A^g \in L^4(\partial Q)$.

Proof of Lemma 4.4.1. Since $A \in \mathfrak{a}_G(\partial Q \cap Q_i))$ and since $\partial Q \cap Q_i$ is bi-Lipschitz homeomorphic to the 4-dimensional ball, there exists $h_i \in W^{1,2}(\partial Q \cap Q_i, G)$ such that $A^{h_i} \in L^4(\partial Q \cap Q_i)$. We consider $\psi_{c,\rho} := (\rho\, \cdot + c) \circ \varphi$ which realizes a bi-Lipschitz homeomorphism from $\mathbb{S}^4$ into ∂Q and we denote U_i the Lipschitz open subset of $\mathbb{S}^4$ given by $U_i := \psi_{c,\rho}^{-1}(\partial Q \cap Q_i)$. Let $(\omega_i)_{i=0\cdots k-1}$ be an open cover of $\mathbb{S}^4$ such that $\overline{\omega_i} \subset U_i$. We then cover each ω_i by finitely many smooth convex geodesic balls $(V_{ij})_{j\in I_i}$ of S^4 with $V_{il}\subset U_i$ so that the whole cover given by the V_{ij} is a good cover of $\mathbb{S}^4$ (i.e. the intersections of the V_{ij}s are again diffeomorphic to balls) by convex geodesic balls. To simplify the notations we re-index the cover $(V_{ij})_{i\in\{0\cdots k-1\},j\in I_i}$ by $(O_l)_{l=1\cdots L}$ and we denote by i_l the index $i\in\{0\cdots k-1\}$ such that $O_l\subset U_i$. Observe that

$$\|\psi^*_{c,\rho}F_A\|_{L^2(\mathbb{S}^4)}\leqslant C\,\|d\varphi\|^2_{L^\infty(\mathbb{S}^4)}\,\|F_A\|_{L^2(\partial Q)}\leqslant C\,\|d\varphi\|^2_\infty\,\sqrt{\varepsilon_G}. \tag{4.4.2}$$

We first choose $\varepsilon_G>0$ in (4.4.1) such that we can apply Proposition 4.A.2 to $\psi^*_{c,\rho}(A^{h_{i_l}})$ on the geodesic ball O_l. Let $g_l\in W^{1,2}(O_l,G)$ given by Proposition 4.A.2 such that

$$B_l:=\left(\psi^*_{c,\rho}(A^{h_{i_l}})\right)^{g_l} \tag{4.4.3}$$

satisfies

$$\|B_l\|_{W^{1,2}(O_l)}\leqslant C\,\|\psi^*_{c,\rho}F_A\|_{L^2(\mathbb{S}^4)}\leqslant C\,\|d\varphi\|^2_{L^\infty(\mathbb{S}^4)}\,\|F_A\|_{L^2(\partial Q)} \tag{4.4.4}$$

and

$$d^{*_{\mathbb{S}^4}}B_l=0. \tag{4.4.5}$$

Using the same argument as in [74, proof of Theorem V.5], we have that the transition functions $\sigma_{mn}:=g_n^{-1}\,(h_{i_n}\circ\psi^{-1}_{c,\rho})^{-1}h_{i_m}\circ\psi^{-1}_{c,\rho}\,g_m$ are continuous on $O_n\cap O_m$. and the co-cycle generated by $(\sigma_{mn})_{m,n\in\{1\cdots L\}}$ is C^0 approximable by a sequence of smooth ones. Hence, σ_{mn} are the transition functions of a smooth bundle and $\psi^*_{c,\rho}A$ defines a $W^{1,2}-$Sobolev connection on this $G-$bundle. Using [92], for ε_G small enough the bundle is trivial and there exists a global $W^{1,2}$ representative of $\psi^*_{c,\rho}A$. Pulling back this representative by $\psi^{-1}_{c,\rho}$ we have the existence of $g\in W^{1,2}(\partial Q,G)$ such that $A^g\in L^4(\partial Q)$. This concludes the proof of Lemma 4.4.1. □

We recall the definition of the Morrey seminorms in $\Omega\subset\mathcal{R}^n$:

$$|f|_{M^s_{p,q}(\Omega)}:=\sup_{\substack{x\in\Omega\\ \rho>0}}\left(\frac{1}{\rho^{n-pq}}\int_{Q_\rho(x)}\sum_{s_1+\cdots+s_n=s}\left|\partial^{s_1}_{x_1}\cdots\partial^{s_n}_{x_n}f\right|^p(x)\,d\mathcal{L}^n\right)^{\frac{1}{p}},$$

for every $s\in\mathbb{N}$, $1\leqslant p<+\infty$, $0<q<\frac{n}{p}$.

Definition 4.4.1. Given an admissible cubic ε-cover $\mathcal{Q}_\varepsilon$, we say that a cube $Q\subset\mathbb{R}^5$ is *uniformly transversal* to $\mathcal{Q}_\varepsilon$ if there are universal constants $0<\alpha\leqslant k$ such that for every $Q'\in\mathcal{Q}_\varepsilon$ we have that $Q\cap Q'$ is either empty or a rectangle with minimum edge length $\alpha\varepsilon$ and maximum edge length $k\varepsilon$.

Theorem 4.4.1 (Approximation under controlled traces of the curvature)**.** *Let G be a compact matrix Lie group and let $\Lambda>0$. There exist $\varepsilon_G\in(0,1)$ and $C_G>0$ depending only on G such that for every $A\in\mathfrak{a}_G(Q^5_1(0))$ satisfying*

$$|F_A|^2_{M^0_{2,2}(Q^5_1(0))}:=\sup_{\substack{x\in Q^5_1(0)\\ \rho>0}}\frac{1}{\rho}\int_{Q_\rho(x)}|F_A|^2\,d\mathcal{L}^5<\varepsilon_G$$

we can find a set $\{\varepsilon_i\}_{i\in\mathbb{N}}\subset\left(0,\frac{1}{4}\right)$ of admissible scales for A (with associated admissible cubic ε_i-covers $\mathcal{Q}_{\varepsilon_i}$) and a family of $\mathfrak{g}$-valued 1-forms $\{A_{i,\Lambda}\}_{i\in\mathbb{N}}\subset L^2(Q^5_1(0))$ such that the following facts hold.

(i) For every $i \in \mathbb{N}$, $F_{A_{i,\Lambda}} \in M^0_{2,2}(Q^5_1(0))$ with

$$|F_{A_{i,\Lambda}}|_{M^0_{2,2}(Q^5_1(0))} < C_G\, |F_A|_{M^0_{2,2}(Q^5_1(0))}.$$

(ii) There exist a universal integer $N \in \mathbb{N}$ and a non-negative $f \in M^0_{1,4}(Q^5_1(0))$ satisfying

$$|f|_{M^0_{1,4}(Q^5_1(0))} \leqslant |F_A|^2_{M^0_{2,2}(Q^5_1(0))}$$

such that for every $i \in \mathbb{N}$, for every $x \in Q^5_1(0)$ and for every $\rho > 0$ for which $Q_\rho(x)$ is uniformly transversal to $\mathcal{Q}_{\varepsilon_i}$ and $\partial Q_\rho(x) \subset Q_{1-\frac{\varepsilon_i}{2}}(0)$ we have

$$\int_{\partial Q_\rho(x)} |F_{A_{i,\Lambda}}|^2\, d\mathscr{H}^4 \leqslant \frac{C_G}{\varepsilon_i} \sum_{k=1}^{N} \int_{Q_{2\varepsilon_i}(x_k)} f\, d\mathcal{L}^5, \tag{4.4.6}$$

for an N-tuple of points $\{x_k = x_k(i,x,\rho)\}_{k=1,\dots,N} \subset Q^5_1(0)$.
Moreover, for every $i \in \mathbb{N}$, for every $x \in Q^5_1(0)$ and for $\mathcal{L}^1$-a.e. $\rho > 0$ for which $Q_\rho(x)$ is uniformly transversal to $\mathcal{Q}_{\varepsilon_i}$ and $\partial Q_\rho(x) \subset Q_{1-\frac{\varepsilon_i}{2}}(0)$ we have $\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda} \in \mathfrak{a}_G(\partial Q_\rho(x))$.

(iii) We have

$$\lim_{i\to+\infty} \|A_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))} \leqslant C_G \lim_{i\to+\infty} \int_{\Omega_{i,\Lambda,A}} |A|^2\, d\mathcal{L}^5,$$

where $\Omega_{i,\Lambda,A} \subset Q^5_1(0)$ is defined as in Lemma 4.3.2.

Remark 4.4.1. It is not claimed in Theorem 4.4.1 that the elements $A_{i,\Lambda}$ are in $\mathfrak{a}_G(Q^5_1(0))$. It could very well be that this is the case and that it could be checked out of the construction given in the proof but, since it is not needed in the sequel, this property is left open.

Before proving Theorem 4.4.1, we shall need the following lemma, which will be used at several steps of the proof of Theorem 4.4.1 and deserves to be stressed separately.

Lemma 4.4.2. *Let $A \in L^2(Q^5_1(0))$, let $Q_\rho(c) \subset Q^5_1(0)$ and let $p \in [1,+\infty)$ and $q \in [1,+\infty]$. Assume moreover that the following facts hold.*

(i) $F_A := dA + A \wedge A \in L^{p,q}(Q_\rho(c)) \cap L^{p,q}(Q^5_1(0) \setminus Q_\rho(c))$.

(ii) There exists an L^2 1-form $\iota^*_{\partial Q_\rho(c)} A$ on $\partial Q_\rho(c)$ such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\rho-\varepsilon}^{\rho+\varepsilon} \int_{\partial Q_\rho(c)} \left| D_r^* \iota^*_{\partial Q_r(c)} A - \iota^*_{\partial Q_\rho(c)} A \right|^2 \, d\mathscr{H}^4\, d\mathcal{L}^1(r) = 0, \tag{4.4.7}$$

where $D_r(x) := \frac{r}{\rho}(x-c) + c$.

Then, $F_A := dA + A \wedge A \in L^{p,q}(Q^5_1(0))$.

Proof of Lemma 4.4.2. Since the proof is identical for $q \neq p$, to fix the ideas we assume that $q = p$, i.e. $L^{p,q}(Q^5_1(0)) = L^p(Q^5_1(0))$. We first claim that $F_A \in L^1(Q^5_1(0))$. Since $A \in L^2(Q^5_1(0))$, automatically we have $A \wedge A \in L^1(Q^5_1(0))$ and it suffices to show that the distributional differential of A is a 2-form in $L^1(Q^5_1(0))$. We claim that the distributional differential of A on $Q^5_1(0)$ is exactly

$dA \in L^1(Q_1^5(0))$. Let $\varphi \in C_c^\infty(Q_1^5(0))$ be any smooth and compactly supported 3-form on $Q_1^5(0)$. Fix any $\varepsilon > 0$ and let $\eta_\varepsilon \in C_c^\infty(Q_1^5(0))$ a cut-off function that vanishes on $\partial Q_\rho(c)$ and satisfies

$$\eta_\varepsilon \equiv 1 \qquad \text{on } \big(Q_1^5(0) \setminus Q_{\rho+\varepsilon}(c)\big) \cup Q_{\rho-\varepsilon}(c), \tag{4.4.8}$$

$$0 \leqslant \eta_\varepsilon \leqslant 1 \qquad \text{on } Q_1^5(0) \tag{4.4.9}$$

and

$$\|d\eta_\varepsilon\|_{L^\infty(Q_1^5(0))} \leqslant \frac{C}{\varepsilon}. \tag{4.4.10}$$

Notice that $\eta_\varepsilon \varphi \in W_0^{1,1}(Q_1^5(0) \setminus \partial Q_\rho(c))$. Hence,

$$\int_{Q_1^5(0)} A \wedge d(\eta_\varepsilon \varphi) = \int_{Q_1^5(0)} \eta_\varepsilon (dA \wedge \varphi). \tag{4.4.11}$$

Moreover,

$$d(\eta_\varepsilon \varphi) = d\eta_\varepsilon \wedge \varphi + \eta_\varepsilon d\varphi. \tag{4.4.12}$$

Thus,

$$\int_{Q_1^5(0)} A \wedge d(\eta_\varepsilon \varphi) = \int_{Q_1^5(0)} \eta_\varepsilon (A \wedge d\varphi) + \int_{Q_1^5(0)} A \wedge d\eta_\varepsilon \wedge \varphi. \tag{4.4.13}$$

This implies that

$$\int_{Q_1^5(0)} \eta_\varepsilon (A \wedge d\varphi) = \int_{Q_1^5(0)} \eta_\varepsilon (dA \wedge \varphi) - \int_{Q_1^5(0)} A \wedge d\eta_\varepsilon \wedge \varphi. \tag{4.4.14}$$

By assumption (ii), we have

$$\int_{Q_1^5(0)} A \wedge d\eta_\varepsilon \wedge \varphi \to 0 \tag{4.4.15}$$

as $\varepsilon \to 0$. Hence, by passing to the limit as $\varepsilon \to 0$ in (4.4.14) we get

$$\int_{Q_1^5(0)} A \wedge d\varphi = \int_{Q_1^5(0)} dA \wedge \varphi, \tag{4.4.16}$$

which implies that $dA \in L^1(Q_1^5(0))$ is the distributional differential of A on $Q_1^5(0)$ and our claim follows.

Let $\varphi \in C_c^\infty(Q_1^5(0))$ a smooth and compactly supported 2-form on $Q_1^5(0)$ such that $\|\varphi\|_{L^{p'}(Q_1^5(0))} \leqslant 1$. Fix any $\varepsilon > 0$ and let

$$\begin{aligned}
\int_{Q_1^5(0)} F_A \wedge *\varphi &= \int_{Q_1^5(0) \setminus Q_{\rho+\varepsilon}(c)} F_A \wedge *\varphi + \int_{Q_{\rho+\varepsilon}(c) \setminus Q_{\rho-\varepsilon}(c)} F_A \wedge *\varphi + \int_{Q_{\rho-\varepsilon}(c)} F_A \wedge *\varphi \\
&\leqslant \int_{Q_1^5(0) \setminus Q_\rho(c)} F_A \wedge *\varphi + \int_{Q_{\rho+\varepsilon}(c) \setminus Q_{\rho-\varepsilon}(c)} F_A \wedge *\varphi + \int_{Q_\rho(c)} F_A \wedge *\varphi \\
&\leqslant \|F_A\|_{L^p(Q_1^5(0) \setminus Q_\rho(c))} + \|F_A\|_{L^p(Q_\rho(c))} + \int_{Q_{\rho+\varepsilon}(c) \setminus Q_{\rho-\varepsilon}(c)} F_A \wedge *\varphi.
\end{aligned}$$

Since we have already shown that $F_A \in L^1(Q_1^5(0))$, by taking the limit as $\varepsilon \to 0^+$ in the above inequality we have

$$\int_{Q_1^5(0)} F_A \wedge *\varphi \leqslant \|F_A\|_{L^p(Q_1^5(0) \setminus Q_\rho(c))} + \|F_A\|_{L^p(Q_\rho(c))}.$$

By taking the supremum over $\varphi \in C_c^\infty(Q_1^5(0))$ a smooth and compactly supported 2-form on $Q_1^5(0)$ such that $\|\varphi\|_{L^2(Q_1^5(0))} \leqslant 1$, we finally get

$$\|F_A\|_{L^p(Q_1^5(0))} \leqslant \|F_A\|_{L^p(Q_1^5(0) \setminus Q_\rho(c))} + \|F_A\|_{L^p(Q_\rho(c))}.$$

This concludes the proof of Lemma 4.4.2. □

Proof of Theorem 4.4.1. Let $\{\varepsilon_i\} \subset \left(0, \frac{1}{4}\right)$ be a set of admissible scales for A with associated admissible cubic ε_i-cover $\mathcal{Q}_{\varepsilon_i}$. We subdivide the family of cubes $\mathcal{Q}_{\varepsilon_i}$ into a uniformly bounded number N of disjoint subfamilies

$$\mathcal{Q}^1_{\varepsilon_i}, ..., \mathcal{Q}^N_{\varepsilon_i}. \tag{4.4.17}$$

We denote by $\mathscr{C}^s_{\varepsilon_i}$ the set of centers of the cubes in $\mathcal{Q}^s_{\varepsilon_i}$. The subfamilies $\mathcal{Q}^s_{\varepsilon_i}$ are chosen such that the union of the cubes in $\mathcal{Q}^j_{\varepsilon_i}$ with same centers and radii multiplied by 2 have no intersections with the union of the cubes in $\mathcal{Q}^k_{\varepsilon_i}$ with same centers and radii multiplied by 2 whenever $j \neq k$.

We claim that for every $s = 0, ..., N$ we can build a $\mathfrak{g}$-valued 1-form $A^s_{i,\Lambda} \in L^2(Q_1^5(0))$ and we can choose a family of radii $\rho^s_i \in \left(\frac{3}{2}\varepsilon_i, 2\varepsilon_i\right)$ for each center $c \in \mathscr{C}^s_{\varepsilon_i}$ in the family of cubes in $\mathcal{Q}^s_{\varepsilon_i}$ such that for some constant $C_s > 0$ depending only on G we have the following.

(a) We have

$$\forall\, c \in \mathscr{C}^s_{\varepsilon_i} \quad A^{s-1}_{i,\Lambda} \in \mathfrak{a}_G(\partial Q_{\rho^s_i}(c)), \tag{4.4.18}$$

and the estimates

$$\int_{\partial Q^s_{\rho_{i,c}}(c)} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \leqslant \frac{C_s}{2\varepsilon_i} \int_{Q^s_{2\varepsilon_i}(c)} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathcal{L}^5, \tag{4.4.19}$$

$$\int_{\partial Q^s_{\rho_{i,c}}(c)} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \leqslant \frac{C_s}{2\varepsilon_i} \int_{Q^s_{2\varepsilon_i}(c)} |A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5. \tag{4.4.20}$$

(b) $F_{A^s_{i,\Lambda}} \in M^0_{2,2}(Q_1^5(0))$ with

$$|F_{A^s_{i,\Lambda}}|_{M^0_{2,2}(Q_{1-3\varepsilon_i}(0))} < C_s \, |F_A|_{M^0_{2,2}(Q_1^5(0))}. \tag{4.4.21}$$

(c) There exist $N_s \in \mathbb{N}$ and a real-valued, non-negative function $f_s \in M^0_{2,2}(Q_1^5(0))$ satisfying

$$|f_s|_{M^0_{1,4}(Q_1^5(0))} \leqslant \, |F_A|^2_{M^0_{2,2}(Q_1^5(0))} \tag{4.4.22}$$

such that, for every $x \in Q_1^5(0)$ and for every $\rho > 0$ for which $Q_\rho(x)$ is uniformly transversal to $\mathcal{Q}_{\varepsilon_i}$, we have

$$\int_{\partial Q_\rho(x)\cap\Omega_i^s} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \leqslant \frac{C_s}{\varepsilon_i} \sum_{k=1}^{N_s} \int_{Q_{2\varepsilon_i}(x_k)} f_s \, d\mathcal{L}^5 \tag{4.4.23}$$

for a family of N_s points $\{x_k^s = x_k^s(i,x,\rho)\}_{k=1,...,N_s} \subset Q_1^5(0)$, where $\Omega_i^s \subset Q_1^5(0)$ is the open set given by

$$\Omega_i^s := \operatorname{int}\Bigg(\bigcup_{k=1}^{s} \bigcup_{Q\in\mathcal{Q}_{\varepsilon_i}^k} \overline{Q}\Bigg). \tag{4.4.24}$$

(d) It holds that

$$\lim_{i\to+\infty} \|A_{i,\Lambda}^s - A\|^2_{L^2(Q_1^5(0))} \leqslant C_s \lim_{i\to+\infty} \int_{\Omega_{i,\Lambda,A}} |A|^2 \, d\mathcal{L}^5, \tag{4.4.25}$$

where $\Omega_{i,\Lambda,A} \subset Q_1^5(0)$ is as in Lemma 4.3.2.

