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Weak holomorphic structures over Kähler manifolds and strong approximation results of curvatures under constraints

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Abstract

The main goal of this thesis is to analyse weak holomorphic structures over closed Kähler manifolds and to positively several answer strong approximation questions. For X^d closed Kähler manifolds with associated Hermitian bundles we investigate weak structures in $W^{1,d}$-critical regularity and establish a one to one correspondence between unitary connections satisfying $F_\nabla^{0,2} = 0$ and the existence of holomorphic structures. In the smooth framework this is the celebrated Newlander-Nirenberg theorem. In an article together with my advisor Tristan Rivière, we have proven the analogue for weak structures over closed Kähler surfaces [25]. In order to answer this question we deeply investigate results in the field of complex analysis of several variables.

Moreover, the calculus of variations yields the question of smooth approximations of such structures. We can establish strong approximations of weak holomorphic structures with smooth holomorphic ones.

This thesis goes beyond the case of Kähler surfaces and analyses the case of closed Kähler manifolds in general dimensions. We positively answer the questions of existence of weak holomorphic structures and strong approximation under critical regularity regime.

Lastly, our techniques naturally lead us to research the case of flat connections under non-compactness of the structure group. The question of strong approximation of such connections is known in the literature assuming compactness of the structure group. However, the difficulty of the noncompactness of the structure group is non-trivial. In collaboration with my advisor and Mircea Petrache we have developed a general framework that enables us to answer strong approximation questions for a large class of weak solutions to PDEs. The case of flat connections is one such application.

Zusammenfassung

Das Hauptziel dieser Arbeit ist es, schwache holomorphe Strukturen über geschlossenen Kähler-Mannigfaltigkeiten zu analysieren und mehrere Fragen zur starken Approximation positiv zu beantworten. Für geschlossene Kähler-Mannigfaltigkeiten X^d mit assoziiertem hermitischen Bündel untersuchen wir schwache Strukturen mit $W^{1,d}$-kritischer Regularität und stellen eine Eins-zu-Eins-Korrespondenz zwischen unitären Zusammenhängen, die $F_\nabla^{0,2} = 0$ erfüllen, und der Existenz holomorpher Strukturen her. Im glatten Rahmen ist dies das berühmte Newlander-Nirenberg-Theorem. In einem Artikel zusammen mit meinem Betreuer Tristan Rivière haben wir das Analogon für schwache Strukturen über geschlossenen Kähler-Oberflächen [25] bewiesen. Um diese Frage zu beantworten, haben wir im Detail Resultate der komplexen Analysis mehrerer Variablen untersucht.

Darüber hinaus wirft die Variationsrechnung die Frage nach glatten Approximationen solcher Strukturen auf. Wir können starke Approximationen schwacher holomorpher Strukturen mit glatten holomorphen Strukturen herleiten.

Diese Arbeit geht über den Fall von Kähler-Oberflächen hinaus und analysiert den Fall von geschlossenen Kähler-Mannigfaltigkeiten in allgemeinen Dimensionen. Wir beantworten die Fragen der Existenz schwacher holomorpher Strukturen und einer starken Approximation unter Annahme kritischer Regularität positiv.

Schliesslich führen uns unsere Techniken natürlicherweise dazu, den Fall von flachen Zusammenhängen bei fehlender Kompaktheit der Strukturgruppe zu untersuchen. Die Frage der starken Approximation solcher Zusammenhänge ist in der Literatur unter der Annahme der Kompaktheit der Strukturgruppe bekannt. Die Schwierigkeit bei fehlender Kompaktheit der Strukturgruppe ist jedoch nicht trivial. In Zusammenarbeit mit meinem Betreuer und Mircea Petrache haben wir einen allgemeinen Rahmen entwickelt, der es uns ermö-

glicht, starke Approximationsfragen für eine grosse Klasse schwacher Lösungen von PDEs zu beantworten. Der Fall von flachen Zusammenhängen ist eine solche Anwendung.

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Contents

Chapter 1

Introduction

The calculus of variations of Yang-Mills in 4-dimensions has naturally lead to the definition of Sobolev connections [10]. In the framework of Sobolev connections defined on a smooth vector bundle over a base manifold M, we analyse the case of weak holomorphic structures and the cousin problem of flat connections.

The strong approximation of Sobolev connections by smooth ones has been proven in the case of Riemannian manifolds without any other topological constraints. This is less involved and hence, one of the novelties of this thesis is exploring how the approximation can be achieved by adding certain topological constraints as we will see in the next sections.

We note that there has been a definition of weak connections with L^2 bounded curvature given by T. Rivière in collaboration with M. Petrache in [26] and [27]. This definition was motivated in a search of the closure of Sobolev connections below a Yang-Mills energy level. Roughly speaking a weak connection in real 5-dimensions is defined as being an L^2 1-form into a Lie algebra $\mathfrak{g}$ such that its restriction on a.e. 4-sphere is a Sobolev connection. In higher dimensions weak connections are defined in an iterative way. That is, for $n > 5$, a weak connection in n-dimensions is an L^2 form A into the Lie Algebra such that when restricted to a.e $n-1$ spheres is itself a weak connection. This space has been proved to be weakly sequentially closed under Yang-Mills Energy control. This was one of the main results in [26] and [27].

1.1 Weak holomorphic structures

We consider the above notions in the following complex framework. Let E be a C^∞ complex vector bundle of rank n over a d-dimensional closed Kähler manifold X^d and h_0 be some reference Hermitian inner product in the fibers of E: i.e. (E, h_0) defines a Hermitian vector bundle. We shall sometimes consider E issued from its associated $GL_n(\mathbb{C})$ principal bundle or from its associated unitary principal bundle.

The classical Newlander-Nirenberg theorem [23] states that given an almost-complex structure J over an even dimensional smooth manifold X then the torsion of J (also called the Nijenhuis tensor) vanishes if and only if J defines a complex structure. Let F_∇ be the curvature 2-form associated to a connection ∇ of (E, h_0) over a complex manifold X. We will be interested in the "bundle" version of the Newlander-Nirenberg theorem (see [16, Theorem 2.1.53], [17, Chapter 1, Section 3, Proposition p. 9]). It states that unitary connections satisfying $F_\nabla^{0,2} = 0$ are in one to one correspondance with holomorphic structures:

Let ∇ be a smooth unitary connection of a C^∞ hermitian bundle (E, h_0) over a complex manifold X. Then X has a holomorphic structure if and only if $F_\nabla^{0,2} = 0$.

One of the goals of this thesis is to extend this identification to Sobolev connections. More precisely, we analyse what we call *weak holomorphic structures*, that is Sobolev connections (see the definition 1.1 below) satisfying the integrability condition $F_\nabla^{0,2} = 0$ and study the analogue of the Newlander-Nirenberg theorem in this regime. Since

$$F_\nabla^{0,2} = 0$$

gives us the integrability of the complex manifold (in the sense that it gives the existence of a complex structure), then we call such a vanishing condition the *integrability condition*.

In addition, because the problem of weak closures naturally appears in the calculus of variations, we will look at the strong approximation of such weak holomorphic structures by smooth ones.

We are interested in the space of Sobolev $W^{k,p}$ connections of E which are defined as follows:

Definition 1.1. *Let ∇_0 be a smooth connection of E, we denote*

$$\mathcal{S}^{k,p}(E,h_0) := \left\{\nabla := \nabla_0 + \eta \quad \textit{where} \quad \eta \in W^{k,p}(\Omega^1(ad_{h_0}(E)))\right\}$$

where $W^{k,p}(\Omega^1(ad_{h_0}(E)))$ is the space of Sobolev $W^{k,p}$ 1-form sections into the sub-bundle of the endomorphism bundle $End(E)$ made of the unitary endomorphisms for the reference metric h_0.

Then $\mathcal{S}^{k,p}(E,h_0)$ is called the space of Sobolev unitary $W^{k,p}$-connections of (E,h_0).

We will heavily use gauge theory in order to obtain our results. To this extent we find it useful to recall to the reader the notion of a gauge transformation of a connection $\nabla \in \mathcal{S}^{k,p}(E,h_0)$. Let g be a section of the Hermitian vector bundle E, then the gauge transformation of $\nabla = \nabla_0 + \eta$ by g is defined as $\nabla^g = \nabla_0 + \eta^g$, where

$$\eta^g := g^{-1}dg + g^{-1}\eta g.$$

Let $\{U_i\}_i$ be a cover of a Kähler manifold X. Then such a connection $\nabla \in \mathcal{S}^{k,p}(E,h_0)$ can be represented in each U_i as

$$\nabla \approx d + A_i,$$

where d is the exterior derivative and $A_i \in W^{k,p}(\Omega^1 U_i \otimes \mathfrak{u}(n))$. Hence, for a local gauge transformation $g : U_i \to U(n)$ on ∇ we have:

$$A_i^g = g^{-1}dg + g^{-1}A_i g \qquad \text{in } U_i.$$

In this framework we study the convergence of Sobolev unitary connections, and their respective Sobolev structures in the case of closed Kähler manifolds X^d. In particular, we consider unitary connections belonging to the space $\mathcal{S}^{1,d}(E,h_0)$ over X^d.

Convergence of Sobolev connections is a subtle issue that deserves to be detailed before going forward. It is customary to define the distance of two Sobolev connections by fixing a system of charts and trivialisations of the connections and measuring the Sobolev distance in each trivialisation. In the calculus of variations of Yang-Mills, however, it is of interest to study the compactness of the space of connections with bounded L^p curvature (see for example [30], [16] and [40]). In this context, the convergence of Sobolev connections is always taken modulo gauge transformations due to the non-coerciveness of the Yang-Mills functional. To this extent, we define the gauge invariant distance between two given connections.

Definition 1.2. *Let X^d be a d-dimensional closed Kähler manifold and $p \geq 1$ with its associated Kähler 2-form ω. We define the gauge invariant distance as the functional*

$$\mathrm{dist}_p : \mathcal{S}^{1,p} \times \mathcal{S}^{1,p} \to \mathbb{R}_{\geq 0}$$

given by:

$$\mathrm{dist}_p(\nabla_1, \nabla_2) := \inf_{\sigma \in \mathcal{G}^{1,d}(GL(n,\mathbb{C}))} \int_{X^d} |\nabla_1 - \nabla_2^\sigma|^p \, \omega^d + \int_{X^d} |F_{\nabla_1} - F_{\nabla_2{}^\sigma}|^p \, \omega^d$$

where $\mathcal{G}^{1,d}(GL_n(\mathbb{C}))$ is the space of $W^{1,d}$ gauge transformations on E for the group $GL_n(\mathbb{C})$.

We point out to the reader that the problem of existence of weak holomorphic structures has been studied in the case of one dimensional complex spaces. Under the assumption that ∇ has $L^{2,1}$ regularity ($L^{2,1}$ is the Lorentz space $2,1$) for one complex dimensional spaces, F. Helein gives a solution in [13, Lemma 4.1.7]. The case of $L^{2,1}$ regularity is, however, subcritical. The critical case of L^2 connections has been proven by B. Sharp in [34]. Analysing higher-dimensional cases leads to non-trivial difficulties and through this work we are building a few mathematical frameworks that allow us to tackle this problem in the case of critical regularity.

Thus, we positively answer the analogue of the Newlander-Nirenberg theorem for $W^{1,d}$ unitary connections and the question of strong convergence. More precisely, the first main result I obtained in collaboration with my advisor Tristan Rivière is the following:

Theorem 1.1. *Let ∇ be a unitary $W^{1,2}$ connection of an hermitian bundle (E, h_0) over a closed Kähler surface X^2. Assume ∇ satisfies the integrability condition*

$$F_\nabla^{0,2} = 0 \tag{1.1}$$

then there exists a <u>smooth</u> *holomorphic structure $\mathcal{E}$ on E and a $\bigcap_{q<2} W^{2,q}$ section h of the bundle of positive Hermitian endomorphisms of E such that*

$$\nabla = \partial_0 + h^{-1}\partial_0 h + \overline{\partial}_{\mathcal{E}} \tag{1.2}$$

where $\overline{\partial}_{\mathcal{E}}$ is the $\overline{\partial}-$operator associated to the holomorphic bundle $\mathcal{E}$ and ∂_0 is the 1-0 part of the Chern connection associated[1] to the holomorphic structure $\mathcal{E}$ and the chosen reference hermitian product h_0.

[1] These connections are not necessarily unitary with respect to h_0 anymore.

Recall that the Dolbeault operator $\overline{\partial}$ induces the Dolbeault cohomology groups $\mathcal{H}^{p,q}_{\overline{\partial}}(X^2)$ (see [12, Chapter 0, p. 25]) which "measure" the failure of $\overline{\partial}$ closed forms to be $\overline{\partial}$ exact. Under the assumption that $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$, the second main result of this thesis obtained in collaboration with my advisor asserts that Sobolev holomorphic structures associated to Sobolev unitary connections are strongly approximable by smooth ones in 2 complex dimensions (the dimension for which the Yang-Mills energy is critical).

Theorem 1.2. *Under the assumptions of Theorem 1.1 and $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$, there exists a sequence of* smooth *connections ∇_k on smooth holomorphic bundles $\mathcal{E}_k$ satisfying*

$$F^{0,2}_{\nabla_k} = 0 \quad ,$$

and converging to ∇ in the sense that for any $p < 2$:

$$\mathrm{dist}_p(\nabla_k, \nabla) = \inf_{\sigma \in \mathcal{G}^{1,2}(GL(n,\mathbb{C}))} \int_{X^2} |\nabla_k - \nabla^\sigma|^p \, \omega^2 + \int_{X^2} |F_{\nabla_k} - F_{\nabla^\sigma}|^p \, \omega^2 \to 0. \tag{1.3}$$

Moreover, there exists a family of isomorphisms $\mathcal{H}_k$ such that

$$\overline{\partial}_{\mathcal{E}_k} = \mathcal{H}_k^{-1} \circ \overline{\partial}_{\mathcal{E}} \circ \mathcal{H}_k.$$

That is, the sequence of connections ∇_k act on equivalent bundles to E. □

Remark 1.1. *We have formulated these theorems by considering closed Kähler manifolds. This consideration has been done for simplicity, since it allows us to use the fact that $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is locally equal to the Hodge-Laplace operator $\Delta_d = dd^* + d^*d$. The reader should take into account the fact that the results are generalisable to closed complex manifolds by carefully dealing with error between the $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and Δ_d operators.*

In higher even dimensions, for the *weak connections* defined in [27] over a complex manifold and satisfying in addition the integrability condition $F^{0,2}_A = 0$ with $W^{1,2}$ regularity on A, we expect theorems 1.1 and 1.2 to extend in the following way: We expect to have necessary singularities and the *smooth holomorphic structures* should be replaced by the more general notion of *coherent sheaves*. The question remains to know how smooth these sheaves can be and if a weak holomorphic structure defines a *reflexive sheaf* or not.

The motivation for addressing these questions takes its roots in a paper of G. Tian [39] in which the closure of the space of smooth Yang-Mills fields has been studied. It leads naturally to the study of Yang-Mills fields on a bundle well defined away from a co-dimension 4 closed rectifiable set in the basis. The attempt in [26] and [27] was to give a suitable notion of such singular bundles together with a singular connection that enjoys a *sequential weak closure property*. The attached singular "bundle" to these singular connections could be thought as a *real version of coherent sheaves*. The goal of mixing the notion of weak connection with the integrability condition $F_\nabla^{0,2} = 0$ is to check whether the corresponding singular bundle coincide with the classical notion of *reflexive sheaves* in the complex framework. The present thesis is bringing a positive answer to this question when the basis is a Kähler surface.

1.1.1 Higher dimensions

We can further extend our results over closed Kähler manifolds while increasing the regularity assumptions under the same integrability condition requirements. For X^d, a d-dimensional closed Kähler manifold, we assume connections of $W^{1,d}$ Sobolev regularity. In order to solve the strong approximation question, in collaboration with my advisor and Mircea Petrache, we have developed a more general framework which studies the invertibility of general Fredholm operators under a certain class of perturbations (see for example Section 4.3.1).

There are a few differences in our approach compared to the case of Kähler surfaces. The main difficulty is that we will not be able to directly obtain the integrability condition $F_\nabla^{0,2} = 0$ for the approximating sequence. We introduce the idea of *extended integrability condition* (4.5) and show that this implies the integrability condition under the assumption that X^d is closed.

Another difference will be the fact that due to the lack of ellipticity of the $\overline{\partial}$ operator, surprisingly we will not be able to prove the equivalent of theorem 1.2 in a low energy regime in d-dimensions on unit balls B^{2d}. This is solely due to the fact that B^{2d} has non-empty boundary. Thus, generalising our result to bounded domains is more delicate and is an open problem.

The results we obtain are the following:

Theorem 1.3. *Let ∇ be a unitary $W^{1,d}$ connection of an hermitian bundle (E, h_0) over a closed Kähler manifold X^d. Assume ∇ satisfies the integrability condition*

$$F_\nabla^{0,2} = 0 \tag{1.4}$$

then there exists a <u>*smooth*</u> *holomorphic structure* $\mathcal{E}$ *on* E *and a* $\bigcap_{q<d} W^{2,q}$ *section* h *of the bundle of positive Hermitian endomorphisms of* E *such that*

$$\nabla = \partial_0 + h^{-1}\partial_0 h + \overline{\partial}_{\mathcal{E}} \tag{1.5}$$

where $\overline{\partial}_{\mathcal{E}}$ *is the* $\overline{\partial}-$*operator associated to the holomorphic bundle* $\mathcal{E}$ *and* ∂_0 *is the* 1-0 *part of the Chern connection associated*[2] *to the holomorphic structure* $\mathcal{E}$ *and the chosen reference hermitian product* h_0.

and:

Theorem 1.4. *Under the assumptions of Theorem 1.3 and* $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$, *there exists a sequence of* <u>*smooth*</u> *connections* ∇_k *on smooth holomorphic bundles* $\mathcal{E}_k$ *satisfying*

$$F^{0,2}_{\nabla_k} = 0 \quad ,$$

and converging to ∇ *in the sense that for all* $p < d$:

$$\mathrm{dist}_p(\nabla_k, \nabla) = \inf_{\sigma \in \mathcal{G}^{1,d}(GL(n,\mathbb{C}))} \int_{X^d} |\nabla_k - \nabla^\sigma|^p \, \omega^d + \int_{X^d} |F_{\nabla_k} - F_{\nabla^\sigma}|^p \, \omega^d \to 0. \tag{1.6}$$

Moreover, there exists a family of isomorphisms $\mathcal{H}_k$ *such that*

$$\overline{\partial}_{\mathcal{E}_k} = \mathcal{H}_k^{-1} \circ \overline{\partial}_{\mathcal{E}} \circ \mathcal{H}_k.$$

That is, the sequence of connections ∇_k *act on equivalent bundles to* E. □

1.2 Flat connections

A problem parallel to obtaining the strong approximation of weak holomorphic structures is that of strongly approximating flat Sobolev connections. As described in Chapter 5, this problem can be solved in several ways (see for example [4]) assuming compactness of the structure Lie groups (such as $O(n), SO(n)$ etc.).

In physics, however, there is a rich range of applications that consider non-compact Lie groups [38]. We can consider the groups $O(p,q)$ which are also called indefinite orthogonal groups. They are non-compact and are structure groups of pseudo-Riemannian manifolds. One of the particular examples is

[2]These connections are not necessarily unitary with respect to h_0 anymore.

the Lorentz group $O(3,1)$ which is found in special relativity and electromagnetism. In particular, the Minkowski pseudo-Riemannian manifold $\mathbb{R}^{3,1}$ with associated structure group $O(3,1)$ has vanishing curvature.

The novelty of this work is to be able to smoothly approximate flat Sobolev connections without assuming the compactness of the structure group of the base manifold. In particular, we will positively answer the problem of approximating flat $W^{1,n/2}$ Sobolev connections in the case of B^n as the base space. Let $\mathfrak{g}$ be the Lie algebra induced by the Lie group G. We further assume that $\mathfrak{g}$ is a subspace of $M_r(\mathbb{R})$, the space of $r \times r$ square matrices. Then the result obtained in collaboration with my advisor and Mircea Petrache is the following:

Theorem 1.5. *A is a $\mathfrak{g}$-valued 1-form in $W^{1,n/2}$ over the ball B^n, and we assume that $F_A = dA + A \wedge A = 0$. Then there exist a sequence of smooth 1-forms A_k with $k \in \mathbb{N}$ such that*

$$A_k \to A \qquad \text{in } W^{1,n/2},$$

furthermore satisfying

$$F_{A_k} = 0 \quad \text{for all } k \in \mathbb{N}.$$

In proving this we used the same framework of inverting Fredholm operators mentioned in section 1.1.1, under appropriate perturbations. This problem brings a similar difficulty as in the case of closed Kähler manifolds. Indeed, we will not be able to achieve the flatness condition $F_{A_k} = 0$ directly, but we will obtain it through what we will define as the extended flatness condition (5.3).

1.3 Structure of the thesis

In order to introduce the reader to our ideas and strategies, we start by analysing the case of abelian bundles. Thus, in Chapter 2 we prove theorems 1.1 and 1.2 in the simplified case of $U(1)$ bundles. This is easier, since the integrability condition locally translates to $F_A^{0,2} = \overline{\partial} A^{0,1} = 0$.

With these ideas in hand, we proceed to Chapter 3 and prove theorems 1.1 and 1.2 in full generality over $U(n)$ bundles. The chapter is structured as follows:

Section 3.1 is devoted to the proof of theorem 1.2 in the case of small $W^{1,2}$ norm. This proof is not going to be used for proving the theorem in its full generality. However, we thought that it could be useful for the reader to expose a different approach in this particular case and the scheme of the proof we are giving in this section is going to be used in later ones.

Under the smallness condition of the $W^{1,2}$ norm of the connection ∇, in Section 3.2 we prove that connections satisfying the integrability condition (1.1) are locally *holomorphically trivialisable*, meaning that in any geodesic ball embedded in our manifold X^2 where we have the local representation $\nabla \simeq d + A$, we show the existence of $g \in GL_n(\mathbb{C})$ such that $A^{0,1} = -\overline{\partial} g \cdot g^{-1}$. Using this result, we prove theorem 1.1 in section 3.3.

Sections 3.4 and 3.5 are dedicated to proving theorem 1.2. The former section proves the strong approximation result, and the latter concludes the statement by proving that the connections ∇_k act on equivalent bundles to E.

Afterwards, in Chapter 4 we prove theorems 1.3 and 1.4. In the first section of the chapter we introduce the idea of *extended integrability condition* (4.5) and describe how we will use it throughout. In section 4.2 we prove the existence of holomorphic trivialisations by adapting the approach to $W^{1,d}$ regularity. This section has a very similar approach to Section 3.2, in the case of closed Kähler surfaces. We note however, that we will already make use of the idea of *extended integrability condition* (4.5) in order to positively answer theorem 1.3.

Section 4.3 discusses density under high energy regime and takes a different approach compared to what we have done for the case of closed Kähler surfaces in Section 3.4. We start by analysing the invertibility of Fredholm operators, and with these results in hand we are able to solve the strong approximation question. We emphasize again that we firstly obtain density under the extended integrability condition and using a topological argument we conclude the integrability condition as required.

We conclude with Chapter 5 which tackles the related problem of strongly approximating flat Sobolev connections and positively answers theorem 1.5.

Appendix A contains a few results on complex several variables where we also introduce some regularity results in particular case of the domain B^4. In Appendix C we prove certain results regarding linear operators and lastly,

Appendix D brings some regularity results and bootstrapping techniques for the classes of PDEs we will encounter.

1.4 Open Problems

We introduce a few open problems we find interesting.

The strong approximation results relies on the cohomological constraint: $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$. It would be interesting to understand this condition further:

Open Problem. *Without assuming the cohomological constraint* $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$*, prove or find a counterexample to theorems 1.2 and 1.4. In particular, one can consider the case where* X^2 *is a K3 surface.*

When dealing with answering the strong approximation question in theorem 1.2, we take a global approach. It would be interesting to analyse the problem from a different perspective:

Open Problem. *Under the assumptions of theorem 1.2, obtain local strongly approximating sequences satisfying the integrability condition and glue them in such a way that we obtain a sequence of smooth connections* ∇_k *strongly approximating* ∇ *in* $W^{1,2}$ *and satisfying the integrability condition* $F^{0,2}_{\nabla_k} = 0$.

The difficulty of this question lies in the lack of ellipticity of the $\overline{\partial}$-operator. Hence, related to the above open problem we suggest another question:

Open Problem. *Let* ∇ *be a unitary* $W^{1,d}$ *connection over a Kähler manifold* X^d *with non-empty boundary satisfying the integrability condition* $F^{0,2}_{\nabla} = 0$. *Assume that in a geodesic ball* B^{2d}_ρ ∇ *is represented by* $d + A$*, where* A *is a* $W^{1,d}$ 1*-form. Prove or disprove whether one can obtain sequences of smooth* 1*-forms* A_k *converging strongly in* $W^{1,d}$ *to* A *and* $F^{0,2}_{A_k} = 0$ *over* B^{2d}.

A natural question arising from smoothly approximating flat Sobolev connections is to consider the setting of two connection 1-forms A and B satisfying $F_A = F_B$. We want to smoothly approximate A and B while keeping the curvature equality condition. Thus, we propose the following open problem:

Open Problem. A, B *are* $\mathfrak{g}$*-valued* 1*-forms in* $W^{1,n/2}$ *over the ball* B^n*, and we assume that* $F_A = F_B$. *Show whether there exist two sequences of smooth*

1-*forms* A_k, B_k *with* $k \in \mathbb{N}$ *such that*

$$A_k \to A \quad \textit{and} \quad B_k \to B \quad \textit{in } L^n,$$

and furthermore satisfying

$$F_{A_k} = F_{B_k} \quad \textit{for all } k \in \mathbb{N}.$$

The difficulty of tackling this problem arises from the fact that when approximating A and B smoothly, we cannot rely on the Bianchi identity, in particular there isn't necessarily a 1-form C such that $d_C(F_B - F_A) = 0$, where

$$d_C \cdot = d \cdot + [C, \cdot].$$

Such an identity is crucial to solving the flat connection case.

1.5 Notations

$M_n(\mathbb{C})$	-	the space of $n \times n$ complex valued square matrices
$\mathfrak{u}(n)$	-	the Lie algebra associated to the unitary group $U(n)$, i.e. matrices satisfying $A = -\overline{A}^T$
$Sym(n)$	-	the space of symmetric matrices $A = \overline{A}^T$. This is shorthand for the space $i\mathfrak{u}(n)$.
$\mathcal{H}^{p,q}_{\overline{\partial}}(X)$	-	Dolbeault cohomology (p,q)-group over the complex manifold X
$\mathcal{G}^{p,q}(GL_n(\mathbb{C}))$	-	the space of $W^{p,q}$ gauge transformations on E for the group $GL_n(\mathbb{C})$
$\Gamma(E)$	-	the space of global smooth sections of the vector bundle E
$\Gamma_{W^{p,q}}(E)$	-	the space of global $W^{p,q}$ sections of the vector bundle E
$\mathcal{A}^{p,q}(X)$	-	the space of global (p,q)-sections defined over the manifold X
$\Omega^{p,q}U \otimes \mathfrak{g}$	-	the space of $\mathfrak{g}$-valued (p,q)-forms on U
$W^{2,p}_D$	-	the space of Sobolev functions $W^{2,p}$ that vanish on the boundary of the domain
$\vartheta\omega$	-	$\vartheta = - * \partial *$, formal adjoint of $\overline{\partial}$ (see [9, p. 83])
$N\omega$	-	N is the inverse operator of $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$
B^{2d}	-	$2d$-dimensional unit open ball
B^{2d}_r	-	$2d$-dimensional open ball of radius $r > 0$
$[A, B]$	-	$[A, B] = A \wedge B + (-1)^{p+q} B \wedge A$, if A is a p form and B a q-form

$\overline{\partial}_A$	-	$\overline{\partial}_A \cdot = \overline{\partial} \cdot + [A^{0,1}, \cdot]$ (see Proposition C.3)
$\overline{\partial}_A^*$	-	$\overline{\partial}_A^* \cdot = \overline{\partial}^* \cdot + * [* \overline{\partial} \cdot, \overline{A^{0,1}}^T]$ (see Proposition C.3)
ϑ_A	-	$\vartheta_A \cdot = \vartheta \cdot + * [* \overline{\partial} \cdot, \overline{A^{0,1}}^T]$ (see Remark C.1)
$(A^{0,1})^g$	-	$g^{-1} \overline{\partial} g + g^{-1} A^{0,1} g$
$\lVert\lvert T \rvert\rVert$	-	the norm of the operator $T : X \to Y$, for X, Y Banach spaces
$\sigma(T)$	-	the spectrum of the operator T
$\rho(T)$	-	the resolvent of the operator T, defined as $\mathbb{C} \setminus \sigma(T)$
$A(x)$	-	for k-forms $A = \sum_{\lvert I \rvert = k} x_I dx_I$ denote $A(x) = \sum_I a_i(x) dx_I$
$\mathrm{dist}_p(\nabla_k, \nabla)$	-	$\inf\limits_{\sigma \in \mathcal{G}^{1,d}(GL(n,\mathbb{C}))} \int_{X^d} \lvert \nabla_k - \nabla^\sigma \rvert^p \, \omega^d + \int_{X^d} \lvert F_{\nabla_k} - F_{\nabla^\sigma} \rvert^p \, \omega^d$

Chapter 2

Abelian $U(1)$ Bundles

In order to obtain a good understanding of the equivalence between the integrability condition $F_\nabla^{0,2} = 0$ and holomorphic local trivialisations in the weak setting we have outlined in the introductory chapter, we first start by investigating the simpler case of $U(1)$ bundles for 2-dimensional closed Kähler manifolds. Even though this does not capture the difficulties of the non-abelian case, it shows the steps we should follow later on and what needs to be changed.