Base of the induction. For $s = 0$, since $\Omega_i^0 = \emptyset$ we have that $A_{i,\Lambda}^0 := A$ satisfies (a), (b), (c) and (d).

Induction step. Let $s \geqslant 1$. Using one more time Fubini theorem combined with the mean value theorem we adjust the radii $\rho_{i,c} \in \big(\frac{3}{2}\varepsilon_i, 2\varepsilon_i\big)$ in $\mathcal{Q}_{\varepsilon_i}^s$ such that $A_{i,\Lambda}^{s-1}$ satisfies the properties in (a). From now on, in the induction procedure, the family $\mathcal{Q}_{\varepsilon_i}^s$ is fixed.
By assumptions (1) and (2) in Definition 4.3.3, if $\varepsilon_G > 0$ is small enough then $\iota_{\partial Q}^* A_{i,\Lambda}^{s-1} \in \mathfrak{a}_G(\partial Q)$ satisfies the hypotheses of Corollary 4.2.1 for every Λ-good cube $Q \in \mathcal{Q}_{\varepsilon_i}^{s,g}$. Moreover, by our choice of the cubic cover (see Lemma 4.3.1-(i)), we have that $\iota_{\partial Q}^* A_{i,\Lambda}^{s-1} \in \mathfrak{a}_G(\partial Q)$ satisfies the hypotheses of Corollary 4.2.2 for every $Q \in \mathcal{Q}_{\varepsilon_i}^s$.
For every $Q \in \mathcal{Q}_{\varepsilon_i}^s$, if Q is Λ-good then we let A_Q^s be the $\mathfrak{g}$-valued 1-form given by applying Corollary 4.2.1–(ii) to $\iota_{\partial Q}^* A_{i,\Lambda}^{s-1}$. On the other hand, if Q is Λ-bad we let A_Q^s be the $\mathfrak{g}$-valued 1-form given by applying Corollary 4.2.2–(ii) to $\iota_{\partial Q}^* A_{i,\Lambda}^{s-1}$.
Let $A_{i,\Lambda}^s$ be defined on $Q_1^5(0)$ by

$$A_{i,\Lambda}^s := \begin{cases} A_Q^s & \text{on } Q \text{ for every } Q \in \mathcal{Q}_{\varepsilon_i}^s, \\ A_{i,\Lambda}^{s-1} & \text{otherwise.} \end{cases}$$

By construction and using Lemma 4.4.2, it is clear that $A_{i,\Lambda}^s, F_{A_{i,\Lambda}^s} \in L^2(Q_1^5(0))$. First, we show that $A_{i,\Lambda}^s$ satisfies (b). Fix any point $x \in Q_1^5(0)$ and let $\rho \in \big(0, \operatorname{dist}(x, \partial Q_1^5(0))\big)$. First we assume that $\rho \leqslant \varepsilon_i$. Then, $Q_\rho(x)$ intersects at most $\tilde{N}$ cubes in $\mathcal{Q}_{\varepsilon_i}^s$, say $Q_1, ..., Q_{\tilde{N}}$, with $\tilde{N} \in \mathbb{N}$ depending only on the choice of the cubic cover and independent on i, x and ρ. Hence, by Hölder inequality, by the estimates (4.2.66) for the good Q_ℓs and (4.2.96) for the bad ones for each A_Q^s, as well as (4.3.2) and the inductive assumption (b) on $A_{i,\Lambda}^{s-1}$, we get

$$\frac{1}{\rho}\int_{Q_\rho(x)\cap(\Omega_i^s\smallsetminus\Omega_i^{s-1})} |F_{A_{i,\Lambda}^s}|^2 \, d\mathcal{L}^5 \leqslant \|F_{A_{i,\Lambda}^s}\|^2_{L^{\frac{5}{2}}(Q_\rho(x)\cap(\Omega_i^s\smallsetminus\Omega_i^{s-1}))}$$

$$
\begin{aligned}
&\leqslant C \sum_{\ell=1}^{\tilde N} \left\| F_{A^s_{i,\Lambda}} \right\|^2_{L^{\frac{5}{2}}(Q_\ell)} = C \sum_{i=1}^{\tilde N} \| F_{A^s_{Q_\ell}} \|^2_{L^{\frac{5}{2}}(Q_\ell)} \\
&\leqslant C \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \Big(\| F_{A^{s-1}_{i,\Lambda}} \|^2_{L^2(\partial Q_\ell)} + \varepsilon_i^{-2} \| A^{s-1}_{i,\Lambda} \|^2_{L^2(\partial Q_\ell)} \Big) + \sum_{Q_\ell \in \mathcal{Q}^b_{\varepsilon_i,\Lambda}} \| F_{A^{s-1}_{i,\Lambda}} \|^2_{L^2(\partial Q_\ell)} \\
&\leqslant C \, |F_{A^{s-1}_{i,\Lambda}}|^2_{M^0_{2,2}(Q^5_1(0))} \leqslant C \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))},
\end{aligned} \tag{4.4.26}
$$

for some constant $C > 0$ depending only on G.

Assume now that $\varepsilon_i < \rho < 1$. Then there exists a universal constant $k > 1$ such that $Q_\rho(x) \cap (\Omega^s_i \smallsetminus \Omega^{s-1}_i)$ can be covered with a finite number of cubes $\{Q_\ell\}_{\ell=1,\dots,N_{i,x,\rho}}$ in $\mathcal{Q}^s_{\varepsilon_i}$ such that

$$
\bigcup_{\ell=1}^{N_{i,x,\rho}} Q_{2\varepsilon_i}(c_{Q_\ell}) \subset Q_{k\rho}(x).
$$

Notice that now the number of cubes $N_{i,x,\rho}$ may depend on i, x and ρ. Then, by the estimates (4.2.66) and (4.2.96) we obtain

$$
\begin{aligned}
\int_{Q_\rho(x)\cap(\Omega^s_i \smallsetminus \Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 &\leqslant \sum_{\ell=1}^{N_{i,x,\rho}} \int_{Q_\ell} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 = \sum_{\ell=1}^{N_{i,x,\rho}} \int_{Q_\ell} |F_{A_{Q_\ell}}|^2 \, d\mathcal{L}^5 \\
&\leqslant C \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i^{-4} \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \;\; \varepsilon_i \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \\
&\quad + C \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i \left(\varepsilon_i^{-2} \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \right)^3 + C\varepsilon_i \sum_{\ell=1}^{N_{i,x,\rho}} \int_{\partial Q_\ell} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathscr{H}^4,
\end{aligned} \tag{4.4.27}
$$

for some constant $C > 0$ depending only on G. By property (4) in Definition 4.3.3, by Remark 4.3.2 and by inductive hypothesis (a) on $A^{s-1}_{i,\Lambda}$, we get

$$
\begin{aligned}
\sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} & \varepsilon_i^{-4} \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \;\; \varepsilon_i \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \\
&\leqslant C\Lambda^{-1} \left| F_{A^{s-1}_{i,\Lambda}} \right|^2_{M_{2,2}(Q^5_1(0))} \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5, \\
&\leqslant C\Lambda^{-1} \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))} \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5,
\end{aligned}
$$

for some constant $C > 0$ depending only on G. By property (5) in Definition 4.3.3 and by our choice of $k > 0$, we get

$$
\sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5 \leqslant C\Lambda \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} (2\varepsilon_i)^5 \leqslant C\Lambda (k\rho)^5,
$$

for some constant $C > 0$ depending only on G. Hence, since by assumption $\rho \in (0,1)$, we have obtained the estimate

$$
\sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i^{-4} \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \;\; \varepsilon_i \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \leqslant C \, k^5 \rho \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))}, \tag{4.4.28}
$$

for some constant $C > 0$ depending only on G. By Remark 4.3.2 and by property (5) in Definition 4.3.3, provided $i \geqslant i_0$ is large enough so that $\varepsilon_i < \Lambda^{-3} |F_A|^2_{M^0_{2,2}(Q^5_1(0))}$, we obtain

$$
\begin{aligned}
\sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i \left(\varepsilon_i^{-2} \int_{\partial Q_\ell} |A^{s-1}_{i,\Lambda}|^2 \, d\mathscr{H}^4 \right)^3 &\leqslant \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i \left(\varepsilon_i^{-3} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5 \right)^3 \\
&\leqslant \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i (\varepsilon_i^{-3} \Lambda \varepsilon_i^5)^3 \qquad (4.4.29) \\
&\leqslant \varepsilon_i^2 \Lambda^3 \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i^5 \leqslant C \varepsilon_i^2 \Lambda^3 \leqslant C\rho \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))},
\end{aligned}
$$

for some constant $C > 0$ depending only on G. By (4.3.2) and the inductive hypothesis (a) on $A^{s-1}_{i,\Lambda}$, we get

$$
\begin{aligned}
\varepsilon_i \sum_{\ell=1}^{N_{i,x\rho}} \int_{\partial Q_\ell} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathscr{H}^4 &\leqslant C \sum_{\ell=1}^{N_{i,x\rho}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \\
&\leqslant C \int_{Q_{k\rho}(x)} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \leqslant C \, k \, \rho \, \left| F_{A^{s-1}_{i,\Lambda}} \right|^2_{M^0_{2,2}(Q^5_1(0))} \qquad (4.4.30) \\
&\leqslant C \, k \, \rho \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))},
\end{aligned}
$$

for some constant $C > 0$ depending only on the choice of the cubic cover. Hence, provided $i \geqslant i_0$ is sufficiently large, combining (4.4.27), (4.4.28), (4.4.29) and (4.4.30) we get this time for $\varepsilon_i < \rho < 1$

$$
\frac{1}{\rho} \int_{Q_\rho(x) \cap (\Omega^s_i \smallsetminus \Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \leqslant C \, |F_A|^2_{M^0_{2,2}(Q^5_1(0))}, \qquad (4.4.31)
$$

for some constant $C > 0$ depending only on G. Since

$$
\begin{aligned}
\frac{1}{\rho} \int_{Q_\rho(x)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 &= \frac{1}{\rho} \int_{Q_\rho(x) \cap (\Omega^s_i \smallsetminus \Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 + \frac{1}{\rho} \int_{Q_\rho(x) \smallsetminus (\Omega^s_i \smallsetminus \Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \\
&\leqslant \frac{1}{\rho} \int_{Q_\rho(x) \cap (\Omega^s_i \smallsetminus \Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2 \, d\mathcal{L}^5 + \frac{1}{\rho} \int_{Q_\rho(x)} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathcal{L}^5,
\end{aligned}
$$

property (b) for $A^s_{i,\Lambda}$ follows by (4.4.31) and inductive hypothesis b) for $A^{s-1}_{i,\Lambda}$.

Now we turn to show property (c) for $A^s_{i,\Lambda}$. Notice that

$$
\begin{aligned}
\|A^s_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))} &\leqslant \|A^s_{i,\Lambda} - A^{s-1}_{i,\Lambda}\|^2_{L^2(Q^5_1(0))} + \|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))} \\
&\leqslant \sum_{Q \in \mathcal{Q}^s_{\varepsilon_i}} \int_Q |A^s_Q - A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5 + \|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))} \\
&\leqslant \sum_{Q \in \mathcal{Q}^s_{\varepsilon_i} \cap \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A^s_Q - A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5 + \|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))}
\end{aligned}
$$

$$+ \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^b_{\varepsilon_i,\Lambda}} \int_Q |A^s_Q - A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5$$
$$\leqslant 2\, I_{i,\Lambda,s} + 2\, J_{i,\Lambda,s} + K_{i,\Lambda,s} + \|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))},$$

with

$$I_{i,\Lambda,s} := \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A^s_Q - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5,$$
$$J_{i,\Lambda,s} := \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5,$$
$$K_{i,\Lambda,s} := \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^b_{\varepsilon_i,\Lambda}} \int_Q |A^s_Q - A^{s-1}_{i,\Lambda}|^2 \, d\mathcal{L}^5.$$

By using (4.2.70) with $\bar{A} = (A^{s-1}_{i,\Lambda})_Q$ for every good cube Q, by Lemma 4.3.1 (in particular (4.3.4)) and by properties (1),(2) in Definition 4.3.3, we have

$$\begin{aligned}
I_{i,\Lambda,s} &= \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A^s_Q - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5 \\
&\leqslant C\,\varepsilon_i \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{\partial Q} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5 + \varepsilon_i^3 \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{\partial Q} |F_{A^{s-1}_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \\
&\quad + \varepsilon_i^{-1} \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \|A^{s-1}_{i,\Lambda}\|^4_{L^2(\partial Q)} \\
&\leqslant C\,\varepsilon_i \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{\partial Q} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5 + C\,\varepsilon_i^{6+1/2} \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} |F_{A^{s-1}_{i,\Lambda}}|^2_{M^0_{2,2}(Q_1(0))} \\
&\quad + C\,\varepsilon_i^6 \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} |F_{A^{s-1}_{i,\Lambda}}|^4_{M^0_{2,2}(Q_1(0))} \\
&\leqslant C\,\varepsilon_i \sum_{Q\in\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{\partial Q} |A^{s-1}_{i,\Lambda} - (A^{s-1}_{i,\Lambda})_Q|^2 \, d\mathcal{L}^5 + O(\varepsilon_i) \;\longrightarrow 0
\end{aligned} \tag{4.4.32}$$

as $\varepsilon_i \to 0^+$ for a fixed Λ. Now, we estimate $J_{i,\Lambda,s}$. We write

$$\begin{aligned}
J_{i,\Lambda,s} &\leqslant 3 \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A^{s-1}_{i,\Lambda} - A|^2 \; d\mathcal{L}^5 + 3 \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |A - (A)_Q|^2 \; d\mathcal{L}^5 \\
&\quad + 3 \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_Q |(A^{s-1}_{i,\Lambda})_Q - (A)_Q|^2 \; d\mathcal{L}^5 \\
&\leqslant 6\,\|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q_1(0))} + 3 \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}} \int_Q |A - (A)_Q|^2 \; d\mathcal{L}^5.
\end{aligned} \tag{4.4.33}$$

Using the induction hypothesis together with (4.3.5) we obtain that

$$\lim_{i\to+\infty} J_{i,\Lambda,s} \leqslant C \lim_{i\to+\infty} \int_{\Omega^{s-1}_{i,\Lambda,A}} |A|^2 \, d\mathcal{L}^5. \tag{4.4.34}$$

Finally, we bound $K_{i,\Lambda,s}$. Using respectively (4.2.76) together with the induction hypothesis on $(Q^s_{\varepsilon_i}, A^{s-1}_{i,\Lambda})$ (4.4.19) and (4.4.20), we obtain

$$\begin{aligned}
K_{i,\Lambda,s} &\leqslant \sum_{Q\in\mathcal{Q}^{s,b}_{\varepsilon_i,\Lambda}} \left(\int_Q |A^s_Q|^2\, d\mathcal{L}^5 + \int_Q |A^{s-1}_{i,\Lambda}|^2\, d\mathcal{L}^5\right)\\
&\leqslant C \sum_{Q\in\mathcal{Q}^{s,b}_{\varepsilon_i,\Lambda}} \left(\varepsilon_i \int_{\partial Q} |A^{s-1}_{i,\Lambda}|^2\, d\mathscr{H}^4 + \varepsilon_i^3 \int_{\partial Q} |F_{A^{s-1}_{i,\Lambda}}|\, d\mathscr{H}^4 + \int_Q |A^{s-1}_{i,\Lambda}|^2\, d\mathcal{L}^5\right)\\
&\leqslant C \sum_{Q\in\mathcal{Q}^{s,b}_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_Q)} |A^{s-1}_{i,\Lambda}|^2\, d\mathcal{L}^5 + C\,\varepsilon_i^2 \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}\cap\mathcal{Q}^b_{\varepsilon_i,\Lambda}} \int_Q |F_{A^{s-1}_{i,\Lambda}}|\, d\mathcal{L}^5\\
&\leqslant C \sum_{Q\in\mathcal{Q}^{s,b}_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon}(c_Q)} |A^{s-1}_{i,\Lambda}|^2\, d\mathcal{L}^5 + C\,\varepsilon_i^2 \int_{Q^5_1(0)} |F_{A^{s-1}_{i,\Lambda}}|^2\, d\mathcal{L}^5\\
&\leqslant C \sum_{Q\in\mathcal{Q}^{s,b}_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon}(c_Q)} |A|^2\, d\mathcal{L}^5 + \|A^{s-1}_{i,\Lambda} - A\|^2_{L^2(Q^5_1(0))} + o_{\varepsilon_i}(1). \qquad (4.4.35)
\end{aligned}$$

Hence, by Lemma 4.3.2 and by inductive hypothesis (d) on $A^{s-1}_{i,\Lambda}$, property (d) for $A^s_{i,\Lambda}$ follows.

We are just left to show property (c) for $A^s_{i,\Lambda}$. Fix any $x \in Q^5_1(0)$ and $\rho > 0$ such that $Q_\rho(x)$ is uniformly transversal to $\mathcal{Q}^l_{\varepsilon_i}$ for $l \leqslant s$. Notice that,

$$\begin{aligned}
\|F_{A^s_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(x)\cap\Omega^s_i)} &= \|F_{A^s_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(x)\cap\Omega^{s-1}_i)} + \|F_{A^s_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(x)\cap(\Omega^s_i\smallsetminus\Omega^{s-1}_i))}\\
&= \|F_{A^{s-1}_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(x)\cap\Omega^{s-1}_i)} + \|F_{A^s_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(x)\cap(\Omega^s_i\smallsetminus\Omega^{s-1}_i))}. \qquad (4.4.36)
\end{aligned}$$

By inductive hypothesis (c) on $A^{s-1}_{i,\Lambda}$, there exists $N_{s-1} \in \mathbb{N}$ independent on x, ρ and i such that

$$\int_{\partial Q_\rho(x)\cap\Omega^{s-1}_i} |F_{A^{s-1}_{i,\Lambda}}|^2\, d\mathscr{H}^4 \leqslant \frac{C_{s-1}}{\varepsilon_i} \sum_{k=1}^{N_{s-1}} \int_{Q_{2\varepsilon_i}(x_k)} f_{s-1}\, d\mathcal{L}^5, \qquad (4.4.37)$$

for $C_{s-1} > 0$ depending only on G, for an N_{s-1}-tuple $\{x^{s-1}_k = x^{s-1}_k(x,\rho,i)\}_{k=1,\dots,N_{s-1}} \subset Q^5_1(0)$ and for a real-valued, non-negative $f_{s-1} \in M^0_{1,4}(Q^5_1(0))$ independent on i such that

$$|f_{s-1}|_{M^0_{1,4}(Q^5_1(0))} \leqslant |F_A|^2_{M^0_{2,2}(Q^5_1(0))}.$$

Hence, in order to prove property (c) for $A^s_{i,\Lambda}$, it remains to control $\|F_{A^s_{i,\Lambda}}\|^2_{L^2(\partial Q_\rho(0)\cap(\Omega^s_i\smallsetminus\Omega^{s-1}_i))}$. Notice that, since $\rho \in \left(\frac{3}{2}\varepsilon_i, \frac{7}{4}\varepsilon_i\right)$, there exists $\hat{N}$ depending only on the choice of the cubic cover such that ∂Q_ρ intersects just $\hat{N}$ cubes in $\mathcal{Q}^s_{\varepsilon_i}$, say $\{Q_1, ..., Q_{\hat{N}}\}$. Then, by construction, we have

$$F_{A^s_{i,\Lambda}} \mathbb{1}_{\partial Q_\rho(x)\cap(\Omega^s_i\smallsetminus\Omega^{s-1}_i)} = \sum_{Q\in\mathcal{Q}^s_{\varepsilon_i}} F_{A^s_Q} \mathbb{1}_{\partial Q_\rho(x)\cap Q} = \sum_{\ell=1}^{\hat{N}} F_{A_{Q_\ell}} \mathbb{1}_{\partial Q_\rho(x)\cap Q_\ell}.$$

Thus, by (4.2.68) and (4.2.98) we get

$$\int_{\partial Q_\rho(x)\cap(\Omega^s_i\smallsetminus\Omega^{s-1}_i)} |F_{A^s_{i,\Lambda}}|^2\, d\mathscr{H}^4 \leqslant \sum_{\ell=1}^{\hat{N}} \int_{\partial Q_\rho(x)\cap Q_\ell} |F_{A^s_{Q_\ell}}|^2\, d\mathscr{H}^4$$

$$\leqslant C \sum_{\ell=1}^{\hat{N}} \int_{\partial Q_\ell} |F_{A_{i,\Lambda}^{s-1}}|^2 \, d\mathscr{H}^4 + C\, \varepsilon_i^{-6} \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \left(\int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1}|^2 \, d\mathscr{H}^4 \right)^3$$

$$+ C \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i^{-4} \int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1} - (A_{i,\Lambda}^{s-1})_{Q_\ell}|^2 \, d\mathscr{H}^4 \int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1}|^2 \, d\mathscr{H}^4. \quad (4.4.38)$$

Fix any $\ell \in \{1, ..., \hat{N}\}$. Notice that we have have chosen $Q^s_{\varepsilon_i}$ so that (4.4.19) holds. Then, we have

$$\int_{\partial Q_\ell} |F_{A_{i,\Lambda}^{s-1}}|^2 \, d\mathscr{H}^4 \leqslant \frac{C}{\varepsilon_i} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |F_A|^2 \, d\mathcal{L}^5, \quad (4.4.39)$$

for some constant $C > 0$ independent of ε_i. Moreover, by properties (2), (4)and (5) in Definition 4.3.3 (characterizing good cubes), by Remark 4.3.2 and by (4.4.20), we obtain the estimates

$$\begin{aligned} \varepsilon_i^{-6} \left(\int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1}|^2 \, d\mathscr{H}^4 \right)^3 &\leqslant \varepsilon_i^{-6} \left(\varepsilon_i^{3+1/2} \int_{Q_1^5(0)} |F_{A_{i,\Lambda}^{s-1}}|^2 \, d\mathcal{L}^5 \right)^3 && (4.4.40) \\ &\leqslant \varepsilon_i^{3+3/2} \, |F_{A_{i,\Lambda}^{s-1}}|^6_{M^0_{2,2}(Q_1^5(0))} \\ &\leqslant C \, \frac{1}{\varepsilon_i} \int_{Q_\ell} \varepsilon_i^{1/2} \, |F_{A_{i,\Lambda}^{s-1}}|^6_{M^0_{2,2}(Q_1^5(0))} \, d\mathcal{L}^5 && (4.4.41) \end{aligned}$$

and

$$\begin{aligned} &\varepsilon_i^{-4} \int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1} - (A_{i,\Lambda}^{s-1})_{Q_\ell}|^2 \, d\mathscr{H}^4 \int_{\partial Q_\ell} |A_{i,\Lambda}^{s-1}|^2 \, d\mathscr{H}^4 \\ &\qquad \leqslant \Lambda^{-1} \, |F_{A_{i,\Lambda}^{s-1}}|^2_{M^0_{2,2}(Q_1^5(0))} \, \varepsilon_i^4 \, \Lambda \, |F_{A_{i,\Lambda}^{s-1}}|^2_{M^0_{2,2}(Q_1^5(0))} \\ &\qquad \leqslant \frac{C_{s-1}}{\varepsilon_i} \int_{Q_{2\varepsilon}(c_{Q_\ell})} |F_A|^4_{M^0_{2,2}(Q_1^5(0))} \, d\mathcal{L}^5. \quad (4.4.42) \end{aligned}$$

By combining (4.4.38), (4.4.39), (4.4.40) and (4.4.42), we get

$$\int_{\partial Q_\rho(x) \cap (\Omega_i^s \smallsetminus \Omega_i^{s-1})} |F_{A_{i,\Lambda}^s}|^2 \, d\mathscr{H}^4 \leqslant \frac{C}{\varepsilon_i} \sum_{\ell=1}^{\hat{N}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} g_s \, d\mathcal{L}^5 \quad (4.4.43)$$

with $g_s := |F_A|^2 + |F_A|^4_{M^0_{2,2}(Q_1^5(0))} + \varepsilon_i^{1/2} \, |F_A|^6_{M^0_{2,2}(Q_1^5(0))}$. The required estimate (4.4.23) for $F_{A_{i,\Lambda}^s}$ then follows by (4.4.36), (4.4.37) and (4.4.43), letting

$$f_s := \max\{g_s, f_{s-1}\}.$$

Finally, let $A_{i,\Lambda} := A_{i,\Lambda}^N \in L^2(Q_1^5(0))$. By construction, $F_{A_{i,\Lambda}} \in L^2(Q_1^5(0))$ and $A_{i,\Lambda}$ satisfies the properties (i) and (iii). Moreover, notice that $\Omega_i^N \supset Q_{1-\frac{\varepsilon_i}{2}}(0)$. In particular, $\partial Q_\rho(x) \subset \Omega_i^N$. Hence, by property (c) for $A_{i,\Lambda}^N$, there exists $\tilde{N} \in \mathbb{N}$ independent on x, ρ and i such that

$$\int_{\partial Q_\rho(x)} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \leqslant \frac{C}{\varepsilon_i} \sum_{k=1}^{\tilde{N}} \int_{Q_{2\varepsilon_i}(x_k)} f \, d\mathcal{L}^5, \quad (4.4.44)$$

for $C>0$ depending only on G, for an N-tuple of points $\{x_k = x_k(x,\rho,i)\}_{k=1,\dots,N} \subset Q_1^5(0)$ and for some real-valued, non-negative $f \in M^0_{1,4}(Q_1^5(0))$ independent on i such that

$$|f|_{M^0_{1,4}(Q_1^5(0))} \leqslant |F_A|^2_{M^0_{2,2}(Q_1^5(0))} \tag{4.4.45}$$

We are just left to check that, given any $x \in Q_1^5(0)$, for $\mathcal{L}^1$-a.e. choice of $\rho > 0$ such that $Q_\rho(x)$ is uniformly transversal to $\mathscr{Q}_{\varepsilon_i}$ and $\partial Q_\rho(x) \subset Q_{1-\frac{\varepsilon_i}{2}}(0)$ we have $\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda} \in \mathfrak{a}_G(\partial Q_\rho(x))$. First, note that for $\mathcal{L}^1$-a.e. $\rho > 0$ we have $\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda} \in L^2(\partial Q_\rho(x))$. Moreover, by (4.4.44), we get

$$\begin{aligned}\int_{\partial Q_\rho(x)} |F_{\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 &= \int_{\partial Q_\rho(x)} |\iota^*_{\partial Q_\rho(x)} F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \\ &\leqslant \int_{\partial Q_\rho(x)} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \leqslant C\varepsilon_G, \end{aligned} \tag{4.4.46}$$

for some constant $C > 0$ depending only on G. By construction, we can find a covering $(R_l)_{\ell=1,\dots,N_{\rho,x}}$ of $\partial Q_\rho(x)$ where R_ℓ is a rectangle included in some $Q_l \in \mathscr{Q}^{s(l)}_{\varepsilon_i}$ for some $s(l) \in \{0 \cdots N\}$ depending on l such that

$$A_{i,\Lambda} = A^{s(\ell)}_{Q_\ell} \qquad \text{on } R_\ell$$

By the properties of the extensions on good and bad cubes implying that we have respectively (4.2.69) and (4.2.99), if (4.4.46) holds there exist $g_\ell \in W^{1,2}(R_\ell, G)$ such that

$$(\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda})^{g_\ell} \in L^4(\partial Q_\rho(x) \cap R_\ell).$$

Then, we can apply Lemma 4.4.1 to conclude there exists $g \in W^{1,2}(\partial Q_\rho(x), G)$ such that

$$(\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda})^{g} \in L^4(\partial Q_\rho(x)). \tag{4.4.47}$$

Hence, $\iota^*_{\partial Q_\rho(x)} A_{i,\Lambda} \in \mathfrak{a}_G(\partial Q_\rho(x))$. This concludes the proof of Theorem 4.4.1. ☐

4.5. The second smoothification: strong L^2-approximation by smooth connections

The goal of this section is to prove that any weak connection with curvature having a small Morrey norm is strongly approximable in L^2 by smooth connections with small Morrey norm of the curvature as well. More precisely the main result obtained in this section is to prove the following theorem.

Theorem 4.5.1 (Smooth approximation under controlled Morrey norm)**.** *Let G be a compact matrix Lie group. There exists $\varepsilon_G \in (0,1)$ such that for every $A \in \mathfrak{a}_G(Q_1^5(0))$ satisfying*

$$|F_A|^2_{M^0_{2,2}(Q_1^5(0))} < \varepsilon_G$$

there exists a sequence of $\mathfrak{g}$-valued 1-forms $\{A_i\}_{i\in\mathbb{N}} \subset C^\infty_c(Q_1^5(0))$ such that:

(i) for every $i \in \mathbb{N}$ we have

$$|F_{A_i}|_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant C_G \, |F_A|_{M^0_{2,2}(Q_1^5(0))}.$$

for some constant $C_G > 0$ depending on G;

(ii) $\|A_i - A\|_{L^2(Q_{\frac{1}{2}}(0))} \to 0$ *as* $i \to +\infty$.

In order to prove Theorem 4.5.1 we shall need some preliminary results. Recall that we have defined

$$\mathscr{C}_\varepsilon := (\varepsilon\mathbb{Z})^5 \cap \overline{Q_{1-2\varepsilon}(0)}$$

for every $\varepsilon \in \left(0, \frac{1}{4}\right)$ (see the beginning of Section 3). Under the same assumption and notation of Theorem 4.4.1, for every $i \in \mathbb{N}$ and for every $t \in Q_{\frac{\varepsilon_i}{8}}(0)$ the cubes in the grid $\{Q_{\varepsilon_i}(c+t)\}_{c\in\mathscr{C}_{\varepsilon_i}}$ are uniformly transverse to the cubes in $\mathscr{Q}_{\varepsilon_i}$ and such that $\partial Q_{\varepsilon_i}(c+t) \subset Q_{1-\frac{\varepsilon_i}{2}}(0)$ for every $c \in \mathscr{C}_{\varepsilon_i}$. Since uniform transversality is stable by small perturbations, there exists $\alpha > 0$ small enough and independent of ε_i so that for every $r \in \big((1-\alpha)\varepsilon_i, (1+\alpha)\varepsilon_i\big)$ and every $t \in Q_{\frac{r}{8}}(0)$ the cubes in the grid $\{Q_r(c+t)\}_{c\in\mathscr{C}_r}$ are still uniformly transverse to the cubes in $\mathscr{Q}_{\varepsilon_i}$ and such that $\partial Q_r(c+t) \subset Q_{1-\frac{\varepsilon_i}{2}}(0)$ for every $c \in \mathscr{C}_r$.

Lemma 4.5.1 (Choice of an admissible cubic grid)**.** *Under the same assumption and notation of Theorem 4.4.1, for every* $i \in \mathbb{N}$ *there exist* $r_i \in \big((1-\alpha)\varepsilon_i, (1+\alpha)\varepsilon_i\big)$ *and a translation* $t_i \in Q_{\frac{r_i}{8}}(0)$ *for which the following facts hold.*

1. $\iota^*_{\partial Q_{r_i}(c+t_i)} A_{i,\Lambda} \in \mathfrak{a}_G(\partial Q_{r_i}(c+t_i))$ *for every* $c \in \mathscr{C}_{r_i}$.
2. *There exist* $N \in \mathbb{N}$ *and a real-valued, non-negative* $f \in M^0_{1,4}(Q^5_1(0))$ *satisfying*

$$|f|_{M^0_{1,4}(Q^5_1(0))} \leqslant |F_A|^2_{M^0_{2,2}(Q^5_1(0))}$$

such that for every $c \in \mathscr{C}_{r_i}$ *we have*

$$\int_{\partial Q_{r_i}(c+t_i)} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 \leqslant \frac{C_G}{\varepsilon_i} \sum_{k=1}^N \int_{Q_{2\varepsilon_i}(x_k)} f \, d\mathcal{L}^5,$$

for an N*-tuple of points* $\{x_k = x_k(i,x,r_i)\}_{k=1,\dots,N} \subset Q^5_1(0)$.

3. *It holds that*

$$\lim_{i\to+\infty} r_i \sum_{c\in\mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t_i)} \left|A_{i,\Lambda} - (A_{i,\Lambda})_{Q_{r_i}(c+t_i)}\right|^2 \, d\mathscr{H}^4 = 0, \tag{4.5.1}$$

$$\lim_{i\to+\infty} r_i \sum_{c\in\mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t_i)} \left|F_{A_{i,\Lambda}} - (F_{A_{i,\Lambda}})_{Q_{r_i}(c+t_i)}\right|^2 \, d\mathscr{H}^4 = 0. \tag{4.5.2}$$

Proof of Lemma 4.5.1. By our assumption on α and by Theorem 4.4.1-(ii), arguing exactly as in the proof of [23, Lemma 2.1] (with $p = 2$ and $q = 1$) or [22, Lemma 3.1] we can show that there exists a full measure subset $E \subset \big((1-\alpha)\varepsilon_i, (1+\alpha)\varepsilon_i\big)$ such that for every $r \in E$ we have that for a.e. translation $t \in Q_{\frac{r}{8}}(0)$ 1. and 2. in the statement hold. We fix $r_i \in E$ and again, exactly as in the proof of [23, Lemma 2.1] (with $p = 2$ and $q = 1$) or [22, Lemma 3.1] we have that

$$I_{r_i} := \int_{Q_{\frac{r_i}{8}}(0)} \sum_{c\in\mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^5 = o\big(r_i^4\big) \tag{4.5.3}$$

$$J_{r_i} := \int_{Q_{\frac{r_i}{8}}(0)} \sum_{c\in\mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |F_{A_{i,\Lambda}} - (F_{A_{i,\Lambda}})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^5 = o\big(r_i^4\big) \tag{4.5.4}$$

as $r_i \to 0^+$. Consider the sets

$$T_{r_i,K,1} := \left\{ t \in Q_{\frac{r_i}{8}}(0) \text{ s.t. } \varphi_1(t) \geqslant K \fint_{Q_{\frac{r_i}{8}}(0)} \varphi_1(t)\, d\mathcal{L}^5 \right\}$$

$$T_{r_i,K,2} := \left\{ t \in Q_{\frac{r_i}{8}}(0) \text{ s.t. } \varphi_2(t) \geqslant K \fint_{Q_{\frac{r_i}{8}}(0)} \varphi_2(t)\, d\mathcal{L}^5 \right\},$$

with

$$\varphi_1(t) := \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 \qquad \forall\, t \in Q_{\frac{r_i}{8}}(0),$$

$$\varphi_2(t) := \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |F_{A_{i,\Lambda}} - (F_{A_{i,\Lambda}})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 \qquad \forall\, t \in Q_{\frac{r_i}{8}}(0).$$

By integration on $T_{r_i,K,1}$ and $T_{r_i,K,2}$ we get

$$\mathcal{L}^5(T_{r_i,K,1}) \leqslant \frac{r_i^5}{K}$$

$$\mathcal{L}^5(T_{r_i,K,2}) \leqslant \frac{r_i^5}{K}.$$

Moreover, for every $t \in Q_{\frac{r_i}{8}}(0) \smallsetminus (T_{r_i,K,1} \cup T_{r_i,K,2})$ it holds that

$$r_i \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 < r_i K \fint_{Q_{r_i}(0)} \varphi_1 \, d\mathcal{L}^5 = K \frac{I_{r_i}}{r_i^4}, \tag{4.5.5}$$

$$r_i \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t)} |F_{A_{i,\Lambda}} - (F_{A_{i,\Lambda}})_{Q_{r_i}(c+t)}|^2 \, d\mathscr{H}^4 < r_i K \fint_{Q_{r_i}(0)} \varphi_2 \, d\mathcal{L}^5 = K \frac{J_{r_i}}{r_i^4}. \tag{4.5.6}$$

Now, we notice that

$$\mathcal{L}^5(T_{r_i,4,1} \cup T_{r_i,4,2}) \leqslant \frac{r_i^5}{2},$$

which implies

$$\mathcal{L}^5(T_{r_i,A}) \geqslant \frac{r_i^5}{2} > 0.$$

with $T_{r_i} := T_{r_i,4,1}^c \cap T_{r_i,4,2}^c$. Hence, we fix $t_i \in T_{r_i}$. By (4.5.3)-(4.5.4) and (4.5.5)-(4.5.6), we have

$$\lim_{r_i \to 0^+} r_i \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t_{r_i})} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_{r_i}(c+t_{r_i})}|^2 \, d\mathscr{H}^4 = 0,$$

$$\lim_{r_i \to 0^+} r_i \sum_{c \in \mathscr{C}_{r_i}} \int_{\partial Q_{r_i}(c+t_{r_i})} |F_{A_{i,\Lambda}} - (F_{A_{i,\Lambda}})_{Q_{r_i}(c+t_{r_i})}|^2 \, d\mathscr{H}^4 = 0.$$

The statement follows. This concludes the proof of Lemma 4.5.1. □

Definition 4.5.1. Under the same notation that we have used in the previous Lemma 4.5.1, we say that the collection of cubes $\mathscr{Q}_{r_i} := \{Q_{r_i}(c + t_i)\}_{c \in \mathscr{C}_{r_i}}$ is a *admissible cubic r_i-grid* relative to $A_{i,\Lambda}$.