2.1 Existence of holomorphic trivialisations

We consider the setting of theorem 1.1. Since the theorem we want to prove is local, we can solve the problem of the existence of local holomorphic trivialisations by restricting to geodesic balls $B_\rho(x)$ embedded in the given closed Kähler surface X^2. Without loss of generality, we shall work on unit balls B^4.

We observe that for any given connection 1-form $A \in W^{1,2}(\Omega^1 B^4, \mathbb{C})$, there exists a perturbation $\omega \in W^{2,2}(\Omega^{0,2} B^4, \mathbb{C})$ such that we obtain the integrability condition

$$F^{0,2}_{A+\overline{\partial}^*\omega} = 0$$

and there exists $C > 0$ independent of A such that

$$\|\omega\|_{W^{2,2}(B^4)} \leq C \|A\|_{W^{1,2}(B^4)} .$$

Indeed, since we are working with abelian bundles, the integrability condition reduces to solving the linear problem:

$$\overline{\partial}\overline{\partial}^*\omega = -F_A^{0,2} = -\overline{\partial} A^{0,1}.$$

According to the $\overline{\partial}$-Hodge decomposition (A.5), we have

$$A^{0,1} = \Delta_{\overline{\partial}} N A^{0,1} = \overline{\partial}\overline{\partial}^* N A^{0,1} + \overline{\partial}^* \overline{\partial} N A^{0,1},$$

where N is the operator $\Delta_{\overline{\partial}}^{-1}$ mapping $W^{1,2}(\Omega^{0,1}B^4)$ to $W^{2,2}(\Omega^{0,1}B^4)$ such that the $\overline{\partial}$-Neumann boundary conditions are satisfied:

$$\sigma(\vartheta, dr) N A^{0,1} = 0 \quad \text{and} \quad \sigma(\vartheta, dr)\overline{\partial} N A^{0,1} = 0,$$

as defined in Appendix A. We have $\overline{\partial} A^{0,1} = \overline{\partial}\overline{\partial}^* \overline{\partial} N A^{0,1}$. Define $\omega = -\overline{\partial} N A^{0,1}$. Then by definition it follows

$$\overline{\partial}\overline{\partial}^* \omega = -\overline{\partial} A^{0,1}, \tag{2.1}$$

and hence the integrability condition is satisfied:

$$F^{0,2}_{A+\overline{\partial}^*\omega} = \overline{\partial}(A^{0,1} + \overline{\partial}^* \omega) = 0.$$

We still have to show that ω has $W^{2,2}$ regularity. A-priori, we only have by the sub-elliptic estimates of $\overline{\partial}$ (A.3), that

$$\|\omega\|_{H^{3/2}(B^4)} = \left\|\overline{\partial} N A^{0,1}\right\|_{H^{3/2}(B^4)} \leq C \left\|A^{0,1}\right\|_{W^{1,2}(B^4)}.$$

It remains to bootstrap to $W^{2,2}$ regularity. Since ω is a $(0,2)$-form in B^4 then $\overline{\partial}^* \omega = 0$. Thus, the equation (2.1) which ω solves is elliptic. In particular:

$$2\Delta_d \omega = \Delta_{\overline{\partial}} \omega = \overline{\partial}\overline{\partial}^* \omega = -\overline{\partial} A^{0,1}. \tag{2.2}$$

Moreover, we know that from $\omega = -\overline{\partial} N A^{0,1}$ and the $\overline{\partial}$-Neumann boundary conditions satisfied by $NA^{0,1}$, it follows that $\sigma(\vartheta, dr)\omega = 0$ on ∂B^4 by Appendix A. According to the equation (A.2) in Appendix A, this is equivalent to

$$\overline{\partial} r \wedge *\overline{\omega} = 0 \quad \text{on } \partial B^4.$$

We need to understand what this condition means. ω is a $(0,2)$ form and can be written as $\omega = f \; \mathrm{d}\overline{z}_1 \wedge \mathrm{d}\overline{z}_2 = f \; \overline{\tau} \wedge \overline{\partial} r$, where f is a function on B^4. Then $*\overline{\omega} = \overline{f} \; \tau \wedge \partial r$ and it follows that the boundary condition implies $f = 0$ on ∂B^4, which in turn implies that

$$\omega = 0 \text{ on } \partial B^4. \tag{2.3}$$

Remark 2.1. *Unlike the case of differential forms whose normal components (containing dr) vanish on the boundary ∂B^4, the complex normal components* *(containing either* $\overline{\partial}r$ *or* ∂r*) do not vanish. Indeed, by* (A.10) *we have* $\overline{\partial}r = dr + iJdr$ *and* $i_{\partial B^4}\overline{\partial}r = iJdr$*.*

Putting (2.2) and (2.3) together, we have a solution ω to the PDE:

$$\begin{cases} \overline{\partial}\,\overline{\partial}^*\omega & = -\overline{\partial}A^{0,1} \qquad \text{in } B^4 \\ i^*_{\partial B^4}\omega & = 0 \qquad\qquad \text{on } \partial B^4 \end{cases}$$

Since this is an elliptic problem with vanishing boundary condition, we have the classic estimate:

$$\|\omega\|_{W^{2,2}(B^4)} \leq C \left\|\overline{\partial}A^{0,1}\right\|_{L^2(B^4)},$$

for some constant $C > 0$. Hence, we have shown that there exists $\omega \in W^{2,2}(\Omega^{0,2}B^4, \mathbb{C})$ such that

$$F^{0,2}_{A+\overline{\partial}^*\omega} = 0.$$

Denote the form $A + \overline{\partial}^*\omega$ by $\tilde{A}$. We show that for $\tilde{A}$ we can find a gauge $g : B^4 \to \mathbb{C}^*$ such that

$$\overline{\partial}g = \tilde{A}^{0,1}g.$$

Proposition 2.1. *There exists* $\varepsilon_0 > 0$ *such that if* $\tilde{A} \in W^{1,2}(\Omega^1 B^4, \mathbb{C})$ *with* $F^{0,2}_{\tilde{A}} = 0$ *satisfies* $\left\|\tilde{A}\right\|_{W^{1,2}} \leq \varepsilon_0$*, then there exists* $r \in (0,1)$ *and* $g \in \bigcap_{p<2} W^{2,p}(B^4_r, \mathbb{C}^*)$ *such that* $\overline{\partial}g = -\tilde{A}^{0,1}g$*.*

Proof of Proposition 2.1. Since $F^{0,2}_{\tilde{A}} = \overline{\partial}\tilde{A}^{0,1} = 0$, then by the $\overline{\partial}$-Hodge decomposition (A.5) there exists a function

$$U := \overline{\partial}^* N\tilde{A}^{0,1} \in H^{3/2}(B^4, \mathbb{C})$$

such that

$$\tilde{A}^{0,1} = \overline{\partial}U.$$

Moreover, there exists $r_0 \in (0,1)$ such that

$$\|U\|_{W^{2,2}(B^4_{r_0})} \leq C \|A\|_{W^{1,2}(B^4)},$$

for some constant $C > 0$. Thus, by the embedding [3, Theorem 4.6] $W^{2,2} \hookrightarrow BMO$, we obtain the estimate

$$\|U\|_{BMO} \leq C \|A\|_{W^{1,2}(B^4)} \leq C\varepsilon_0.$$

We choose $\varepsilon_0 > 0$ such that $C\varepsilon_0 < \frac{1}{4}2^4 e$. It follows that $c := 4 < (2^4 eC\varepsilon_0)^{-1} \leq (2^4 e \|U\|_{BMO})^{-1}$ and as a consequence of [11, Corollary 3.1.7], for any compact set $K \subseteq B^4 \subset \mathbb{R}^4$ we have

$$\int_K \exp c|U| dx < \infty.$$

In particular, for $K = B^4_{r_0}$, we have

$$\int_{B^4_{r_0}} \exp 4|U| < \infty. \tag{2.4}$$

Define $g = \exp(-U)$. By the above estimate (2.4), we have $g \in L^4(B^4_{r_0})$. Moreover, g solves the required PDE:

$$\overline{\partial} g = -\overline{\partial} U \cdot \exp(-U) = -\tilde{A}^{0,1} g.$$

By Lemma D.1 it follows that there exists $r \in (0, r_0)$ such that

$$g \in \bigcap_{p<2} W^{2,p}(B^4_r, \mathbb{C}^*).$$

□

Thus, putting everything together we can prove theorem 1.1 in the case of abelian bundles. We formulate the statement for $U(1)$ bundles:

Theorem 1.1. *Let ∇ be a unitary $W^{1,2}$ connection of an hermitian $U(1)$ bundle (E, h_0) over a closed Kähler surface X^2. Assume ∇ satisfies the integrability condition*

$$F^{0,2}_\nabla = 0 \tag{2.5}$$

then there exists a smooth holomorphic structure $\mathcal{E}$ on E and a $\bigcap_{q<2} W^{2,q}$ section h of the bundle of positive Hermitian endomorphisms of E such that

$$\nabla = \partial_0 + h^{-1}\partial_0 h + \overline{\partial}_{\mathcal{E}} \tag{2.6}$$

where $\overline{\partial}_{\mathcal{E}}$ is the $\overline{\partial}-$operator associated to the holomorphic bundle $\mathcal{E}$ and ∂_0 is the 1-0 *part of the Chern connection associated*[1] *to the holomorphic structure $\mathcal{E}$ and the chosen reference hermitian product h_0.*

[1]These connections are not necessarily unitary with respect to h_0 anymore.

Proof of Theorem 1.1. We pick geodesic balls $B^4_r(x_i)$ covering X^2 on which the connection can be trivialised as $\nabla \simeq d + A_i$ and

$$\|A_i\|_{W^{1,2}(B^4_r(x_i))} \leq \varepsilon_0,$$

where ε_0 is given by Proposition 2.1. Because X^2 is a compact manifold, there are finitely many such balls covering X^2. By Proposition 2.1 there exists $r' \in (0, r)$, $\sigma_i \in W^{2,p}(B^4_{r'}(x_i), \mathbb{C}^*)$ and $h_i = \overline{\sigma_i}\sigma_i \in W^{2,p}(B^4_{r'}, \mathbb{R}_{>0})$, for any $p < 2$ so that $A_i^{\sigma_i} = h_i^{-1}\partial h_i$. Hence

$$\nabla^{\sigma_i} \simeq d + h_i^{-1}\partial h_i \qquad \text{in } B^4_{r'}(x_i). \tag{2.7}$$

It is worth noting that since A_i is unitary ($A_i = -\overline{A_i}$), then A_i is purely imaginary. However, due to the fact that h_i is a positive real valued function then $A_i^{\sigma_i}$ is only complex valued, not necessarily unitary.

It remains to show that ∇ defines a connection on a holomorphic vector bundle structure $\mathcal{E}$. In order to achieve this, it is enough to find holomorphic transition functions. On the initial bundle E, there exists gauge transition functions $g_{ij} \in W^{2,2}(B^4_r(x_i) \cap B^4_r(x_j), U(1))$ such that

$$A_i^{g_{ij}} = A_j.$$

Define the transition functions

$$\sigma_{ij} = \sigma_i^{-1} g_{ij} \sigma_j. \tag{2.8}$$

We show that these functions are holomorphic and consequently since they define a cocycle, they define a holomorphic vector bundle structure $\mathcal{E}$ over the Kähler manifold X^2:

$$\begin{aligned}
\overline{\partial}\sigma_{ij} &= \overline{\partial}\sigma_i^{-1} \cdot g_{ij}\sigma_j + \sigma_i^{-1}\overline{\partial} g_{ij}\sigma_j + \sigma_i^{-1} g_{ij}\overline{\partial}\sigma_j \\
&= \sigma_i^{-1} A_i^{0,1} g_{ij}\sigma_j + \sigma_i^{-1}\overline{\partial} g_{ij}\sigma_j - \sigma_i^{-1} g_{ij} A_j^{0,1}\sigma_j \\
&= \sigma_i^{-1}\left(g_{ij}\left(A_i^{0,1} - A_j^{0,1}\right) + \overline{\partial} g_{ij}\right)\sigma_j,
\end{aligned}$$

where we have used the abelian property of our bundle. Since

$$g_{ij}^{-1}\overline{\partial} g_{ij} + A_i^{0,1} = A_j^{0,1},$$

the above equation gives:

$$\overline{\partial}\sigma_{ij} = 0.$$

This shows that the transition functions are holomorphic. Thus, there exists a holomorphic vector bundle structure $\mathcal{E}$ which is compatible with ∇ since $(\nabla^{\sigma_i})^{\sigma_{ij}} = \nabla^{\sigma_j}$ in local coordinates. From the local representation (2.7) we obtain:

$$\nabla^{0,1} = \overline{\partial}_{\mathcal{E}}.$$

□

2.2 Global approximation

It remains to prove theorem 1.2 in the context of $U(1)$ bundles. In order to do so, we focus on proving the following result:

Lemma 2.1. *Given ∇ a unitary connection over X^2 with $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$ satisfying*

$$F^{0,2}_{\nabla} = 0,$$

then there exists ∇_k a sequence of smooth connections with $F^{0,2}_{\nabla_k} = 0$ and

$$\nabla_k \to \nabla \text{ as } k \to \infty \text{ in } W^{1,2}(X^2),$$

It is worth noting that in this section we will emphasize a few different approaches. Firstly, we introduce the global approach we will also take in Chapters 3 and 4. Secondly, we show that under the setting of $U(1)$ bundles we can achieve a gluing procedure and obtain the required strong approximation. Lastly, we draw a parallel to an analogue construction using sheaf theoretical methods.

We warn the reader that in order to prove Lemma 2.1 we will use the "classical" notion of $W^{1,2}$ convergence of connections. We fix a system of finitely many charts $\{U_i\}_{i=1}^N$ covering X^2 and trivialisations $\nabla \approx d + A_i$ in U_i. Then the distance function we use is

$$\mathrm{cdist}(\nabla, \nabla') := \sum_{i=1}^{N} \|A_i - A_i'\|_{W^{1,2}(U_i)}, \tag{2.9}$$

for any $W^{1,2}$ connections ∇, ∇' over X^2. Thus, when we say $\nabla_k \to \nabla$ in $W^{1,2}$ as $k \to \infty$, we mean $\mathrm{cdist}(\nabla, \nabla') \to 0$.

The distance between ∇ and a connection 1-form A_i' on a chart U_i is naturally defined as:

$$\mathrm{cdist}(\nabla, A_i') = \|A_i - A_i'\|_{W^{1,2}(U_i)},$$

where $\nabla \approx d + A_i$ in U_i.

Later on, when discussing convergence of connections attaining weak holomorphic structures, we will use the gauge invariant distance dist_p as defined in Definition 1.2.

Global Approach

We start with a global approach which is the abelian analogue to how we will solve the more difficult problem in the non-abelian settings.

Proof of Lemma 2.1. Let $\tilde{\nabla}_k$ be a sequence of smooth connections over X^2 converging to ∇ in $W^{1,2}(X^2)$. Such a smooth sequence exists by [15]. In the case of $U(1)$ bundles, we have $\overline{\partial} F^{0,2}_{\tilde{\nabla}_k} = 0$, where $\overline{\partial}$ is the operator mapping $\mathcal{A}^{p,q}(X^2)$ to $\mathcal{A}^{p,q+1}(X^2)$. Thus, by the condition $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$ and the $\overline{\partial}$-Hodge decomposition (A.4), there exists $(0,1)$ forms

$$u_k \in C^\infty(\Omega^{0,1} X^2 \otimes M_n(\mathbb{C}))$$

such that $F^{0,2}_{\tilde{\nabla}_k} = \overline{\partial} u_k$.

This sequence converges by construction in L^2 to $0 = F^{0,2}_{\nabla}$. Since the decomposition is taken over a compact manifold X^2, then $\overline{\partial}$ is an elliptic operator [12, p. 93] and hence the sequence of maps u_k is uniformly bounded in $W^{1,2}$. Thus, define

$$\nabla_k := \tilde{\nabla}_k - u_k + \overline{u_k},$$

where we have added $\overline{u_k}$ in order to have that ∇_k is unitary. Then by construction

$$F^{0,2}_{\nabla_k} = 0$$

and $\nabla_k \to \nabla$ in $W^{1,2}$ as $k \to \infty$ (convergence taken in the sense of (2.9)).

Moreover, the convergence of $F^{0,2}_{\tilde{\nabla}_k} \to 0$ in L^2 gives that $u_k \to 0$ in $W^{1,2}$ as $k \to \infty$. □

Gluing Approach

We construct a global approximation result using a gluing approach. The general idea is having two smooth connections $\tilde{\nabla}_U$ and $\tilde{\nabla}_V$ on intersecting open sets $U, V \subseteq X^2$ that are close in $W^{1,2}$ distance (2.9) to the given connection ∇ on X^2, then there is a smooth connection $\tilde{\nabla}$ on U, V that is close in $W^{1,2}$ distance to $\tilde{\nabla}_U$, $\tilde{\nabla}_V$ and ∇. Thus, by constructing approximating smooth connection 1-forms on fixed charts and gluing them, we obtain global smooth connections on X^2 that are close to ∇ in $W^{1,2}$ distance. The main difficulty is keeping the integrability condition (1.1) throughout the procedure.

We assume the existence of ∇ a unitary $W^{1,2}$ connection over X^2. Moreover, we fix coordinate charts $\{B^4_{\rho_i}(x_i)\}_{i=1}^N$ over X^2 and trivialisations $\nabla \approx d + A_i$ in each $B^4_{\rho_i}(x_i)$. The following proposition gives the existence result we have just described:

Proposition 2.2. *Let $U = \bigcup_{i\in I} B^4_{\rho_i}(x_i) \subseteq X^2$ an open set and $B^4_{\rho_{i_0}}(x_{i_0})$ such that $B^4_{\rho_{i_0}}(x_{i_0}) \cap U \neq \emptyset$. Moreover, let $\tilde{\nabla}_U \in W^{1,2} \cap C^\infty(U)$ be a connection over U and $\tilde{A} \in C^\infty(\Omega^1 B^4_{\rho_{i_0}}(x_{i_0}) \otimes \mathbb{C})$ a connection 1-form such that*

$$F^{0,2}_{\tilde{\nabla}_U} = F^{0,2}_{\tilde{A}} = 0.$$

Then there exists $\rho_0 \in (0, \rho_{i_0})$ and a connection

$$\tilde{\nabla} \in C^\infty \cap W^{1,2}(U \cup B^4_{\rho_0}(x_{i_0}), \mathbb{C})$$

satisfying the integrability condition $F^{0,2}_{\tilde{\nabla}} = 0$ and the estimate

$$\operatorname{cdist}(\tilde{\nabla}, \tilde{\nabla}_U) + \operatorname{cdist}(\tilde{\nabla}, \tilde{A}) \leq C \left(\operatorname{cdist}(\nabla, \tilde{\nabla}_U) + \operatorname{cdist}(\nabla, \tilde{A}) \right)$$

for some constant $C > 0$.

Proof of Proposition 2.2. In $B^4_{\rho_{i_0}}(x_{i_0})$, we have the representation

$$\nabla \approx d + A_{i_0},$$

where $A_{i_0} \in W^{1,2}(\Omega^1 B_{\rho_{i_0}}(x_{i_0}) \otimes \mathbb{C})$. In $B^4_{\rho_{i_0}}(x_{i_0}) \cap U \neq \emptyset$, we have the representation

$$\nabla \approx d + B$$

such that there exists a gauge change $g : B^4_{\rho_{i_0}}(x_{i_0}) \cap U \mapsto U(1)$ satisfying the gluing compatibility $B^g = A$. Moreover, we have

$$\nabla_U \approx d + \tilde{B}$$

in $B^4_{\rho_{i_0}}(x_{i_0}) \cap U$. We want to find a perturbation of $\tilde{A}$ such that we satisfy $\tilde{B}^g = \tilde{A}$. Due to the abelian-ness of the bundle, the gluing compatibility we need to find translates as

$$\tilde{A} = \tilde{B} + g^{-1}dg.$$

Denote

$$\eta = \tilde{A}^{0,1} - \tilde{B}^{0,1} - g^{-1}\overline{\partial} g \quad \text{on } B^4_{\rho_{i_0}}(x_{i_0}) \cap U, \tag{2.10}$$

and $\eta \in C^\infty \cap W^{1,2}(\Omega^{0,1} B^4_{\rho_{i_0}}(x_{i_0}), \mathbb{C})$. Moreover,

$$\overline{\partial}\eta = \overline{\partial}\tilde{A}^{0,1} - \overline{\partial}\tilde{B}^{0,1} - \overline{\partial}(g^{-1}\overline{\partial} g).$$

We have

$$\overline{\partial}(g^{-1}\overline{\partial} g) = \overline{\partial} g^{-1} \wedge \overline{\partial} g = -g^{-1}\overline{\partial} g \wedge g^{-1}\overline{\partial} g.$$

Because $g \in U(1)$, then $\overline{\partial} g$ and g^{-1} commute and $\overline{\partial} g \wedge \overline{\partial} g = 0$. Hence,

$$g^{-1}\overline{\partial} g \wedge g^{-1}\overline{\partial} g = g^{-1} \cdot g^{-1}\overline{\partial} g \wedge \overline{\partial} g = 0$$

and

$$\overline{\partial}(g^{-1}\overline{\partial} g) = 0.$$

Putting this together with the integrability conditions $F^{0,2}_{\tilde{A}} = 0$ and $F^{0,2}_{\tilde{B}} = 0$ satisfied by $\tilde{A}$ and $\tilde{B}$ respectively, we obtain

$$\overline{\partial}\eta = F^{0,2}_{\tilde{A}} - F^{0,2}_{\tilde{B}} = 0.$$

By the $\overline{\partial}$-Hodge decomposition A.5 on $B^4_{\rho_{i_0}}(x_{i_0}) \cap U$, this implies that there exists

$$\alpha = \overline{\partial}^* N\eta \in C^\infty \cap H^{3/2}(B^4_{\rho_{i_0}}(x_{i_0}) \cap U, \mathbb{C})$$

satisfying:

$$\eta = \overline{\partial}\alpha \tag{2.11}$$

with estimate

$$\|\alpha\|_{H^{3/2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)} \le C_1 \|\eta\|_{W^{1,2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)}, \tag{2.12}$$

for some constant $C_1 > 0$. However, since α is a function, there exists $\rho_0 \in (0, \rho_{i_0})$ such that we have the elliptic estimate:

$$\|\alpha\|_{W^{2,2}(B^4_{\rho_0}(x_{i_0})\cap U)} \le C_2 \left(\left\|\overline{\partial}\alpha\right\|_{W^{1,2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)} + \|\alpha\|_{W^{1,2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)} \right) \tag{2.13}$$

for some constant $C_2 > 0$. Putting together (2.11),(2.12), (2.13) and the Sobolev embedding $H^{3/2} \hookrightarrow W^{1,2}$, the estimate follows:

$$\|\alpha\|_{W^{2,2}(B^4_{\rho_0}(x_{i_0})\cap U)} \le C \|\eta\|_{W^{1,2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)}, \tag{2.14}$$

for some constant $C > 0$. In order to obtain a good bound on η, we can rewrite equation (2.10) as:

$$\begin{aligned}\eta &= \tilde{A}^{0,1} - A_{i_0}^{0,1} - \tilde{B}^{0,1} + B^{0,1} + A_{i_0}^{0,1} - B^{0,1} - g^{-1}\overline{\partial} g \\ &= (\tilde{A}^{0,1} - A_{i_0}^{0,1}) - (\tilde{B}^{0,1} - B^{0,1}).\end{aligned}$$

Thus for some constant $C > 0$ we have:

$$\begin{aligned}\|\alpha\|_{W^{2,2}(B^4_{\rho_0}(x_{i_0})\cap U)} &\leq C\,\|\eta\|_{W^{1,2}(B^4_{\rho_{i_0}}(x_{i_0})\cap U)} \\ &\leq 2C\left(\operatorname{cdist}(\nabla, \tilde{\nabla}_U) + \operatorname{cdist}(\nabla, \tilde{A})\right).\end{aligned}$$

Using the extension result Proposition B.1, it follows that there exists an extension $\tilde{\alpha}$ on $B^4_{\rho_0}(x_{i_0})$ such that $\tilde{\alpha} = \alpha$ on $B^4_{\rho_0}(x_{i_0}) \cap U$ and

$$\|\tilde{\alpha}\|_{W^{2,2}(B^4_{\rho_0}(x_{i_0}))} \leq 2C\left(\operatorname{cdist}(\nabla, \tilde{\nabla}_U) + \operatorname{cdist}(\nabla, \tilde{A})\right),$$

for some constant $C > 0$. Thus, we redefine

$$\tilde{A} := \tilde{A} - \overline{\partial}\tilde{\alpha} - \partial\overline{\tilde{\alpha}} \in C^\infty \cap W^{1,2}(\Omega^1 B^4_{\rho_0}(x_{i_0}), \mathbb{C})$$

and by construction satisfies the gluing compatibility $\tilde{A} = \tilde{B}^g = \tilde{B} + g^{-1}dg$ and the integrability condition $F^{0,2}_{\tilde{A}} = 0$. Hence, it yields a well-defined smooth connection $\tilde{\nabla}$ on $B^4_{\rho_0}(x_{i_0}) \cup U$ with the required estimate:

$$\operatorname{cdist}(\tilde{\nabla}, \tilde{\nabla}_U) + \operatorname{cdist}(\tilde{\nabla}, \tilde{A}) \leq C\left(\operatorname{cdist}(\nabla, \tilde{\nabla}_U) + \operatorname{cdist}(\nabla, \tilde{A})\right)$$

for some constant $C > 0$. This finishes the proof of Proposition 2.2. □

It is crucial to note that in Proposition 2.2 we have that $\eta = \overline{\partial}\alpha$. In the general case of $U(n)$ bundles, we can only decompose η as $\overline{\partial}\alpha + \overline{\partial}^*\beta$ for some $(0,2)$-form β. Hence, we cannot glue in a similar way and the gluing problem is open for non-abelian bundles.

Using the Proposition 2.2 above, we prove Lemma 2.1 which gives us a strong $W^{1,2}$ global approximation of ∇ under the integrability condition (1.1) constraint.

Proof of Lemma 2.1. By [15], there exists a smooth sequence of connections $\tilde{\nabla}^0_k$ such that $\tilde{\nabla}^0_k \to \nabla$ in $W^{1,2}$. However, unlike [15], there is no need to smooth out the bundle since we only work in the setting of weak connections and not weak bundles. Hence, the transition functions between charts are smooth.

We want to perturb $\tilde{\nabla}_k^0$ so that we obtain a smooth sequence which satisfies the integrability condition. We cover X^2 by finitely many geodesic balls $\{B^4_{\rho_i}(x_i)\}_{i=1}^N$, where $N < \infty$ and $\rho > 0$ not necessarily small. On each $B^4_{\rho_i}(x_i)$ we fix the representations:

$$\tilde{\nabla}_k^0 \approx d + \tilde{A}_k^i \quad \text{and} \quad \nabla \approx d + A^i.$$

We have shown that for each i we can perturb $\tilde{A}_k^i$ such that we obtain $\tilde{A}_k^i$, satisfying $F^{0,2}_{\tilde{A}_k^i} = 0$, while preserving the convergence: $\tilde{A}_k^i \to A^i$ in $W^{1,2}(B^4_{\rho_i}(x_i))$. We construct ∇_k inductively using Proposition 2.2.

Without loss of generality we will assume that we do not shrink the domains as in Proposition 2.2. We can assume this since we have a finite cover and hence the induction will finish in a finite number of steps.