Definition 4.5.2 (Good and bad cubes)**.** Under the same notation that we have used in the previous Lemma 4.5.1, given an admissible cubic r_i-grid relative to $A_{i,\Lambda}$ we say that $Q \in \mathscr{Q}_{r_i}$ is a Λ-*good* cube if all the following conditions hold:

(1) $\displaystyle \frac{1}{r_i^3}\int_{\partial Q}|F_{A_{i,\Lambda}}|^2\,d\mathscr{H}^4 \leqslant r_i^{\frac{1}{2}}\int_{Q_1^5(0)}|F_{A_{i,\Lambda}}|^2\,d\mathcal{L}^5,$

(2) $\displaystyle \frac{1}{r_i^3}\int_{\partial Q}|A_{i,\Lambda}|^2\,d\mathscr{H}^4 \leqslant r_i^{\frac{1}{2}}\int_{Q_1^5(0)}|F_{A_{i,\Lambda}}|^2\,d\mathcal{L}^5,$

(3) $\displaystyle \int_{\partial Q}|A_{i,\Lambda}-(A_{i,\Lambda})_Q|^2\,d\mathscr{H}^4 \leqslant \frac{1}{r_i}\int_Q|A_{i,\Lambda}|^2\,d\mathcal{L}^5,$

(4) $\displaystyle \frac{1}{r_i^4}\int_{\partial Q}|A_{i,\Lambda}-(A_{i,\Lambda})_Q|^2\,d\mathscr{H}^4 \leqslant \Lambda^{-1}\int_{Q_1^5(0)}|F_{A_{i,\Lambda}}|^2\,d\mathcal{L}^5,$

(5) $\displaystyle ⨍_Q|A_{i,\Lambda}|^2\,d\mathcal{L}^5 \leqslant \Lambda.$

Otherwise, we say that that Q is a Λ-*bad* cube. We denote by $\mathscr{Q}^g_{r_i,\Lambda}$ the set of all the Λ-good cubes and $\mathscr{Q}^b_{r_i,\Lambda} := \mathscr{Q}_{r_i} \smallsetminus \mathscr{Q}^g_{r_i,\Lambda}$.

Lemma 4.5.2. *Under the same notation that we have used in the previous Lemma 4.5.1, let $\mathscr{Q}_{r_i}$ be a good r_i-grid relative to $A_{i,\Lambda}$. We have*

$$\lim_{i\to+\infty}\sum_{Q\in\mathscr{Q}^b_{r_i,\Lambda}}\int_Q|A_{i,\Lambda}|^2\,d\mathcal{L}^5 \leqslant C\lim_{i\to+\infty}\int_{\Omega_{i,\Lambda,A_{i,\Lambda}}}|A_{i,\Lambda}|^2\,d\mathcal{L}^5,$$

where $\Omega_{i,\Lambda,A_{i,\Lambda}} \subset Q_1^5(0)$ is given by

$$\Omega_{i,\Lambda,A_{i,\Lambda}} := \bigcup\left\{Q\in\mathscr{Q}_{r_i}\ \ s.t.\ \ ⨍_Q|A_{i,\Lambda}|^2\,d\mathcal{L}^5 > \Lambda\right\}.$$

Proof of Lemma 4.5.2. The proof is identical to the one of Lemma 4.3.2. □

Remark 4.5.1. Notice that we have

$$\begin{aligned} r_i\int_{Q\in\mathscr{Q}^b_{r_i,\Lambda}}\int_{\partial Q}|A_{i,\Lambda}|^2\,d\mathscr{H}^4 &\leqslant r_i\sum_{Q\in\mathscr{Q}_{r_i,\Lambda}}\int_{\partial Q}|A_{i,\Lambda}-(A_{i,\Lambda})_Q|^2\,d\mathscr{H}^4 \\ &\quad + r_i\sum_{Q\in\mathscr{Q}^b_{r_i,\Lambda}}\int_{\partial Q}|(A_{i,\Lambda})_Q|^2\,d\mathscr{H}^4 \\ &\leqslant r_i\sum_{Q\in\mathscr{Q}_{r_i,\Lambda}}\int_{\partial Q}|A_{i,\Lambda}-(A_{i,\Lambda})_Q|^2\,d\mathscr{H}^4 + \sum_{Q\in\mathscr{Q}^b_{r_i,\Lambda}}\int_Q|A_{i,\Lambda}|^2\,d\mathcal{L}^5. \end{aligned}$$

Thus, by (4.5.1) and by Lemma 4.5.2 we have

$$\lim_{i\to+\infty} r_i\int_{Q\in\mathscr{Q}^b_{r_i,\Lambda}}\int_{\partial Q}|A_{i,\Lambda}|^2\,d\mathscr{H}^4 \leqslant C\lim_{i\to+\infty}\int_{\Omega_{i,\Lambda,A_{i,\Lambda}}}|A_{i,\Lambda}|^2\,d\mathcal{L}^5,$$

where $\Omega_{i,\Lambda,A_{i,\Lambda}} \subset Q_1^5(0)$ is given as in Lemma 4.5.2.

We now have all the tools that are needed to prove the following strong L^2-approximation result by smooth connections under controlled Morrey norm of their curvatures.

Proof of Theorem 4.5.1. We assume that $\varepsilon_G > 0$ is small enough so that we can apply Theorem 4.4.1 to A. Hence, we find a sequence of admissible scales $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0^+$ as $i \to +\infty$ and a sequence $\{A_{i,\Lambda}\}_{i\in\mathbb{N}\subset L^2(Q_1^5(0))}$ satisfying (i), (ii) and (iii) in Theorem 4.4.1.

Let $\mathcal{Q}_{r_i}$ be an admissible cubic r_i-grid relative to $A_{i,\Lambda}$ in the sense of Definition 4.5.1. Recall that by definition $r_i \in \big((1-\alpha)\varepsilon_i, (1+\alpha)\varepsilon_i\big)$, for some fixed $\alpha \in \big(0, \frac{1}{2}\big)$ sufficiently small. We fix $i \in \mathbb{N}$ big enough so that

$$\overline{Q_{\frac{1}{2}}(0)} \subset \bigcup_{Q\in\mathcal{Q}_{r_i}} Q.$$

Step 1: construction of the approximating 1-forms.

Let N_i be the number of cubes in $\mathcal{Q}_{r_i}$. We first enumerate the family

$$\mathcal{Q}_{r_i} = \big\{Q_1, ..., Q_{N_i}\big\}$$

in such a way that for every for every $n \in \{1, ..., N_i\}$ we have

$$\Omega_i^n := \operatorname{int}\bigg(\bigcup_{k=1}^{n} \overline{Q_k}\bigg).$$

satisfies $\emptyset \neq H_i^n := \partial\Omega_i^n \cap \partial Q_{n+1} \neq \partial Q_{n+1}$ and Ω_i^n is bi-Lipschitz equivalent to a 5-dimensional ball, for every $n = 1, ..., N_i$.[14]

If $\varepsilon_G > 0$ is small enough, by 1. and 2. in Lemma 4.5.1 we have that $\iota_{\partial Q}^* A_{i,\Lambda}$ satisfies the assumptions of Corollary 4.2.2 on ∂Q, for every $Q \in \mathcal{Q}_{r_i}$. Moreover, if $Q \in \mathcal{Q}_{r_i}$ is Λ-good then $\iota_{\partial Q}^* A_{i,\Lambda}$ satisfies the assumptions of Corollary 4.2.1 on ∂Q. For every $k \in \{1, ..., N_i\}$ we denote by $A_{Q_k} \in L^5(Q_k)$ the extension given by applying Corollary 4.2.1–(i) (if Q_k is Λ-good) or Corollary 4.2.2–(i) (if Q_k is Λ-bad) to $\iota_{\partial Q_k}^* A_{i,\Lambda}$. Hence, for every $k \in \{1, ..., N_i\}$ such that Q_k is Λ-good there exists $g_k \in W^{1,2}(\partial Q_k, G)$ such that for every 4-dimensional face F of ∂Q_k we have

$$\begin{cases} \|(\iota_{\partial Q_k}^* A_{i,\Lambda})^{g_k}\|_{W^{1,2}(F)} \leqslant C\big(\|F_{A_{i,\Lambda}}\|_{L^2(\partial Q_k)} + r_i^{-1}\|A_{i,\Lambda}\|_{L^2(\partial Q_k)}\big) \\ \iota_F^* A_{Q_k} = (\iota_F^* A_{i,\Lambda})^{g_{Q_k}} \end{cases} \tag{4.5.7}$$

and the estimates

$$\begin{aligned} \|F_{A_{Q_k}}\|_{L^{\frac{5}{2}}(Q_k)}^2 &\leqslant C(\|F_{A_{i,\Lambda}}\|_{L^2(\partial Q_k)}^2 + r_i^{-2}\|A_{i,\Lambda}\|_{L^2(\partial Q_k)}^2\|A_{i,\Lambda} - (A_{i,\Lambda})_{Q_k}\|_{L^2(\partial Q_k)}^2 \\ &\quad + r_i^{-4}\|A_{i,\Lambda}\|_{L^2(\partial Q_k)}^6) \end{aligned} \tag{4.5.8}$$

$$\|A_{Q_k} - \bar{A}\|_{L^2(Q_k)}^2 \leqslant C\big(r_i\|F_{A_{i,\Lambda}}\|_{L^2(\partial Q_k)}^2 + r_i^{-1}\|A_{i,\Lambda} - (A_{i,\Lambda})_{Q_k}\|_{L^2(\partial Q_k)}^2 + r_i^{-3}\|A_{i,\Lambda}\|_{L^2(\partial Q_k)}^2\big) \tag{4.5.9}$$

[14]Notice that, by construction, H_n^i is always non-empty and bi-Lipschitz equivalent to a 4-dimensional closed ball.

for every constant $\mathfrak{g}$-valued 1-form $\bar{A}$ on $\mathbb{R}^5$, where $C > 0$ is a constant depending only on G. On the other hand, for every $k \in \{1, ..., N_i\}$ such that Q_k is Λ-bad there exists $g_k \in W^{1,2}(\partial Q_k, G)$ such that for every 4-dimensional face F of ∂Q_k we have

$$\begin{cases} \|(\iota^*_{\partial Q_k} A_{i,\Lambda})^{g_k}\|_{W^{1,2}(F)} \leqslant C \|F_{A_{i,\Lambda}}\|_{L^2(\partial Q_k)} \\ \iota^*_F A_{Q_k} = (\iota^*_F A_{i,\Lambda})^{g_{Q_k}}, \end{cases} \tag{4.5.10}$$

and the estimate

$$\|F_{A_{Q_k}}\|^2_{L^{\frac{5}{2}}(Q_k)} \leqslant C \|F_{A_{i,\Lambda}}\|^2_{L^2(\partial Q_k)}, \tag{4.5.11}$$

where $C > 0$ is again a constant depending only on G. Lastly, for every $k = 1, ..., N_i$, let $\tilde{g}_k \in W^{1,2}(Q_k, G)$ be the extension of g_k given by Corollary 4.2.1-(ii), if Q_k is Λ-good, or Corollary 4.2.2-(ii), if Q_k is Λ-bad.

In order to construct the approximating 1-forms, we proceed by induction on $n = 1, ..., N_i$. In particular, for every $n = 1, ..., N_i$ we build a $\mathfrak{g}$-valued 1-form $\tilde{A}^n_{i,\Lambda} \in L^{5,\infty}(\Omega^n_i)$ with $F_{\tilde{A}^n_{i,\Lambda}} \in L^{\frac{5}{2},\infty}(\Omega^n_i)$ such that there exists a gauge transformation $\sigma_n \in W^{1,2}(\Omega^n_i, G) \cap W^{1,2}(\partial\Omega^n_i, G)$ satisfying

$$\iota^*_{\partial\Omega^n_i} \tilde{A}^n_{i,\Lambda} = (\iota^*_{\partial\Omega^n_i} A_{i,\Lambda})^{\sigma_n} \in L^4(\partial\Omega^n_i)$$

Base of the induction. At the initial step $n = 1$, we have $\Omega^1_i = Q_1$ and we set $\tilde{A}^0_{i,\Lambda} := A_{Q_1} \in L^5(\Omega^1_i)$. By (4.5.8) (if Q_1 is Λ-good) or (4.5.11) (if Q_1 is Λ-bad) we get $F_{\tilde{A}^1_{i,\Lambda}} \in L^{\frac{5}{2}}(\Omega^1_i)$. Let

$$\sigma_1 := \tilde{g}_1 \in W^{1,2}(\Omega^1_i, G) \cap W^{1,2}(\partial\Omega^1_i, G),$$

we have

$$\iota^*_{\partial\Omega^1_i} \tilde{A}^1_{i,\Lambda} = \iota^*_{\partial\Omega^1_i} A_{Q_1} = (\iota^*_{\partial\Omega^1_i} A_{i,\Lambda})^{g_1} = (\iota^*_{\partial\Omega^1_i} A_{i,\Lambda})^{\sigma_1}.$$

Hence, we have proved our claim for $n = 1$.

Induction step. Let $\eta_n := g_n^{-1}\sigma_{n-1} \in W^{1,4}(H^{n-1}_i, G)$ and notice that, by the inductive assumption, we have

$$\big((\iota^*_{H^{n-1}_i} A_{i,\Lambda})^{g_n}\big)^{\eta_n} = (\iota^*_{H^{n-1}_i} A_{i,\Lambda})^{\sigma_{n-1}} = \iota^*_{H^{n-1}_i} \tilde{A}^{n-1}_{i,\Lambda} \in L^4(H^{n-1}_i). \tag{4.5.12}$$

Hence, we have

$$d\eta_n = \eta_n \big(\iota^*_{H^{n-1}_i} A_{i,\Lambda}\big)^{g_n} - \big(\iota^*_{H^{n-1}_i} \tilde{A}^{n-1}_{i,\Lambda}\big)\eta_n \in L^4(H^{n-1}_i). \tag{4.5.13}$$

Thus, by Corollary 4.2.3, if $\varepsilon_G > 0$ is small enough we can find an extension $h_n \in W^{1,(5,\infty)}(Q_n, G)$ of η_n. Let $\tilde{A}^n_{i,\Lambda} \in L^{5,\infty}(\Omega^n_i)$ and $\sigma_n \in W^{1,2}(\Omega^n_i, G) \cap W^{1,2}(\partial\Omega^n_i, G)$ be given by

$$\tilde{A}^n_{i,\Lambda} := \begin{cases} \tilde{A}^{n-1}_{i,\Lambda} & \text{on } \Omega^{n-1}_i \\ (A_{Q_n})^{h_n} & \text{on } Q_n \end{cases} \tag{4.5.14}$$

and

$$\sigma_n := \begin{cases} \sigma_{n-1} & \text{on } \Omega_i^{n-1} \\ \tilde{g}_n h_n & \text{on } Q_n. \end{cases} \tag{4.5.15}$$

By construction and by Lemma 4.4.2, we have $F_{\tilde{A}^n_{i,\Lambda}} \in L^{\frac{5}{2},\infty}(\Omega_i^n)$. Moreover, since $\partial\Omega_i^n \smallsetminus \partial\Omega_i^{n-1} \subset \partial Q_n$ and we have

$$\iota^*_{H_i^n} \tilde{A}^n_{i,\Lambda} = \iota^*_{H_i^n}(A^{h_n}_{Q_n}) = (\iota^*_{H_i^n} A_{i,\Lambda})^{g_n h_n} = (\iota^*_{H_i^n} A_{i,\Lambda})^{\sigma_n},$$

we conclude that

$$\iota^*_{\partial\Omega_i^n} \tilde{A}^n_{i,\Lambda} = (\iota^*_{\partial\Omega_i^n} A_{i,\Lambda})^{\sigma_n} \in L^4(\partial\Omega_i^n).$$

This concludes the proof of the induction step and of our claim at once.

Now, to lighten the notation let $\Omega_i := \Omega_i^{N_i}$ and define

$$\tilde{A}_{i,\Lambda} := \tilde{A}^{N_i}_{i,\Lambda} \in L^{5,\infty}(\Omega_i).$$

$$\sigma_{i,\Lambda} := \sigma_{N_i} \in W^{1,2}(\Omega_i, G).$$

Step 2: Morrey norm control on the curvatures.

Note that, by construction, we have $\overline{Q_{\frac{1}{2}}(0)} \subset \Omega_i$. Fix any point $x \in Q_{\frac{1}{2}}(0)$ and let $r \in \big(0, \operatorname{dist}(x, \partial Q_{\frac{1}{2}}(0))\big)$. Assume that $r \leqslant \varepsilon_i$. Then, $Q_r(x)$ intersects at most $\tilde{N}$ cubes in $\mathcal{Q}^s_{r_i}$, say $Q_1, ..., Q_{\tilde{N}}$, with $\tilde{N} \in \mathbb{N}$ depending only on the choice of the cubic cover. Hence, by Hölder inequality, by the estimates (4.2.65), (4.2.95) for each A_Q, by the property (1) of Λ-good cubes and by the property (i) of $A_{i,\Lambda}$ given by Theorem 4.4.1, we get

$$\begin{aligned} \frac{1}{r}\int_{Q_r(x)} |F_{\tilde{A}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 &\leqslant \|F_{\tilde{A}_{i,\Lambda}}\|^2_{L^{\frac{5}{2}}(Q_r(x))} \\ &\leqslant C \sum_{\ell=1}^{\tilde{N}} \|F_{\tilde{A}_{i,\Lambda}}\|^2_{L^{\frac{5}{2}}(Q_\ell)} = C \sum_{i=1}^{\tilde{N}} \|F_{A_{Q_\ell}}\|^2_{L^{\frac{5}{2}}(Q_\ell)} \\ &\leqslant C \sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \big(\|F_{A_{i,\Lambda}}\|^2_{L^2(\partial Q_\ell)} + \varepsilon_i^{-2}\|A_{i,\Lambda}\|^2_{L^2(\partial Q_\ell)}\big) \\ &\quad + \sum_{Q_\ell \in \mathcal{Q}^b_{r_i,\Lambda}} \|F_{A_{i,\Lambda}}\|^2_{L^2(\partial Q_\ell)} \\ &< C|F_{A_{i,\Lambda}}|^2_{M^0_{2,2}(Q^5_1(0))} \leqslant C|F_A|^2_{M^0_{2,2}(Q^5_1(0))}, \end{aligned}$$

for some constant $C > 0$ depending only on G.

Assume now that $\varepsilon_i < r < 1$. Then there exists a universal constant $k > 1$ such that $Q_r(x)$ can be covered with a finite number of cubes $\{Q_\ell\}_{\ell=1,...,N_{i,x,r}}$ in $\mathcal{Q}^s_{r_i}$ such that

$$\bigcup_{\ell=1}^{N_{i,x,r}} Q_{2\varepsilon_i}(c_{Q_\ell}) \subset Q_{kr}(x).$$

Notice that now the number of cubes $N_{i,x,r}$ may depend on i, x and r. Then, by the estimates (4.2.66) and (4.2.96) we obtain

$$
\begin{aligned}
\int_{Q_r(x)} |F_{\tilde{A}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 &\leqslant \sum_{\ell=1}^{N_{i,x,r}} \int_{Q_\ell} |F_{\tilde{A}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 = \sum_{\ell=1}^{N_{i,x,r}} \int_{Q_\ell} |F_{A_{Q_\ell}}|^2 \, d\mathcal{L}^5 \\
&\leqslant C \sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \bigg(\varepsilon_i^{-4} \int_{\partial Q_\ell} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \bigg) \bigg(\varepsilon_i \int_{\partial Q_\ell} |A_{i,\Lambda}|^2 \, d\mathscr{H}^4 \bigg) \\
&\quad + \varepsilon_i \bigg(\varepsilon_i^{-2} \int_{\partial Q_\ell} |A_{i,\Lambda}|^2 \, d\mathscr{H}^4 \bigg)^3 \bigg) + C\varepsilon_i \sum_{\ell=1}^{N_{i,x,r}} \int_{\partial Q_\ell} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4,
\end{aligned}
$$

for some constant $C > 0$ depending only on G. By property (4) in Definition 4.5.2 and by the property (i) of $A_{i,\Lambda}$ given by Theorem 4.4.1 we get

$$
\begin{aligned}
\sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} & \bigg(\bigg(\varepsilon_i^{-4} \int_{\partial Q_\ell} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \bigg) \bigg(\varepsilon_i \int_{\partial Q_\ell} |A_{i,\Lambda}|^2 \, d\mathscr{H}^4 \bigg) \\
&\leqslant C\Lambda^{-1} [F_{A_{i,\Lambda}}]^2_{M^0_{2,2}(Q^5_1(0))} \sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A_{i,\Lambda}|^2 \, d\mathcal{L}^5, \\
&\leqslant C\Lambda^{-1} |F_A|^2_{M^0_{2,2}(Q^5_1(0))} \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A_{i,\Lambda}|^2 \, d\mathcal{L}^5,
\end{aligned}
$$

for some constant $C > 0$ depending only on G. By property (5) in Definition 4.5.2 and by our choice of $k > 0$, we get

$$
\sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A_{i,\Lambda}|^2 \, d\mathcal{L}^5 \leqslant C\Lambda \sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} (2\varepsilon_i)^5 \leqslant C\Lambda(kr)^5,
$$

for some constant $C > 0$ depending only on G. Hence, since by assumption $r \in (0,1)$, we have obtained the estimate

$$
\sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \bigg(\bigg(\varepsilon_i^{-4} \int_{\partial Q_\ell} |A_{i,\Lambda} - (A_{i,\Lambda})_{Q_\ell}|^2 \, d\mathscr{H}^4 \bigg) \bigg(\varepsilon_i \int_{\partial Q_\ell} |A_{i,\Lambda}|^2 \, d\mathscr{H}^4 \bigg) \leqslant Ck^5 r |F_A|^2_{M^0_{2,2}(Q^5_1(0))},
$$

for some constant $C > 0$ depending only on G. By property (5) in Definition 4.5.2, provided $i \geqslant i_0$ is large enough so that $\varepsilon_i < \Lambda^{-3} |F_A|^2_{M^0_{2,2}(Q^5_1(0))}$ we obtain

$$
\begin{aligned}
\sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i \bigg(\varepsilon_i^{-2} \int_{\partial Q_\ell} |A_{i,\Lambda}|^2 \, d\mathscr{H}^4 \bigg)^3 &\leqslant \sum_{Q_\ell \in \mathcal{Q}^g_{r_i,\Lambda}} \varepsilon_i \bigg(\varepsilon_i^{-3} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |A_{i,\Lambda}|^2 \, d\mathcal{L}^5 \bigg)^3 \\
&\leqslant \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i (\varepsilon_i^{-3} \Lambda \varepsilon_i^5)^3 \leqslant \varepsilon_i^2 \Lambda^3 \sum_{Q_\ell \in \mathcal{Q}^g_{\varepsilon_i,\Lambda}} \varepsilon_i^5 \\
&\leqslant C\varepsilon_i^2 \Lambda^3 \leqslant Cr |F_A|^2_{M^0_{2,2}(Q^5_1(0))},
\end{aligned}
$$

for some constant $C>0$ depending only on G. By Lemma 4.5.1–(ii) and by the property (i) of $A_{i,\Lambda}$ given by Theorem 4.4.1 we get

$$
\begin{aligned}
\varepsilon_i \sum_{\ell=1}^{N_{i,xr}} \int_{\partial Q_\ell} |F_{A_{i,\Lambda}}|^2 \, d\mathscr{H}^4 &\leqslant C \sum_{\ell=1}^{N_{i,x,r}} \int_{Q_{2\varepsilon_i}(c_{Q_\ell})} |F_{A_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \\
&\leqslant C \int_{Q_{kr}(x)} |F_{A_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \leqslant Ckr|F_{A_{i,\Lambda}}|^2_{M^0_{2,2}(Q^5_1(0))} \\
&\leqslant Ckr|F_A|^2_{M^0_{2,2}(Q^5_1(0))},
\end{aligned}
$$

for some constant $C>0$ depending only on the choice of the cubic cover. Hence, provided $i \geqslant i_0$ i sufficiently large we get

$$
\frac{1}{r} \int_{Q_r(x)} |F_{\tilde{A}_{i,\Lambda}}|^2 \, d\mathcal{L}^5 \leqslant C|F_A|^2_{M^0_{2,2}(Q^5_1(0))}, \tag{4.5.16}
$$

for some constant $C>0$ depending only on G.

Step 3: strong L^2-convergence of the connections.

Notice that, by construction, we have

$$
\begin{aligned}
\left\| \tilde{A}_{i,\Lambda}^{\sigma_{i,\Lambda}^{-1}} - A \right\|^2_{L^2(\Omega_i)} &= \sum_{n=1}^{N_i} \left\| A_{Q_n}^{\tilde{g}_n^{-1}} - A \right\|^2_{L^2(Q_n)} \\
&\leqslant C\left(\sum_{n=1}^{N_i} \left\| A_{Q_n}^{\tilde{g}_n^{-1}} - A_{i,\Lambda} \right\|^2_{L^2(Q_n)} + \sum_{n=1}^{N_i} \|A_{i,\Lambda} - A\|^2_{L^2(Q_n)} \right) \\
&\leqslant C\left(\sum_{n=1}^{N_i} \left\| A_{Q_n}^{\tilde{g}_n^{-1}} - A_{i,\Lambda} \right\|^2_{L^2(Q_n)} + \|A_{i,\Lambda} - A\|^2_{L^2(Q_1(0))} \right)
\end{aligned}
$$

for some constant $C>0$ depending on G and A. By the same procedure that we have used in the proof of Theorem 4.4.1, we get

$$
\lim_{i\to+\infty} \sum_{n=1}^{N_i} \left\| A_{Q_n}^{\tilde{g}_n^{-1}} - A_{i,\Lambda} \right\|_{L^2(Q_n)} \leqslant C \lim_{i\to+\infty} \int_{\Omega_{i,\Lambda,A_{i,\Lambda}}} |A_{i,\Lambda}|^2 \, d\mathcal{L}^5,
$$

where $C>0$ is a universal constant depending only on G and $\Omega_{i,\Lambda,A_{i,\Lambda}} \subset Q^5_1(0)$ is given as in Lemma 4.5.2. Now we notice that

$$
\begin{aligned}
\mathcal{L}^5(\Omega_{i,\Lambda,A_{i,\Lambda}}) &= r_i^5 \,\mathrm{card}\left(\left\{ Q \in \mathcal{Q}_{r_i} \text{ s.t. } \fint_Q |A_{i,\Lambda}|^2 \, dx^5 > \Lambda \right\} \right) \\
&\leqslant r_i^5 \,\mathrm{card}\left(\left\{ Q \in \mathcal{Q}_{r_i} \text{ s.t. } \fint_Q |A - A_{i,\Lambda}|^2 \, dx^5 + \fint_Q |A|^2 \, d\mathcal{L}^5 > \Lambda \right\} \right) \leqslant \frac{C}{\Lambda}
\end{aligned}
$$

and, recalling the definition of $\Omega_{i,\Lambda,A}$ from Lemma 4.3.2, we have

$$
\mathcal{L}^5(\Omega_{i,\Lambda,A}) \leqslant \frac{C}{\Lambda}
$$

for every given i, where $C > 0$ is independent on i. Thus, by Theorem 4.4.1-(iii), we get that for every $j \in \mathbb{N}$ there exists $i(j) > 0$ big enough so that be letting $\tilde{A}_j := \tilde{A}_{i(j),j}$ and $\sigma_j := \sigma_{i(j),j}^{-1} \in W^{1,2}(\Omega_{i(j)})$ we have

$$\Big\|\tilde{A}_j^{\sigma_j^{-1}} - A_{i(j),j}\Big\|_{L^2(\Omega_{i(j)})}^2 \leqslant C \int_{\Omega_{i(j),j,A_{i(j),j}}} |A_{i(j),j}|^2 \, d\mathcal{L}^5$$
$$\leqslant C\bigg(\int_{\Omega_{i(j),j,A_{i(j),j}}} |A|^2 \, d\mathcal{L}^5 + \int_{Q_1^5(0)} |A_{i(j),j} - A|^2 \, d\mathcal{L}^5\bigg)$$
$$\leqslant C \int_{\Omega_{i(j),j,A_{i(j),j}}} |A|^2 \, d\mathcal{L}^5 + \int_{\Omega_{i(j),j,A}} |A|^2 \, d\mathcal{L}^5,$$

where $C > 0$ doesn't depend on j. Finally, we get

$$\Big\|\tilde{A}_j^{\sigma_j^{-1}} - A\Big\|_{L^2(\Omega_{i(j)})}^2 \leqslant C \int_{\Omega_{i(j),j,A_{i(j),j}}} |A|^2 \, d\mathcal{L}^5 \to 0$$

as $j \to +\infty$.

Notice that, by assumption, $\Omega_{i(j)} \supset \overline{Q_{\frac{1}{2}}(0)}$ for every $j \in \mathbb{N}$. Since $\pi_2(G) = 0$, we can find a sequence $\{\tilde{\sigma}_j\}_{j\in\mathbb{N}} \subset C^\infty\big(\overline{Q_{\frac{1}{2}}(0)}\big)$ such that

$$\|\tilde{\sigma}_j - \sigma_j\|_{W^{1,2}(Q_{\frac{1}{2}}(0))} \to 0$$

and $\tilde{\sigma}_j - \sigma_j \to 0$ pointwise $\mathcal{L}^5$-a.e. on $Q_{\frac{1}{2}}(0)$ as $j \to +\infty$. Moreover, since we have the improved integrability properties $A \in L^{5,\infty}(Q_{\frac{1}{2}}(0))$ and $F_A \in L^{\frac{5}{2},\infty}(Q_{\frac{1}{2}}(0))$, by standard convolution with a smooth mollifying kernel we can find a sequence $\{\hat{A}_j\}_{j\in\mathbb{N}} \in C_c^\infty\big(Q_{\frac{1}{2}}(0)\big)$ such that

$$|F_{\hat{A}_j}|^2_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant 2|F_{\tilde{A}_j}|^2_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant 2C|F_A|^2_{M^0_{2,2}(Q^5_1(0))} \qquad \forall\, j \in \mathbb{N},$$

$$\|\hat{A}_j - \tilde{A}_j\|_{L^2(Q_{\frac{1}{2}}(0))} \to 0$$

and $\hat{A}_j - \tilde{A}_j \to 0$ pointwise $\mathcal{L}^5$-a.e. on $Q_{\frac{1}{2}}(0)$ as $j \to +\infty$. Now define

$$A_j := \hat{A}_j^{\tilde{\sigma}_j^{-1}} \in C^\infty\big(\overline{Q_{\frac{1}{2}}(0)}\big) \qquad \forall\, j \in \mathbb{N}.$$

By construction and Ad-invariance of the norm on $\mathfrak{g}$, we have

$$|F_{\tilde{A}_j}|^2_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant 2|F_{\hat{A}_j}|^2_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant 2C|F_A|^2_{M^0_{2,2}(Q^5_1(0))} \qquad \forall\, j \in \mathbb{N}.$$

By construction and dominated convergence, we have

$$\|A_j - A\|_{L^2(Q_{\frac{1}{2}}(0))} \leqslant \big\|A_j - \tilde{A}_j^{\sigma_j^{-1}}\big\|_{L^2(Q_{\frac{1}{2}}(0))} + \big\|\tilde{A}_j^{\sigma_j^{-1}} - A\big\|_{L^2(Q_{\frac{1}{2}}(0))} \to 0$$

as $j \to +\infty$. This concludes the proof of Theorem 4.5.1. $\square$

4.6. Partial regularity for stationary weak Yang–Mills fields

In this section we prove our main result about the partial regularity of stationary weak Yang–Mills fields on the unit cube $Q_1^5(0)$. In order to get that, we first use Theorem 4.5.1 and the main result in [57] to obtain the following Coulomb gauge extraction theorem for weak connections whose curvature has sufficiently small Morrey norm.

Theorem 4.6.1 (Coulomb gauge extraction under small Morrey norm assumption)**.** *Let G be a compact matrix Lie group. There exists $\varepsilon_G \in (0,1)$ such that for every weak connection $A \in \mathfrak{a}_G(Q_1^5(0))$ satisfying*

$$|F_A|^2_{M^0_{2,2}(Q_1^5(0))} < \varepsilon_G$$

there exist $g \in W^{1,2}(Q_{\frac{1}{2}}(0), G)$ such that $A^g \in (M^1_{2,2} \cap M^0_{4,1})(Q_{\frac{1}{2}}(0))$ satisfies

$$\begin{cases} d^* A^g = 0, \\ \iota^*_{\partial Q_1^5(0)}(*A^g) = 0, \\ |\nabla A^g|_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} + |A^g|_{M^0_{4,1}(Q_{\frac{1}{2}}(0))} \leqslant C_G |F_A|_{M^0_{2,2}(Q_{\frac{1}{2}}(0))}, \end{cases}$$

for some constant $C_G > 0$ depending only on G.

Proof of Theorem 4.6.1. By Theorem 4.5.1, if $\varepsilon_G > 0$ is small enough we can find a sequence of $\mathfrak{g}$-valued 1-forms $\{A_i\}_{i\in\mathbb{N}} \subset C_c^\infty(Q_{\frac{1}{2}}(0))$ such that

1. for every $i \in \mathbb{N}$ we have
$$|F_{A_i}|^2_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} \leqslant K_G |F_A|^2_{M^0_{2,2}(Q_1^5(0))}.$$
for some constant $K_G > 0$ depending on G;
2. $\|A_i - A\|_{L^2(Q_{\frac{1}{2}}(0))} \to 0$ as $i \to +\infty$.

By choosing $\varepsilon_G > 0$ possibly smaller, we can make sure that $\sqrt{K_G \varepsilon_G}$ is smaller than the constant given by [57, Theorem 1.3] and we can apply such statement to conclude that for every $i \in \mathbb{N}$ there exists a gauge g_i such that $d^* A_i^{g_i} = 0$ on $Q_{\frac{1}{2}}(0)$, $\iota^*_{\partial Q_1^5(0)}(*A^g) = 0$ and

$$|\nabla A_i^{g_i}|_{M^0_{2,2}(Q_{\frac{1}{2}}(0))} + |A_i^{g_i}|_{M^0_{4,1}(Q_{\frac{1}{2}}(0))} \leqslant C_G |F_{A_i}|_{M^0_{2,2}(Q_{\frac{1}{2}}(0))}, \tag{4.6.1}$$

for some constant $C_G > 0$ depending only on G. By standard Sobolev embedding theorems, there exists a subsequence (not relabeled) such that $A_i^{g_i} \rightharpoonup \tilde{A} \in M^0_{4,1}$ weakly in L^4 and $\nabla A_i^{g_i} \rightharpoonup \nabla\tilde{A} \in M^0_{2,2}$ weakly in L^2. Clearly, we have $d^*\tilde{A} = 0$. Notice that

$$dg_i = g_i A_i^{g_i} - A_i g_i.$$

By (4.6.1) and 1, we get that $\{g_i\}_{i\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(Q_{\frac{1}{2}}(0), G)$. Hence, there exists a subsequence (not relabeled) such that $g_i \rightharpoonup g$ weakly in $W^{1,2}$. By Sobolev embedding theorems, we have $g_i \to g$ strongly in L^2. Thus, since $A_i \to A$ strongly in L^2 and G is bounded we conclude that $A^g = \tilde{A}$ on $Q_{\frac{1}{2}}(0)$. This concludes the proof of Theorem 4.6.1. □

Definition 4.6.1 (Weak Yang–Mills connections)**.** Let G be a compact matrix Lie group. We say that $A \in \mathfrak{a}_G(Q_1^5(0))$ is *weak Yang–Mills connection* on $Q_1^5(0)$ if

$$d_A^* F_A = 0 \qquad \text{distrubutionally on } Q_1^5(0),$$

i.e.

$$\int_{Q_1^5(0)} F_A \cdot d_A\varphi = 0 \quad \text{ for every } \varphi \in C_c^\infty(\wedge^1 Q_1^5(0) \otimes \mathfrak{g}).$$

Definition 4.6.2 (Stationary weak Yang–Mills connections)**.** Let G be a compact matrix Lie group. We say that a weak Yang–Mills connection $A \in \mathfrak{a}_G(Q_1^5(0))$ on $Q_1^5(0)$ is *stationary* if

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{YM}(\Phi_t^* A) = 0, \tag{4.6.2}$$

for every smooth 1-parameter group of diffeomorphisms Φ_t of $Q_1^5(0)$ with compact support.

Remark 4.6.1. Note that if A is a stationary weak Yang–Mills connection and $\tilde{A} \in W^{1,2} \cap L^4$ is another $\mathfrak{g}$-valued 1-form on $Q_1^5(0)$ such that $\tilde{A} = A^g$ for some $g \in W^{1,2}(Q_1^5(0), G)$, then $\tilde{A}$ is a stationary weak Yang–Mills connection as well. The proof of such fact follows by direct computation exploiting the gauge invariance of the Yang–Mills functional.