In the first step we define:

$$\varepsilon_0 := \left\| \tilde{A}_k^1 - A^1 \right\|_{W^{1,2}(B^4_{\rho_1}(x_1))} + \left\| \tilde{A}_k^2 - A^2 \right\|_{W^{1,2}(B^4_{\rho_2}(x_2))},$$

$\nabla := \tilde{\nabla}_k^0$, $U := B^4_{\rho_1}(x_1)$, $\nabla_U := d + A_k^1$, $\tilde{A} := A_k^2$. By Proposition 2.2 there exists a smooth connection $\tilde{\nabla}_k^1$ on $B^4_{\rho_1}(x_1) \cup B^4_{\rho_2}(x_2)$ satisfying the integrability condition (1.1) such that

$$\operatorname{cdist}(\tilde{\nabla}_k^1, \tilde{\nabla}_k^0) \leq C\varepsilon_0,$$

for some $C > 0$.

Assume we have obtained a connection $\tilde{\nabla}_k^n$ on $U_n = \bigcup_{i=1}^n B^4_{\rho_i}(x_i)$ and we construct a connection $\tilde{\nabla}_k^{n+1}$ over $U_{n+1} = \bigcup_{i=1}^{n+1} B^4_{\rho_i}(x_i)$. We apply Proposition 2.2 with

$$\varepsilon_n := \operatorname{cdist}(\tilde{\nabla}_k^n, \tilde{\nabla}_k^0) + \left\| \tilde{A}_k^{n+1} - A^1 \right\|_{W^{1,2}(B^4_{\rho_{n+1}}(x_{n+1}))},$$

$\nabla := \tilde{\nabla}_k^n$, $U := U_n$, $\nabla_U = \tilde{\nabla}_k^n$, $\tilde{A} := A_k^{n+1}$. Then there exists a smooth connection $\tilde{\nabla}_k^{n+1}$ on U_{n+1} such that:

$$\operatorname{cdist}(\tilde{\nabla}_k^{n+1}, \tilde{\nabla}_k^n) + \operatorname{cdist}(\tilde{\nabla}_k^{n+1}, \tilde{A}_k^{n+1}) \leq C\varepsilon_n,$$

for some constant $C > 0$. Thus, we have inductively constructed a smooth connection $\nabla_k := \tilde{\nabla}_k^N$ on X^2 such that:

$$\mathrm{cdist}(\nabla_k, \tilde{\nabla}_k^0) \leq C \sum_{i=1}^{N} \left\| \tilde{A}_k^i - A^i \right\|_{W^{1,2}(B^4_{\rho_i}(x_i))} \to 0$$

and $F^{0,2}_{\nabla_k} = 0$. Since $\tilde{\nabla}_k^0 \to \nabla$ as $k \to \infty$, then we conclude that $\nabla_k \to \nabla$ in $W^{1,2}$. □

Using the result of Lemma 2.1, the proof of theorem 1.2 in the case of $U(1)$ bundles follows line by line as in section 3.5.

Sheaf Approach

We conclude this chapter by making a geometrical observation. It is useful to make a connection between the previous construction and sheaf theory. In order to have a concrete example, we take $X^2 = \mathbb{CP}^2$. The following approach gives a sheaf theoretical argument on why we are able to perform a gluing argument. However, it also outlines the difficulties such a proof would bring regarding the global convergence result.

In essence the above Lemma 2.1 is a gluing result which can be reformulated as such: there exists $\rho > 0$ such that we can cover $X^2 = \mathbb{CP}^2$ by finitely many geodesic balls $B_\rho(x_i)$ and in each such ball we have the representation

$$\nabla \approx d + A^i$$

where A^i are $W^{1,2}$ connection 1-forms satisfying the integrability condition

$$F^{0,2}_{A^i} = 0 = \overline{\partial}(A^i)^{0,1}.$$

Moreover, we have shown that we can strongly approximate A^i by sequences $\tilde{A}_k^i$ such that

$$\tilde{A}_k^i \to A^i \text{ in } W^{1,2} \quad \text{and} \quad F^{0,2}_{\tilde{A}_k^i} = 0.$$

We need to find perturbations to each $\tilde{A}_k^i$ to obtain A_k^i such that there exist gauge transition functions g_k^{ij} satisfying the gluing compatibility $A_k^i = \left(g_k^{ij}\right)^{-1} dg_k^{ij} + A_k^j$, the strong $W^{1,2}$ convergence $A_k^i \to A^i$ as $k \to \infty$ for each i, and $F^{0,2}_{A_k^i} = 0$. Thus, the error of the gluing compatibility of $\tilde{A}_k^i$ can be denoted by:

$$\eta_k^{ij} = \tilde{A}_k^i - \left(g_k^{ij}\right)^{-1} dg_k^{ij} - \tilde{A}_k^j \in C^\infty \cap W^{1,2}(\Omega^1 B_\rho(x_i) \cap B_\rho(x_j), \mathbb{C}).$$

We have $\overline{\partial}(\eta_k^{ij})^{0,1} = 0$, which implies that $(\eta_k^{ij})^{0,1}$ are holomorphic. Moreover, they satisfy the cocycle condition:

$$(\eta_k^{ij})^{0,1} + (\eta_k^{j\ell})^{0,1} + (\eta_k^{\ell i})^{0,1} = 0 \quad \text{on } B_\rho(x_i) \cap B_\rho(x_j) \cap B_\rho(x_\ell). \tag{2.15}$$

Since $X^2 = \mathbb{CP}^2$ we have that $H^1(X^2, \mathcal{O}) \simeq H^{0,1}_{\overline{\partial}}(X^2) = 0$ by [12, Dolbeault Theorem, p.45], where $\mathcal{O}$ is the sheaf of holomorphic functions. In particular, for any indices $s_1, \ldots, s_j$ we have

$$\mathcal{H}^{0,q}_{\overline{\partial}}(B_\rho(x_{s_1}) \cup \ldots \cup B_\rho(x_{s_j})) \simeq H^q(B_\rho(x_{s_1}) \cup \ldots \cup B_\rho(x_{s_j}), \mathcal{O}) = 0$$

i.e. our cover is acyclic. Thus, by Leray Theorem [12, p. 40] it follows that

$$\check{H}^1(\{B_\rho(x_i)\}_{i=1}^N), \mathcal{O}) \simeq H^1(X^2, \mathcal{O}) \simeq H^{0,1}_{\overline{\partial}}(X^2) = 0.$$

Hence, we have obtained that the first Cech coholomology is trivial. It follows that the cocycle condition (2.15) implies exactness as well, i.e. there exists functions η_k^i defined on $B_\rho(x_i)$ such that

$$\eta_k^{ij} = \eta_k^i - \eta_k^j \quad \text{on } B_\rho(x_i) \cap B_\rho(x_j),$$

and $\overline{\partial}\eta_k^i = 0$ for each i. Moreover, by definition we have $\eta_k^{ij} \to 0$ in $W^{1,2}$ as $k \to \infty$. Hence, it is enough to define

$$A_k^i = \tilde{A}_k^i - \eta_k^i,$$

since these 1-forms satisfy the gluing compatibility and integrability condition $F^{0,2}_{A_k^i} = 0$. From this we obtain a sequence of smooth connections ∇_k. Although, this sheaf theoretical result strengthens our constructive gluing, it fails to reveal whether the connections ∇_k we obtain preserve the $W^{1,2}$ convergence to ∇. This is because the co-exactness of

$$\eta_k^{ij} = \eta_k^i - \eta_k^j$$

does not give $W^{1,2}$ convergence of η_k^i and η_k^j even though η_k^{ij} converges to 0.

Chapter 3

$U(n)$ Bundles in Hilbert Spaces

Having seen the ideas behind solving the problem in the setting of $U(1)$ bundles, we are now in a position to analyse the case of $U(n)$ bundles. This Chapter will look at the case of closed Kähler surfaces which is yet another intermediary step to understanding the problem on closed Kähler manifolds. Here we will develop the first methods that help us deal with the non-linearities that occur from lacking the abelian property of the bundle.

3.1 Density under low energy

Given a unitary $W^{1,2}$ connection ∇ of the hermitian bundle (E, h_0) over a closed Kähler surface X^2 satisfying $F_\nabla^{0,2} = 0$, we assume without loss of generality that B^4 is a geodesic ball in X^2 and that ∇ trivialises as $\nabla \simeq d+A$ in B^4, where A is a connection 1-form. Moreover, in this section we will work with low $W^{1,2}$ connection norm in B^4 i.e. A satisfies the smallness condition

$$\|A\|_{W^{1,2}(B^4)} \leq \varepsilon_0(X^2, \omega)$$

for some $\varepsilon_0(X^2, \omega) > 0$ depending on the surface X^2 and the Kähler form ω. We will use the smallness assumption throughout this section. Moreover, to fix ideas we will assume that B^4 is the flat closed unit ball.

We start by showing how to smooth 1-forms, keeping the approximating sequence unitary. This method however, does not ensure the integrability condition (1.1). Let $p > 1$ and $A \in W^{1,p}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ then we can always find a sequence of unitary smooth 1-forms $A_k \in C^\infty(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ such that

$$A_k \to A \text{ in } W^{1,p}(B^4).$$

Indeed, we can write A as $A = A^{0,1} - \overline{A^{0,1}}^T$, where $A^{0,1} = \alpha_1 d\bar{z}_1 + \alpha_2 d\bar{z}_2$. Since for each $i = 1, 2$, we have $\alpha_i \in W^{1,p}(B^4, M_n(\mathbb{C}))$, then by the density of C^∞ maps into $W^{1,p}$, there exist sequences

$$\alpha_{1,k} \to \alpha_1 \text{ in } W^{1,p}(B^4)$$

and

$$\alpha_{2,k} \to \alpha_2 \text{ in } W^{1,p}(B^4).$$

By defining $A_k^{0,1} := \alpha_{1,k} d\bar{z}_1 + \alpha_{2,k} d\bar{z}_2$ and $A_k := A_k^{0,1} - \overline{A_k^{0,1}}^T$, we obtain by construction the convergence of A_k to our initial form A in $W^{1,p}$. Moreover, A_k is a unitary 1-form.

The next lemma helps us prove that we can always find a perturbation of a given connection 1-form $A \in W^{1,p}$ with low norm such that the integrability condition (1.1) is satisfied.

Lemma 3.1. *Let $p \geq 2$. There exists $\varepsilon > 0$ depending on p such that for any $A \in W^{1,p}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ satisfying $\|A\|_{W^{1,p}(B^4)} \leq \varepsilon$, there exists a 1-form*

$$\tilde{A} \in W^{1,p}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$$

that satisfies the integrability condition

$$F_{\tilde{A}}^{0,2} = 0$$

and

$$\left\|\tilde{A} - A\right\|_{W^{1,p}(B^4)} \leq C \left\|F_A^{0,2}\right\|_{L^p(B^4)},$$

for some constant $C > 0$ depending on p. Moreover, if A is smooth then $\tilde{A}$ is also.

In the proof we will use Sobolev embeddings under the assumption that $p \in [2, 4)$ - which is the more delicate case. If $p \geq 4$, the results hold by considering the corresponding Sobolev embeddings.

Proof of Lemma 3.1. In order to obtain a form satisfying the integrability condition, we want to perturb the A with a 1-form $V \in C^\infty(\Omega^1 B^4, \mathfrak{u}(n))$ such that $F_{A+V}^{0,2} = 0$. We express V by $V = v - \overline{v}^T$, where $v \in W^{1,p}(\Omega^{0,1} B^4 \otimes M_n(\mathbb{C}))$. We expand $F_{A+V}^{0,2} = 0$ as such:

$$\begin{aligned} 0 = F_{A+V}^{0,2} = & \ (d(A+V) + (A+V) \wedge (A+V))^{0,2} \\ = & \ (d(A+V))^{0,2} + ((A+V) \wedge (A+V))^{0,2} \end{aligned}$$

$$
\begin{aligned}
&= \left((\overline{\partial}+\partial)(A^{0,1}+V^{0,1}+A^{1,0}+V^{1,0})\right)^{0,2} \\
&\quad + ((A^{0,1}+V^{0,1}+A^{1,0}+V^{1,0}) \wedge (A^{0,1}+V^{0,1}+A^{1,0}+V^{1,0}))^{0,2} \\
&= \overline{\partial}A^{0,1} + \overline{\partial}V^{0,1} + A^{0,1}\wedge A^{0,1} + [A^{0,1}, V^{0,1}] + V^{0,1}\wedge V^{0,1} \\
&= F_A^{0,2} + \overline{\partial}V^{0,1} + [A^{0,1}, V^{0,1}] + V^{0,1}\wedge V^{0,1} \\
&= F_A^{0,2} + \overline{\partial}v + [A^{0,1}, v] + v\wedge v.
\end{aligned}
$$

Thus, we get the following equation

$$\overline{\partial}v + [v, A^{0,1}] + v \wedge v = -F_A^{0,2}.$$

By the fact that we work on a Kähler manifold, we know that $\overline{\partial\partial}^* \cdot = \frac{1}{2}\Delta \cdot d\overline{z_1} \wedge d\overline{z_2}$ on the space $\Omega^{0,2}B^4$. Thus, we want to transform the PDE above into an elliptic one by taking v of the form

$$v = \overline{\partial}^* \omega$$

with $\omega = 0$ on ∂B^4 - so that $\overline{\partial}^*$ is well defined. We solve the following elliptic system:

$$\overline{\partial\partial}^*\omega + [\overline{\partial}^*\omega, A^{0,1}] + \overline{\partial}^*\omega \wedge \overline{\partial}^*\omega = -F_A^{0,2}.$$

Since $A^{0,1}$ and $F_A^{0,2}$ have small norms, we can use a fixed point argument. We consider the following Dirichlet problem:

$$
\begin{cases}
\overline{\partial\partial}^*\omega &= -[\overline{\partial}^*\omega, A^{0,1}] - \overline{\partial}^*\omega \wedge \overline{\partial}^*\omega - F_A^{0,2} \quad \text{in } B^4 \\
\\
\omega &= 0 \quad \text{on } \partial B^4.
\end{cases}
$$

We fix k and we build the following sequence $\{\omega_j\}_{j=1}^{\infty}$ of forms that solve the PDEs:

$$
\begin{aligned}
\overline{\partial\partial}^*\omega_1 &= -F_A^{0,2} \\
\overline{\partial\partial}^*\omega_2 &= -[\overline{\partial}^*\omega_1, A^{0,1}] - \overline{\partial}^*\omega_1 \wedge \overline{\partial}^*\omega_1 - F_A^{0,2} \\
&\cdots \\
\overline{\partial\partial}^*\omega_{j+1} &= -[\overline{\partial}^*\omega_j, A^{0,1}] - \overline{\partial}^*\omega_j \wedge \overline{\partial}^*\omega_j - F_A^{0,2} \\
&\cdots
\end{aligned}
$$

where $\omega_j = 0$ on ∂B^4 for all $j \geq 1$.

Claim. $\{\omega_j\}_{j=1}^{\infty}$ exists and is a bounded sequence in $W^{2,p}$.

By classical elliptic theory, since $F_A^{0,2} \in L^p$, there exists a constant $C_1 > 0$ depending only on p such that

$$\|\omega_1\|_{W^{2,p}(B^4)} \leq C_1 \left\| \overline{\partial}\overline{\partial}^* \omega_1 \right\|_{L^p(B^4)} = C_1 \left\| F_A^{0,2} \right\|_{L^p(B^4)} < 2C_1 \left\| F_A^{0,2} \right\|_{L^p(B^4)}.$$

By induction we prove that ω_j exists and satisfies the uniform bound

$$\|\omega_j\|_{W^{2,p}(B^4)} \leq 2C_1 \left\| F_A^{0,2} \right\|_{L^p(B^4)}.$$

We assume that ω_j exists and $\|\omega_j\|_{W^{2,p}(B^4)} \leq 2C_1 \left\| F_A^{0,2} \right\|_{L^p(B^4)}$ and prove that ω_{j+1} exists with the same $W^{2,p}$ bound. By the Sobolev embedding $W^{1,p} \hookrightarrow L^{4p/(4-p)}$, there exist constants $C_2 > 0, C_3 > 0$ so that

$$\begin{aligned} \left\| \overline{\partial}^* \omega_j \right\|_{L^{4p/(4-p)}(B^4)} &\leq C_2 \|\nabla \omega_j\|_{W^{1,p}(B^4)} \\ &\leq C_2 \|\omega_j\|_{W^{2,p}(B^4)} \\ &\leq 2C_1 \cdot C_2 \left\| F_A^{0,2} \right\|_{L^p(B^4)} \end{aligned}$$

and

$$\left\| A^{0,1} \right\|_{L^{4p/(4-p)}(B^4)} \leq C_3 \left\| A^{0,1} \right\|_{W^{1,p}(B^4)} \leq C_3 \varepsilon$$

In addition, since $4p/(4-p) \geq 2p$ for any $p \geq 2$, then $W^{1,p}$ continuously embeds into L^{2p} and we can bound $\|F_A\|_{L^p(B^4)}$ as such:

$$\begin{aligned} \|F_A\|_{L^p(B^4)} &\leq \|dA\|_{L^p(B^4)} + \|A\|_{L^{2p}(B^4)}^2 \leq \|A\|_{W^{1,p}(B^4)} + C_3 \|A\|_{W^{1,p}(B^4)}^2 \\ &\leq \|A\|_{W^{1,p}(B^4)} + C_3 \varepsilon \|A\|_{W^{1,p}(B^4)} \\ &= (1 + C_3 \varepsilon) \|A\|_{W^{1,p}(B^4)}. \end{aligned}$$

Define the constant $C_4 := 1 + C_3\varepsilon$. Moreover, since $p \geq 2$, we have the embedding $L^{2p/(4-p)} \hookrightarrow L^p$. Denote

$$f_j := -[\overline{\partial}^* \omega_j, A^{0,1}] - \overline{\partial}^* \omega_j \wedge \overline{\partial}^* \omega_j - F_A^{0,2}.$$

Using the estimates above we obtain:

$$\|f_j\|_{L^p(B^4)} \leq \left\| [\overline{\partial}^* \omega_j, A^{0,1}] \right\|_{L^p(B^4)} + \left\| \overline{\partial}^* \omega_j \wedge \overline{\partial}^* \omega_j \right\|_{L^p(B^4)} + \left\| F_A^{0,2} \right\|_{L^p(B^4)}$$

$$
\begin{aligned}
&\leq \left\|[\overline{\partial}^*\omega_j, A^{0,1}]\right\|_{L^{2p/(4-p)}(B^4)} + \left\|\overline{\partial}^*\omega_j \wedge \overline{\partial}^*\omega_j\right\|_{L^{2p/(4-p)}(B^4)} \\
&\quad + \left\|F_A^{0,2}\right\|_{L^p(B^4)} \\
&\leq \left\|\overline{\partial}^*\omega_j\right\|_{L^{4p/(4-p)}(B^4)} \left\|A^{0,1}\right\|_{L^{4p/(4-p)}(B^4)} + \left\|\overline{\partial}^*\omega_j\right\|^2_{L^{4p/(4-p)}(B^4)} \\
&\quad + \left\|F_A^{0,2}\right\|_{L^p(B^4)} \\
&\leq C_2C_3\left\|F_A^{0,2}\right\|_{L^p(B^4)}\varepsilon + 4C_1^2\cdot C_2^2\left\|F_A^{0,2}\right\|^2_{L^p(B^4)} + \left\|F_A^{0,2}\right\|_{L^p(B^4)} \\
&\leq C_2C_3\left\|F_A^{0,2}\right\|_{L^p(B^4)}\varepsilon + 4C_1^2\cdot C_2^2C_4\left\|F_A^{0,2}\right\|_{L^p(B^4)}\|A\|_{W^{1,p}(B^4)} \\
&\quad + \left\|F_A^{0,2}\right\|_{L^p(B^4)} \\
&\leq (C_2C_3\varepsilon + 4C_1^2C_2^2C_4\varepsilon + 1)\left\|F_A^{0,2}\right\|_{L^p(B^4)}
\end{aligned}
$$

Hence, $-[\overline{\partial}^*\omega_j, A^{0,1}] - \overline{\partial}^*\omega_j \wedge \overline{\partial}^*\omega_j - F_A^{0,2} \in L^p$ and the solution ω_{j+1} to the PDE

$$
\begin{cases}
\overline{\partial}\overline{\partial}^*\omega_{j+1} = -[\overline{\partial}^*\omega_j, A^{0,1}] - \overline{\partial}^*\omega_j \wedge \overline{\partial}^*\omega_j - F_A^{0,2} & \text{in } B^4 \\
\omega_{j+1} = 0 & \text{on } \partial B^4
\end{cases}
\tag{3.1}
$$

exists. Choosing $\varepsilon > 0$ such that

$$
C_2C_3\varepsilon + 4C_1^2C_2^2C_4\varepsilon < 1
$$

is satisfied, it follows that we can obtain the required bound:

$$
\|\omega_{j+1}\|_{W^{2,p}(B^4)} \leq C_1\left\|\overline{\partial}^*\overline{\partial}\omega_{j+1}\right\|_{L^p(B^4)} = C_1\|f_j\|_{L^p(B^4)} \leq 2C_1\left\|F_A^{0,2}\right\|_{L^p(B^4)}.
$$

Hence, by induction, we have proven the claim.

Claim. $\{\omega_j\}_{j=1}^{\infty}$ is a Cauchy sequence.

Since each ω_j satisfies the elliptic PDE (3.1), we can estimate the difference

$\omega_{j+1} - \omega_j$ as such:

$$\begin{aligned}\|\omega_{j+1} - \omega_j\|_{W^{2,p}(B^4)} &\leq C\Big(\|\omega_j - \omega_{j-1}\|_{W^{2,p}(B^4)} \|A^{0,1}\|_{W^{1,p}(B^4)} \\ &\quad + \|F_A^{0,2}\|_{L^p} \|\omega_j - \omega_{j-1}\|_{W^{2,p}(B^4)}\Big) \\ &\leq 2C\varepsilon \|\omega_j - \omega_{j-1}\|_{W^{2,p}(B^4)}\end{aligned}$$

where $C > 0$ is a constant depending on p. Choosing ε such that in addition $2C\varepsilon < 1$ is satisfied, it then follows that the sequence is Cauchy. Because $W^{2,p}$ is a Banach space and the sequence $\{\omega_j\}_{j=1}^{\infty}$ is Cauchy, we have that the sequence converges strongly in $W^{2,p}$ to a limit which we denote by ω. Moreover, by construction ω satisfies the PDE:

$$\begin{cases} \overline{\partial}\overline{\partial}^*\omega &= -[\overline{\partial}^*\omega, A^{0,1}] - \overline{\partial}^*\omega \wedge \overline{\partial}^*\omega - F_A^{0,2} \quad \text{in } B^4 \\ \omega &= 0 \quad \text{on } \partial B^4. \end{cases}$$

Define $\tilde{A} = A + \overline{\partial}^*\omega - \overline{\overline{\partial}^*\omega}^T$. Then $F_{\tilde{A}}^{0,2} = 0$ and

$$\left\|\tilde{A} - A\right\|_{W^{1,p}} = \left\|\overline{\partial}^*\omega - \overline{\overline{\partial}^*\omega}^T\right\|_{W^{1,p}} \leq 4C_1 \left\|F_A^{0,2}\right\|_{L^p}.$$

□

Using the result above, we can prove the following main theorem of this section:

Theorem 3.1. *There exists $\varepsilon_0 > 0$ such that if $A \in W^{1,2}(\Omega^1 B^4, \mathfrak{u}(n))$ satisfies the smallness condition*

$$\|A\|_{W^{1,2}(B^4)} \leq \varepsilon_0$$

and the integrability condition $F_A^{0,2} = 0$, then there exists a sequence $A_k \in C^\infty(\Omega^1 B^4, \mathfrak{u}(n))$ so that:

$$A_k \to A \text{ in } W^{1,2}(\Omega^1 B^4, \mathfrak{u}(n))$$

and satisfies the integrability condition $F_{A_k}^{0,2} = 0$.

Proof of Theorem 3.1. As we have discussed at the start of this section, we can always construct a sequence of smooth 1-forms $\hat{A}_k$ that converge in $W^{1,2}$

to A and satisfy$F^{0,2}_{\hat{A}_k} \to 0 = F^{0,2}_A$ in L^2 as $k \to \infty$. Let $\varepsilon > 0$ be the constant given by Lemma 3.1 and pick $\varepsilon_0 = \varepsilon/2$. Then there exists $k_0 \geq 0$ such that $\left\|\hat{A}_k - A\right\|_{W^{1,2}(B^4)} \leq \varepsilon_0$ for all $k \geq k_0$ and:

$$\left\|\hat{A}_k\right\|_{W^{1,2}(B^4)} \leq \left\|\hat{A}_k - A\right\|_{W^{1,2}(B^4)} + \|A\|_{W^{1,2}(B^4)} \leq 2\varepsilon_0 = \varepsilon.$$

Thus, for each $k \geq k_0$ we can apply Lemma 3.1 in order to obtain a perturbed sequence A_k satisfying the integrability condition $F^{0,2}_{A_k} = 0$, such that there exists a constant $C > 0$

$$\left\|A_k - \hat{A}_k\right\|_{W^{1,2}} \leq C \left\|F^{0,2}_{\hat{A}_k}\right\|_{L^2} \to 0.$$

Thus,

$$\begin{aligned}\|A_k - A\|_{W^{1,2}} &\leq \left\|A_k - \hat{A}_k\right\|_{W^{1,2}} + \left\|\hat{A}_k - A\right\|_{W^{1,2}} \\ &\leq C \left\|F^{0,2}_{\hat{A}_k}\right\|_{L^2} + \left\|A - \hat{A}_k\right\|_{W^{1,2}} \to 0\end{aligned}$$

as $k \to \infty$. This concludes the statement. □

3.2 Existence of holomorphic trivialisations

In this section we prove that under the integrability condition $F^{0,2}_A = 0$ we obtain the existence of local holomorphic trivialisations assuming low $W^{1,2}$ norm for A as in the section before. We state the result:

Theorem 3.2. *There exists $\varepsilon_0 > 0$ such that if $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ satisfies*

$$\|A\|_{W^{1,2}(B^4)} \leq \varepsilon_0,$$

and the integrability condition $F^{0,2}_A = 0$, then there exists $r > 0$ and $g \in W^{2,q}(B^4_r, GL_n(\mathbb{C}))$ for all $q < 2$ such that

$$A^{0,1} = -\overline{\partial} g \cdot g^{-1} \qquad \text{in } B^4_r. \tag{3.2}$$

Moreover, there exists a constant $C_q > 0$ such that the following estimates hold:

$$\|g - id\|_{W^{2,q}(B^4_r)} \leq C_q \|A\|_{W^{1,2}(B^4)} \quad \text{and} \quad \left\|g^{-1} - id\right\|_{W^{2,q}(B^4_r)} \leq C_q \|A\|_{W^{1,2}(B^4)}.$$

It follows that $A^g = h^{-1}\partial h$ *where* $h = \overline{g}^T g$.

This result is an analogue of the following real case framework: Let P be a principal G- bundle over B^4, where G is a compact Lie group. Assume A is a connection 1-form on B^4. The flatness condition $F_A = 0$ together with the compactness of G imply that $A = -dg \cdot g^{-1}$ where $g \in W^{2,2}(B^4) \cap L^\infty(B^4)$. This can be easily concluded by using Uhlenbeck's gauge extraction procedure [41]. In the complex framework, however, due to the lack of compactness of the group $GL_n(\mathbb{C})$, we fail to obtain $W^{2,2} \cap L^\infty$ regularity of g and g^{-1}.

We can assume that the ball of radius 2 B_2^4 equipped with the canonical complex structure is holomorphically embedded into $\mathbb{CP}^2$ (simply take the embedding $(z, w) \mapsto [z, w, 1]$).

Strategy:
Since this proof is quite technical, we start by describing the strategy. We will first prove in Proposition 3.2 that we can extend a small perturbation $A + V$ of our connection 1-form A from B^4 to $\mathbb{CP}^2$ while also keeping the integrability condition (1.1). Secondly, Lemma 3.2 shows that this extended form is holomorphically trivialisable in the sense of (3.2). Thirdly, Lemma 3.3 proves a technical result which shows the existence of holomorphic trivialisations of forms that are more regular than $W^{1,2}$. This will help us later to cancel the initial perturbation V we have added.

By combining all these steps, theorem 3.2 proves the existence of holomorphic trivialisation of our initial form $A^{0,1}$ in B_r^4 for some $r > 0$. We conclude the section with Corollary 3.1 which proves a stability result of the trivialisations we obtain.