If A is a stationary Yang–Mills connection, it can be shown that the following *monotonicity property* holds true: for every given $x \in Q_1^5(0)$ the function

$$\big(0, \operatorname{dist}(x, \partial Q_1^5(0))\big) \ni \rho \to \frac{e^{c\Lambda\rho}}{\rho} \int_{Q_\rho(x)} |F_A|^2 \, d\mathcal{L}^5 \tag{4.6.3}$$

is non-decreasing, where $c > 0$ is a universal constant and Λ depends on $Q_1(x)$. In particular, we have

$$|F_A|_{M^0_{2,2}(Q_1^5(0))} \leqslant C \|F_A\|_{L^2(Q_1^5(0))},$$

for some constant $C > 0$ independent on A. Thanks to (4.6.3) and to Theorem 4.6.1, we can derive the following ε-regularity statement by using the same arguments presented in [57, Section 4]. This is a crucial initial step in building a regularity theory for YM-energy minimizers, following the same path that R. Schoen and K. Uhlenbeck walked in their work on energy-minimizing harmonic maps in general supercritical dimension $n > 2$ (as documented in [79]).

Theorem 4.6.2 (ε-regularity)**.** *Let G be a compact matrix Lie group. There exists $\varepsilon_G \in (0,1)$ such that for every stationary weak Yang–Mills $A \in \mathfrak{a}_G(Q_1^5(0))$ satisfying*

$$\mathrm{YM}(A) = \int_{Q_1^5(0)} |F_A|^2 \, d\mathcal{L}^5 < \varepsilon_G$$

there exist $g \in W^{1,2}(Q_{\frac{1}{2}}(0), G)$ such that $A^g \in C^\infty(Q_{\frac{1}{2}}(0))$.

Standard covering arguments (see e.g. [39, Proposition 9.21]) together with Theorem 4.6.2 and (4.6.3) give the following bound on the singular set of stationary weak Yang–Mills connections, which is the main result of the present paper.

Theorem 4.6.3. *Let G be a compact matrix Lie group and let $A \in \mathfrak{a}_G(Q^5_1(0))$ be a stationary weak Yang–Mills connection on $Q^5_1(0)$. Then*

$$\mathscr{H}^1(\mathrm{Sing}(A)) = 0,$$

where $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure on $Q^5_1(0)$ and $\mathrm{Sing}(A) \subset Q^5_1(0)$ is the singular set of A, given by $\mathrm{Sing}(A) := Q^5_1(0) \setminus \mathrm{Reg}(A)$ where

$$\mathrm{Reg}(A) := \{x \in Q^5_1(0) \text{ s.t. } \exists \rho > 0, g \in W^{1,2}(B_\rho(x), G) \text{ s.t. } A^g \in C^\infty(B_\rho(x))\}.$$

4.A. Adapted Coulomb gauge extraction statements in critical dimension

Proposition 4.A.1. *Let G be any compact matrix Lie group. There exist constants $\varepsilon_G, C_G > 0$ depending only on G such that for every $\mathfrak{g}$-valued 1-form $A \in W^{1,2}(\mathbb{S}^4)$ on $\mathbb{S}^4$ such that*

$$\int_{\mathbb{S}^4} |F_A|^2 \, d\mathscr{H}^4 < \varepsilon_G$$

we can find a gauge $g \in W^{1,4}(\mathbb{S}^4, G)$ satisfying $A^g \in W^{1,2}(\mathbb{S}^4)$, $d^* A^g = 0$ and

$$\|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{S}^4)}$$

Proof of Proposition 4.A.1. Let π be the stereographic projection from $\mathbb{S}^4 \setminus \{N\}$ into $\mathbb{R}^4$ which is sending the north pole N of $\mathbb{S}^4$ to infinity and which is conformally invariant. Introduce

$$D := (\pi^{-1})^* A$$

Because of conformal invariance of the Hodge operator on 2-forms one has

$$\begin{aligned}\int_{\mathbb{R}^4} |F_D|^2 d\mathcal{L}^4 &= -\int_{\mathbb{R}^4} \mathrm{tr}(F_D \wedge *F_D) = -\int_{\mathbb{R}^4} (\pi^{-1})^* (\mathrm{tr}(F_A \wedge *F_A)) \\ &= -\int_{\mathbb{S}^4} \mathrm{tr}(F_A \wedge *F_A) = \int_{\mathbb{S}^4} |F_A|^2 d\mathscr{H}^4 < \varepsilon_0.\end{aligned}$$

For every $R > 0$, on $B_R(0)$ one chooses the Uhlenbeck Coulomb gauge $D_R := (D)^{g_R}$ (see [90]) that satisfies

$$d^* D_R = 0$$

and

$$\|D_R\|_{L^4(B_R(0))} + \|dD_R\|_{L^2(B_R(0))} \leqslant C_G \, \|F_{D_R}\|_{L^2(B_R(0))} \leqslant C_G \, \|F_A\|_{L^2(\mathbb{S}^4)}.$$

What is crucial here is that all the norms involved are scaling invariant and then the constant $C_G > 0$ is independent of R.

Consider now on $\pi^{-1}(B_R(0)) = \mathbb{S}^4 \setminus B_{\rho_R}(N)$ where $\rho_R \simeq R/(1+R^2) \to 0$ as $R \to +\infty$ the 1-form

$$\pi^*(D_R) = A^{h_R}$$

where $h_R := g_R \circ \pi$. Using the conformal invariance of the L^4 norms on 1-forms as well as the L^2 norms on 2-forms one has

$$\|A^{h_R}\|_{L^4(\pi^{-1}(B_R(0))} + \|d(A^{h_R})\|_{L^2(\pi^{-1}(B_R(0))} \leqslant C_G \; \|F_A\|_{L^2(\mathbb{S}^4)} \tag{4.A.1}$$

There exists obviously a sequence $R_k \to +\infty$ such that for some $\Omega \in L^4_{loc}(\mathbb{S}^4 \setminus \{N\})$ we have

$$A^{h_{R_k}} \rightharpoonup \Omega \text{ weakly in } L^4_{loc}(\mathbb{S}^4 \setminus \{N\})$$

and

$$h_{R_k} \rightharpoonup h_\infty \text{ weakly in } W^{1,4}_{loc}(\mathbb{S}^4 \setminus \{N\}).$$

Moreover

$$d(A^{h_{R_k}}) \rightharpoonup d\Omega \text{ weakly in } L^2_{loc}(\mathbb{S}^4 \setminus \{N\})$$

By (4.A.1) we have in particular that

$$\|\Omega\|_{L^4(\mathbb{S}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{S}^4)} \quad \text{ and } \quad \Omega = A^{h_\infty} \in L^4(\mathbb{S}^4) \tag{4.A.2}$$

Following Uhlenbeck continuity type argument introduced in [70] and adapted in [55] to the 4-dimensional case we construct $g \in W^{1,4}(\mathbb{S}^4, G)$ such that

$$d^*(g^{-1}dg + g^{-1}\Omega g) = 0 \quad \text{ and } \quad \|dg\|_{L^4(\mathbb{S}^4)} \leqslant C_G \; \|\Omega\|_{L^4(\mathbb{S}^4)}.$$

Hence we have the existence of $g \in W^{1,4}(\mathbb{S}^4, G)$ such that $(A^{h_\infty})^g = \Omega^g$ satisfies

$$d^*((A^{h_\infty})^g) = 0 \quad \text{ and } \|(A^{h_\infty})^g\|_{L^4(\mathbb{S}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{S}^4)}$$

which gives

$$\|(A^{h_\infty})^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{S}^4)}$$

This proves the existence of a controlled global Coulomb gauge $\tilde{g} := h_\infty g \in W^{1,4}(\mathbb{S}^4, G)$ under small curvature assumption. This concludes the proof of Proposition 4.A.1. □

We recall here the main statement that we will use to extract Coulomb gauges in critical dimension. This is an adaptation of the main Theorem in [90].

Proposition 4.A.2. *Let G be a compact matrix Lie group. There exist constants $\varepsilon_G, C_G > 0$ depending only on G such that for every $\mathfrak{g}$-valued 1-form $A \in L^4(\mathbb{B}^4)$ on $\mathbb{B}^4$ such that*

$$\|F_A\|_{L^2(\mathbb{S}^4)} < \varepsilon_G$$

we can find a gauge $g \in W^{1,4}(\mathbb{B}^4, G)$ satisfying $A^g \in W^{1,2}(\mathbb{B}^4)$, $d^* A^g = 0$ and

$$\|A^g\|_{W^{1,2}(\mathbb{B}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{B}^4)}$$

Proof of Proposition 4.A.2. Fix any $4 < p < 8$. For every $\varepsilon, C > 0$, we define

$$
\begin{aligned}
X &:= \{A \in L^p(\mathbb{B}^4, \wedge^1\mathbb{B}^4 \otimes \mathfrak{g}) \text{ s.t. } dA \in L^{\frac{p}{2}}(\mathbb{B}^4, \wedge^2\mathbb{B}^4 \otimes \mathfrak{g})\} \\
\mathscr{U}^\varepsilon &:= \{A \in X \text{ with } \|F_A\|_{L^2(\mathbb{B}^4)} < \varepsilon\} \\
\mathscr{V}^\varepsilon_C &:= \{A \in \mathscr{U}^\varepsilon : \exists\, g \in W^{1,p}_{\mathrm{id}_G}(\mathbb{B}^4, G) : d^*A^g = 0, \|A^g\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_A\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}, \, q = 4, p\}.
\end{aligned}
$$

On X, we consider the topology induced by the following norm:

$$
\|A\|_X := \|A\|_{L^p(\mathbb{B}^4)} + \|dA\|_{L^{\frac{p}{2}}(\mathbb{B}^4)}, \qquad \forall\, A \in X.
$$

Notice that X is a Banach vector space with respect to such norm.

Claim. We claim that there exist $\varepsilon, C > 0$ such that $\mathscr{V}^\varepsilon_C = \mathscr{U}^\varepsilon$. In order to achieve such result, we prove separately the following facts.

- *$\mathscr{U}^\varepsilon$ is path-connected.* Notice that $0 \in \mathscr{U}^\varepsilon$. Given any $A \in \mathscr{U}^\varepsilon$, we define

$$
A(t) := tA(t\,\cdot\,), \qquad \forall\, t \in [0,1].
$$

  Notice that $A(0) = 0$ and $A(1) = A$. Moreover,

$$
F_{A(t)} = dA(t) + A(t) \wedge A(t) = t^2 dA(t\,\cdot\,) + t^2 A(t\,\cdot\,) \wedge A(t\,\cdot\,) = t^2 F_A(t\,\cdot\,)
$$

  which implies that

$$
\begin{aligned}
\int_{\mathbb{B}^4} |F_{A(t)}|^{\frac{p}{2}}\, d\mathcal{L}^4 &= \int_{\mathbb{B}^4} t^p |F_A(t\,\cdot\,)|^{\frac{p}{2}}\, d\mathcal{L}^4 = t^{p-4} \int_{t\mathbb{B}^4} |F_A|^{\frac{p}{2}}\, d\mathcal{L}^4 \\
&\leqslant t^{p-4} \int_{\mathbb{B}^4} |F_A|^{\frac{p}{2}}\, d\mathcal{L}^4 < +\infty
\end{aligned}
\tag{4.A.3}
$$

  and

$$
\int_{\mathbb{B}^4} |F_{A(t)}|^2\, d\mathcal{L}^4 = \int_{\mathbb{B}^4} t^4 |F_A(t\,\cdot\,)|^2\, d\mathcal{L}^4 = \int_{t\mathbb{B}^4} |F_A|^2\, d\mathcal{L}^4 \leqslant \int_{\mathbb{B}^4} |F_A|^2\, d\mathcal{L}^4 < \varepsilon.
$$

  Hence, $A(t) \in \mathscr{U}^\varepsilon$ for every $t \in [0,1]$. At the same time we have

$$
\int_{\mathbb{B}^4} |A(t)|^p\, d\mathcal{L}^4 = \int_{\mathbb{B}^4} t^p |A(t\,\cdot\,)|^4\, d\mathcal{L}^4 \leqslant t^{p-4} \int_{\mathbb{B}^4} |A|^4\, d\mathcal{L}^4 \to 0
$$

  as $t \to 0^+$. Hence, $A(t) \to 0$ in $L^p(\mathbb{B}^4)$. As a byproduct, we have $A(t) \wedge A(t) \to 0$ in $L^{\frac{p}{2}}(\mathbb{B}^4)$. Moreover, by (4.A.3) we have $F_{A(t)} \to 0$ in $L^{\frac{p}{2}}(\mathbb{B}^4)$. This implies $dA(t) \to 0$ in $L^{\frac{p}{2}}(\mathbb{B}^4)$.
  It follows that $[0,1] \ni t \to A(t)$ is a continuous path in $\mathscr{U}^\varepsilon$ joining A and 0.
- *For every $\varepsilon, C > 0$, $\mathscr{V}^\varepsilon_C$ is closed in $\mathscr{U}^\varepsilon$.* Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathscr{V}^\varepsilon_C$ be such that $A_k \to A \in \mathscr{U}_\varepsilon$. We want to show that $A \in \mathscr{V}^\varepsilon_C$.
  Notice that, by assumption, we have $F_{A_k} \to F_A$ strongly in $L^{\frac{p}{2}}(\mathbb{B}^4)$. Moreover, for every $k \in \mathbb{N}$ there exists $g_k \in W^{1,p}_{\mathrm{id}_G}(\mathbb{B}^4, G)$ such that $d^*A_k^{g_k} = 0$ and

$$
\|A_k^{g_k}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_{A_k}\|_{L^2(\mathbb{B}^4)} \leqslant C\varepsilon
$$

for $q = p, 4$. Since $\{A_k^{g_k}\}_{k\in\mathbb{N}}$ is bounded in $W^{1,\frac{p}{2}}$, there exists a subsequence (not relabeled) such that $A_k^{g_k} \rightharpoonup A_\infty$ in $W^{1,\frac{p}{2}}$. Hence, for $q = p, 4$, we have

$$\|A_\infty\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant \liminf_{k\to+\infty} \|A_k^{g_k}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C \lim_{k\to+\infty} \|F_{A_k}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} = C\|F_A\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}.$$

Notice that $d^*A_\infty = 0$ and that, by Sobolev embedding theorem, we have $A_k^{g_k} \to A_\infty$ strongly in L^p. Since

$$dg_k = g_k A_k^{g_k} - A_k g_k, \tag{4.A.4}$$

we get that $\{g_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}_{\mathrm{id}_G}(\mathbb{B}^4, G)$. Thus, there exists a subsequence (not relabeled) such that $g_k \rightharpoonup g$ in $W^{1,p}_{\mathrm{id}_G}(\mathbb{B}^4, G)$. This is enough to pass to the limit in (4.A.4) and we get $A_\infty = A^g$. Since g is a gauge satisfying all the required properties for A, we conclude that $A \in \mathscr{V}_C^\varepsilon$.

- *For some choice of $\varepsilon, C > 0$, $\mathscr{V}_C^\varepsilon$ is open in $\mathscr{U}^\varepsilon$.* Let $A \in \mathscr{V}_C^\varepsilon$. It is clear that if we find an open neighborhood of its Coulomb gauge A^g for the topology of X in $\mathscr{V}_C^\varepsilon$, then A posses also such a neighborhood. So we can assume right away that $d^*A = 0$ and

$$\|A\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_A\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} < C\varepsilon.$$

Notice that, in such a way, automatically we have $A \in W^{1,\frac{p}{2}}(\mathbb{B}^4)$.

$$Y := \left\{U \in W^{1,p}(\mathbb{B}^4, \mathfrak{g}) : \int_{\mathbb{B}^4} U \, d\mathcal{L}^4 = 0\right\}. \tag{4.A.5}$$

$$Z := \left\{(f, \alpha) \in W^{-1,p}(\mathbb{B}^4, \mathfrak{g}) \times W^{1-\frac{1}{p},p}(\partial\mathbb{B}^4, \wedge^3 T^*\partial\mathbb{B}^4 \otimes \mathfrak{g}) : \int_{\mathbb{B}^n} f \, d\mathcal{L}^4 = -\int_{\partial\mathbb{B}^4} \alpha\right\}. \tag{4.A.6}$$

Notice that Y and Z are Banach spaces, as they are closed subspaces of $W^{1,p}(\mathbb{B}^4, \mathfrak{g})$ and of the product $W^{-1,p}(\mathbb{B}^n, \mathfrak{g}) \times W^{-\frac{1}{p},p}(\partial\mathbb{B}^4, \wedge^3 T^*\partial\mathbb{B}^4 \otimes \mathfrak{g})$ respectively. We introduce the map

$$\mathscr{F}_A : X \times Y \to Z$$

given by

$$\mathscr{F}_A(\omega, U) := \Big(d^*\big((A+\omega)^{\exp(U)}\big), \iota^*_{\partial\mathbb{B}^n}\big(*(A+\omega)^{\exp(U)}\big)\Big)$$

By direct computation we can show that $\mathscr{F}_A$ is a C^1-map between Banach spaces. The partial derivative $\partial_U\mathscr{F}_A(0,0) : Y \to Z$ is the following linear and continuous operator between Banach spaces:

$$\begin{aligned}\partial_U\mathscr{F}_A(0,0)[V] &= \big(\Delta V + d^*(AV - VA), \partial_r V\big)\\ &= \big(\Delta V + d^*([A,V]), \partial_r V\big), \qquad \forall\, V \in Y.\end{aligned}$$