Before we start we need to prove the following technical proposition:

Proposition 3.1. *There exists* $\varepsilon > 0$ *such that for any* $\tilde{A} \in W^{1,2}(\Omega^1\mathbb{CP}^2 \otimes \mathfrak{u}(n))$ *satisfying the bound* $\left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq \varepsilon$, *the operator*

$$L_{\tilde{A}} : W^{2,2}(\Omega^2\mathbb{CP}^2 \otimes M_n(\mathbb{C})) \to L^2(\Omega^2\mathbb{CP}^2 \otimes M_n(\mathbb{C}))$$

defined by

$$L_{\tilde{A}}(\omega) = \overline{\partial\partial}^* \omega + [\tilde{A}^{0,1}, \overline{\partial}^* \omega] \tag{3.3}$$

is Fredholm and invertible.

Proof of Proposition 3.1. It follows from Gårding's Inequality, that the operator $\overline{\partial\partial}^* = \frac{1}{2}\Delta_d$ is elliptic over $\mathbb{CP}^2$ (see [12, p. 93]). Because $\mathbb{CP}^2$ is a compact manifold, then $\mathrm{Ker}\frac{1}{2}\Delta_d$ and $\mathrm{Coker}\frac{1}{2}\Delta_d$ are finite dimensional spaces. By definition it follows that $\overline{\partial\partial}^*$ is Fredholm.

Let $\varepsilon > 0$ be as defined in [32, Theorem 4.4.2 (ii), p.185] such that

$$\left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq \varepsilon$$

is small in $W^{1,2}$ norm. It follows that the operator $[\tilde{A}^{0,1}, \overline{\partial}^* \cdot]$ has small operator norm $W^{2,2}$ to L^2:

$$\begin{aligned}\left|\left|\left|[\tilde{A}^{0,1}, \overline{\partial}^* \cdot]\right|\right|\right| &= \sup_{\|\omega\|_{W^{2,2}}=1} \left\|[\tilde{A}^{0,1}, \overline{\partial}^* \omega]\right\|_{L^2(\mathbb{CP}^2)} \\ &\leq \left\|\tilde{A}^{0,1}\right\|_{L^4(\mathbb{CP}^2)} \left\|\overline{\partial}^* \omega\right\|_{L^4(\mathbb{CP}^2)} \\ &\leq C \left\|\tilde{A}^{0,1}\right\|_{W^{1,2}(\mathbb{CP}^2)} \|\omega\|_{W^{2,2}(\mathbb{CP}^2)} \\ &\leq C\varepsilon,\end{aligned}$$

for some constant $C > 0$ coming from the Sobolev embedding $W^{1,2} \hookrightarrow L^4$. Hence, from the continuity of the index maps [32, Theorem 4.4.2, p.185], we have that $L_{\tilde{A}}$ is Fredholm and has the same index as $\overline{\partial\partial}^*$ mapping $W^{2,2}(\mathbb{CP}^2, M_n(\mathbb{C}))$ to $L^2(\mathbb{CP}^2, M_n(\mathbb{C}))$.

It is well-known that there are no global nonzero holomorphic $(0, 2)$-forms on $\mathbb{CP}^2$ [12, p. 118]. The lack of holomorphic $(0, 2)$-forms implies that $\overline{\partial\partial}^*$ defined from $W^{2,2}(\mathbb{CP}^2)$ to $L^2(\mathbb{CP}^2)$ is an invertible operator on the space of $(0, 2)$-forms and consequently has index 0. Thus, it follows that $\mathrm{index}(L_{\tilde{A}}) = \mathrm{index}(\overline{\partial\partial}^*) = 0$.

It remains to show that $L_{\tilde{A}}$ has trivial kernel. Once we have shown this, we can use the zero index of $L_{\tilde{A}}$ in order to conclude that $L_{\tilde{A}}$ is invertible. Assume $\omega \in KerL_{\tilde{A}}$. Hence, ω satisfies

$$\overline{\partial\partial}^* \omega = -[\tilde{A}^{0,1}, \overline{\partial}^* \omega].$$

By the Fredholm Lemma [32, Lemma 4.3.9] and the Sobolev embedding $W^{1,2} \hookrightarrow L^4$, we obtain

$$\begin{aligned}\|\omega\|_{W^{2,2}} \leq C \left\|\overline{\partial\partial}^* \omega\right\|_{L^2} &\leq C(\|L_{\tilde{A}}(\omega)\|_{L^2} + \left\|[\tilde{A}^{0,1}, \overline{\partial}^* \omega]\right\|_{L^2}) \\ &\leq C \|L_{\tilde{A}}(\omega)\|_{L^2} + \left\|\tilde{A}^{0,1}\right\|_{L^4} \left\|\overline{\partial}^* \omega\right\|_{L^4}\end{aligned}$$

$$\leq C \left\| L_{\tilde{A}}(\omega) \right\|_{L^2} + C'\varepsilon \left\| \omega \right\|_{W^{2,2}}$$

for some constants $C, C' > 0$. We can take the term $C'\varepsilon \left\| \omega \right\|_{W^{2,2}}$ on the left hand side of the inequality:

$$(1 - C'\varepsilon) \left\| \omega \right\|_{W^{2,2}} \leq C \left\| L_{\tilde{A}}(\omega) \right\|_{L^2}.$$

Choosing $\varepsilon > 0$ such that $1 - C'\varepsilon > \frac{1}{2}$, then we can divide by the positive factor $1 - C'\varepsilon$. We obtain the bound:

$$\left\| \omega \right\|_{W^{2,2}} \leq \frac{C}{1 - C'\varepsilon} \left\| L_{\tilde{A}}(\omega) \right\|_{L^2}.$$

Because $\omega \in KerL_{\tilde{A}}$, we have that $\omega = 0$. Since ω was arbitrarily chosen from the kernel, it follows that the kernel of $L_{\tilde{A}}$ is trivial: $KerL_{\tilde{A}} = \{0\}$. This finishes the proof. □

Having this result at our disposal, we can prove the existence of a $\mathbb{CP}^2$ extension of our connection form A, keeping the integrability condition (1.1). We assume the holomorphic embedding of B^4 in $\mathbb{CP}^2$.

Proposition 3.2. *There exists $\varepsilon > 0$ such that for any $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ satisfying $F_A^{0,2} = 0$ and $\|A\|_{W^{1,2}(B^4)} < \varepsilon$, there exists $\tilde{A} \in W^{1,2}(\Omega^1 \mathbb{CP}^2 \otimes \mathfrak{u}(n))$ that satisfies*

$$F_{\tilde{A}}^{0,2} = 0$$

in $\mathbb{CP}^2$ and $\omega \in W^{2,2}(\Omega^{0,2}\mathbb{CP}^2 \otimes M_n(\mathbb{C}))$ such that $\tilde{A}^{0,1} = A^{0,1} + \vartheta\omega$ in B^4.

Moreover, ω satisfies the estimate

$$\left\| \omega \right\|_{W^{2,2}(\mathbb{CP}^2)} \leq C \left\| A \right\|_{W^{1,2}(B^4)}$$

for some constant $C > 0$.

We recall to the reader that according to equation (A.1), we have $\vartheta = -*\partial*$, which is the formal adjoint of $\overline{\partial}$.

Proof of Proposition 3.2. Step 1. Since A is unitary, we can decompose A into its $(0,1)$ and $(1,0)$ parts as such: $A = A^{0,1} - \overline{A^{0,1}}^T$ where

$$A^{0,1} = \alpha_1 d\overline{z_1} + \alpha_2 d\overline{z_2}$$

and $\alpha_i \in W^{1,2}(B^4, \mathfrak{u}(n))$ for $i = 1, 2$. We extend each α_i into B_2^4 to a compactly supported map $\hat{\alpha}_i$, so that $\hat{\alpha}_i = 0$ in $B_2^4 \setminus B_{3/2}^4$. For constructing

$\hat{\alpha}_i$, for each $i = 1, 2$ we solve:

$$\begin{cases} \Delta\phi_i = 0 & \text{in } B^4_{3/2} \setminus B^4_1 \\ \phi_i = \alpha_i & \text{on } \partial B^4_1 \\ \phi_i = 0 & \text{on } \partial B^4_{3/2} \end{cases}$$

Such solutions exist by [20, Remark 7.2, Chapter 2] and satisfy

$$\|\phi_i\|_{W^{1,2}(B^4_{3/2}\setminus B^4_1)} \leq C\,\|\alpha_i\|_{H^{1/2}(\partial B^4_1)} \leq C'\,\|\alpha_i\|_{W^{1,2}(B^4_1)}$$

for some constants $C, C' > 0$. We can now define the following extensions on B^4_2:

$$\hat{\alpha}_i = \begin{cases} \alpha_i & \text{in } B^4_1 \\ \phi_i & \text{in } B^4_{3/2} \setminus B^4_1 \\ 0 & \text{in } B^4_2 \setminus B^4_{3/2}. \end{cases}$$

By the construction of ϕ_i, the functions $\hat{\alpha}_i$ are well-defined $W^{1,2}(B^4_2)$ Sobolev maps that satisfy the estimate:

$$\|\hat{\alpha}_i\|_{W^{1,2}(B^4_2)} \leq C\,\|\alpha_i\|_{W^{1,2}(B^4_1)}\,.$$

Define the $(0,1)$-form $\hat{A}^{0,1} = \hat{\alpha}_1 d\overline{z_1} + \hat{\alpha}_2 d\overline{z_2}$ and

$$\hat{A} := \hat{A}^{0,1} - \overline{\hat{A}^{0,1}}^T \in W^{1,2}(\Omega^1 B^4_2 \otimes \mathfrak{u}(n)).$$

By covering $\mathbb{CP}^2 \setminus B^4_2$ with coordinate charts, we can trivially extend $\hat{A}$ by 0 on $\mathbb{CP}^2 \setminus B^4_2$. Thus, we have obtained $\hat{A} \in W^{1,2}(\Omega^1\mathbb{CP}^2 \otimes \mathfrak{u}(n))$ and there exists a constant $\hat{C} > 0$ such that $\left\|\hat{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq \hat{C}\,\|A\|_{W^{1,2}(B^4)}$.

Step 2. It remains to perturb the form $\hat{A}$ so that we obtain the integrability condition. This can be done by finding a $(0,2)$-form solution ω to the integrability condition:

$$F^{0,2}_{\hat{A}+\overline{\partial}^*\omega-\overline{\overline{\partial}^*\omega}^T} = 0.$$

This amounts to solving the following PDE globally on the complex projective space $\mathbb{CP}^2$:

$$\overline{\partial}\overline{\partial}^*\omega + [\hat{A}^{0,1}, \overline{\partial}^*\omega] = -\overline{\partial}^*\omega \wedge \overline{\partial}^*\omega - F^{0,2}_{\hat{A}} \tag{3.4}$$

where ω is a $(0,2)$ form on $\mathbb{CP}^2$. Using the invertibility of the operator $L_{\hat{A}}$ proven in Proposition 3.1, we can solve equation (3.4) using a fixed point method. This is done by mimicking the procedure we have employed before, in Lemma 3.1. Indeed, consider the sequence given by:

$$\begin{aligned} L_{\hat{A}}(\omega_1) &= -F^{0,2}_{\hat{A}} \\ L_{\hat{A}}(\omega_2) &= -\overline{\partial}^* \omega_1 \wedge \overline{\partial}^* \omega_1 - F^{0,2}_{\hat{A}} \\ &\dots \\ L_{\hat{A}}(\omega_k) &= -\overline{\partial}^* \omega_{k-1} \wedge \overline{\partial}^* \omega_{k-1} - F^{0,2}_{\hat{A}} \\ &\dots \end{aligned} \tag{3.5}$$

By showing that the sequence ω_k converges strongly in $W^{2,2}$, we will obtain a $W^{2,2}$ solution to the required equation (3.4). Since $L_{\hat{A}}$ is invertible as an operator from $W^{2,2}$ to L^2, it is clear that existence holds for each ω_k, $k \geq 1$. We need to show that the sequence $\{\omega_k\}_{k=1}^{\infty}$ is Cauchy in $W^{2,2}$.

Claim. $\{\omega_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $W^{2,2}(\mathbb{CP}^2)$.

Let $\varepsilon_0 := C \left\| F^{0,2}_{\hat{A}} \right\|_{L^2(\mathbb{CP}^2)}$, where $C > 0$ is the constant appearing in Fredholm inequality:

$$\|\phi\|_{W^{2,2}(\mathbb{CP}^2)} \leq C \|L_{\hat{A}}(\phi)\|_{L^2(\mathbb{CP}^2)}.$$

We first show by induction the uniform bound on the sequence $\|\omega_k\|_{W^{2,2}(\mathbb{CP}^2)} \leq 2\varepsilon_0$. By the Fredholm Lemma [32, Lemma 4.3.9] we have that

$$\|\omega_1\|_{W^{2,2}(\mathbb{CP}^2)} \leq C \|L_{\hat{A}}(\omega_1)\|_{L^2(\mathbb{CP}^2)} = C \left\| F^{0,2}_{\hat{A}} \right\|_{L^2(\mathbb{CP}^2)} = \varepsilon_0 < 2\varepsilon_0$$

Let $k \geq 1$. By the Sobolev embedding $W^{1,2}(\mathbb{CP}^2) \hookrightarrow L^4(\mathbb{CP}^2)$ there exists a constant $C_1 > 0$ so that

$$\left\| \overline{\partial}^* \omega_k \right\|_{L^4(\mathbb{CP}^2)} \leq C_1 \left\| \overline{\partial}^* \omega_k \right\|_{W^{1,2}(\mathbb{CP}^2)} \leq C_1^2 \|\omega_k\|_{W^{2,2}(\mathbb{CP}^2)}.$$

Thus, the inequalities follow:

$$\begin{aligned} \|\omega_{k+1}\|_{W^{2,2}(\mathbb{CP}^2)} &\leq C \|L_{\hat{A}}(\omega_{k+1})\|_{L^2(\mathbb{CP}^2)} \\ &\leq C \left\| \overline{\partial}^* \omega_k \wedge \overline{\partial}^* \omega_k \right\|_{L^2(\mathbb{CP}^2)} + C \left\| F^{0,2}_{\hat{A}} \right\|_{L^2(\mathbb{CP}^2)} \\ &\leq C \|\omega_k\|^2_{L^4(\mathbb{CP}^2)} + \varepsilon_0 \end{aligned}$$

$$\leq C \cdot C_1^4 \left\| \omega_k \right\|^2_{W^{2,2}(\mathbb{CP}^2)} + \varepsilon_0$$

By the induction hypothesis, we have the bound $\left\| \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)} < 2\varepsilon_0$. Thus,

$$\left\| \omega_{k+1} \right\|_{W^{2,2}(\mathbb{CP}^2)} \leq 4(C \cdot C_1^4)\varepsilon_0^2 + \varepsilon_0. \tag{3.6}$$

Having chosen $\varepsilon > 0$ such that $4(C \cdot C_1^4)\varepsilon < 1$ and

$$\left\| \hat{A} \right\|_{W^{1,2}(\mathbb{CP}^2)} \leq \hat{C} \left\| A \right\|_{W^{1,2}(B^4)} \leq \varepsilon,$$

it follows that $4(C \cdot C_1^4)\varepsilon_0 < 1$. We conclude from (3.6) the estimate:

$$\left\| \omega_{k+1} \right\|_{W^{2,2}(\mathbb{CP}^2)} \leq 2\varepsilon_0.$$

By induction, we prove that there is a uniform bound for the sequence of 2-forms $\{\omega_k\}$:

$$\left\| \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)} \leq 2\varepsilon_0.$$

for all $k \geq 1$.

It remains to show that $\{\omega_k\}$ is a Cauchy sequence. Let $k \geq 2$. Thus, we derive the following bounds from the recurrence relation (3.5) satisfied by the sequence:

$$\begin{aligned}
\left\| \omega_{k+1} - \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)} &\leq C \left\| L_{\hat{A}}(\omega_{k+1} - \omega_k) \right\|_{L^2(\mathbb{CP}^2)} \\
&\leq C \left\| \overline{\partial}^*(\omega_k - \omega_{k-1}) \wedge \overline{\partial}^* \omega_k \right\|_{L^2(\mathbb{CP}^2)} + \\
&\quad C \left\| \overline{\partial}^* \omega_{k-1} \wedge \overline{\partial}^*(\omega_k - \omega_{k-1}) \right\|_{L^2(\mathbb{CP}^2)} \\
&\leq 4C \cdot C_1^2 \varepsilon_0 \left\| \omega_k - \omega_{k-1} \right\|_{W^{2,2}(\mathbb{CP}^2)}
\end{aligned}$$

To simplify notation, denote $\varepsilon_1 := 4C \cdot C_1^2 \varepsilon_0 < 1$. We further expand our estimate above:

$$\begin{aligned}
\left\| \omega_{k+1} - \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)} &\leq \varepsilon \left\| \omega_k - \omega_{k-1} \right\|_{W^{2,2}(\mathbb{CP}^2)} \\
&\leq \ldots \leq \varepsilon_1^k \left\| \omega_1 - \omega_0 \right\|_{W^{2,2}(\mathbb{CP}^2)} \\
&\leq 4\varepsilon_1^k \varepsilon_0^2
\end{aligned}$$

Let $\ell > k > 0$. It follows that

$$\left\| \omega_\ell - \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)} \leq \left\| \omega_\ell - \omega_{\ell-1} \right\|_{W^{2,2}(\mathbb{CP}^2)} + \left\| \omega_{\ell-1} - \omega_k \right\|_{W^{2,2}(\mathbb{CP}^2)}$$

$$\begin{aligned}&\leq 4\varepsilon_1^{\ell-1}\varepsilon_0^2 + \|\omega_{\ell-1} - \omega_{\ell-2}\|_{W^{2,2}(\mathbb{CP}^2)} + \|\omega_{\ell-2} - \omega_k\|_{W^{2,2}(\mathbb{CP}^2)}\\ &\leq 4\varepsilon_1^{\ell-1}\varepsilon_0^2 + 4\varepsilon_1^{\ell-2}\varepsilon_0^2 + \ldots + 4\varepsilon_1^k\varepsilon_0^2\\ &= 4\varepsilon_0^2 \cdot \varepsilon_1^k \cdot \frac{1-\varepsilon_1^{\ell-k}}{1-\varepsilon_1} \leq \varepsilon_1^k\end{aligned}$$

This is clearly a Cauchy sequence by the inequality above and the claim is proven.

Hence, since $\{\omega_k\}_{k=1}^{\infty}$ is a Cauchy sequence in the Banach space of $(0,2)$ forms $W^{2,2}(\Omega^{0,2}\mathbb{CP}^2 \otimes M_n(\mathbb{C}))$, it has a limit ω and converges strongly in $W^{2,2}$ to it. Hence, by defining $\tilde{A} = \hat{A} + \overline{\partial}^*\omega - \overline{\overline{\partial}^*\omega}^T$, we obtain a skew-Hermitian 1-form, satisfying $F_{\tilde{A}}^{0,2} = 0$ such that $\tilde{A}^{0,1} = \hat{A}^{0,1} + \vartheta\omega = A^{0,1} + \vartheta\omega$ in B^4.

Moreover, by convergence, we have that the uniform bound is satisfied by the limiting form ω, indeed $\|\omega\|_{W^{2,2}} < 2\varepsilon_0 = 2C\,\|F_{\hat{A}}\|_{L^2}$. By construction of $\hat{A}$, it is clear that there exists a constant $C' > 0$ so that $\|F_{\hat{A}}\|_{L^2} \leq C'\,\|A\|_{W^{1,2}}$. This leads to the required estimate on ω, $\|\omega\|_{W^{2,2}(\mathbb{CP}^2)} \leq C\,\|A\|_{W^{1,2}(B^4)}$, where $C > 0$ is some constant. □

We prove that on $\mathbb{CP}^2$ we can obtain holomorphic trivialisations of $\tilde{A}$, a connection 1-form.

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any form $\tilde{A} \in W^{1,2}(\Omega^1\mathbb{CP}^2 \otimes \mathfrak{u}(n))$ satisfying the integrability condition (1.1) and $\left\|\tilde{A}\right\|_{W^{1,2}} < \varepsilon$, then there exist gauges $\tilde{g}, \tilde{g}^{-1} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$ for all $q < 2$ such that*

$$\tilde{A}^{0,1} = -\overline{\partial}\tilde{g} \cdot \tilde{g}^{-1}.$$

Futhermore for each $q < 2$ there exists a constant $C_q > 0$ such that

$$\|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} \tag{3.7}$$

and

$$\left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)}. \tag{3.8}$$

Proof of Lemma 3.2. We divide the proof into three steps. Using a fixed point argument the first two steps show the existence of a map $\tilde{g}$ satisfying $\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g}$. Step 3 shows the existence of $\tilde{g}^{-1}$ and proves the estimates (3.7) and (3.8).

Step 1. Consider the linear operator:

$$T : W^{1,2}(\mathbb{CP}^2, M_n(\mathbb{C})) \to W^{1,2}(\mathbb{CP}^2, M_n(\mathbb{C}))$$

given by

$$T(\tilde{g}) = -\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) + id,$$

where id is the constant identity matrix and N is the inverse operator of $\Delta_{\overline{\partial}}$ as defined in (A.3). We verify that T is well-defined. It follows from the Gårding inequality on $\mathbb{CP}^2$ (see for example [12, p. 93]) that we have the elliptic estimate

$$\left\|\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq C \left\|\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{L^2(\mathbb{CP}^2)} \tag{3.9}$$

for some constant C. Moreover, using the $\overline{\partial}$-Hodge decomposition (A.5) we can decompose $\tilde{A}^{0,1}\tilde{g}$ as such:

$$\tilde{A}^{0,1}\tilde{g} = \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) + \overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) \tag{3.10}$$

and because $\overline{\partial}^*$ and $\overline{\partial}$ are orthogonal with respect to the L^2 inner product, it follows that

$$\left\|\tilde{A}^{0,1}\tilde{g}\right\|_{L^2} = \left\|\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})\right\|_{L^2} + \left\|\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{L^2}.$$

Consequently,

$$\left\|\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{L^2(\mathbb{CP}^2)} \leq \left\|\tilde{A}^{0,1}\tilde{g}\right\|_{L^2(\mathbb{CP}^2)}. \tag{3.11}$$

Putting (3.9) and (3.11) together, we obtain:

$$\left\|\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq C \left\|\tilde{A}^{0,1}\tilde{g}\right\|_{L^2(\mathbb{CP}^2)} \leq C \left\|\tilde{A}^{0,1}\right\|_{L^4(\mathbb{CP}^2)} \|\tilde{g}\|_{L^4(\mathbb{CP}^2)}.$$

Furthermore, using the Sobolev embedding in 4-dimensions $W^{1,2} \hookrightarrow L^4$, there exists a constant C' so that

$$\left\|\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{W^{1,2}} \leq C' \left\|\tilde{A}^{0,1}\right\|_{W^{1,2}} \|\tilde{g}\|_{W^{1,2}}. \tag{3.12}$$

Thus, the operator T is well-defined, mapping $W^{1,2}$ to $W^{1,2}$.

We can now show that T has a unique fixed point. Consider $\tilde{g}_1, \tilde{g}_2 \in$

$W^{1,2}(\mathbb{CP}^2, M_n(\mathbb{C}))$. Then

$$\|T(\tilde{g}_1) - T(\tilde{g}_2)\|_{W^{1,2}(\mathbb{CP}^2)} = \left\|\overline{\partial}^* N(\tilde{A}^{0,1}(\tilde{g}_1 - \tilde{g}_2))\right\|_{W^{1,2}(\mathbb{CP}^2)}.$$

Using the inequality (3.12) above, we obtain

$$\left\|\overline{\partial}^* N(\tilde{A}^{0,1}(\tilde{g}_1 - \tilde{g}_2))\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq C' \left\|\tilde{A}^{0,1}\right\|_{W^{1,2}(\mathbb{CP}^2)} \|\tilde{g}_1 - \tilde{g}_2\|_{W^{1,2}(\mathbb{CP}^2)}$$

and we can choose $\varepsilon > 0$ for the bound $\left\|\tilde{A}^{0,1}\right\|_{W^{1,2}(\mathbb{CP}^2)} < \varepsilon$ so that the factor $C' \left\|\tilde{A}^{0,1}\right\|_{W^{1,2}(\mathbb{CP}^2)}$ is strictly smaller than 1. Hence, T is a contraction operator and there exists a unique fixed point $\tilde{g} \in W^{1,2}(\mathbb{CP}^2, M_n(\mathbb{C}))$, $T(\tilde{g}) = \tilde{g}$. Thus, we have

$$\overline{\partial}\tilde{g} = -\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}). \tag{3.13}$$

Step 2. We can now prove that the above equation (3.13) coupled with the integrability condition $F^{0,2}_{\tilde{A}^{0,1}} = 0$ satisfied by $\tilde{A}^{0,1}$, imply that $\tilde{g}$ solves the required PDE: $\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g}$. The $\overline{\partial}$-Hodge decomposition (3.10) gives

$$\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g} + \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}). \tag{3.14}$$

Since the operators N and $\overline{\partial}$ compute, $N\overline{\partial} = N\overline{\partial}$ (see [8, Theorem 4.4.1 (3)]), we can further compute the term $\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})$:

$$\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) = \overline{\partial}^* N\overline{\partial}(\tilde{A}^{0,1}\tilde{g}) = \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}\tilde{g}).$$

Using the above equation (3.14), then

$$\begin{aligned}\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) &= \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}\tilde{g}) \\ &= \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} + \tilde{A}^{0,1} \wedge \tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})).\end{aligned}$$

Since $\tilde{A}$ satisfies the integrability condition $F^{0,2}_{\tilde{A}} = 0$, we have the recurrence relation:

$$\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) = -\overline{\partial}^* N(\tilde{A}^{0,1} \wedge \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})). \tag{3.15}$$

Thus, it is natural to consider the operator

$$\mathcal{L} : L^2(\Omega^1\mathbb{CP}^2 \otimes M_n(\mathbb{C})) \to L^2(\Omega^1\mathbb{CP}^2 \otimes M_n(\mathbb{C}))$$

$$V \mapsto -\overline{\partial}^* N(\tilde{A}^{0,1} \wedge V).$$

We need to establish whether $\mathcal{L}$ is a well-defined operator and find its fixed points in order to analyse equation (3.15). By the Sobolev embedding $W^{1,4/3} \hookrightarrow L^2$ it follows that

$$\|\mathcal{L}(V)\|_{L^2(\mathbb{CP}^2)} \leq C \|\mathcal{L}(V)\|_{W^{1,4/3}(\mathbb{CP}^2)}$$

for some constant $C > 0$. We also have that

$$\|\nabla \mathcal{L}(V)\|_{L^{4/3}(\mathbb{CP}^2)} \leq C \left\| \nabla^2 N(\tilde{A}^{0,1} \wedge V) \right\|_{L^{4/3}(\mathbb{CP}^2)}$$

and consequently, since $N(\tilde{A}^{0,1} \wedge V)$ is a $(0,2)$-form in 4-dimensions, the elliptic estimate holds:

$$\begin{aligned} \left\| \nabla^2 N(\tilde{A}^{0,1} \wedge V) \right\|_{L^{4/3}(\mathbb{CP}^2)} &\leq C \left\| \overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1} \wedge V) \right\|_{L^{4/3}(\mathbb{CP}^2)} \\ &= C \left\| \tilde{A}^{0,1} \wedge V \right\|_{L^{4/3}(\mathbb{CP}^2)}. \end{aligned}$$

By the Hölder inequality and the estimates above, it immediately follows that:

$$\|\mathcal{L}(V)\|_{L^2(\mathbb{CP}^2)} \leq C \left\| \tilde{A}^{0,1} \right\|_{L^4(\mathbb{CP}^2)} \|V\|_{L^2(\mathbb{CP}^2)}.$$

Similarly as before, this means that $\mathcal{L}$ is a well-defined contraction operator and has a unique fixed point. In particular, 0 is its fixed point. We know from equation (3.15) that $\overline{\partial}^* \overline{\partial} N(\tilde{A}^{0,1}\tilde{g})$ is also a fixed point for $\mathcal{L}$ and, thus, we have that the term $\overline{\partial}^* \overline{\partial} N(\tilde{A}^{0,1}\tilde{g})$ vanishes. In particular, from (3.14) the equation is solved:

$$\overline{\partial} \tilde{g} = -\tilde{A}^{0,1} \tilde{g}.$$

Step 3. It remains to show that $\tilde{g} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$ for all $q < 2$ and that it satisfies the required estimate (3.7). Moreover, we need to show that its inverse satisfies the estimate (3.8). Let $q < 2$. We know that $\tilde{g}$ is a $W^{1,2}$ map and satisfies:

$$\tilde{g} - id = \overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}).$$

Since $\tilde{g}$ is a fixed point of T, then it satisfies the estimate (3.12), which means:

$$\begin{aligned} \|\tilde{g} - id\|_{W^{1,2}(\mathbb{CP}^2)} &\leq C \left\| \tilde{A}^{0,1} \right\|_{W^{1,2}(\mathbb{CP}^2)} \|\tilde{g}\|_{W^{1,2}} \\ &\leq C \left\| \tilde{A}^{0,1} \right\|_{W^{1,2}(\mathbb{CP}^2)} \|\tilde{g} - id\|_{W^{1,2}} + C \left\| \tilde{A}^{0,1} \right\|_{W^{1,2}(\mathbb{CP}^2)}. \end{aligned}$$

Because $\left\| \tilde{A}^{0,1} \right\|_{W^{1,2}(\mathbb{CP}^2)} < \varepsilon$, where ε is small, then there exists a constant

$C > 0$ such that

$$\|\tilde{g} - id\|_{W^{1,2}(\mathbb{CP}^2)} \leq C \left\| \tilde{A}^{0,1} \right\|_{W^{1,2}(\mathbb{CP}^2)} .$$

Since $\tilde{g}$ satisfies this estimate, we can bootstrap using Lemma D.1 and Remark D.1(i), from which follows the required estimate and regularity (3.7)

$$\|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\| \tilde{A} \right\|_{W^{1,2}(\mathbb{CP}^2)} , \tag{3.16}$$

for some constant $C_q > 0$.