In order to apply the implicit function theorem to $\mathscr{F}_A$ we need to show that $\partial_U\mathscr{F}_A(0,0)$ is invertible. First we show that it is injective. Indeed, by standard L^p-theory for the Laplacian and since A is in a controlled Coulomb gauge, for every $V \in Y$ we get

$$\|V\|_Y \leqslant \tilde{C}\Big(\|\Delta V\|_{W^{-1,p}(\mathbb{B}^4)} + \|\partial_r V\|_{W^{-\frac{1}{p},p}(\partial\mathbb{B}^4)}\Big)$$

$$
\begin{aligned}
&\leqslant \tilde{C}\Big(\|\partial_U \mathscr{F}_A(0,0)[V]\|_{W^{-1,p}(\mathbb{B}^4)} + \|d^*([A,V])\|_{W^{-1,p}(\mathbb{B}^4)}\Big)\\
&\leqslant \tilde{C}\Big(\|\partial_U \mathscr{F}_A(0,0)[V]\|_{W^{-1,p}(\mathbb{B}^4)} + \|[A,V]\|_{L^p(\mathbb{B}^4)}\Big)\\
&\leqslant \tilde{C}\Big(\|\partial_U \mathscr{F}_A(0,0)[V]\|_{W^{-1,p}(\mathbb{B}^4)} + \|A\|_{L^p(\mathbb{B}^4)}\|V\|_{L^\infty(\mathbb{B}^4)}\Big)\\
&\leqslant \tilde{C}\Big(\|\partial_U \mathscr{F}_A(0,0)[V]\|_{L^{\frac{p}{2}}(\mathbb{B}^4)} + C\varepsilon\|V\|_{W^{1,p}(\mathbb{B}^4)}\Big),
\end{aligned}
$$

for some constant $\tilde{C} > 0$ which just depends on G. By choosing $\varepsilon, C > 0$ in such a way that $C\varepsilon < \frac{1}{2\tilde{C}}$, we obtain

$$
\|V\|_Y \leqslant 2\tilde{C}\|\partial_U \mathscr{F}_A(0,0)[V]\|_{W^{-1,p}(\mathbb{B}^4)}.
$$

which implies that $\partial_U \mathscr{F}_A(0,0)$ has trivial kernel. Classical Calderon-Zygmund theory asserts that the operator $\mathcal{L}_0 : Y \to Z$ given by

$$
\mathcal{L}_0(V) := \Delta V, \qquad \forall\, V \in Y
$$

is invertible and therefore it has zero index. For every $t \in [0,1]$ we define the operator $\mathcal{L}_t : Y \to Z$ by

$$
\mathcal{L}_t(V) := \Delta V - t(*[*A, dV]), \qquad \forall\, V \in Y.
$$

and we notice that $[0,1] \ni t \to \mathcal{L}_t$ is a continuous path of bounded operators joining $\mathcal{L}_0$ and $\mathcal{L}_1 = \partial_U \mathscr{F}_A(0,0)$. By continuity of the Fredholm index, we conclude that $\partial_U \mathscr{F}_A(0,0)$ has zero index. This, together with $\ker\big(\partial_U \mathscr{F}_A(0,0)\big) = 0$, implies that $\partial_U \mathscr{F}_A(0,0)$ is invertible. Hence, we can apply the implicit function theorem in order to get that there exist an open neighbourhood $\mathcal{O}$ of 0 in Y and $\delta > 0$ such that for every $\omega \in X$ such that $\|\omega\|_X < \delta$ there exists $U_\omega \in \mathcal{O}$ such that

$$
0 = \mathscr{F}_A(\omega, U_\omega) = d^*\big((A+\omega)^{g_\omega}\big),
$$

where we have set $g_\omega := \exp(U_\omega) \in W^{1,p}(\mathbb{B}^4, G)$. We just need to establish the bounds

$$
\|(A+\omega)^{g_\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_{A+\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}
$$

for $q = 4, p$. Notice that, by triangular inequality and Sobolev embedding theorem, we get

$$
\begin{aligned}
\|(A+\omega)^{g_\omega}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} &= \|(A+\omega)^{g_\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} + \|d((A+\omega)^{g_\omega})\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}\\
&\leqslant \hat{C}\big(\|F_{A+\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} + \|(A+\omega)^{g_\omega}\|^2_{L^q(\mathbb{B}^4)}\big)\\
&\leqslant \hat{C}\Big(\|F_{A+\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} + \|(A+\omega)^{g_\omega}\|_{L^q(\mathbb{B}^4)}\|(A+\omega)^{g_\omega}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)}\Big).
\end{aligned}
$$

for some constant $\hat{C} > 0$ depending only on G. Now we see that

$$
\begin{aligned}
\|(A+\omega)^{g_\omega}\|_{L^q(\mathbb{B}^4)} &\leqslant K\|A\|_{L^q(\mathbb{B}^4)} + K\big(\|\omega\|_{L^q(\mathbb{B}^4)} + \|dg_\omega\|_{L^q(\mathbb{B}^4)}\big)\\
&\leqslant K\|A\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} + K\big(\|\omega\|_{L^q(\mathbb{B}^4)} + \|dg_\omega\|_{L^q(\mathbb{B}^4)}\big),
\end{aligned}
$$

where again $K > 0$ depends only on G. Notice that by possibly reducing $\delta > 0$ and the size of the neighbourhood $\mathcal{O}$ we bring the quantity $\|\omega\|_{L^q(\mathbb{B}^4)} + \|dg_\omega\|_{L^q(\mathbb{B}^4)}$ to be arbitrary small. Then, we choose such parameters possibly depending on A in such a way that

$$\|\omega\|_{L^q(\mathbb{B}^4)} + \|dg_\omega\|_{L^q(\mathbb{B}^4)} \leqslant \|A\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)}$$

which implies

$$\|(A+\omega)^{g_\omega}\|_{L^q(\mathbb{B}^4)} \leqslant 2K\|A\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant 2KC\varepsilon$$

and finally

$$\|(A+\omega)^{g_\omega}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant \hat{C}\|F_{A+\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)} + 2\hat{C}KC\varepsilon\|(A+\omega)^{g_\omega}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)}\Big).$$

Assuming that $\varepsilon, C > 0$ are such that $2\hat{C}KC\varepsilon \leqslant \frac{1}{2}$ and $C \geqslant 2\hat{C}$ we get

$$\|(A+\omega)^{g_\omega}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_{A+\omega}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}$$

and it follows that

$$B_\delta^X(A) := \big\{A+\omega\,:\,\|\omega\|_X < \delta\big\} \subset \mathscr{V}_C^\varepsilon.$$

This concludes the proof of the openness of $\mathscr{V}_C^\varepsilon$ in $\mathcal{U}_\varepsilon$ and our claim follows.

Now that the previous claim is proved, we proceed by approximation in the following way. Let A satisfy the hypothesis of the statement and let $\{A_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{B}^4)$ be such that $A_k \to A$ strongly in L^4 and $dA_k \to dA$ strongly in L^2. Hence, we have that $F_{A_k} \to F_A$ strongly in L^2 and, for $k \in \mathbb{N}$ large enough, we get

$$\int_{\mathbb{B}^4} |F_{A_k}|^2\, d\mathcal{L}^4 < \varepsilon.$$

Hence, for $k \in \mathbb{N}$ large enough we have $A_k \in \mathscr{U}^\varepsilon$. We choose $\varepsilon, C > 0$ so that $\mathscr{V}_C^\varepsilon = \mathscr{U}^\varepsilon$. Thus, we get that for every $k \in \mathbb{N}$ large enough there exists $g_k \in W^{1,p}(\mathbb{B}^4, G)$ such that $d^*A_k^{g_k} = 0$ and

$$\|A_k^{g_k}\|_{W^{1,\frac{q}{2}}(\mathbb{B}^4)} \leqslant C\|F_{A_k}\|_{L^{\frac{q}{2}}(\mathbb{B}^4)}$$

for $q = 4, p$. Hence we find a subsequence (not relabeled) such that $A_k^{g_k} \rightharpoonup A_\infty$ weakly in $W^{1,\frac{p}{2}}$. Clearly, we have $d^*A_\infty = 0$. Moreover, $A_k^{g_k} \rightharpoonup A_\infty$ weakly in $W^{1,2}$ and we get

$$\|A_\infty\|_{W^{1,2}(\mathbb{B}^4)} \leqslant \liminf_{k\to+\infty} \|A_k^{g_k}\|_{W^{1,2}(\mathbb{B}^4)} \leqslant C \lim_{k\to+\infty} \|F_{A_k}\|_{L^2} = C\|F_A\|_{L^2(\mathbb{B}^4)}.$$

Since $p > 4$, the weak convergence of $A_k^{g_k}$ in $W^{1,\frac{p}{2}}$ implies that $A_k^{g_k} \to A_\infty$ strongly in L^4. In particular, by

$$dg_k = g_k A_k^{g_k} - A_k g_k,$$

we get that $\{g_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,4}(\mathbb{B}^4)$. Thus, there exists a subsequence (not relabeled) such that $g_k \rightharpoonup g \in W^{1,4}(\mathbb{B}^4)$ weakly in $W^{1,4}$. By passing to the limit in the previous equality, we eventually get

$$dg = gA_\infty - Ag,$$

i.e. $A_\infty = A^g$. This concludes the proof of Proposition 4.A.2. □

Remark 4.A.1. A statement that is completely equivalent to Proposition 4.A.2 also applies to every open and bounded domain D contained in $\mathbb{R}^4$ with a sufficiently smooth boundary. This is necessary to use the standard tools from elliptic regularity theory in the proof. For the purposes of this chapter, it suffices to note that all the elliptic regularity theory can be applied to cubes using standard reflection methods. As a result, Proposition 4.A.2 can be extended to open 4-cubes in $\mathbb{R}^4$.

Finally we extend Proposition 4.A.2 to the sphere case following the same proof of Proposition 4.A.1.

Proposition 4.A.3. *Let G be any compact and connected Lie group. There exist constants $\varepsilon_G, C_G > 0$ depending only on G such that for every $\mathfrak{g}$-valued 1-form $A \in L^4(\mathbb{S}^4)$ on $\mathbb{S}^4$ such that*

$$\int_{\mathbb{S}^4} |F_A|^2 \, d\mathscr{H}^4 < \varepsilon_G$$

we can find a gauge $g \in W^{1,4}(\mathbb{S}^4, G)$ satisfying $A^g \in W^{1,2}(\mathbb{S}^4)$, $d^*A^g = 0$ and

$$\|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leqslant C_G \|F_A\|_{L^2(\mathbb{S}^4)}.$$

4.B. G-valued map extensions of traces in $W^{1,2}(\partial\mathbb{B}^5, G)$

The aim of the present appendix is to show the following extension result for general compact Lie groups.

Proposition 4.B.1. *Let G be a compact Lie group. We can find a smooth Riemannian manifold M_G such that for every $g \in W^{1,2}(\partial\mathbb{B}^5, G)$ there exists a measurable map*

$$\begin{aligned} \operatorname{Ext}(g) : M_G &\longrightarrow W^{1,\frac{5}{2}}(\mathbb{B}^5, G) \\ p &\longmapsto g_p \end{aligned}$$

such that for vol_{M_G}-a.e. $p \in M_G$ we have

$$g_p = g \quad \text{on } \mathbb{S}^4 = \partial\mathbb{B}^5 \tag{4.B.1}$$

and for $g \equiv g_0 \in G$

$$g_p \equiv g_0 \tag{4.B.2}$$

We have

$$\int_{M_G} \int_{\mathbb{B}^5} |d_x g_p|^{\frac{5}{2}} d\mathcal{L}^5 \, d\operatorname{vol}_{M_G}(p) \leqslant \|dg\|_{L^2(\mathbb{S}^4)}^{\frac{5}{2}} \tag{4.B.3}$$

and

$$\int_{M_G} \left(\int_{\partial\Omega\cap\mathbb{B}^5} |d_x g_p|^2 \, d\mathscr{H}^4 \right) d\operatorname{vol}_{M_G}(p) \leqslant C_G(\Omega) \; \|dg\|_{L^2(\mathbb{S}^4)}^2 \tag{4.B.4}$$

for every C^2-domain $\Omega \subset \mathbb{R}^5$ of $\mathbb{R}^5$.
Moreover, for every C^2-domain $\Omega \subset \mathbb{R}^5$ of $\mathbb{R}^5$, Ext *is continuous from $H^{\frac{1}{2}}(\mathbb{S}^4, G)$ into $L^2(\partial\Omega \times M_G)$. More precisely, for every pair of maps $(g^1, g^2) \in H^{\frac{1}{2}}(\mathbb{S}^4, G) \times H^{\frac{1}{2}}(\mathbb{S}^4, G)$ we have*

$$\int_{M_G} \left(\int_{\partial\Omega\cap\mathbb{B}^5} |g_p^1 - g_p^2|^2 \, d\mathscr{H}^4 \right) d\operatorname{vol}_{M_G}(p) \leqslant C_G(\Omega) \; \|g^1 - g^2\|_{H^{\frac{1}{2}}(\mathbb{S}^4)}^2, \tag{4.B.5}$$

where $C_G(\Omega)$ only depend on G and the C^2-norm of the domain Ω. In particular for any $g \in H^{\frac{1}{2}}(\mathbb{S}^4, G)$ there holds

$$\int_{M_G} \left(\int_{\partial\Omega\cap\mathbb{B}^5} |g_p - \mathrm{id}_G|^2 \, d\mathscr{H}^4 \right) d\,\mathrm{vol}_{M_G}(p) \leqslant C_G(\Omega) \|g - \mathrm{id}_G\|^2_{H^{\frac{1}{2}}(\mathbb{S}^4)}, \tag{4.B.6}$$

where $C_G(\Omega)$ only depend on G and the C^2-norm of the domain Ω.

Remark 4.B.1. Notice that, for every compact Lie group, we can write G is

$$G = \bigsqcup_{i=1}^{n} G_i \tag{4.B.7}$$

where the G_i are connected, compact Lie groups themselves and the one above is a disjoint union. Hence, up to working on each connected component G_i of G separately, without losing generality we can assume that G is compact and connected in the statement of Proposition 4.B.1.

Lemma 4.B.1. *Let G_1 and G_2 be Lie groups. Assume that we have proved Proposition 4.B.1 for G_1 and G_2. Then, Proposition 4.B.1 holds also for $G_1 \times G_2$ with $M_{G_1\times G_2} := M_{G_1} \times M_{G_2}$.*

Proof. Let $g = (g^1, g^2) \in W^{1,2}(\mathbb{S}^4, G_1 \times G_2)$. Applying Proposition 4.B.1 to g^1 and g^2 separately we find the two maps

$$\begin{aligned} \mathrm{Ext}(g^1) : \ M_{G_1} &\longrightarrow W^{1,\frac{5}{2}}(\mathbb{B}^5, G_1) \\ p_1 &\longmapsto g^1_{p_1} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ext}(g^2) : \ M_{G_2} &\longrightarrow W^{1,\frac{5}{2}}(\mathbb{B}^5, G_2) \\ p_2 &\longmapsto g^2_{p_2} \end{aligned}$$

and we define

$$\begin{aligned} \mathrm{Ext}(g) : \ M_{G_1\times G_2} := M_{G_1} \times M_{G_2} &\longrightarrow W^{1,\frac{5}{2}}(\mathbb{B}^5, G_1 \times G_2) \\ p := (p_1, p_2) &\longmapsto g_p := (g^1_{p_1}, g^2_{p_2}). \end{aligned}$$

Straightforward computations show that $\mathrm{Ext}(g)$ satisfies (4.B.1), (4.B.4) and (4.B.6). □

Lemma 4.B.2. *Let G, H be Lie groups and let $\pi : G \to H$ be a smooth covering map. Assume that we have proved Proposition 4.B.1 for G. Then, Proposition 4.B.1 holds also for H with $M_H := M_G$.*

Proof. Let $g \in W^{1,2}(\mathbb{S}^4, H)$. By [15, Theorem 1], we can find a lift $\tilde{g} \in W^{1,2}(\mathbb{S}^4, G)$ such that $\pi \circ \tilde{g} = g$. Now we use Proposition 4.B.1 to find the map

$$\begin{aligned} \mathrm{Ext}(\tilde{g}) : \ M_G &\longrightarrow W^{1,2}(\mathbb{B}^5, G) \\ p &\longmapsto \tilde{g}_p \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Ext}(g) : \ M_G &\longrightarrow W^{1,2}(\mathbb{B}^5, H) \\ p &\longmapsto g_p := \pi \circ \tilde{g}_p. \end{aligned}$$

Straightforward computations show that $\mathrm{Ext}(g)$ satisfies (4.B.1), (4.B.4) and (4.B.6). □

We recall the following structure theorem for compact Lie groups, whose proof can be found in [82, §5.2.2 and §5.2.4].

Theorem 4.B.1. *Let G be a compact and connected Lie group. Then, G is diffeomorphic $(G' \times \mathbb{T}^n)/F$, where:*

- *G' is a simply connected compact Lie group.*
- *$\mathbb{T}^n = U(1) \times ... \times U(1)$ is an n-dimensional torus, called the abelian component of G.*
- *F is a finite abelian normal subgroup of $G' \times \mathbb{T}^n$.*

Recall that if H is a discrete normal subgroup of G, then the projection map $\pi : G \to G/H$ is a covering map. Hence, by Lemma 4.B.1 and Lemma 4.B.2, Theorem 4.B.1 implies that if we can show Proposition 4.B.1 for any simply connected compact Lie group and for the group $U(1)$, then we have proved it for every compact and connected Lie group.

We start by facing the case $G = U(1)$.

Proposition 4.B.2. *Proposition 4.B.1 holds for $G = U(1)$.*

Proof. Let $g \in W^{1,2}(\mathbb{S}^4, U(1))$. For every $u = u_1 + iu_2 \in W^{1,2}(\mathbb{S}^4, \mathbb{C})$, by standard approximation results we get the identity

$$\begin{cases} d(\bar{u}du) = d\bar{u} \wedge du = 2i(du_1 \wedge du_2) & \text{distributionally on } \mathbb{S}^4 \\ d|u|^2 = 2u_1 du_1 + 2u_2 du_2 & \text{in } L^2(\mathbb{S}^4). \end{cases} \tag{4.B.8}$$

Notice that (4.B.8) immediately implies that $\bar{u}du$ is always a purely imaginary 1-form on $\mathbb{S}^4$. Since $g = g_1 + ig_2 \in W^{1,2}(\mathbb{S}^4, U(1)) \subset W^{1,2}(\mathbb{S}^4, \mathbb{C})$, by using (4.B.8) and the constraint $|g|^2 \equiv 1$, we get

$$0 = g_1 dg_1 + g_2 dg_2 = 2(dg_1 \wedge dg_2) = d(\bar{g}dg) \quad \text{distributionally on } \mathbb{S}^4.$$

Hence, $\bar{g}dg$ is a purely imaginary closed 1-form on $\mathbb{S}^4$. Since $H^1_{dR}(\mathbb{S}^4) = 0$, standard L^2-Hodge decomposition on $\mathbb{S}^4$ (see for instance) gives the existence of $\varphi \in W^{1,2}(\mathbb{S}^4, \mathbb{R})$ such that

$$id\varphi = \bar{g}dg.$$

Vol'pert chain rule in $W^{1,2}(\mathbb{S}^4, \mathbb{R})$ gives

$$d(e^{-i\varphi}) = -ie^{-i\varphi}d\varphi = -e^{-i\varphi}\bar{g}dg \in L^2(\mathbb{S}^4, U(1)). \tag{4.B.9}$$

By Leibniz rule in the algebra $(L^\infty \cap W^{1,2})(\mathbb{S}^4, \mathbb{C})$ and by (4.B.9) we get

$$d(e^{-i\varphi}g) = d(e^{-i\varphi})g + e^{-i\varphi}dg = -e^{-i\varphi}\bar{g}dgg + e^{-i\varphi}dg = -e^{-i\varphi}|g|^2dg + e^{-i\varphi}dg = 0,$$

where in the last equality we have used that $|g|^2 \equiv 1$. Hence, there exists a constant $g_0 \in U(1)$ such that

$$g = g_0 e^{i\varphi} \quad \text{a.e. on } \mathbb{S}^4.$$

Now let $\tilde{\varphi} \in W^{\frac{3}{2},2}(\mathbb{B}^5, \mathbb{C})$. be the standard harmonic extension of φ, i.e. the solution of the following PDE

$$\begin{cases} \Delta\tilde{\varphi} = 0 & \text{in } \mathbb{B}^5 \\ \tilde{\varphi} = \varphi & \text{on } \mathbb{S}^4. \end{cases} \tag{4.B.10}$$

Let $M_{U(1)} = \{p\}$ and $g_p := g_0 e^{i\tilde{\varphi}}$. By construction, (4.B.1) and (4.B.3) are satisfied. Moreover, by standard trace theory for Sobolev functions $W^{\frac{3}{2},2}(\mathbb{B}^5, \mathbb{C})$, the estimates (4.B.4), (4.B.6) and hence the statement follow immediately. □

We now tackle the case of a general simply-connected compact Lie group G. Recall the following lemma, whose prove can be found in [44, Lemma 6.1].