We need to show that $\tilde{g}$ is in $GL_n(\mathbb{C})$ over $\mathbb{CP}^2$ and that its inverse satisfies a similar estimate as (3.16). Arguing in a similar way to Step 1, 2 and the way we obtained the regularity estimates for g in (3.16), we can show that there exists $\tilde{u} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$ for any $q < 2$ such that

$$\overline{\partial}\tilde{u} = \tilde{u}\tilde{A}^{0,1}$$

and

$$\|\tilde{u} - id\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\| \tilde{A} \right\|_{W^{1,2}(\mathbb{CP}^2)} \tag{3.17}$$

for some constant $C_q > 0$. In particular, we have that $\overline{\partial}(\tilde{u}\tilde{g}) = 0$. Hence, there exists a holomorphic function $\tilde{h}$ such that $\tilde{u}\tilde{g} = \tilde{h}$. However, since the only holomorphic functions on $\mathbb{CP}^2$ are the constant ones [12, p. 118], then $\tilde{h}$ is a constant.

We can pick $3/2 < q_0 < 2$ so that we obtain the Sobolev embedding $W^{2,q_0} \hookrightarrow L^\infty$ on any 3-dimensional hypersurface. Moreover, by the Sobolev products regularity results in [31, Section 4.8.2, Theorem 1], there exists $q_1 \in (q_0, 2)$ such that for $\tilde{u}, \tilde{g} \in W^{2,q_1}(\mathbb{CP}^2, M_n(\mathbb{C}))$, $\tilde{u}\tilde{g} \in W^{2,q_0}$ and

$$\|\tilde{u}\tilde{g} - id\|_{W^{2,q_0}(\mathbb{CP}^2)} \leq C_{q_0} \left\| \tilde{A} \right\|_{W^{1,2}(\mathbb{CP}^2)}$$

for some constant $C_{q_0} > 0$. By Fubini, there exists a radius $r > 0$ and $z_0 \in \mathbb{CP}^2$ such that

$$\|\tilde{u}\tilde{g} - id\|_{W^{2,q_0}(\partial B^4_r(z_0))} < 2C'_{q_0} \left\| \tilde{A} \right\|_{W^{1,2}(\mathbb{CP}^2)}$$

where $B^4_r(z_0)$ is a ball in $\mathbb{CP}^2$. Thus, from the embedding of W^{2,q_0} into L^∞ in 3-dimensions, there exists a constant $C''_{q_0} > 0$ so that

$$\|\tilde{u}\tilde{g} - id\|_{L^\infty(\partial B^4_r(z_0))} \leq C''_{q_0} \left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)}. \tag{3.18}$$

We can choose a possibly smaller $\varepsilon > 0$ than we have done for the estimate $\left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} < \varepsilon$ such that we obtain that $\tilde{h} = \tilde{u}\tilde{g} \in GL_n(\mathbb{C})$ over $\partial B_r(z_0)$ from (3.18). However, because $\tilde{h}$ is a constant, then $\tilde{h} \in GL_n(\mathbb{C})$ over $\mathbb{CP}^2$ and satisfies the estimate:

$$\left\|\tilde{h} - id\right\|_{L^\infty(\mathbb{CP}^2)} \leq C \left\|\tilde{A}\right\|_{W^{1,2}(B^4)},$$

for some constant $C > 0$.

Hence, we can define $\tilde{g}^{-1} := \tilde{h}^{-1}\tilde{u}$. Since $\tilde{g}^{-1}\tilde{g} = id$ by construction, we obtain that $\tilde{g}$ maps into $GL_n(\mathbb{C})$. Moreover, from the fact that $\tilde{h}^{-1}$ is a constant and from the estimate (3.17) it follows that $\tilde{g}^{-1} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$ for all $q < 2$. In particular, we obtain that for each $q < 2$ there exists a constant $C_q > 0$ such that

$$\left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \|A\|_{W^{1,2}(\mathbb{CP}^2)}.$$

This concludes the proof of Lemma 3.2. □

Before proving the existence of a local holomorphic trivialisation for our initial $W^{1,2}$ form A defined on B^4, we need to show a stronger version of existence of such trivialisations. We consider 1-forms of small norm in $W^{1,p}$, $p > 3$. This will be a useful result for our final theorem.

Lemma 3.3. *Let $p > 3$. There exists $\varepsilon > 0$ such that for any*

$$\omega \in W^{1,p}(\Omega^{0,1}B^4 \otimes M_n(\mathbb{C}))$$

satisfying $F^{0,2}_\omega = 0$ and $\|\omega\|_{W^{1,p}(B^4)} \leq \varepsilon$, there exists $r \in (1/2, 1)$ and gauges $u, u^{-1} \in W^{2,p}(B^4_r, GL_n(\mathbb{C}))$ so that

$$\omega = -\overline{\partial}u \cdot u^{-1} \qquad \text{in } B^4_r,$$

with estimates

$$\begin{gathered}\|u - id\|_{W^{2,p}(B^4_r)} \leq C \|\omega\|_{W^{1,p}(B^4)} \\ \text{and} \\ \left\|u^{-1} - id\right\|_{W^{2,p}(B^4_r)} \leq C \|\omega\|_{W^{1,p}(B^4)}.\end{gathered} \tag{3.19}$$

Remark 3.1. *The reader can note the fact that the technique to solve this lemma is similar to the ideas used in the Lemma 3.2. However, since we work on B^4, the proof will rely more on regularity results from the literature on the analysis of several complex variables.*

Proof of Lemma 3.3. Step 1. Let $q = 4p/(4-p)$. We show the existence of a gauge $u \in GL_n(\mathbb{C})$ that "almost" solves our equation modulo a perturbation term. Indeed, in Step 2 we can show that the perturbation term vanishes and consequentially u is the solution. Let T_1, T_2 be the operators defined as in (A.7) and (A.8). Note that we can extend T_1 and T_2 to operators defined on Sobolev spaces by density arguments.

We define the operator

$$\mathcal{H} : L^\infty(B^4, M_n(\mathbb{C})) \cap \{f : \overline{\partial} f \in L^q\} \to L^\infty(B^4, M_n(\mathbb{C})) \cap \{f : \overline{\partial} f \in L^q\}$$

given by

$$\mathcal{H}(u) = id + T_1(-\omega \cdot u),$$

where id is the constant identity matrix.

Claim. $\mathcal{H}$ is well-defined.

Since T_1 takes $(0,1)$-forms to maps we only need to check that $\mathcal{H}$ maps $L^\infty \cap \{f : \overline{\partial} f \in L^q\}$to $L^\infty \cap \{f : \overline{\partial} f \in L^q\}$. By the Sobolev embedding $W^{1,p} \hookrightarrow L^q$, there exists a constant $C_1 > 0$ so that

$$\|\omega\|_{L^q(B^4)} \leq C_1 \|\omega\|_{W^{1,p}(B^4)}.$$

The assumption $p > 3$ implies that $q = 4p/(4-p) > 12$. Consequently, for $u \in L^\infty(B^4, M_n(\mathbb{C})) \cap \{f : \overline{\partial} f \in L^q\}$ we have $\omega \cdot u \in L^q$. Moreover $\overline{\partial}(\omega \cdot u) \in L^{q/2}$. We prove this.

$$\begin{aligned}
\left\|\overline{\partial}(\omega \cdot u)\right\|_{L^{q/2}(B^4)} &\leq \left\|\overline{\partial}\omega\right\|_{L^{q/2}(B^4)} \|u\|_{L^\infty(B^4)} + \left\|\omega \wedge \overline{\partial} u\right\|_{L^{q/2}(B^4)} \\
&\leq \left\|\overline{\partial}\omega\right\|_{L^{q/2}(B^4)} \|u\|_{L^\infty(B^4)} + \|\omega\|_{L^q(B^4)} \left\|\overline{\partial} u\right\|_{L^q(B^4)}
\end{aligned}$$

Crucially, we have that $F_\omega^{0,2} = \overline{\partial}\omega + \omega \wedge \omega = 0$. Because $\omega \in L^q(B^4)$, then $F_\omega^{0,2} = 0$ gives $\overline{\partial}\omega \in L^{q/2}$. Thus,

$$\begin{aligned}
\left\|\overline{\partial}(\omega \cdot u)\right\|_{L^{q/2}(B^4)} &\leq \|\omega\|_{L^q(B^4)}^2 \|u\|_{L^\infty(B^4)} + \|\omega\|_{L^q(B^4)} \left\|\overline{\partial} u\right\|_{L^q(B^4)} \\
&\leq C_1 \varepsilon \|\omega\|_{L^q(B^4)} \|u\|_{L^\infty(B^4)} + \|\omega\|_{L^q(B^4)} \left\|\overline{\partial} u\right\|_{L^q(B^4)} \\
&\leq C_1^2 \|\omega\|_{W^{1,p}(B^4)} \left(\|u\|_{L^\infty(B^4)} + \left\|\overline{\partial} u\right\|_{L^q(B^4)} \right). \qquad (3.20)
\end{aligned}$$

where we have implicitly used the fact that we can choose $\varepsilon < 1$ so that $\|\omega\|_{W^{1,p}(B^4)} \leq \varepsilon$. Hence, we have shown that $\overline{\partial}(\omega \cdot u) \in L^{q/2}$. Taking into account that $q/2 > 6$ and the embedding $W^{1,q/2} \hookrightarrow L^q$, we can apply Proposition A.2 to the $(0,1)$-form $\omega \cdot u$ and obtain the estimate:

$$\begin{aligned} &\|T_1(\omega \cdot u)\|_{L^\infty(B^4)} + \left\|\overline{\partial} T_1(\omega \cdot u)\right\|_{L^q(B^4)} \\ &\qquad \leq C\left(\|\omega \cdot u\|_{L^q(B^4)} + \left\|\overline{\partial}(\omega \cdot u)\right\|_{L^{q/2}(B^4)}\right). \end{aligned} \tag{3.21}$$

This shows that, $\mathcal{H}$ is well-defined, since the operator $T_1(\omega\cdot)$ is a well-defined map from $L^\infty \cap \{f : \overline{\partial} f \in L^q\}$ to $L^\infty \cap \{f : \overline{\partial} f \in L^q\}$. We have proven the claim.

Next, we show that $\mathcal{H}$ has a fixed point. From (3.21), it follows that

$$\begin{aligned} &\|T_1(\omega \cdot u)\|_{L^\infty(B^4)} + \left\|\overline{\partial} T_1(\omega \cdot u)\right\|_{L^q(B^4)} \\ &\qquad \leq C\left(\|\omega\|_{L^q(B^4)} \|u\|_{L^\infty(B^4)} + \left\|\overline{\partial}(\omega \cdot u)\right\|_{L^{q/2}(B^4)}\right). \end{aligned}$$

Since $\|\omega\|_{W^{1,p}(B^4)} \leq \varepsilon$ and using (3.20), for any $u_1, u_2 \in L^\infty(B^4, M_n(\mathbb{C})) \cap \{f : \overline{\partial} f \in L^q\}$ we have:

$$\begin{aligned} &\|\mathcal{H}(u_1) - \mathcal{H}(u_2)\|_{L^\infty(B^4)} + \left\|\overline{\partial}\mathcal{H}(u_1) - \overline{\partial}\mathcal{H}(u_2)\right\|_{L^q(B^4)} \\ &= \|T_1(-\omega \cdot (u_1 - u_2))\|_{L^\infty(B^4)} + \left\|\overline{\partial} T_1(-\omega \cdot (u_1 - u_2))\right\|_{L^q(B^4)} \\ &\leq C\varepsilon\left(\|u_1 - u_2\|_{L^\infty(B^4)} + \left\|\overline{\partial}(u_1 - u_2)\right\|_{L^q(B^4)}\right), \end{aligned}$$

where C depends on the $W^{1,p}$ norm of ω. Choosing $\varepsilon > 0$ such that $C\varepsilon < 1$, we obtain that $\mathcal{H}$ is a contraction and therefore there exists $u \in L^\infty(B^4, M_n(\mathbb{C})) \cap \{f : \overline{\partial} f \in L^q\}$ satisfying

$$u = id + T_1(-\omega \cdot u) = \mathcal{H}(u).$$

This fixed point "almost" solves the required equation. We will show in the next step that the error we obtain vanishes in light of the integrability condition $F_\omega^{0,2} = 0$.

Step 2. Having obtained this fixed point, we show that u satisfies $\overline{\partial} u = -\omega \cdot u$. Since we have proven that $u - id = T_1(-\omega \cdot u)$, we get $\overline{\partial} u = \overline{\partial} T_1(-\omega \cdot u) \in L^q$. We can apply Theorem A.1 from the Appendix to get the integral representation of $-\omega \cdot u$:

$$-\omega \cdot u = \overline{\partial} T_1(-\omega \cdot u) + T_2(\overline{\partial}(-\omega \cdot u))$$

and expand the last term in the following way:

$$\begin{aligned} T_2(\overline{\partial}(-\omega \cdot u)) &= T_2(-\overline{\partial}\omega \cdot u + \omega \wedge \overline{\partial} u) \\ &= T_2(-\overline{\partial}\omega \cdot u + \omega \wedge \overline{\partial} T_1(-\omega \cdot u)) \\ &= T_2(-\overline{\partial}\omega \cdot u + \omega \wedge (-\omega \cdot u - T_2(\overline{\partial}(-\omega \cdot u))) \\ &= T_2(-(\overline{\partial}\omega + \omega \wedge \omega)u - \omega \wedge T_2(\overline{\partial}(-\omega \cdot u))). \end{aligned}$$

By using the fact that ω satisfies the integrability condition

$$F_\omega^{0,2} = \overline{\partial}\omega + \omega \wedge \omega = 0,$$

we obtain:

$$T_2(\overline{\partial}(-\omega \cdot u)) = T_2(\omega \wedge T_2(\overline{\partial}(-\omega \cdot u))). \tag{3.22}$$

We want to show that this recurrence equation implies that $T_2(\overline{\partial}(-\omega \cdot u)) = 0$. From Proposition A.1, T_2 is a well-defined operator mapping L^s to $W^{1,s}$ for any $s > 1$ and the following estimate holds:

$$\begin{aligned} \left\| T_2(\overline{\partial}(-\omega \cdot u)) \right\|_{L^\infty(B^4)} &\leq C \left\| T_2(\overline{\partial}(-\omega \cdot u)) \right\|_{W^{1,q}(B^4)} \\ &= C \left\| T_2(\omega \wedge T_2(\overline{\partial}(-\omega \cdot u))) \right\|_{W^{1,q}(B^4)} \\ &\leq C \left\| \omega \wedge T_2(\overline{\partial}(-\omega \cdot u)) \right\|_{L^q(B^4)} \\ &\leq C \left\| \omega \right\|_{L^q(B^4)} \left\| T_2(\overline{\partial}(-\omega \cdot u)) \right\|_{L^\infty(B^4)} \\ &\leq C C_1 \varepsilon \left\| T_2(\overline{\partial}(-\omega \cdot u)) \right\|_{L^\infty(B^4)}, \end{aligned}$$

where C is the Sobolev constant given by the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ ($q > 6$) in 4 dimensions. Moreover, for $1 - C \cdot C_1 \varepsilon > 0$ there is a contradiction unless $T_2(\overline{\partial}(-\omega \cdot u)) = 0$ and we can conclude that the $\overline{\partial}$-equation is solved:

$$\overline{\partial} u = -\omega \cdot u \qquad \text{in } B^4.$$

Step 3. It remains to show that $u \in GL_n(\mathbb{C})$ and satisfies the required estimates (3.19). We have:

$$\| u - id \|_{L^\infty(B^4)} = \| \mathcal{H}(u) - id \|_{L^\infty(B^4)}$$

$$\begin{aligned}&\le C\,\|\omega\|_{W^{1,p}(B^4)}\,\|u\|_{L^\infty(B^4)}\\&\le C\varepsilon\,\|u-id\|_{L^\infty(B^4)}+C\,\|\omega\|_{W^{1,p}(B^4)}\,.\end{aligned}$$

Thus, since $\varepsilon > 0$ is small, we get the L^∞ bound:

$$\|u-id\|_{L^\infty(B^4)} \le \frac{C}{1-C\varepsilon}\,\|\omega\|_{W^{1,p}(B^4)}\,.$$

Because we can assume that $1 - C\varepsilon > \frac{1}{2}$ for ε small enough, then

$$\|u-id\|_{L^\infty(B^4)} \le 2C\,\|\omega\|_{W^{1,p}(B^4)}\,. \tag{3.23}$$

This implies that $u \in GL_n(\mathbb{C})$. Remark D.1(iii) gives the existence of $r \in (1/2, 1)$ and a constant $C > 0$ such that

$$\|u-id\|_{W^{2,p}(B^4_r)} \le C\,\|\omega\|_{W^{1,p}(B^4)}\,. \tag{3.24}$$

Moreover, since u^{-1} exists, we have the following L^∞ estimate:

$$\begin{aligned}\left\|u^{-1}-id\right\|_{L^\infty(B^4)} &= \left\|u^{-1}-u^{-1}u\right\|_{L^\infty(B^4)} \le \left\|u^{-1}\right\|_{L^\infty(B^4)}\|u-id\|_{L^\infty(B^4)}\\&\le \left\|u^{-1}-id\right\|_{L^\infty(B^4)}\|u-id\|_{L^\infty(B^4)}+\|u-id\|_{L^\infty(B^4)}\end{aligned}$$

The estimate (3.23) on u also implies that the norm $\|u-id\|_{L^\infty(B^4)}$ is small. Hence, the estimate of u^{-1} then follows:

$$\left\|u^{-1}-id\right\|_{L^\infty(B^4)} \le \frac{\|u-id\|_{L^\infty(B^4)}}{1-\|u-id\|_{L^\infty(B^4)}} \le C\,\|\omega\|_{W^{1,p}(B^4)}\,,$$

for some constant $C > 0$. By Remark D.1 applied to u^{-1}, we obtain a similar estimate. This finishes the proof of Lemma 3.3. □

Having the results above at our disposal, we are ready to proceed at showing the existence of local holomorphic trivialisations in B^4_r for some $r > 0$.

Theorem 3.2. *There exists $\varepsilon_0 > 0$ such that if $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ satisfies*

$$\|A\|_{W^{1,2}(B^4)} \le \varepsilon_0,$$

and the integrability condition $F^{0,2}_A = 0$, then there exists $r > 0$ and $g \in W^{2,q}(B^4_r, GL_n(\mathbb{C}))$ for all $q < 2$ such that

$$A^{0,1} = -\overline{\partial} g \cdot g^{-1} \qquad \text{in } B^4_r. \tag{3.25}$$

Moreover, there exists a constant $C_q > 0$ such that the following estimates hold:

$$\|g - id\|_{W^{2,q}(B^4_r)} \leq C_q \|A\|_{W^{1,2}(B^4)} \quad \text{and} \quad \left\|g^{-1} - id\right\|_{W^{2,q}(B^4_r)} \leq C_q \|A\|_{W^{1,2}(B^4)}. \tag{3.26}$$

It follows that $A^g = h^{-1}\partial h$ where $h = \overline{g}^T g$.

Proof of Theorem 3.2. From Proposition 3.2, there exists a 1-form

$$\tilde{A} \in W^{1,2}(\Omega^1\mathbb{CP}^2 \otimes \mathfrak{u}(n))$$

satisfying the integrability condition so that $\tilde{A}^{0,1} = A^{0,1} + \vartheta\omega$ in B^4, where $\omega \in W^{2,2}(\Omega^{0,2}\mathbb{CP}^2 \otimes M_n(\mathbb{C}))$ with estimate $\|\omega\|_{W^{2,2}(\mathbb{CP}^2)} \leq \|A\|_{W^{1,2}(B^4)}$. This implies that

$$\left\|\tilde{A}^{0,1}\right\|_{W^{1,2}(\mathbb{CP}^2)} \leq C \|A\|_{W^{1,2}(B^4)} \tag{3.27}$$

for some constant $C > 0$.

Lemma 3.2 applied to the form $\tilde{A}$ gives the existence of a gauge

$$\tilde{g} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$$

for all $q < 2$ so that

$$\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g} \qquad \text{in } \mathbb{CP}^2$$

and for each $q < 2$ there exists $C_q > 0$ such that

$$\|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)} \quad \text{and} \quad \left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^2)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,2}(\mathbb{CP}^2)}. \tag{3.28}$$

On the unit ball B^4 we can rewrite $(A^{0,1})^{\tilde{g}}$ as such:

$$\left(A^{0,1}\right)^{\tilde{g}} = \tilde{g}^{-1}\overline{\partial}\tilde{g} + \tilde{g}^{-1}A^{0,1}\tilde{g} = \tilde{g}^{-1}\overline{\partial}\tilde{g} + \tilde{g}^{-1}\tilde{A}^{0,1}\tilde{g} - \tilde{g}^{-1}\vartheta\omega\tilde{g} = -\tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g}.$$

In order to find a gauge g for $A^{0,1}$ that gives a holomorphical trivialisation, it remains to find a gauge change u that cancels the perturbation term $-\tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g}$:

$$\overline{\partial}u = \tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g} \cdot u. \tag{3.29}$$

We claim that the composition of gauges $\tilde{g} \cdot u$ satisfies the statement.

Since the Sobolev embedding $W^{2,q} \hookrightarrow L^{2q/(2-q)}$ holds for any $q < 2$, it implies that $\tilde{g}, \tilde{g}^{-1} \in \bigcap_{q<\infty} L^q$. The fact that A and $\tilde{A}$ satisfy the integrability condition on B^4: $F_A^{0,2} = 0$ and $F_{\tilde{A}}^{0,2} = 0$, implies that $\omega \in W^{2,2}(\Omega^{0,2}B^4)$ satisfies the following PDE:

$$\frac{1}{2}\Delta\omega = -[A^{0,1}, \vartheta\omega] - \vartheta\omega \wedge \vartheta\omega. \tag{3.30}$$

Proposition D.2 applied to this PDE allows to bootstrap the regularity of ω inside B^4. Indeed, we have the much better regularity $\omega \in W^{2,q}_{loc}(B^4, M_n(\mathbb{C}))$ for any $q < 4$. Sobolev embeddings yield:

$$\vartheta\omega \in \bigcap_{q<4} W^{1,q}_{loc}(B^4, M_n(\mathbb{C})) \hookrightarrow \bigcap_{q<\infty} L^q_{loc}.$$

Putting together the regularity of $\vartheta\omega$, $\tilde{g}$ and $\tilde{g}^{-1}$ we can obtain the regularity of $\tilde{g}^{-1}(\vartheta\omega)\tilde{g}$:

$$\tilde{g}^{-1}(\vartheta\omega)\tilde{g} \in \bigcap_{q<4} W^{1,q}_{loc} \hookrightarrow \bigcap_{q<\infty} L^q_{loc}. \tag{3.31}$$

Fix $p > 3$ and $\delta > 0$ small. There exists $r_0 \in (0,1)$ so that

$$\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,p}(B^4_{r_0})} < \delta. \tag{3.32}$$

This $(0,1)$-form also solves $F^{0,2}_{\tilde{g}^{-1}(\vartheta\omega)\tilde{g}} = 0$ in $B^4_{r_0}$. Hence, we apply Lemma 3.3 to $\tilde{g}^{-1}(\vartheta\omega)\tilde{g}$ in $B^4_{r_0}$ (by rescaling) to get the existence of $r \in (r_0/2, r_0)$ and $u \in W^{2,p}(B^4_r, GL_n(\mathbb{C}))$ that solves the $\overline{\partial}$-equation above (3.29):

$$\overline{\partial}u = \tilde{g}^{-1}(\vartheta\omega)\tilde{g} \cdot u \qquad \text{in } B^4_r.$$

and satisfies the estimates

$$\|u - id\|_{W^{2,p}(B^4_r)} \leq C\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,2}(B^4_r)} \leq C\|A\|_{W^{1,2}(B^4)} \tag{3.33}$$

and

$$\left\|u^{-1} - id\right\|_{W^{2,p}(B^4_r)} \leq C\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,2}(B^4_r)} \leq C\|A\|_{W^{1,2}(B^4)}, \tag{3.34}$$

for some constant $C > 0$.

Define $g := \tilde{g}u$ in B^4_r. By construction, the required $\overline{\partial}$-equation is solved:

$$\overline{\partial}g = -A^{0,1}g \qquad \text{in } B^4_r \tag{3.35}$$

We show that g and g^{-1} satisfy the required estimates (3.26). Let $q < 2$ arbitrary. The triangle inequality applied on the norm $W^{2,q}$ gives:

$$\|g - id\|_{W^{2,q}(B_r^4)} \leq \|(\tilde{g} - id)(u - id)\|_{W^{2,q}(B_r^4)} + \|\tilde{g} - id\|_{W^{2,q}(B_r^4)} + \|u - id\|_{W^{2,q}(B_r^4)} .$$

Using the Sobolev product results of [31, Section 4.8.2, Theorem 1] and the regularity of $\tilde{g} - id \in \bigcap_{q<2} W^{2,q}(B_r^4)$ and $u - id \in W^{2,p}(B_r^4)$, it follows that

$$(\tilde{g} - id)(u - id) \in \bigcap_{q<2} W^{2,q}(B_r^4)$$

with

$$\|(\tilde{g} - id)(u - id)\|_{W^{2,q}(B_r^4)} \leq C \|\tilde{g} - id\|_{W^{2,q_1}(B_r^4)} \cdot \|u - id\|_{W^{2,p}(B_r^4)} ,$$

for some $q_1 \in (q, 2)$ and constant $C > 0$. Hence, from (3.27), (3.28) and (3.33) it immediately follows that there exists a constant $C_q > 0$ such that:

$$\|g - id\|_{W^{2,q}(B_r^4)} \leq C_q \|A\|_{W^{1,2}(B^4)} . \tag{3.36}$$

By arguing in a completely analogous way to how the obtained (3.36), using (3.28) and (3.34), we obtain

$$\left\|g^{-1} - id\right\|_{W^{2,q}(B_r^4)} \leq C_q \|A\|_{W^{1,2}(B^4)} .$$

It remains to show the existence of h such that $A^g = h^{-1}\partial h$. We apply g to A in B_r^4 to get:

$$A^g = g^{-1}(\overline{\partial} g + \partial g) + g^{-1}A^{0,1}g - g^{-1}\overline{A^{0,1}}^T g = g^{-1}\partial g - g^{-1}\overline{A^{0,1}}^T g.$$

Since (3.35) holds, then $\partial \overline{g}^T = -\overline{g}^T \overline{A^{0,1}}^T$. Hence, $\left(\overline{g}^T\right)^{-1} \partial \overline{g}^T = -\overline{A^{0,1}}^T$. By plugging this into the equation above, we get

$$A^g = g^{-1}\partial g + g^{-1} \left(\overline{g}^T\right)^{-1} \partial \overline{g}^T g = (\overline{g}^T g)^{-1}\partial(\overline{g}^T g).$$

We conclude the proof of Theorem 3.2 by defining $h := \overline{g}^T g$, and $h \in W^{2,q}(B_r^4, i\mathfrak{u}(n))$ for any $q < 2$. □

Remark 3.2. *From the proof of the theorem 3.2 above the radius $r > 0$ can be chosen to be the same under small perturbations of the 1-form A. We shortly* *justify this fact. We have shown that there exists a 2-form ω associated to A satisfying (3.30) in B^4. Moreover, from Proposition D.2 for each $q < 4$, there exists a constant $C_q > 0$ such that*

$$\|\omega\|_{W^{2,q}_{loc}(B^4)} \leq C_q \|A\|_{W^{1,2}(B^4)} .$$

Over domains B^4_r we have the estimate

$$\|\omega\|_{W^{2,q}(B^4_r)} \leq C_{q,r} \|A\|_{W^{1,2}(B^4)} .$$

for constants $C_{q,r} > 0$ which depend on r and $C_{q,r} \to 0$ as $r \to 0$. Let $\delta > 0$ and $p > 3$ such that (3.32) holds. Consider perturbations A_ε of A with $F^{0,2}_{A_\varepsilon} = 0$ over B^4 such that

$$\|A - A_\varepsilon\|_{W^{1,2}(B^4)} \leq \frac{1}{2} \|A_\varepsilon\|_{W^{1,2}(B^4)} . \tag{3.37}$$

Choose $C > 0$ such that

$$C < \frac{\delta}{2 \|A\|_{W^{1,2}(B^4)}} \leq \frac{\delta}{\|A_\varepsilon\|_{W^{1,2}(B^4)}}.$$

Since $C_{p,r} \to 0$ as $r \to 0$, choose $r_0 > 0$ such that $C_{p,r_0} < C$, where $p > 3$ was fixed. Then any ω_ε associated to A_ε through (3.30) satisfies

$$\|\omega_\varepsilon\|_{W^{2,p}(B^4_{r_0})} \leq C_{p,r_0} \|A_\varepsilon\|_{W^{1,2}(B^4)} < \delta.$$

Thus, up to constants we obtain the estimate (3.32) for ω_ε and we conclude that under perturbations A_ε of A satisfying (3.37), we can always choose $r := r_0 > 0$.