Lemma 4.B.3. *Let $N \subset \mathbb{R}^k$ be an n-dimensional smooth submanifold of $\mathbb{R}^k$ such that*

$$\pi_0(N) = \pi_1(N) = \pi_2(N) = 0.$$

Then, there exists a compact $(k-4)$-dimensional Lipschitz polyhedron $X \subset \mathbb{R}^k$ such that a locally Lipschitz retraction $Q : \mathbb{R}^k \setminus X \to N$ such that

$$\int_{B_r(0)} |dQ|^p \, d\mathcal{L}^k < +\infty, \tag{4.B.11}$$

for every $p \in (1, 4)$ and $r \in (0, +\infty)$. Moreover, the projection map Q is smooth of constant rank n near the manifold $G \subset \mathbb{R}^k$ and coincides with the identity on N.

Lastly, we use Lemma 4.B.3 to prove the following.

Proposition 4.B.3. *Proposition 4.B.1 holds if G is a simply connected, compact Lie group.*

Proof. Notice that $\pi_2(G) = 0$ for every compact Lie group and moreover, by Nash embedding theorem, there exists $k \in \mathbb{N}$ such that G is isometrically embedded in $\mathbb{R}^k$. Hence, without losing generality, we can assume that $G \subset \mathbb{R}^k$ is a smooth submanifold of $\mathbb{R}^k$ satisfying the assumptions of Lemma 4.B.3. We denote by $Q : \mathbb{R}^k \setminus X \to G$ the projection map given by applying Lemma 4.B.3 to G. Recall that both G and X are compact subsets of $\mathbb{R}^k$ and let $B \subset \mathbb{R}^k$ be any open ball in $\mathbb{R}^k$ containing $G \cup X$. Exactly as in the proof of [44, Theorem 6.2], for a small positive number $\sigma \in (0, \operatorname{dist}(G, \partial B))$ chosen so that Q is smooth on $G + B^k_\sigma(0)$ and any arbitrary point $p \in B^k_\sigma(0) \subset \mathbb{R}^k$ we define $X_p := X + p = \{y + a : y \in X\}$ and the translated projection $Q_p : \mathbb{R}^k \setminus X_a \to G$ given by

$$Q_p(y) := Q(y - p) \qquad \forall\, y \in \mathbb{R}^k \setminus X_p. \tag{4.B.12}$$

For such a sufficiently small σ,

$$\Lambda := \sup_{p \in B^k_\sigma(0)} \operatorname{Lip}(Q_p|_G)^{-1}$$

is, by the inverse function theorem, a finite number depending only on G.

Fix any $g \in W^{1,2}(\mathbb{S}^4, G)$ and let $\tilde{g} \in W^{\frac{3}{2},2}(\mathbb{B}^5, \mathbb{R}^k)$ be the solution of the following PDE:

$$\begin{cases} \Delta \tilde{g} = 0 & \text{in } \mathbb{B}^5, \\ \tilde{g} = g & \text{on } \mathbb{S}^4. \end{cases}$$

Notice that, by standard elliptic regularity, $\tilde{g} \in C^\infty(\mathbb{B}^5, \mathbb{R}^k)$ and it satisfies the estimate

$$\|\tilde{g} - g_0\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C\|g - g_0\|_{W^{1,2}(\mathbb{S}^4)}, \tag{4.B.13}$$

where $g_0 \in G$ is any constant in G. Define

$$\bar{g} := \fint_{\mathbb{S}^4} g \, d\mathscr{H}^4$$

and notice that, for every C^2-domain $\Omega \subset \mathbb{R}^5$ the trace theorem for C^2-domains (see [1, Chapter 1, Section 5.1]) (4.B.13) with $g_0 = \bar{g}$ and Poincaré–Wirtinger inequality give the estimate

$$\int_{\partial\Omega \cap \mathbb{B}^5} |\iota_{\partial\Omega}^* d\tilde{g}|^2 \, d\mathscr{H}^4 = \int_{\partial\Omega \cap \mathbb{B}^5} |\iota_{\partial\Omega}^* d(\tilde{g} - \bar{g})|^2 \, d\mathscr{H}^4 \tag{4.B.14}$$

$$\leqslant C\|\tilde{g} - \bar{g}\|^2_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C\|g - \bar{g}\|^2_{W^{1,2}(\mathbb{S}^4)} \leqslant C(\Omega)\|dg\|^2_{L^2(\mathbb{S}^4)}, \tag{4.B.15}$$

for a constant $C(\Omega) > 0$ depending only on the C^2-norm of the domain Ω. We have also obviously using classical Sobolev embeddings

$$\|d\tilde{g}\|_{L^{\frac{5}{2}}(\mathbb{B}^5)} \leqslant C \|\tilde{g} - \bar{g}\|_{W^{\frac{3}{2},2}(\mathbb{B}^5)} \leqslant C \|dg\|_{L^2(\mathbb{S}^4)}. \tag{4.B.16}$$

For every $p \in B^k_\sigma(0)$, we define

$$g_p := Q_p \circ \tilde{g} : \mathbb{B}^5 \to G$$

and we notice that, by the area formula and chain rules in Sobolev Spaces $g_p \in W^{1,\frac{5}{2}}(\mathbb{B}^5, G)$ for $\mathcal{L}^k$-a.e. $p \in B^k_\sigma(0)$ moreover there holds $g_p|_{\mathbb{S}^4} = g$ for for $\mathcal{L}^k$-a.e. $p \in B^k_\sigma(0)$. Using the fact that by the maximum principle $\|\tilde{g}\|_\infty \leqslant \|g\|_\infty \leqslant \operatorname{diam}(G)$, we have

$$\begin{aligned}
\int_{p \in B^k_\sigma(0)} \int_{\mathbb{B}^5} |dg_p|^{\frac{5}{2}} \, d\mathcal{L}^5 \, d\mathcal{L}^k(p) &\leqslant \int_{\mathbb{B}^5} |d\tilde{g}(x)|^{\frac{5}{2}} \int_{B^k_\sigma(0)} |dQ(\tilde{g}(x) - p)|^{\frac{5}{2}} \, d\mathcal{L}^k(p) \, d\mathcal{L}^5(x) \\
&\leqslant \int_{\mathbb{B}^5} |d\tilde{g}(x)|^{\frac{5}{2}} \int_{B^k_{\sigma + \|g\|_\infty}(0)} |dQ(y)|^{\frac{5}{2}} \, d\mathcal{L}^k(y) \, d\mathcal{L}^5(x) \\
&\leqslant C_G \int_{\mathbb{B}^5} |d\tilde{g}(x)|^{\frac{5}{2}} \, d\mathcal{L}^5(x) \\
&\leqslant C_G \|dg\|^{\frac{5}{2}}_{L^2(\mathbb{S}^4)}.
\end{aligned} \tag{4.B.17}$$

Given now any C^2-domain $\Omega \subset \mathbb{R}^5$, by using Fubini's theorem, the chain rule and (4.B.14) we infer that

$$\begin{aligned}
\int_{B^k_\sigma(0)} \int_{\partial\Omega \cap \mathbb{B}^5} |\iota^*_{\partial\Omega} dg_p|^2 \, d\mathscr{H}^4 \, d\mathcal{L}^k &\leqslant \int_{\partial\Omega \cap \mathbb{B}^5} |\iota^*_{\partial\Omega} d\tilde{g}(x)|^2 \int_{B^k_\sigma(0)} |dQ_p(\tilde{g}(x))|^2 \, d\mathcal{L}^k(p) \, d\mathscr{H}^4(x) \\
&\leqslant \int_{\partial\Omega \cap \mathbb{B}^5} |\iota^*_{\partial\Omega} d\tilde{g}(x)|^2 \int_{B^k_\sigma(0)} |dQ(\tilde{g}(x) - p)|^2 \, d\mathcal{L}^k(p) \, d\mathscr{H}^4(x) \\
&\leqslant \int_{\partial\Omega \cap \mathbb{B}^5} |\iota^*_{\partial\Omega} d\tilde{g}(x)|^2 \left(\int_B |dQ(y)|^2 \, d\mathcal{L}^k(y) \right) d\mathscr{H}^4(x) \\
&\leqslant C \int_{\partial\Omega \cap \mathbb{B}^5} |\iota^*_{\partial\Omega} d\tilde{g}(x)|^2 \, d\mathscr{H}^4 \leqslant C(\Omega) \int_{\mathbb{S}^4} |dg|^2 \, d\mathscr{H}^4,
\end{aligned} \tag{4.B.18}$$

and we have shown (4.B.4). It remains to prove the continuity of Ext from $H^{\frac{5}{2}}(\mathbb{S}^4, G)$ into $L^2(\partial\Omega \times B^k_\sigma(0))$. Let g^1 and g^2 in $W^{1,2}(\mathbb{S}^4, G)$, we first have

$$\|\tilde{g}^1 - \tilde{g}^2\|_{W^{1,2}(\mathbb{B}^5)} \leqslant C\, \|g^1 - g^2\|_{H^{\frac{1}{2}}(\mathbb{S}^4)} \tag{4.B.19}$$

We write using again Fubini's theorem

$$\begin{aligned}&\int_{B^k_\sigma(0)} \left(\int_{\partial\Omega\cap\mathbb{B}^5} |g^1_p - g^2_p|^2 \, d\mathscr{H}^4 \right) d\mathcal{L}^k(p)\\ &\qquad \leqslant \int_{\partial\Omega\cap\mathbb{B}^5} \int_{B^k_\sigma(0)} |Q(\tilde{g}^1(x) - p) - Q(\tilde{g}^2(x) - p)|^2 \, d\mathcal{L}^k(p)\, d\mathscr{H}^4(x)\end{aligned} \tag{4.B.20}$$

Recall the Lusin–Lipschitz inequality

$$\begin{aligned}&|Q(\tilde{g}^1(x) - p) - Q(\tilde{g}^2(x) - p)|^2\\ &\qquad \leqslant C\, |\tilde{g}^1(x) - \tilde{g}^2(x)|^2 \, \left[M(|dQ|)^2(\tilde{g}^1(x) - p) + M(|dQ|)^2(\tilde{g}^2(x) - p) \right]\end{aligned} \tag{4.B.21}$$

where $M(|dQ|)$ denotes the Hardy–Littlewood maximal function of dQ. Combining (4.B.20), (4.B.21) and the trace theorem we get

$$\begin{aligned}&\int_{B^k_\sigma(0)} \left(\int_{\partial\Omega\cap\mathbb{B}^5} |g^1_p - g^2_p|^2 \, d\mathscr{H}^4 \right) d\mathcal{L}^k(p)\\ &\qquad \leqslant C \int_{\partial\Omega\cap\mathbb{B}^5} |\tilde{g}^1(x) - \tilde{g}^2(x)|^2 \;\, d\mathscr{H}^4(x)\; \|dQ\|^2_{L^2(B^k_\sigma(0))}\\ &\qquad \leqslant C\; \|\tilde{g}^1 - \tilde{g}^2\|^2_{W^{1,2}(\mathbb{B}^5)} \leqslant C \|g^1 - g^2\|^2_{H^{\frac{1}{2}}(\mathbb{S}^4)}.\end{aligned} \tag{4.B.22}$$

Hence, the statement follows by letting $M_G := B^k_\sigma(0)$. □

General notation

$\mathbb{B}^n$	open, unit n-dimensional ball in $\mathbb{R}^n$
B_ρ	open, n-dimensional ball in $\mathbb{R}^n$ of radius ρ centered at the origin
$B_\rho(x)$	open, n-dimensional ball in $\mathbb{R}^n$ of radius ρ centered at x
$Q_\rho(x)$	open, n-dimensional cube in $\mathbb{R}^n$ of edge-length ρ centered at x
$\mathbb{S}^n$	open, unit n-dimensional sphere in $\mathbb{R}^{n+1}$
$\Omega^k(M)$	smooth differential k-forms on the manifold M
$\mathcal{D}^k(M)$	smooth and compactly supported differential k-forms on the manifold M
$\|\omega\|_*$	comass of the differential form ω
$\mathcal{D}_k(M)$	general k-currents on the manifold M
$\mathbb{M}(T)$	mass of the current T
∂T	boundary of the current T
$\mathcal{M}_k(M)$	k-currents with finite mass on the differentiable manifold M
$\mathcal{N}_k(M)$	normal k-currents on the differentiable manifold M
$\mathcal{R}_k(M)$	integer-multiplicity rectifiable k-current on the differentiable manifold M
ι_M	standard inclusion map $\iota_M : M \to \mathbb{R}^n$, for any $M \subset \mathbb{R}^n$.
$\| \cdot \|_\infty$	L^∞-norm on $\mathbb{R}^n$.
$L^p(M)$	real-valued, L^p-integrable functions on the Riemannian manifold M.
$L^p_{loc}(M)$	real-valued, locally L^p-integrable functions on the Riemannian manifold M.
	real-valued, $W^{1,p}$-Sobolev functions on the Riemannian manifold M.
$W^{1,p}(M)$	real-valued, locally $W^{1,p}$-Sobolev functions on the Riemannian manifold M.
$W^{1,p}_{loc}(M)$	real-valued, compactly supported smooth functions on the manifold M.
	distributions on the manifold M.
$C^\infty_c(M)$	current of integration on some oriented k-submanifold $\Sigma \subset M$ of the mani-
$\mathcal{D}'(M)$	fold M.
$[\Sigma]$	

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Riccardo Caniato

Curriculum Vitae

ETH Zürich
Department of Mathematics
Rämistrasse 101, 8092 Zürich (CH)
✉ *riccardo.caniato@math.ethz.ch*
people.math.ethz.ch/~rcaniato

Personal information

Date of birth	09/11/1995
Citizenship	Italian citizen
Languages	Italian (mother tongue), English (proficient)
Civil status	Celibate

Education

Sep. 2019 - present: **Ph.D. student** at *ETH Zürich.*
Advisor: Prof. Tristan Rivière.

Jul. 2019: **Master degree** in *Matematica* at *Università degli studi di Trieste* and at *Scuola Internazionale di Studi Superiori Avanzati* (SISSA).
Title of the thesis: *About optimal transport in a Lorentzian setting and its applications to General Relativity.*
Advisor: Prof. Nicola Gigli.
Grade: 110/110 cum laude.

Jul. 2017: **Bachelor degree** in *Matematica per l'ingegneria* at *Politecnico di Torino.*
Title of the thesis: *Operatori illimitati e Teorema di Stone (Unbounded operators and Stone's theorem).*
Advisor: Prof. Maria Vallarino.
Grade: 110/110 cum laude.

Fellowships & Awards

Feb. 2020: Winner of the prize *con.Scienze*, awarded by *Conferenza Nazionale dei Presidenti e dei Direttori delle Strutture Universitarie di Scienze e Tecnologie.*

Sep. 2020 - Aug. 2023: Fellowship by *Swiss Nation Foundation* (SNF).
Project title: *Variational Analysis in Geometry.*
Project number: `200020_192062`.

Sep. 2014 - Jul. 2017: Full fellowship provided by the scholarship programme *Percorso per giovani talenti*, a joint project funded by *Politecnico di Torino* and *Fondazione Cassa di Risparmio di Torino* (CRT).

Teaching experiences

Autumn Semester 2022: Course Assistant for the class of *Functional Analysis I* at *ETH Zürich.*
Lecturer: Prof. Peter Hintz.

Spring Semester 2022: Course Assistant for the class of *Analysis II* at *ETH Zürich.*
Lecturer: Prof. Tristan Rivère.

Spring Semester 2021: Course Assistant for the class of *Functional Analysis II* at *ETH Zürich.*
Lecturer: Prof. Alessandro Carlotto.

Autumn Semester 2021	Teaching Assistant for the class of *Functional Analysis I* at *ETH Zürich*. Lecturer: Prof. Alessandro Carlotto.
Autumn Semester 2020	Teaching Assistant for the class of *Functional Analysis I* at *ETH Zürich*. Lecturer: Prof. em. Michael Struwe.

Research interests

I'm mainly interested in geometric analysis, calculus of variations and analysis of elliptic partial differential equations (PDEs).
My research is currently about the regularity of currents and varifolds, both on Euclidean spaces and on manifolds, with a particular focus on the almost complex framework and on geometric variational problems arising in Physics (e.g. Yang-Mills theories).

Publications and preprints

My papers can be downloaded from my personal webpage
`https://people.math.ethz.ch/~rcaniato`
or from ArXiv.org
`https://arxiv.org/search/?query=caniato+riccardo&searchtype=all`

Publications

The unique tangent cone property for weakly holomorphic maps into projective algebraic varieties, in collaboration with Tristan Rivière
`https://arxiv.org/abs/2108.10371`
Accepted for publication on ***Duke Mathematical Journal***

The strong L^p-closure of vector fields with finitely many integer singularities on B^3
`https://arxiv.org/abs/2009.01050`
Journal of Functional Analysis, *281 (2020), no. 6, paper no. 109095*

Preprints

Weak and strong L^p-limits of vector fields with finitely many integer singularities in dimension n, in collaboration with Filippo Gaia
`http://arxiv.org/abs/2210.04730`

An ε-regularity result for weak Yang-Mills fields in supercritical dimension
In preparation

Invited talks and conferences attended

Given talks

Apr. 2023	Analysis and Geometric Analysis Seminar at Cornell University, Ithaca (NY), United States
Jun. 2022	Workshop/Summer School Geometric analysis and calibrated geometries at ETH Zürich, Zürich
Oct. 2021	Analysis Seminar at ETH Zürich, Zürich

Schools and conferences attended

Jan. 2022	Variational Aspect of Minimal Surfaces at Institut Henri Poincaré and University of Paris, Paris
Dec. 2021	Geometric PDEs in Freiburg at University of Freiburg, Freiburg

References

Prof. Maria Vallarino, *Politecnico di Torino*.
Contact: maria.vallarino@polito.it.

Prof. Nicola Gigli, *SISSA*.
Contact: ngigli@sissa.it.

Prof. Guido De Philippis, *Courant Institute of Mathematical Sciences, NYU*
Contact: guido@cims.nyu.edu.

Prof. Tristan Rivière, *ETH Zürich*
Contact: tristan.riviere@math.ethz.ch.

Prof. Costante Bellettini, *University College London*
Contact: c.bellettini@ucl.ac.uk.

Prof. Gang Tian, *Peking University and Princeton University*
Contact: tian@math.princeton.edu.