Having made the above remark, we end the section by proving a stability result for holomorphic trivialisations. This Corollary will be used to show the convergence of holomorphic structures later on in Section 3.5.

Corollary 3.1. *Let $A_1 \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ and $r > 1$ so that*

$$g_1 \in W^{2,q}(B^4_r, GL_n(\mathbb{C}))$$

satisfies theorem 3.2. There exists $\delta > 0$ such that for all $A_2 \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ with $F^{0,2}_{A_2} = 0$ satisfying

$$\|A_1 - A_2\|_{W^{1,2}(B^4)} \leq \delta,$$

there exists a radius $r_0 \in (r/2, r)$ depending only on A_1 and a gauge

$$g_2 \in \bigcap_{q<2} W^{2,q}(B^4_{r_0}, GL_n(\mathbb{C}))$$

that trivialises A_2 in the sense that:

$$A_2 = -\overline{\partial} g_2 \cdot g_2^{-1} \qquad \text{in } B^4_{r_0}$$

with the following estimates: for any $q < 2$ there exists $C_q > 0$ such that

$$\|g_2 - id\|_{W^{2,q}(B^4_{r_0})} \leq C_q \left(\|A_1\|_{W^{1,2}(B^4)} + \|A_2\|_{W^{1,2}(B^4)} \right)$$

and there exists $C > 0$ such that

$$\|g_1 - g_2\|_{L^p(B^4_{r_0})} \leq C \, \|A_1 - A_2\|_{W^{1,2}(B^4)}$$

for any $p < 12$.

Proof of Corollary 3.1. Choose $\delta > 0$ such that A_2 is a small perturbation of A_1. By Remark 3.2 and Theorem 3.2 applied to the forms A_1 and A_2 we obtain the existence of $r > 0$ and gauges $g_1, g_2 \in W^{2,q}(B^4_r, GL_n(\mathbb{C}))$ for all $q < 2$ so that

$$\overline{\partial} g_1 = -A_1^{0,1} \cdot g_1 \quad \text{and} \quad \overline{\partial} g_2 = -A_2^{0,1} \cdot g_2 \text{ in } B^4_r \tag{3.38}$$

and there exists a constant $C_q > 0$ such that

$$\begin{aligned} \|g_1 - id\|_{W^{2,q}(B^4_r)} &\leq C_q \, \|A_1\|_{W^{1,2}(B^4)} \, , \\ \|g_2 - id\|_{W^{2,q}(B^4_r)} &\leq C_q \, \|A_2\|_{W^{1,2}(B^4)} \\ &\leq C_q \left(\|A_1\|_{W^{1,2}(B^4)} + \|A_1 - A_2\|_{W^{1,2}(B^4)} \right) \quad \text{and} \\ \left\|g_2^{-1} - id\right\|_{W^{2,q}(B^4_r)} &\leq C_q \, \|A_2\|_{W^{1,2}(B^4)} \\ &\leq C_q \left(\|A_1\|_{W^{1,2}(B^4)} + \|A_1 - A_2\|_{W^{1,2}(B^4)} \right) \end{aligned} \tag{3.39}$$

Since g_1 and g_2 holomorphically trivialise A_1 and A_2 respectively (3.38), we can relate the *transition gauge* $g_2^{-1} g_1$ with the difference 1-form $A_2 - A_1$ through the following $\overline{\partial}$-equation:

$$\overline{\partial}(g_2^{-1} g_1) = g_2^{-1} (A_2 - A_1)^{0,1} g_2 \cdot (g_2^{-1} g_1). \tag{3.40}$$

We first estimate $g_2^{-1}g_1 - id$ using the inequalities (3.39) and then use equation (3.40) to show that $g_2^{-1}g_1 - id$ is only bounded by the norm of $A_2 - A_1$. Fix $q < 2$. The triangle inequality gives:

$$\begin{aligned} \left\|g_2^{-1}g_1 - id\right\|_{W^{2,q}(B_r^4)} \;\leq\; & \left\|(g_2^{-1} - id)(g_1 - id)\right\|_{W^{2,q}(B_r^4)} \\ & + \left\|g_2^{-1} - id\right\|_{W^{2,q}(B_r^4)} + \|g_1 - id\|_{W^{2,q}(B_r^4)} \end{aligned}$$

Hence, by the results of [31, Section 4.8.2, Theorem 1] applied to the product

$$(g_2^{-1} - id)(g_1 - id)$$

and estimates (3.39), there exists a constant $C_q > 0$ so that

$$\begin{aligned} \left\|g_2^{-1}g_1 - id\right\|_{W^{2,q}(B_r^4)} &\leq C_q(\|A_1\|_{W^{1,2}} + \|A_2\|_{W^{1,2}}) \\ &\leq 2C_q(\|A_1\|_{W^{1,2}} + \|A_1 - A_2\|_{W^{1,2}}). \end{aligned} \tag{3.41}$$

for any $q < 2$. We can use equation (3.40) in order to find an a-posteriori estimate of $g_2^{-1}g_1 - id$ involving *only* the 1-form $A_2 - A_1$. Let $s < 4$. By the regularity of $\overline{\partial}$ in L^s (see [18, Theorem 1(b)]) there exists a holomorphic function h and a constant $C_s > 0$ such that

$$\begin{aligned} \left\|g_2^{-1}g_1 - h\right\|_{L^{6s/(6-s)}(B_r^4)} &\leq C_s \left\|\overline{\partial}(g_2^{-1}g_1)\right\|_{L^s(B_r^4)} \\ &\leq C_s \left\|g_2^{-1}(A_2 - A_1)^{0,1}g_2\right\|_{L^{sp/(p-s)}(B_r^4)} \left\|g_2^{-1}g_1\right\|_{L^p(B_r^4)}, \end{aligned}$$

where $p \in (s, \infty)$ arbitrary. Hence, it follows that there exists $C > 0$ depending on A_1 such that

$$\begin{aligned} \left\|g_2^{-1}g_1 - h\right\|_{L^{6s/(6-s)}(B_r^4)} \leq & C \|A_1 - A_2\|_{W^{1,2}(B^4)} \left\|g_2^{-1}g_1 - id\right\|_{L^p(B_r^4)} \\ & + C \|A_1 - A_2\|_{W^{1,2}(B^4)}. \end{aligned}$$

There exists $q < 2$ such that $W^{2,q} \hookrightarrow L^p$. Since $g_2^{-1}g_1 - id$ is bounded in $W^{2,q}$ as in (3.41), then it is also bounded in L^p. Hence,

$$\left\|g_2^{-1}g_1 - h\right\|_{L^{6s/(6-s)}(B_r^4)} \leq C \|A_1 - A_2\|_{W^{1,2}(B^4)}.$$

Since this holds for any $s < 4$, there exists a constant $C > 0$ such that

$$\left\|g_2^{-1}g_1 - h\right\|_{L^p(B_r^4)} \leq C \|A_1 - A_2\|_{W^{1,2}(B^4)},$$

for any $p < 12$. Having this inequality at our disposal, we can turn to estimate $g_1 - g_2 \cdot h$. Let $p < 12$, then:

$$\|g_1 - g_2 \cdot h\|_{L^p(B^4_r)} = \left\|(g_2 - id)(h - g_2^{-1}g_1) + h - g_1^{-1}g_2\right\|_{L^p(B^4_r)}.$$

For $v \in (p, 12)$, we get:

$$\begin{aligned}\|g_1 - g_2 \cdot h\|_{L^p(B^4_r)} \leq &\|g_2 - id\|_{L^{vp/(v-p)}(B^4_r)} \left\|g_2^{-1}g_1 - h\right\|_{L^v(B^4_r)}\\ &+ \left\|g_2^{-1}g_1 - h\right\|_{L^p(B^4_r)}.\end{aligned}$$

Thus, there exists a constant $C_{v,p} > 0$ depending on v, p and A_1 such that

$$\|g_1 - g_2 \cdot h\|_{L^p(B^4_r)} \leq C_{vp} \|A_1 - A_2\|_{W^{1,2}(B^4)}.$$

Moreover, $g_2 \cdot h$ solves the equation:

$$\overline{\partial}(g_2 \cdot h) = A_2^{0,1}(g_2 \cdot h) \tag{3.42}$$

in a distributional sense. It remains to show that the $g_2 \cdot h$ is bounded in $W^{2,q}$ by the norms of A_1 and A_2 in a possible slightly smaller ball. Let $r_0 \in (r/2, r)$, then there exists a constant $C > 0$ such that

$$\|g_2 \cdot h - id\|_{W^{1,2}(B^4_{r_0})} \leq C \left(\left\|\overline{\partial} g_2 \cdot h\right\|_{L^2(B^4_r)} + \|g_2 \cdot h - id\|_{L^2(B^4_r)}\right).$$

Consequently, by using the $\overline{\partial}$-equation (3.42) satisfied by $g_2 \cdot h$, it follows that:

$$\begin{aligned}\|g_2 \cdot h - id\|_{W^{1,2}(B^4_{r_0})} \leq &C \Big(\|A_2\|_{L^4(B^4_r)} \|g_2 \cdot h\|_{L^4(B^4_r)}\\ &+ \|g_2 \cdot h - g_1\|_{L^2(B^4_r)} + \|id - g_1\|_{L^2(B^4_r)}\Big).\end{aligned}$$

Having shown that $g_2 \cdot h \in L^p$ for all $p < 12$, we obtain in particular

$$\|g_2 \cdot h - id\|_{L^4(B^4_{r_0})} \leq C \left(\|A_1\|_{W^{1,2}(B^4)} + \|A_2\|_{W^{1,2}(B^4)}\right).$$

Hence, given that $g_2 \cdot h \in L^4$ and $g_2 \cdot h - id$ is bounded by A_1 and A_2, we get from Lemma D.1 and Remark D.1(ii) the estimate: for any $q < 2$ there exists a constant $C_q > 0$ such that:

$$\|g_2 \cdot h - id\|_{W^{2,q}(B^4_{r_0})} \leq C_q \left(\|A_1\|_{W^{1,2}(B^4)} + \|A_2\|_{W^{1,2}(B^4)}\right).$$

By redefining g_2 as $g_2 \cdot h$, we have proven our stability result. □

3.3 Proof of Theorem 1.1

We pick geodesic balls $B^4_r(x_i)$ covering X^2 on which the connection can be trivialised: $\nabla \simeq d + A_i$ and

$$\|A_i\|_{W^{1,2}(B^4_r(x_i))} \leq \varepsilon_0(X^2, \omega),$$

where $\varepsilon_0(X^2, \omega)$ is given by Theorem 3.2. Because X^2 is a compact manifold, there are finitely many such balls covering X^2. By Theorem 3.2 there exists $r' \in (0, r)$, $\sigma_i \in W^{2,p}(B^4_{r'}(x_i), GL_n(\mathbb{C}))$ and $h_i = \overline{\sigma}_i^T \sigma_i \in W^{2,p}(B^4_{r'}, Sym(n))$ for all $p < 2$ so that

$$A_i^{\sigma_i} = h_i^{-1} \partial h_i,$$

where we recall that $Sym(n)$ is the space of symmetric $n \times n$ matrices. Hence

$$\nabla^{\sigma_i} \simeq d + h_i^{-1} \partial h_i \qquad \text{in } B^4_{r'}(x_i). \tag{3.43}$$

It remains to show that ∇ defines a connection on a holomorphic vector bundle structure $\mathcal{E}$ over X^2. In order to achieve this, it is enough to find holomorphic transition maps. On the initial bundle E, there exists gauge transition maps $g_{ij} \in W^{2,2}(B^4_r(x_i) \cap B^4_r(x_j), U(n))$ such that

$$A_i^{g_{ij}} = A_j.$$

Define the transition maps

$$\sigma_{ij} = \sigma_i^{-1} g_{ij} \sigma_j. \tag{3.44}$$

We show that the σ_{ij} are holomorphic and consequently since they define a cocycle, they define a holomorphic vector bundle structure $\mathcal{E}$ over the Kähler manifold X^2:

$$\begin{aligned}
\overline{\partial} \sigma_{ij} &= \overline{\partial} \sigma_i^{-1} \cdot g_{ij} \sigma_j + \sigma_i^{-1} \overline{\partial} g_{ij} \cdot \sigma_j + \sigma_i^{-1} g_{ij} \overline{\partial} \sigma_j \\
&= \sigma_i^{-1} A_i^{0,1} g_{ij} \sigma_j + \sigma_i^{-1} \overline{\partial} g_{ij} \sigma_j - \sigma_i^{-1} g_{ij} A_j^{0,1} \sigma_j \\
&= \sigma_i^{-1} g_{ij} g_{ij}^{-1} A_i^{0,1} g_{ij} \sigma_j + \sigma_i^{-1} \overline{\partial} g_{ij} \sigma_j - \sigma_i^{-1} g_{ij} A_j^{0,1} \sigma_j \\
&= \sigma_i^{-1} g_{ij} \left(A_j^{0,1} - g_{ij}^{-1} \overline{\partial} g_{ij} \right) \sigma_j + \sigma_i^{-1} \overline{\partial} g_{ij} \sigma_j - \sigma_i^{-1} g_{ij} A_j^{0,1} \sigma_j.
\end{aligned}$$

This equation gives:

$$\overline{\partial}\sigma_{ij} = 0$$

and shows that the transition maps are holomorphic. Thus, there exists a holomorphic vector bundle structure $\mathcal{E}$ which is compatible with ∇ since $(\nabla^{\sigma_i})^{\sigma_{ij}} = \nabla^{\sigma_j}$ in local coordinates. From the local representation (3.43) we finally obtain:

$$\nabla^{0,1} = \overline{\partial}_{\mathcal{E}}.$$

3.4 Density under high energy

Until now we have proven results under the assumption of low $W^{1,2}$ connection norm. In this section we lose this assumption. As before we assume that we work on the flat unit ball B^4. In Section 3.4.4 we show how to generalise our results on the closed Kähler surface X^2. We start by investigating the case when $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ and $\|A\|_{W^{1,2}(B^4)} < \infty$. Furthermore, through-out the section we assume the integrability condition $F_A^{0,2} = 0$ is satisfied.

Difficulty:
If we want to proceed as in the case of low $W^{1,2}$ connection norm, we start by smoothing A inside B^4 by simple convolutions and thus, obtain a sequence of smooth forms A_k converging to A in $W^{1,2}$ as $k \to \infty$. The integrability condition (1.1) is, however, lost for A_k. Furthermore, since we want to preserve the condition for each k, the argument reduces to finding a sequence of perturbations $\omega_k \in C^\infty(\Omega^{0,2} B^4 \otimes M_n(\mathbb{C}))$ uniformly bounded in $W^{2,2}$ such that for each k, ω_k solves

$$\begin{cases} \overline{\partial}\overline{\partial}^* \omega_k &= -\left[\overline{\partial}^* \omega_k, A_k^{0,1}\right] - \overline{\partial}^* \omega_k \wedge \overline{\partial}^* \omega_j^k - F_{A_k}^{0,2} \quad \text{in } B^4 \\ \omega_k &= 0 \quad \text{on } \partial B^4 \end{cases}$$

Since $A_k^{0,1}$ is not small in $W^{1,2}$ norm, we cannot hope to apply a fixed point argument even if $F_{A_k}^{0,2}$ is very small in L^2 norm (it converges to $F_A^{0,2} = 0$). To make the situation worse, the linear operator $\overline{\partial}\overline{\partial}^* \cdot + \left[\overline{\partial}^* \cdot, A_k^{0,1}\right]$ might have non-trivial kernel..

Hence, in this section we have developed a method that deals with the case of A_k having high $W^{1,2}$ norm. We present it below:

Strategy:
We recall that through Proposition 3.1 and Lemma 3.2, we were able to find a perturbation $\overline{\partial}^*\omega$ to the form $\tilde{A}$ in order to obtain the integrability condition (1.1) over $\mathbb{CP}^2$. This method, however, heavily used the fact that the operator $L_{\tilde{A}}$ (3.3) is invertible under the smallness condition of the $W^{1,2}$ norm of $\tilde{A}$. Following this blueprint, our idea is to find a unitary gauge change g of A such that the operator $\overline{\partial\partial}^* \cdot + \left[\overline{\partial}^* \cdot, (A^{0,1})^g\right]$ is invertible.

Firstly, we will need to acquaint ourselves with this idea. We found it natural to start by considering the case of linear perturbations of A and show that we can always find a smooth perturbation U such that the operator $\overline{\partial\partial}^* \cdot + \left[\overline{\partial}^* \cdot, A^{0,1} + \beta\overline{\partial}U\right]$ acting on $(0,2)$ forms has a trivial kernel for some $\beta > 0$.

Having this idea, we search for a unitary gauge change g that forces the operator $\overline{\partial\partial}^* \cdot + \left[\overline{\partial}^* \cdot, (A^{0,1})^g\right]$ to have trivial kernel. Moreover, we show that for k large enough, the <u>same</u> gauge g gives that the operators $\overline{\partial\partial}^* \cdot + \left[\overline{\partial}^* \cdot, \left(A_k^{0,1}\right)^g\right]$ are also invertible. This enables us to find a perturbation ω_k that solves

$$\overline{\partial\partial}^*\omega_k + \left[\overline{\partial}^*\omega_j^k, \left(A_k^{0,1}\right)^g\right] + \overline{\partial}^*\omega_k \wedge \overline{\partial}^*\omega_k = -F^{0,2}_{A_k^g}$$

with $\omega_k = 0$ on ∂B^4, and satisfies the estimate:

$$\|\omega_k\|_{W^{2,2}(B^4)} \leq C \left|\left|\left| T^{-1}_{\left(A_k^{0,1}\right)^g} \right|\right|\right| \left\| F^{0,2}_{A_k^g} \right\|_{L^2(B^4)},$$

for some constant $C > 0$. Moreover, as $k \to \infty$, we show that the $W^{2,2}$ norm of ω_k is uniformly bounded. Using this estimate, together with the convergence of A_k to A in $W^{1,2}$ and $F_A^{0,2} = 0$, we obtain the strong convergence of the sequence ω_k to 0.

Hence, we prove the first theorem of this section by also taking into account that g is a unitary gauge transformation, and we can thus use the invariance of the L^2 norm under the action of g: $\left\|F^{0,2}_{A_k}\right\|_{L^2} = \left\|g^{-1}F^{0,2}_{A_k^g}g\right\|_{L^2} = \left\|F^{0,2}_{A_k^g}\right\|_{L^2}$.

The local theorem on B^4 is stated as follows:

Theorem 3.3. *Let $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$, with $F_A^{0,2} = 0$. There exists a sequence of smooth 1-forms $A_k \in C^\infty(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ satisfying $F^{0,2}_{A_k} = 0$ and $A_k \to A$ in $W^{1,2}(B^4)$.*

Moreover, using the local theorem, we will show that it implies the global existence of an approximating smooth sequence under the cohomological constraint $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$. In particular, we obtain:

Theorem 3.4. *Let ∇ a $W^{1,2}$ unitary connection over X^2 with $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$, satisfying the integrability condition*

$$F^{0,2}_{\nabla} = 0.$$

Then there exists a sequence of smooth unitary connections ∇_k, with $F^{0,2}_{\nabla_k} = 0$ such that

$$\mathrm{dist}_2(\nabla_k, \nabla) \to 0.$$

3.4.1 Linear perturbation

We will be looking for a small linear perturbation that forces the operator

$$L_{A^{0,1}} = \overline{\partial}_A \overline{\partial}^* \cdot = \overline{\partial\partial}^* \cdot + [A^{0,1}, \overline{\partial}^* \cdot]$$

to have trivial kernel, assuming vanishing boundary conditions. Let $U \in C^\infty(B^4, M_n(\mathbb{C}))$. $W^{2,2}_D$ is the space of $(0,2)$ forms vanishing on ∂B^4 and we define the following operators

$$L_0 : W^{2,2}_D(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$$

$$\omega \mapsto \overline{\partial\partial}^* \omega + [A^{0,1}, \overline{\partial}^* \omega]$$

$$L_{\beta,U} : W^{2,2}_D(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$$

$$\omega \mapsto L_0\omega + \beta B_U \omega$$

where $B_U = [\overline{\partial} U, \overline{\partial}^* \cdot]$.

Proposition 3.3. *L_0 and $L_{\beta,U}$ are Fredholm operators of index zero from the space $W^{2,2}_D(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$ to $L^2(B^4, M_n(\mathbb{C}))$.*

Proof of Proposition 3.3. It is sufficient to prove this statement for L_0. We argue that L_0 is Fredholm. We know that L_0 is elliptic on the domain B^4. Thus, by [3] for some $C > 0$ the estimate

$$\|\omega\|_{W^{2,2}(B^4)} \leq C \left(\|L_0 \omega\|_{L^2(B^4)} + \|\omega\|_{L^2(B^4)} \right)$$

holds for all $\omega \in W^{2,2}_D(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$. From this we can deduce that L_0 is Fredholm. Moreover, $\overline{\partial\partial}^* = \frac{1}{2}\Delta d\overline{z_1} \wedge d\overline{z_2}$ is an elliptic operator of Fredholm index zero mapping $W^{2,2}_D(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$ to $L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$.

Let A_k be a sequence of smooth 1-forms converging strongly in $W^{1,2}$ to A. Then the bracket operator

$$\omega \mapsto [A_k^{0,1}, \overline{\partial}^*\omega]$$

is compact from $W^{2,2}(\Omega^{0,2}(B^4))$ to $L^2(\Omega^{0,2}(B^4))$. Indeed, A_k is bounded in L^∞ and hence:

$$\left\|[A_k^{0,1}, \overline{\partial}^*\omega]\right\|_{L^2(B^4)} \leq C \left\|A_k\right\|_{L^\infty(B^4)} \left\|\overline{\partial}^*\omega\right\|_{L^2(B^4)},$$

for some constant $C > 0$, where we have used the fact that $W^{1,2}$ is compactly embedded in L^2 in 4-dimensions, by Rellich-Kondrachov [2]. By the compact embeddedness, it follows that the operators $\omega \mapsto [A_k^{0,1}, \overline{\partial}^*\omega]$ are compact $W^{2,2}$ to L^2 for all k. Hence, using the compactness of these operators and the fact that $\overline{\partial\partial}^*$ is Fredholm, by [32, Theorem 4.4.2, p.185] we have

$$\text{index}(\overline{\partial\partial}^* \cdot + [A_k^{0,1}, \overline{\partial}^* \cdot]) = \text{index}(\overline{\partial\partial}^*) = 0.$$

Moreover, for a fixed $\varepsilon > 0$ given by [32, Theorem 4.4.2, p.185], then there exists $k_0 > 0$ such that for all $k \geq k_0$, we have that $\left|\left|\left|[A_k^{0,1} - A^{0,1}, \overline{\partial}^*]\right|\right|\right| \leq \varepsilon$ since A_k converges strongly to A in $W^{1,2}$. Thus, by applying [32, Theorem 4.4.2, p.185] to the perturbation operator $[A_k^{0,1} - A^{0,1}, \overline{\partial}^*]$ and to L_0, we obtain that

$$\text{index}(L_0) = \text{index}(L_0 + [A_k^{0,1} - A^{0,1}, \overline{\partial}^*]) = \text{index}(\overline{\partial\partial}^* \cdot + [A_k^{0,1}, \overline{\partial}^* \cdot]) = 0$$

This proves the statement. □

We will be working with operators of the form L_0 and $L_{\beta,U}$, where $\beta \in \mathbb{R}$ is small and $U \in W^{2,2}(B^4, M_n(\mathbb{C}))$. By the Proposition above the kernel of L_0 is finite dimensional. On the L^2 orthogonal space of $KerL_0$ denoted by $(KerL_0)^\perp$, there exists a compact operator $S = L_0^{-1}$ from L^2 into $W^{2,2}_D$ such that

$$S : RanL_0 \to (KerL_0)^\perp.$$

By classical spectral theory, S has discrete spectrum with a possible accumulation point at 0 (see for example [29, Theorem VI.15]). This means that the spectrum of S is $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ where $\lambda_n \to 0$.

Thus, on $(KerL_0)^{\perp}$, the spectrum of L_0 is

$$\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}, \dots\right\}$$

where $\lambda_n \to 0$. If we assume $KerL_0 \neq \{0\}$, we have that 0 has to be in the spectrum as well and then

$$\left\{0, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}, \dots\right\}$$

is the spectrum of the operator L_0. In addition, $L_{\beta,U}$ has discrete spectrum by arguing as before with $A^{0,1} + \beta\overline{\partial}U$ instead of $A^{0,1}$.

The following proposition states that the number of eigenvalues near 0 of $L_{\beta,U}$ cannot exceed the multiplicity of the 0 eigenvalue of L_0 for β small enough. This was proven in [7, Theorem 1]. We denote by m be the multiplicity of 0 for the operator L_0.

Proposition 3.4. *There exists $\beta_0 > 0$ so that for each $0 \leq \beta \leq \beta_0$, $L_{\beta,U}$ has at most m repeated according to multiplicity $\lambda_U^{(1)}(\beta), \dots \lambda_U^{(m)}(\beta)$ near 0 that converge to 0 as $\beta \to 0$.*

Proof of Proposition 3.4. Denote $\varepsilon_0 := \left|\frac{1}{\lambda_1}\right|$. and let $\varepsilon < \varepsilon_0$. Since the ball $B_\varepsilon(0) = \{|\lambda| \leq \varepsilon\} \subseteq \mathbb{C}$ is compact, then $L_{\beta,U}$ has finitely many not necessarily distinct m_β eigenvalues in $B_\varepsilon(0)$, $\lambda_U^{(1)}(\beta), \dots, \lambda_U^{(m_\beta)}(\beta)$. The statement follows by [7, Theorem 1]. □

Define the following operator:

$$P_{\beta,U} := -\frac{1}{2\pi i} \oint_{|\lambda|=\varepsilon} (L_{\beta,U} - \lambda)^{-1} \, d\lambda$$

where ε is chosen as in Proposition 3.4. Moreover, from Proposition 3.4, we have isolated branched points of the spectrum. Thus, we can rewrite $P_{\beta,U}$ as

$$P_{\beta,U} := -\frac{1}{2\pi i} \sum_{i=1}^{m} \oint_{|\lambda - \lambda_U^{(i)}(\beta)|=\varepsilon_1} (L_{\beta,U} - \lambda)^{-1} \, d\lambda.$$

for some $\varepsilon_1 > 0$ small enough. Each term of this sum is the projection onto the generalised eigenspace of $L_{\beta,U}$ corresponding to the eigenvalue $\lambda_U^{(i)}(\beta)$ (see [28, Chapter XII]). By Proposition 3.4 on the circle $|\lambda| = \varepsilon$, we have that $\lambda \in \rho(L_{\beta,U})$, where ρ is the resolvent of $L_{\beta,U}$. Thus, the resolvent operator $R_\lambda = (L_{\beta,U} - \lambda)^{-1}$ is analytic in β on the ε circle. It follows that $P_{\beta,U}$ is also analytic in β for β small enough. Similarly, we can define the operator P_0 associated to L_0.

Following [32, Chapter 5], we denote the generalised eigenspace corresponding to 0 for L_0 by

$$G_0 = \bigcup_{k=1}^{\infty} \left\{ v \in W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) | L_0^k v = 0 \right\}$$

and the range of $P_{\beta,U}$ by

$$G_{\beta,U} = \bigcup_{k=1}^{\infty} \bigoplus_{i=1}^{m} \left\{ v \in W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) | (L_{\beta,U} - \lambda_U^{(i)}(\beta))^k v = 0 \right\}$$

respectively. The following proposition will show that these two spaces are isomorphic for small β.

Proposition 3.5. *There exists $\beta_0 > 0$ so that $P_{\beta,U} : G_0 \to G_{\beta,U}$ is an isomorphism for all $0 < \beta < \beta_0$.*

Proof of Proposition 3.5. From [28, Theorem XII.5], $P_{\beta,U}$ and P_0 define two surjective projection operators:

$$P_{\beta,U} : W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to G_{\beta,U}$$

and

$$P_0 : W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to G_0.$$

Claim 1. There exists $\beta_0 > 0$ so that for all $\beta < \beta_0$, $P_{\beta,U}$ is surjective as an operator from G_0 to $G_{\beta,U}$.

$$\begin{aligned} P_{\beta,U} - P_0 = & -\frac{1}{2\pi i} \oint_{|\lambda|=\epsilon} (L_{\beta,U} - \lambda)^{-1} - (L_0 - \lambda)^{-1}\, d\lambda \\ = & -\frac{1}{2\pi i} \oint_{|\lambda|=\epsilon} \left((L_0 - \lambda)^{-1} - \sum_{i=1}^{\infty} \beta^i (L_0 - \lambda)^{-1} (B_U (L_0 - \lambda)^{-1})^i \right) \\ & - (L_0 - \lambda)^{-1}\, d\lambda \\ = & \; O(\beta) \end{aligned} \tag{3.45}$$

Thus, there exists β_0 such that for all $\beta < \beta_0$ we have the following bound for the norm of the operator $P(\beta) - P(0)$:

$$\|P_{\beta,U} - P_0\|_{W_D^{2,2}} < \frac{1}{16}.$$

Since $L_{\beta,U} - \lambda$ is a continuous operator from $W_D^{2,2}$ to L^2 for any $\lambda, \beta \in \mathbb{R}$, we have that

$$(L_{\beta,U} - \lambda)^{-1} G_0$$

is closed, since $RanP_0 = G_0$ is closed (see [28, Theorem XII.5]). This implies that $P_{\beta,U}G_0$ is closed as $P_{\beta,U}$ is a composition of two continuous operators.

Assume that $P_{\beta,U}G_0 \neq G_{\beta,U}$. Since $P_{\beta,U}G_0$ is closed in $G_{\beta,U}$, we can apply Riesz Lemma (see for example [32, Lemma 1.2.13]). Then there exists $u \in G_{\beta,U}$ where $\|u\|_{W_D^{2,2}} = 1$ and

$$\inf_{v \in P_{\beta,U} G_0} \|u - v\|_{W_D^{2,2}} > \tfrac{1}{2}.$$

$P_{\beta,U}$ is a projection operator and $u \in G_{\beta,U}$, then u is its own projection - $u = P_{\beta,U}u$. Moreover, the norm distance between u and P_0u satisfies the following inequality

$$\begin{aligned} \|u - P_0 u\|_{W_D^{2,2}} = \|P_{\beta,U} u - P_0 u\|_{W_D^{2,2}} &\leq \|P_{\beta,U} - P_0\|_{W_D^{2,2}} \|u\|_{W_D^{2,2}} \\ &= \|P_{\beta,U} - P_0\|_{W_D^{2,2}} < \frac{1}{16}. \end{aligned}$$

The last inequality holds because β is chosen to be small. From the above estimate, we can estimate the norm of P_0u:

$$\|P_0 u\|_{W_D^{2,2}} \leq \|u\|_{W_D^{2,2}} + \|u - P_0 u\|_{W_D^{2,2}} < 1 + \tfrac{1}{16}.$$

By further computing, we get

$$\begin{aligned} \|P_{\beta,U} P_0 u - P_0 u\|_{W_D^{2,2}} &= \|P_{\beta,U} P_0 u - P_0 P_0 u\|_{W_D^{2,2}} \\ &\overset{P_0^2 = P_0}{\leq} \|P_{\beta,U} - P_0\|_{W_D^{2,2}} \|P_0 u\|_{W_D^{2,2}} \\ &< \tfrac{1}{16^2} + \tfrac{1}{16}. \end{aligned}$$

Thus,

$$\|u - P_{\beta,U}P_0u\|_{W^{2,2}_D} \leq \|P_{\beta,U}P_0u - P_0u\|_{W^{2,2}_D} + \|u - P_0u\|_{W^{2,2}_D} < \tfrac{2}{16} + \tfrac{1}{16^2}.$$

Since $P_{\beta,U}P_0u \in P_{\beta,U}G_0$, we get a contradiction with $\inf\limits_{v \in P_{\beta,U}G_0} \|u - v\|_{W^{2,2}} > \frac{1}{2}$. Hence, $P_{\beta,U}G_0 = G_{\beta,U}$ and we have proven the claim that $P_{\beta,U}$ is surjective from G_0 to $G_{\beta,U}$.

Claim 2. $P_{\beta,U}$ is injective as an operator from G_0 to $G_{\beta,U}$.

We have shown in (3.45) that

$$P_{\beta,U} - P_0 = \frac{1}{2\pi i} \oint_{|\lambda|=\epsilon} \left(\sum_{i=1}^{\infty} \beta^i (L_0 - \lambda)^{-1} (B_U (L_0 - \lambda)^{-1})^i \right) d\lambda.$$

Define $E_{\beta,U} = P_{\beta,U} - P_0$. Then $E_{\beta,U}$ is a well-defined bounded operator defined on the space of $W^{2,2}$ $(0,2)$ forms. Moreover, there exists $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$ we have

$$|||E_{\beta,U}||| < 1.$$

Let $\omega \in G_0$ such that $P_{\beta,U}\omega = 0$. Without loss of generality we can assume $\|\omega\|_{W^{2,2}} = 1$. Then we obtain:

$$\begin{aligned} 0 = P_{\beta,U}(\omega) &= -\frac{1}{2\pi i} \oint_{|\lambda|=\epsilon} (L_{\beta,U} - \lambda)^{-1} \omega \, d\lambda \\ &= -\frac{1}{2\pi i} \oint_{|\lambda|=\epsilon} \left((L_0 - \lambda)^{-1} - \sum_{i=1}^{\infty} \beta^i (L_0 - \lambda)^{-1} (B_U (L_0 - \lambda)^{-1})^i \right) \omega \, d\lambda \\ &= \omega + E_{\beta,U}\omega. \end{aligned}$$

Hence,

$$1 = \|\omega\|_{W^{2,2}} = \|E_{\beta,U}\omega\|_{W^{2,2}} \leq |||E_{\beta,U}||| \, \|\omega\|_{W^{2,2}} = |||E_{\beta,U}||| < 1.$$

We have obtained a contradiction. Hence, for all $\beta < \beta_0$ we have that $P_{\beta,U}$ is injective from G_0 to $G_{\beta,U}$.

From the two claims above, there exists $\beta_0 > 0$ so that

$$P_{\beta,U} : G_0 \to G_{\beta,U}$$

is an isomorphism for all $\beta < \beta_0$. □

The last ingredient we need to prove in order to obtain the existence of a perturbation that makes $KerL_{\beta,U}$ trivial is the next statement. It gives us the perturbation U which will satisfy the necessary condition to make the kernel trivial. This next Proposition together with the existence of the isomorphism $P_{\beta,U}$ will be key to proving Lemma 3.4.

Proposition 3.6. *There exists a smooth map $U \in C^\infty(B^4, \mathfrak{u}(n))$ such that B_U is injective on $KerL_0$.*

Proof of Proposition 3.6. Since $KerL_0$ is finite dimensional, let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of it.

Let $v \in KerL_0$, $v \neq 0$. We show that for each such v, we can find U_v such that $B_{U_v} v \neq 0$. Assume by contradiction that $B_U v = 0$ for all smooth maps U on B^4. Define the linear operator $H(\omega) := \left([\omega, \overline{\partial}^* v]\right)^{0,2} : C^\infty(\Omega^1 B^4 \otimes M_n(\mathbb{C})) \to C^\infty(\Omega^2 B^4 \otimes M_n(\mathbb{C}))$ which satisfies the fact that $H(\omega(z)) = H(\omega)(z)$, where $\omega(z)$ means that each component of ω is applied to z. Moreover, we have that

$$0 = B_U v = [\overline{\partial} U, \overline{\partial}^* v] = H(dU)$$

for all smooth maps U on B^4. Applying Proposition C.1 to H, we obtain that $H = 0$. By density of smooth $(0,1)$-forms into $W^{1,2}$ $(0,1)$-forms, it follows in particular that

$$H(A^{0,1}) = [A^{0,1}, \overline{\partial}^* v] = 0.$$

Putting this together with the fact that $v \in KerL_0$, we obtain:

$$0 = L_0 v = \overline{\partial}\overline{\partial}^* v.$$

Since $v = 0$ on ∂B^4 and $\overline{\partial}\overline{\partial}^* v = 0$, then $v = 0$ in B^4. This is a contradiction because $\|v\|_{W^{2,2}_D} = 1$. Hence, there exists $U_v \in C^\infty(B^4, M_n(\mathbb{C}))$ so that $B_{U_v} v \neq 0$.

Next, we show that such a U_v can be chosen to be Hermitian. Indeed since $U_v \in M_n(\mathbb{C})$, there exists a decomposition in terms of its Hermitian and anti-Hermitian part:

$$U_v = U_1 + U_2,$$

where $U_1 \in C^\infty(B^4, \mathfrak{u}(n))$ and $U_2 \in C^\infty(B^4, i\mathfrak{u}(n))$. Assume that $B_{U_1} v = 0$, otherwise we redefine $U_v := U_1$. Under this assumption, by linearity it then necessarily follows that

$$B_{U_2} v \neq 0.$$

If this condition holds, then by multiplying with i,

$$iB_{U_2}v = B_{iU_2}v \neq 0.$$

Moreover, $iU_2 \in C^\infty(B^4, \mathfrak{u}(n))$ and in this case we redefine $U_v := iU_2$. Hence, there exists

$$U_v \in C^\infty(B^4, \mathfrak{u}(n)) \text{ so that } B_{U_v}v \neq 0, \text{ for any } v \in KerL_0, v \neq 0. \tag{3.46}$$

Claim. There exists U smooth Hermitian function such that B_U is injective on $KerL_0$.

We formulate the following inductive hypothesis:

$$\mathcal{I}(k) = \begin{cases} \text{there exists } U^k \in C^\infty(B^4, \mathfrak{u}(n)) \text{ supported in } V^k \subsetneq B^4 \text{ such that} \\ \{B_{U^k}e_j\}_{j=1}^k \text{ is linearly independent,} \end{cases}$$

where $k \leq N$. We show by induction that $\mathcal{I}(N)$ holds from which it follows that B_{U^N} is injective on $KerL_0$.

By (3.46), there exists U_1 such that $B_{U_1}e_1 \neq 0$. Without loss of generality, by multiplying with a compactly supported function ρ_1, we can localise U_1 in $V_1 \subsetneq B^4$. Hence $\mathcal{I}(1)$ holds. Assume that for $k < N$, $\mathcal{I}(k)$ holds. We prove that $\mathcal{I}(k+1)$ holds as well.

If $\{B_{U^k}e_j\}_{j=1}^{k+1}$ is linearly independent, then set $U^{k+1} = U^k$. Otherwise there exists $\lambda_1, \dots, \lambda_{k+1}$ not all 0 such that $\sum_{i=1}^{k+1} \lambda_i B_{U^k}e_i = 0$. Notice that $\lambda_{k+1} \neq 0$.

By (3.46) there exists U_{k+1} such that $B_{U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0$. We can choose a neighbourhood V_{k+1} and $\tilde{V}^k \subseteq V^k$ disjoint from V_{k+1} such that $\{B_{U^k}e_j\}_{j=1}^k$ is linearly independent in $\tilde{V}^k$ and

$$B_{\rho_{k+1}U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0 \qquad \text{in } V_{k+1}$$

In particular, we can define functions ρ_{k+1} compactly supported in V_{k+1}, ρ_k compactly supported in $\tilde{V}^k$. Define $U^{k+1} := \rho_k U^k + \rho_{k+1}U_{k+1}$.

It remains to show that $\{B_{U^{k+1}}e_j\}_{j=1}^{k+1}$ is linearly independent. Assume there exists $\beta_1, \dots, \beta_{k+1}$ such that

$$\sum_{j=1}^{k+1} \beta_j B_{\rho_k U^k + \rho_{k+1} U_{k+1}} e_j = \sum_{j=1}^{k+1} \beta_j B_{U^{k+1}} e_j = 0. \tag{3.47}$$

In the neighbourhood $\tilde{V}^k$, we have that

$$\sum_{j=1}^{k+1} \beta_j B_{U^k} e_j = 0.$$

Then $(\beta_1, \ldots, \beta_{k+1}) = c(\lambda_1, \ldots, \lambda_{k+1})$ for some constant c. Hence, in V_{k+1}, we have that

$$c \sum_{j=1}^{k+1} \lambda_j B_{U_{k+1}} e_j = 0.$$

By the choice of U_{k+1}, we obtain that $c = 0$. Hence $\beta_1 = \ldots = \beta_{k+1} = 0$. To conclude, define $V^{k+1} = \tilde{V}^k \cup V_{k+1}$. This proves the induction.

Hence, we have obtained $U = U^N$ such that $\{B_U e_j\}_{j=1}^N$ are linearly independent, where $\{e_j\}_{j=1}^N$ is the orthonormal basis of $KerL_0$ we picked initially. It follows that B_U is injective on $KerL_0$. □

We are now ready to prove the result of this section.

Lemma 3.4. *There exists a small constant $\beta \in [0,1]$ and a smooth map $U \in C^\infty(B^4, M_n(\mathbb{C}))$ such that*

$$KerL_{\beta,U} = \{0\}$$

where

$$L_{\beta,U} : W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})).$$

Hence, $L_{\beta,U}$ is an invertible operator.

Proof of Lemma 3.4. The case when $\beta = 0$ and $KerL_0 = \{0\}$ is trivial. We focus on the case when $KerL_0 \neq \{0\}$.

We assume the worst case scenario $dimG_0 = \infty$. By Proposition 3.6, there exists $U \in C^\infty(B^4, \mathfrak{u}(n))$ so that B_U is injective on $KerL_0$. Furthermore it follows from Proposition 3.5 that there exists an isomorphism $P_{\beta,U}$ between G_0 and $G_{\beta,U}$ for all $\beta < \beta_0$ for some $\beta_0 > 0$. We want to show the existence of β so that $KerL_{\beta,U} = \{0\}$.

Assume that for all $\beta < \beta_0$ we have that $KerL_{\beta,U} \neq \{0\}$. We aim at showing by contradiction that for some $\beta < \beta_0$ we will get that $KerL_{\beta,U} = \{0\}$. Thus, let the space

$$S_{\beta,U} := P_{\beta,U}^{-1}(KerL_{\beta,U}).$$

Since $KerL_{\beta,U}$ is finite dimensional and its dimension is bounded by the dimension of $KerL_0$ by Proposition 3.4. Because $P_{\beta,U}$ is an isomorphism, then $S_{\beta,U}$ is a finite dimensional space in G_0 of dimension at most $dimKerL_0$. Moreover, $KerL_0 \cap S_{\beta,U}$ is also finite dimensional and there exists an orthonormal basis of this space. We can complete it, to obtain an orthonormal basis $\{e_j^{\beta,U}\}_{j=1}^N$ on $S_{\beta,U}$, where $N = dimKerL_{\beta,U} = dimS_{\beta,U}$. Fix $1 \le j \le N$ and $\varepsilon > 0$ small enough such that $\lambda \in \rho(L_0)$ for all $|\lambda| = \varepsilon$. We compute the following:

$$\begin{aligned}
0 = L_{\beta,U}P_{\beta,U}e_j^{\beta_U} &= -\frac{1}{2\pi i}L_{\beta,U}\oint_{|\lambda|=\epsilon}(L_{\beta,U}-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}L_{\beta,U}\oint_{|\lambda|=\epsilon}(L_0+\beta B_U-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}(L_0+\beta B_U)\oint_{|\lambda|=\epsilon}(L_0+\beta B_U-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}(L_0+\beta B_U)\oint_{|\lambda|=\epsilon}\Big((L_0-\lambda)^{-1} \\
&\quad -\sum_{i=1}^{\infty}\beta^i(L_0-\lambda)^{-1}(B_U(L_0-\lambda)^{-1})^i\Big)e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}L_0\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&\quad +\beta\frac{1}{2\pi i}L_0\oint_{|\lambda|=\epsilon}\sum_{i=1}^{\infty}\beta^{i-1}(L_0-\lambda)^{-1}(B_U(L_0-\lambda)^{-1})^i e_j^{\beta,U}\,d\lambda \\
&\quad -\beta\frac{1}{2\pi i}B_U\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&\quad +\frac{1}{2\pi i}\beta B_U\oint_{|\lambda|=\epsilon}\sum_{i=1}^{\infty}\beta^i(L_0-\lambda)^{-1}(B_U(L_0-\lambda)^{-1})^i e_j^{\beta,U}\,d\lambda \\
&= L_0e_j^{\beta,U}-\beta\frac{1}{2\pi i}\left(B_Ue_j^{\beta,U}-L_0\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}B_U(L_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda\right) \\
&\quad +O(\beta^2)
\end{aligned} \tag{3.48}$$

The last equality holds because $e_j^{\beta,U} \in G_0$ and we have

$$P_0 e_j^{\beta,U} = -\frac{1}{2\pi i}\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda = e_j^{\beta,U}.$$

We further discuss two cases:

Case 1. $e_j^{\beta,U} \notin KerL_0$

Because $e_j^{\beta,U}$ is an element of the orthonormal basis, then $e_j^{\beta,U} \in (KerL_0)^{\perp}$. Moreover, since L_0 is Fredholm, we have

$$1 = \left\|e_j^{\beta,U}\right\|_{W_D^{2,2}} \le C\left\|L_0 e_j^{\beta,U}\right\|_{L^2} \le O(\beta)\left\|e_j^{\beta,U}\right\|_{W_D^{2,2}} = O(\beta),$$

where $C>0$ is a constant independent of β. Since $\beta<\beta_0$ is small, we get a contradiction.

Case 2. $e_j^{\beta,U} \in KerL_0$

Because the equation (3.48) vanishes for any $\beta<\beta_0$ and $KerL_0 \neq \{0\}$ it follows that

$$B_U e_j^{\beta,U} = L_0\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}B_U(L_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda. \tag{3.49}$$

Using the invertibility of the operator $L_0-\lambda$, where $\lambda\in\rho(L_0)$ then

$$(L_0-\lambda)^{-1}(L_0-\lambda) = id.$$

Thus, by expanding we obtain that

$$(L_0-\lambda)^{-1}L_0 - id = (L_0-\lambda)^{-1}\lambda.$$

$((L_0-\lambda)^{-1}\frac{L_0}{\lambda}-\frac{id}{\lambda})e_j^{\beta,U} = (L_0-\lambda)^{-1}e_j^{\beta,U}$. Since $e_j^{\beta,U}\in KerL_0$, then

$$(L_0-\lambda)^{-1}e_j^{\beta,U} = -\frac{1}{\lambda}e_j^{\beta,U}.$$

We obtain the following:

$$\begin{aligned}(3.49) &= -L_0\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}\tfrac{1}{\lambda}B_U e_j^{\beta,U}\,d\lambda \\ &= -\oint_{|\lambda|=\epsilon}\left((L_0-\lambda)^{-1}+\tfrac{I}{\lambda}\right)B_U e_j^{\beta,U}\,d\lambda \\ &= -2\pi i B_U e_j^{\beta,U} - \oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}B_U e_j^{\beta,U}\,d\lambda\end{aligned} \tag{3.50}$$

Putting the above equalities (3.49) and (3.50) together, we have that

$$(1+2\pi i)B_U e_j^{\beta,U} = -\oint_{|\lambda|=\epsilon}(L_0-\lambda)^{-1}B_U e_j^{\beta,U}\,d\lambda = 2\pi i P_0 B_U e_j^{\beta,U}$$

We apply P_0 on both sides of the equation to get

$$(1+2\pi i)P_0 B_U e_j^{\beta,U} = 2\pi i P_0^2 B_U e_j^{\beta,U}.$$

Moreover, since P_0 is a projection, and thus satisfies $P_0^2 = P_0$, our computations then give us the following equality

$$(1+2\pi i)P_0 B_U e_j^{\beta,U} = 2\pi i P_0 B_U e_j^{\beta,U}.$$

This can be true only if $P_0 B_U e_j^{\beta,U} = 0$. Together with

$$(1+2\pi i)B_U e_j^{\beta,U} = 2\pi i P_0 B_U e_j^{\beta,U},$$

it implies that $B_U e_j^{\beta,U} = 0$. This is a contradiction by the choice of our initial U.

We conclude that for some $\beta < \beta_0$ we have $Ker L_{\beta,U} = \{0\}$. □

3.4.2 Gauge perturbation

After having acquainted ourselves with the linear perturbation in the section before, we are now in a position to generalise the previous results. First consider operators of the form

$$T_{(A^{0,1})^{g(\beta U)}} = \overline{\partial \partial}^* \cdot + [(A^{0,1})^{g(\beta U)}, \overline{\partial}^* \cdot]$$

where

$$T_{(A^{0,1})^{g(\beta U)}} : W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C})) \to L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$$

and $g(\beta U) := \exp(\beta U) \in C^\infty(B^4, U(n))$. For the following proofs we will denote $T_{A^{0,1}}$ and $T_{A^{0,1 g(\beta U)}}$ by T_0 and $T_{\beta,U}$ respectively. We can remark the fact that $T_0 = L_0$.

Similar to the linear case, we can deduce that $T_{\beta,U}$ has a discrete spectrum for any U and β, with $\beta < 1$ small (so that exp is defined). Moreover, the family of operators have discrete spectrum with no accumulation points and we can express them as

$$\begin{aligned} T_{\beta,U} &= \overline{\partial}\overline{\partial}^* \cdot + [(A^{0,1})^{g(\beta U)}, \overline{\partial}^* \cdot] \\ &= \overline{\partial}\overline{\partial}^* \cdot + \sum_{n=0}^{\infty} [\beta^n A_n, \overline{\partial}^* \cdot] \\ &= \overline{\partial}\overline{\partial}^* \cdot + [A^{0,1}, \overline{\partial}^* \cdot] + \sum_{n=1}^{\infty} \beta^n [A_n, \overline{\partial}^* \cdot] \end{aligned}$$

where $A_1 = A^{0,1}$ and A_n are $(0,1)$-forms. This shows that the resolvent is an analytic function of β. Thus, the operator is analytic in the sense of Kato (see [28]). In a completely analogous way to Proposition 3.4 we have the existence of m not necessarily distinct eigenvalues corresponding to $T_{\beta,U}$, namely $\lambda_U^{(1)}(\beta), \ldots, \lambda_U^{(m)}(\beta)$. There exists $\varepsilon > 0$ so that $|\lambda_U^{(i)}(\beta)| < \varepsilon$ for all $\beta \geq 0$ and $1 \leq i \leq m$. In this section we denote $P_{\beta,U}$ by

$$P_{\beta,U} = -\frac{1}{2\pi i} \oint_{|\lambda|=\varepsilon} (T_{\beta,U} - \lambda)^{-1} \, d\lambda$$

and it is an analytic function of β on $|\lambda| = \varepsilon$. Similarly, we define P_0 for the operator T_0.

We denote the generalised eigenspace corresponding to 0 for T_0 by

$$G_0 = \bigcup_{k=1}^{\infty} \left\{ v \in W_D^{2,2}(\Omega^{0,2} B^4 \otimes M_n(\mathbb{C})) | T_0^k v = 0 \right\}$$

and the range of $P_{\beta,U}$ by

$$G_{\beta,U} = \bigcup_{k=1}^{\infty} \bigoplus_{i=1}^{m} \left\{ v \in W_D^{2,2}(\Omega^{0,2} B^4 \otimes M_n(\mathbb{C})) | (T_{\beta,U} - \lambda_U^{(i)}(\beta))^k v = 0 \right\}$$

respectively.

Since $g(\beta U) = \exp(\beta U)$, we then have the existence of an operator $B_{\beta,U}$ so that

$$\beta B_{\beta,U} \cdot = [(A^{0,1})^{g(\beta U)} - A^{0,1}, \overline{\partial}^* \cdot]$$

and $B_{\beta,U}$ is analytic in β (in particular it does not have any poles). Since U is a smooth Hermitian mapping we can obtain

$$B_{\beta,U} \cdot = [\overline{\partial} U + [A^{0,1}, U], \overline{\partial}^* \cdot] + O(\beta),$$

by expanding $B_{\beta,U}$ in β. Define

$$B_{0,U}\cdot := [\overline{\partial}U + [A^{0,1}, U], \overline{\partial}^*\cdot]$$

as a map from $W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$ to $L^2(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$. Thus, by again using the smoothness of U, we can expand $T_{\beta,U}$ in U and obtain

$$T_{\beta,U} = T_0 + \beta B_{\beta,U} = T_0 + \beta B_{0,U} + O(\beta^2).$$

In a completely analogous to Proposition 3.5 way we obtain that $P_{\beta,U}$ is an isomorphism between G_0 and $G_{\beta,U}$. Hence, we can assume this and prove the reciprocal version of Lemma 3.4. Firstly, we prove an analogue of Proposition 3.6. The proof will follow very similar steps as before.

Proposition 3.7. *There exists a smooth map $U \in C^\infty(B^4, \mathfrak{u}(n))$ such that $B_{0,U}$ is injective on $KerT_0$.*

Remark 3.3. *It is important to remark that this proof will give us a map U that belongs to the Lie algebra $\mathfrak{u}(n)$. This, in turn, will yield a perturbation by a gauge that is unitary, since $g(\beta U) = \exp(\beta U)$. It is crucial to find a unitary gauge, because it will preserve our Hermitian vector bundle structure later on.*

Proof of Proposition 3.7. Since $KerT_0$ is finite dimensional, let $\{e_1, \dots, e_N\}$ be an orthonormal basis of it.

Let $v \in KerT_0$, $v \neq 0$. We show that for each such v, we can find U_v such that $B_{0,U_v}v \neq 0$. Assume by contradiction that $B_{0,U}v = 0$ for all smooth maps U on B^4. Define the linear operators

$$H_0(\omega) := \Big([[A^{0,1}, \omega], \overline{\partial}^* v]\Big)^{0,2} : C^\infty(B^4, M_n(\mathbb{C})) \to C^\infty(\Omega^2 B^4 \otimes M_n(\mathbb{C}))$$

and

$$H_1(\omega) := \Big([\omega, \overline{\partial}^* v]\Big)^{0,2} : C^\infty(\Omega^1 B^4 \otimes M_n(\mathbb{C})) \to C^\infty(\Omega^2 B^4 \otimes M_n(\mathbb{C}))$$

which satisfies the fact that $H_0(\omega(z)) = H_0(\omega)(z)$. Moreover, we have that

$$0 = B_{0,U}v = [\overline{\partial}U + [A^{0,1}, U], \overline{\partial}^* v] = H_1(dU) + H_0(U)$$

for all smooth maps U on B^4. Applying Proposition C.2 to H_1 and H_0, we obtain that $H_1 \circ d = 0$ and $H_0 = 0$. In particular, we have obtained that for all U smooth maps on B^4,

$$H_1(dU) = [\overline{\partial}U, \overline{\partial}^* v] = 0.$$

This is a contradiction by Proposition 3.4. Hence, there exists a map $U_v \in C^\infty(B^4, M_n(\mathbb{C}))$ so that $B_{0,U_v} v \neq 0$.

Next, we show that such a U_v can be chosen to be Hermitian. Indeed since $U_v \in M_n(\mathbb{C})$, there exists a decomposition in terms of its Hermitian and anti-Hermitian part:

$$U_v = U_1 + U_2,$$

where $U_1 \in C^\infty(B^4, \mathfrak{u}(n))$ and $U_2 \in C^\infty(B^4, i\mathfrak{u}(n))$. Assume that $B_{0,U_1} v = 0$, otherwise we redefine $U_v := U_1$. Under this assumption, by linearity it then necessarily follows that

$$B_{0,U_2} v \neq 0.$$

If this condition holds, then by multiplying with i,

$$iB_{0,U_2} v = B_{0,iU_2} v \neq 0.$$

Moreover, $iU_2 \in C^\infty(B^4, \mathfrak{u}(n))$ and in this case we redefine $U_v := iU_2$. Hence, there exists

$$U_v \in C^\infty(B^4, \mathfrak{u}(n)) \text{ so that } B_{0,U_v} v \neq 0, \text{ for any } v \in KerT_0, v \neq 0. \quad (3.51)$$

Claim. There exists U smooth Hermitian function such that $B_{0,U}$ is injective on $KerT_0$.

We formulate the following inductive hypothesis:

$$\mathcal{I}(k) = \begin{cases} \text{there exists } U^k \in C^\infty(B^4, \mathfrak{u}(n)) \text{ supported in } V^k \subsetneq B^4 \text{ such that} \\ \{B_{0,U^k} e_j\}_{j=1}^k \text{ is linearly independent,} \end{cases}$$

where $k \leq N$. We show by induction that $\mathcal{I}(N)$ holds from which it follows that B_{0,U^N} is injective on $KerT_0$.

By (3.51), there exists U_1 such that $B_{0,U_1} e_1 \neq 0$. Without loss of generality, by multiplying with a compactly supported ρ_1, we can localise U_1 in a neighbourhood $V_1 \subsetneq B^4$. Hence $\mathcal{I}(1)$ holds. Assume that for $k < N$, $\mathcal{I}(k)$ holds. We prove that $\mathcal{I}(k+1)$ holds as well.

If $\{B_{0,U^k} e_j\}_{j=1}^{k+1}$ is linearly independent, then set $U^{k+1} = U^k$. Otherwise there exists $\lambda_1, \ldots, \lambda_{k+1}$ not all 0 such that $\sum_{i=1}^{k+1} \lambda_i B_{0,U^k} e_i = 0$. Notice that

$\lambda_{k+1} \neq 0$.

By (3.51) there exists U_{k+1} such that $B_{0,U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0$. We can choose a neighbourhood V_{k+1} and $\tilde{V}^k \subseteq V^k$ disjoint from V_{k+1} such that $\{B_{0,U^k} e_j\}_{j=1}^k$ is linearly independent in $\tilde{V}^k$ and

$$B_{0,\rho_{k+1}U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0 \qquad \text{in } V_{k+1}$$

In particular, we can define functions ρ_{k+1} compactly supported in V_{k+1}, ρ_k compactly supported in $\tilde{V}^k$. Define $U^{k+1} := \rho_k U^k + \rho_{k+1} U_{k+1}$.

It remains to show that $\{B_{0,U^{k+1}} e_j\}_{j=1}^{k+1}$ is linearly independent. Assume there exists $\beta_1, \ldots, \beta_{k+1}$ such that

$$\sum_{j=1}^{k+1} \beta_j B_{0,\rho_k U^k + \rho_{k+1} U_{k+1}} e_j = \sum_{j=1}^{k+1} \beta_j B_{0,U^{k+1}} e_j = 0. \tag{3.52}$$

In the neighbourhood $\tilde{V}^k$, we have that

$$\sum_{j=1}^{k+1} \beta_j B_{0,U^k} e_j = 0.$$

Then $(\beta_1, \ldots, \beta_{k+1}) = c(\lambda_1, \ldots, \lambda_{k+1})$ for some constant c. Hence, in V_{k+1}, we have that

$$c \sum_{j=1}^{k+1} \lambda_j B_{0,U_{k+1}} e_j = 0.$$

By the choice of U_{k+1}, we obtain that $c = 0$. Hence $\beta_1 = \ldots = \beta_{k+1} = 0$. To conclude, define $V^{k+1} = \tilde{V}^k \cup V_{k+1}$. This proves the induction.

Hence, we have obtained $U = U^N$ such that $\{B_{0,U} e_j\}_{j=1}^N$ are linearly independent, where e_j the orthonormal basis of $KerT_0$ we have picked initially. It follows that $B_{0,U}$ is injective on $KerT_0$. □

The following Lemma proves our perturbation result.

Lemma 3.5. *There exists a small constant* $\beta \in [0, 1]$ *and* $U \in C^\infty(B^4, \mathfrak{u}(n))$ *such that*

$$KerT_{\beta,U} = \{0\}$$

where

$$T_{\beta,U} : W_D^{2,2}(\Omega^{0,2} B^4 \otimes M_n(\mathbb{C})) \to L^2(\Omega^{0,2} B^4 \otimes M_n(\mathbb{C})).$$

Hence, $T_{\beta,U}$ is an invertible operator.

Proof of Lemma 3.5. We argue as before:

The case when $\beta = 0$ and $KerT_0 = \{0\}$ is trivial. We focus on the case when $KerT_0 \neq \{0\}$.

We assume the worst case scenario $dimG_0 = \infty$. By Proposition 3.7, there exists $U \in C^\infty(B^4, \mathfrak{u}(n))$ so that $B_{0,U}$ is injective on $KerT_0$. To further set up our proof, it follows from Proposition 3.5 that there exists an isomorphism $P_{\beta,U}$ between G_0 and $G_{\beta,U}$ for all $\beta < \beta_0$ for some $\beta_0 > 0$. We want to show the existence of β so that $KerT_{\beta,U} = \{0\}$.

Assume that for all $\beta < \beta_0$ we have that $KerT_{\beta,U} \neq \{0\}$. We aim at showing by contradiction that for some $\beta < \beta_0$ we will get that $KerT_{\beta,U} = \{0\}$. Thus, let the space

$$S_{\beta,U} := P^{-1}_{\beta,U}(KerL_{\beta,U}).$$

Since $KerT_{\beta,U}$ is finite dimensional, its dimension is bounded by the dimension of $KerT_0$ by Proposition 3.4. Because $P_{\beta,U}$ is an isomorphism, then $S_{\beta,U}$ is a finite dimensional space in G_0 of size at most $dimKerT_0$. Moreover, $KerT_0 \cap S_{\beta,U}$ is also finite dimensional and there exists an orthonormal basis of this space. We can complete it, to obtain an orthonormal basis $\{e_j^{\beta,U}\}_{j=1}^N$ on $S_{\beta,U}$, where $N = dimKerT_{\beta,U} = dimS_{\beta,U}$. Fix $1 \leq j \leq N$ and $\varepsilon > 0$ small enough such that $\lambda \in \rho(T_0)$ for all $|\lambda| = \varepsilon$. We compute the following:

$$\begin{aligned}
0 &= T_{\beta,U}P_{\beta,U}e_j^{\beta,U} = -\frac{1}{2\pi i}T_{\beta,U}\oint_{|\lambda|=\epsilon}(T_{\beta,U}-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}(T_0+\beta B_{\beta,U})\oint_{|\lambda|=\epsilon}(T_0+\beta B_{\beta,U}-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&= -\frac{1}{2\pi i}T_0\oint_{|\lambda|=\epsilon}(T_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda \\
&\quad -\frac{1}{2\pi i}\beta\left(B_{\beta,U}-T_0\oint_{|\lambda|=\epsilon}(T_0-\lambda)^{-1}B_{\beta,U}(T_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda\right)+O(\beta^2) \\
&= T_0e_j^{\beta,U}-\frac{1}{2\pi i}\beta\left(B_{\beta,U}-T_0\oint_{|\lambda|=\epsilon}(T_0-\lambda)^{-1}B_{\beta,U}(T_0-\lambda)^{-1}e_j^{\beta,U}\,d\lambda\right) \\
&\quad +O(\beta^2)
\end{aligned} \tag{3.53}$$

The last equality holds because $e_j^{\beta,U} \in G_0$ and we have

$$P_0 e_j^{\beta,U} = -\frac{1}{2\pi i}\oint_{|\lambda|=\epsilon} (T_0 - \lambda)^{-1} e_j^{\beta,U}\, d\lambda = e_j^{\beta,U}.$$

By further expanding $B_{\beta,U}$ in β we can rewrite the vanishing equation (3.53) as such:

$$0 = T_0 e_j^{\beta,U} - \frac{1}{2\pi i}\beta \left(B_{0,U} - T_0 \oint_{|\lambda|=\epsilon} (T_0 - \lambda)^{-1} B_{0,U} (T_0 - \lambda)^{-1} e_j^{\beta,U}\, d\lambda \right) + O(\beta^2). \tag{3.54}$$

We discuss two cases:

Case 1. $e_j^{\beta,U} \notin KerT_0$

Because $e_j^{\beta,U}$ is an element of the orthonormal basis, then $e_j^{\beta,U} \in (KerT_0)^{\perp}$. Moreover, since T_0 is Fredholm, we have

$$1 = \left\| e_j^{\beta,U} \right\|_{W_D^{2,2}} \leq C \left\| T_0 e_j^{\beta,U} \right\|_{L^2} \leq O(\beta) \left\| e_j^{\beta,U} \right\|_{W_D^{2,2}} = O(\beta),$$

where $C > 0$ is a constant independent of β. Since $\beta < \beta_0$ is small, we get a contradiction.

Case 2. $e_j^{\beta,U} \in KerT_0$

Because the equation (3.54) holds for any $\beta < \beta_0$ and $KerT_0 \neq \{0\}$ we then have that

$$B_{0,U} e_j^{\beta,U} = T_0 \oint_{|\lambda|=\epsilon} (T_0 - \lambda)^{-1} B_{0,U} (T_0 - \lambda)^{-1} e_j^{\beta,U}\, d\lambda.$$

By computing in an analogous way to Lemma 3.4 we get that the equation above implies that $B_{0,U} e_j^{\beta,U} = 0$. This is a contradiction with our initial choice of U. Thus, we found β and U so that $KerT_{\beta,U} = \{0\}$. □

3.4.3 Local density result in the high energy case

We start this section by first proving that under a fixed gauge transformation the operators $T_{A_k^{0,1}}$ corresponding to the approximating smooth 1-forms have trivial kernels. Secondly, we prove the existence of perturbations that give us the integrability condition (1.1) in B^4. Finally, we end this section by proving the main result - that we can always approximate connection forms by smooth ones in B^4 in such a way that we satisfy the integrability condition (1.1) throughout.

Proposition 3.8. *Let $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$ and a sequence of smooth 1-forms $A_k \to A$ in $W^{1,2}$. Then there exists a gauge $g \in C^\infty(B^4, U(n))$ and $k_0 \in \mathbb{N}$ such that*

(i)

$$KerT_{(A^{0,1})^g} = \{0\} \qquad \text{and} \qquad KerT_{\left(A_k^{0,1}\right)^g} = \{0\}$$

for all $k \geq k_0$. In particular, the operators $T_{\left(A_k^{0,1}\right)^g}$ and $T_{(A^{0,1})^g}$ are all invertible.

(ii)

$$\sup_{k \geq k_0} \left|\left|\left| T^{-1}_{\left(A_k^{0,1}\right)^g} \right|\right|\right| \leq 2 \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|.$$

Proof of Proposition 3.8. (i) The existence of a unitary smooth gauge g is given by Lemma 3.5. We have that

$$KerT_{(A^{0,1})^g} = \{0\}.$$

It remains to prove that there exists $k_0 \in \mathbb{N}$ so that $KerT_{\left(A_k^{0,1}\right)^g} = \{0\}$ for all $k \geq k_0$.

In order to prove this statement, we assume by contradiction that

$$KerT_{\left(A_k^{0,1}\right)^g} \neq \{0\}$$

and let $0 \neq \omega_k \in KerT_{\left(A_k^{0,1}\right)^g}$. We can also assume without loss of generality that $\|\omega_k\|_{W_D^{2,2}(B^4)} = 1$. Since $KerT_{(A^{0,1})^g}$ is trivial and $T_{(A^{0,1})^g}$ is Fredholm, we then get (see [32, Lemma 4.3.9]):

$$1 = \|\omega_k\|_{W_D^{2,2}(B^4)} \leq C_{A,g} \left\| T_{(A^{0,1})^g} \omega_k \right\|_{L^2(B^4)},$$

for some constant $C_{A,g} > 0$ depending on the initial 1-form A and on the gauge change g.
We compute this further:

$$\begin{aligned} 1 &\leq C_{A,g} \left\| T_{(A^{0,1})^g} \omega_k \right\|_{L^2(B^4)} \\ &= \left\| T_{(A^{0,1})^g} \omega_k - T_{\left(A_k^{0,1}\right)^g} \omega_k \right\|_{L^2(B^4)} \\ &= \left\| \left[\left(A_k^{0,1}\right)^g - \left(A^{0,1}\right)^g, \overline{\partial}^* \omega_k \right] \right\|_{L^2(B^4)}. \end{aligned}$$

Indeed, we can bound the last bracket above by $\|A_k - A\|_{W^{1,2}(B^4)}$:

$$\begin{aligned}\left\| \left[\left(A_k^{0,1}\right)^g - \left(A^{0,1}\right)^g , \overline{\partial}^* \omega_k \right] \right\|_{L^2(B^4)} &\leq C_g \|A_k - A\|_{L^4(B^4)} \|\omega_k\|_{L^4(B^4)} \\ &\leq C_g \|A_k - A\|_{W^{1,2}(B^4)} \|\omega_k\|_{W_D^{2,2}(B^4)} \\ &= C_g \|A_k - A\|_{W^{1,2}(B^4)} .\end{aligned}$$

for some constant C_g depending on g. Since the constants are independent of k, it follows that

$$1 \leq C_{A,g} \cdot C_g \|A_k - A\|_{W^{1,2}(B^4)} \to 0 \text{ as } k \to \infty.$$

Thus, for k large enough the above inequality yields a contradiction. We then have that there exists k_0 large so that $\omega_k = 0$ and that $KerT_{\left(A_k^{0,1}\right)^g} = \{0\}$ for all $k \geq k_0$. We conclude that since $T_{(A^{0,1})^g}$ and $T_{\left(A_k^{0,1}\right)^g}$ for each $k \geq k_0$ are all operators of index zero and their kernel is trivial, they are invertible.

(ii) Since $T_{(A^{0,1})^g}$ has trivial kernel and is an operator of index zero, its inverse exists mapping L^2 to $W_D^{2,2}$ and we have that $\left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right| < \infty$. Since A_k converges strongly to A in $W^{1,2}$ and g is smooth by construction, then we can assume without loss of generality that $\left|\left|\left| T_{\left(A_k^{0,1}\right)^g} - T_{(A^{0,1})^g} \right|\right|\right| \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right| < \frac{1}{2}$ for any $k \geq k_0$. Hence [32, Theorem 1.5.5(iii)] yields

$$\begin{aligned}\left|\left|\left| T^{-1}_{\left(A_k^{0,1}\right)^g} - T^{-1}_{(A^{0,1})^g} \right|\right|\right| &\leq \frac{\left|\left|\left| T_{\left(A_k^{0,1}\right)^g} - T_{(A^{0,1})^g} \right|\right|\right| \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|}{1 - \left|\left|\left| T_{\left(A_k^{0,1}\right)^g} - T_{(A^{0,1})^g} \right|\right|\right| \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|} \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right| \\ &\leq \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|.\end{aligned}$$

Thus, for $k \geq k_0$, we have

$$\left|\left|\left| T^{-1}_{\left(A_k^{0,1}\right)^g} \right|\right|\right| \leq \left|\left|\left| T^{-1}_{\left(A_k^{0,1}\right)^g} - T^{-1}_{(A^{0,1})^g} \right|\right|\right| + \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right| \leq 2 \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|.$$

Hence, by taking the sup over all $k \geq k_0$, we obtain the result. □

The following Lemma proves the existence of a perturbation under the conditions that $T_{A^{0,1}}$ has trivial kernel and that $F_A^{0,2}$ is small in L^2 norm.

Lemma 3.6. *There exists a constant $C > 0$ such that for every*

$$A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$$

with $T_{A^{0,1}}$ invertible and satisfying $\left\|F_A^{0,2}\right\|_{L^2} \leq \frac{C}{\left|\left|\left|T_{A^{0,1}}^{-1}\right|\right|\right|^2}$, there exists a $(0,2)$ form $\omega \in W_D^{2,2}(\Omega^{0,2}B^4 \otimes M_n(\mathbb{C}))$ solving the PDE:

$$\overline{\partial}\overline{\partial}^*\omega + [A^{0,1}, \overline{\partial}^*\omega] + \overline{\partial}^*\omega \wedge \overline{\partial}^*\omega = -F_A^{0,2}$$

and satisfying the estimate

$$\|\omega_0\|_{W_D^{2,2}(B^4)} \leq C' \left|\left|\left|T_{A^{0,1}}^{-1}\right|\right|\right| \left\|F_A^{0,2}\right\|_{L^2(B^4)} \tag{3.55}$$

where $C' > 0$ is a constant independent of A.

Proof of Lemma 3.6. We construct the following sequence of solutions:

$$\begin{aligned}
T_{A^{0,1}}\omega_0 &= -F_A^{0,2} \\
T_{A^{0,1}}\omega_1 &= -\overline{\partial}^*\omega_0 \wedge \overline{\partial}^*\omega_0 - F_A^{0,2} \\
T_{A^{0,1}}\omega_2 &= -\overline{\partial}^*\omega_1 \wedge \overline{\partial}^*\omega_1 - F_A^{0,2} \\
&\dots \\
T_{A^{0,1}}\omega_k &= -\overline{\partial}^*\omega_{k-1} \wedge \overline{\partial}^*\omega_{k-1} - F_A^{0,2} \\
&\dots
\end{aligned}$$

Claim. $\{\omega_k\}_{k=0}^{\infty}$ is a Cauchy sequence in $W_D^{2,2}$.

We first show by induction the uniform bound on the sequence

$$\|\omega_k\|_{W_D^{2,2}} \leq 2|||T_{A^{0,1}}||| \left\|F_A^{0,2}\right\|_{L^2(B^4)}.$$

Since $T_{A^{0,1}}$ is invertible, then we have the identity:

$$\omega_0 = T_{A^{0,1}}^{-1} T_{A^{0,1}}\omega_0.$$

Hence, from the definition of the norm of operators, it follows that:

$$\begin{aligned}
\|\omega_0\|_{W_D^{2,2}(B^4)} &\leq \left|\left|\left|T_{A^{0,1}}^{-1}\right|\right|\right| \|T_{A^{0,1}}\omega_0\|_{L^2(B^4)} \\
&= \left|\left|\left|T_{A^{0,1}}^{-1}\right|\right|\right| \left\|F_A^{0,2}\right\|_{L^2(B^4)}
\end{aligned}$$

$$< 2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}$$

Let $k > 0$. By the Sobolev embedding of $W^{1,2} \hookrightarrow L^4$ there exists a constant $C_1 > 0$ so that

$$\left\| \overline{\partial}^* \omega_k \right\|_{L^4(B^4)} \leq C_1 \left\| \overline{\partial}^* \omega_k \right\|_{W^{1,2}(B^4)} \leq C_1 \left\| \omega_k \right\|_{W_D^{2,2}(B^4)}.$$

Then we have:

$$\begin{aligned}
\|\omega_k\|_{W_D^{2,2}(B^4)} &\leq \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| T_{A^{0,1}} \omega_k \right\|_{L^2(B^4)} \\
&\leq \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| \overline{\partial}^* \omega_{k-1} \wedge \overline{\partial}^* \omega_{k-1} \right\|_{L^2(B^4)} + \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)} \\
&\leq C_1 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| \omega_{k-1} \right\|_{L^4(B^4)}^2 + \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)} \\
&\leq C_1^2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| \omega_{k-1} \right\|_{W_D^{2,2}(B^4)}^2 + \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}
\end{aligned}$$

By the induction hypothesis we have that

$$\|\omega_{k-1}\|_{W_D^{2,2}(B^4)} < 2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}.$$

Thus,

$$\|\omega_k\|_{W_D^{2,2}(B^4)} \leq 4C_1^2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\|^3 \left\| F_A^{0,2} \right\|_{L^2(B^4)}^2 + \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}.$$

Choosing the constant $C > 0$ such that $\frac{1}{C} < 4C_1^2$, we obtain by assumption

$$4C_1^2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\|^2 \left\| F_A^{0,2} \right\|_{L^2(B^4)} < C \left\| \left| T_{A^{0,1}}^{-1} \right| \right\|^2 \left\| F_A^{0,2} \right\|_{L^2(B^4)} \leq 1,$$

and we conclude that

$$\|\omega_k\|_{W_D^{2,2}} \leq 2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}$$

Hence, by induction it follows that the sequence is uniformly bounded in $W_D^{2,2}$:

$$\|\omega_k\|_{W_D^{2,2}(B^4)} \leq 2 \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| F_A^{0,2} \right\|_{L^2(B^4)}$$

for all $k \geq 0$. It remains to show that $\{\omega_k\}_{k=0}^{\infty}$ is a Cauchy sequence.

Let $k > 0$. It follows that

$$\|\omega_{k+1} - \omega_k\|_{W_D^{2,2}(B^4)} \leq \left\| \left| T_{A^{0,1}}^{-1} \right| \right\| \left\| T_{A^{0,1}}(\omega_{k+1} - \omega_k) \right\|_{L^2(B^4)}$$

$$
\begin{aligned}
&\leq |||T^{-1}_{A^{0,1}}||| \left\|\overline{\partial}^*(\omega_k - \omega_{k-1}) \wedge \overline{\partial}^*\omega_k\right\|_{L^2(B^4)} \\
&+ |||T^{-1}_{A^{0,1}}||| \left\|\overline{\partial}^*\omega_{k-1} \wedge \overline{\partial}^*(\omega_k - \omega_{k-1})\right\|_{L^2(B^4)} \\
&\leq 4C_1 |||T^{-1}_{A^{0,1}}|||^2 \left\|F^{0,2}_A\right\|_{L^2(B^4)} \|\omega_k - \omega_{k-1}\|_{W^{2,2}_D(B^4)}
\end{aligned}
$$

Choosing the constant $C > 0$ such that $\frac{1}{C} < 4C_1$, we obtain by assumption

$$
4C_1 |||T^{-1}_{A^{0,1}}|||^2 \left\|F^{0,2}_A\right\|_{L^2(B^4)} < \frac{1}{C} |||T^{-1}_{A^{0,1}}|||^2 \left\|F^{0,2}_A\right\|_{L^2(B^4)} \leq 1,
$$

and we can conclude that the sequence $\{\omega_k\}_{k=0}^\infty$ is Cauchy. Hence, we have proven the claim.

Moreover, because $W^{2,2}_D$ is a Banach space and $\{\omega_k\}_{k=0}^\infty$ is a Cauchy sequence, there exists ω_∞ such that $\omega_k \to \omega_\infty$ as $k \to \infty$ in $W^{2,2}_D$. Moreover, by the strong convergence and the uniform bound of the sequence we obtain that $\|\omega_\infty\|_{W^{2,2}_D(B^4)} \leq 2|||T^{-1}_{A^{0,1}}||| \left\|F^{0,2}_A\right\|_{L^2(B^4)}$ and

$$
\overline{\partial}\overline{\partial}^*\omega_\infty + [A^{0,1}, \overline{\partial}^*\omega_\infty] + \overline{\partial}^*\omega_\infty \wedge \overline{\partial}^*\omega_\infty = -F^{0,2}_A.
$$

□

We can conclude this section with the main result.

Theorem 3.3. *Let $A \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$, with $F^{0,2}_A = 0$. There exists a sequence of smooth forms $A_k \in C^\infty(\Omega^1 B^4 \otimes \mathfrak{u}(n))$, $F^{0,2}_{A_k} = 0$ and $A_k \to A$ in $W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n))$.*

Proof of Theorem 3.3. By Lemma 3.5 there exists a unitary gauge change $g \in C^\infty(B^4, U(n))$ so that $T_{(A^{0,1})^g}$ is invertible $W^{2,2}_D$ to L^2. Moreover, we can obtain a sequence of smooth 1-forms $\tilde{A}_k$ by simple convolution such that $\left\|F^{0,2}_{\tilde{A}_k}\right\|_{L^2(B^4)} \to 0$ and $\tilde{A}_k \to A$ in $W^{1,2}$ as $k \to \infty$. By Proposition 3.8(i) there exists k_0 so that for all $k \geq k_0$, $T_{\tilde{A}^g_k}$ is invertible and that $\tilde{A}^g_k \to A^g$. Moreover, since the change of gauge is unitary we also have that $\left\|F^{0,2}_{\tilde{A}_k}\right\|_{L^2(B^4)} = \left\|F^{0,2}_{\tilde{A}^g_k}\right\|_{L^2(B^4)}$.

Moreover, by Proposition 3.8(ii) we know that

$$
\sup_{k \geq k_0} \left|\left|\left| T^{-1}_{(\tilde{A}^{0,1}_k)^g} \right|\right|\right| \leq 2 \left|\left|\left| T^{-1}_{(A^{0,1})^g} \right|\right|\right|.
$$

Thus, for all $k \geq k_0$, it follows that $\frac{1}{\left\|\left|T^{-1}_{\left(\tilde{A}^{0,1}_k\right)^g}\right|\right\|^2} \geq \frac{1}{2\left\|\left|T_{(A^{0,1})^g}\right|\right\|^2}$. Let $C > 0$ the constant given by Lemma 3.6. There exists $k_1 > k_0$ such that

$$\left\|F^{0,2}_{\tilde{A}^g_k}\right\|_{L^2(B^4)} \leq \frac{C}{2\left\|\left|T_{(A^{0,1})^g}\right|\right\|^2} \leq \frac{C}{\left\|\left|T^{-1}_{\left(\tilde{A}^{0,1}_k\right)^g}\right|\right\|^2}.$$

for all $k \geq k_1$. Hence, for each $k \geq k_1$, Lemma 3.6 gives the existence of $(0,2)$ forms ω_k satisfying the estimate

$$\|\omega_k\|_{W^{2,2}_D(B^4)} \leq C'\left\|\left|T^{-1}_{\left(A^{0,1}_k\right)^g}\right|\right\| \left\|F^{0,2}_{\tilde{A}^g_k}\right\|_{L^2(B^4)} \leq 2C'\left\|\left|T^{-1}_{(A^{0,1})^g}\right|\right\| \left\|F^{0,2}_{\tilde{A}_k}\right\|_{L^2(B^4)},$$

where $C' > 0$ is a constant independent of k. Moreover, each ω_k solve the PDE:

$$\overline{\partial}\overline{\partial}^*\omega_k + \left[\left(\tilde{A}^{0,1}_k\right)^g, \overline{\partial}^*\omega_k\right] + \overline{\partial}^*\omega_k \wedge \overline{\partial}^*\omega_k = -F^{0,2}_{\tilde{A}^g_k}, \tag{3.56}$$

i.e. $F^{0,2}_{\tilde{A}^g_k + \overline{\partial}^*\omega_k} = 0$. Since $\left\|F^{0,2}_{\tilde{A}_k}\right\|_{L^2(B^4)}$ converges strongly to 0, the estimates on the $(0,2)$ forms ω_k give $\omega_k \to 0$ in $W^{2,2}_D$ as $k \to \infty$. Thus, we obtain the strong convergence

$$\left(\tilde{A}^{0,1}_k\right)^g + \overline{\partial}^*\omega_k \to \left(A^{0,1}\right)^g \text{ in } W^{1,2}.$$

Define the sequence of connection forms

$$A_k := \left((\tilde{A}^{0,1}_k)^g + \overline{\partial}^*\omega_k - \overline{(\tilde{A}^{0,1}_k)^g + \overline{\partial}^*\omega_k}^T\right)^{g^{-1}} \in W^{1,2}(\Omega^1 B^4 \otimes \mathfrak{u}(n)) \cap C^\infty.$$

Because g is a smooth unitary gauge, and $A^g_k \to A^g$ in $W^{1,2}$ by construction, then this sequence of forms are unitary and convergent in $W^{1,2}$. We need to establish that $A_k \to A$ in $W^{1,2}$. Indeed, we obtain the following L^2 convergence:

$$\|A^g_k - A^g\|_{L^2} \to 0 \iff \left\|g^{-1}\left(A_k - A\right)g\right\|_{L^2} \to 0 \overset{g\in U(n)}{\iff} \|A_k - A\|_{L^2} \to 0.$$

Because the limit is unique, then $A_k \to A$ in $W^{1,2}$. Moreover, the smooth sequence A_k satisfies the integrability condition $F^{0,2}_{A_k} = 0$ by the construction of ω_k in (3.56). $\square$

3.4.4 Global density result

In this section we will use the result we have proven in the previous section in order to obtain a global result for a closed Kähler manifold X^2. In order to be able to generalise, we will work on sections of the vector bundle (E, h_0) over X^2. The $\overline{\partial}$ operator over X^2 is well-defined and acts on the space of (p,q)-forms:

$$\overline{\partial} : \mathcal{A}^{p,q}(X^2) \to \mathcal{A}^{p,q+1}(X^2).$$

Its corresponding dual operator, $\overline{\partial}^*$, is defined as a map:

$$\overline{\partial}^* : \mathcal{A}^{p,q}(X^2) \to \mathcal{A}^{p,q-1}(X^2).$$

On the space $\mathcal{A}^{p,q}(X^2)$ the $\overline{\partial}$-Hodge decomposition (A.4) gives the orthogonal L^2 decomposition:

$$\mathcal{A}^{p,q}(X^2) = \overline{\partial}\mathcal{A}^{p,q-1}(X^2) + \overline{\partial}^*\mathcal{A}^{p,q+1}(X^2) + \mathcal{H}^{p,q}(X^2), \tag{3.57}$$

where $\mathcal{H}^{p,q}(X^2)$ is the space of holomorphic (p,q)-sections. Since X^2 is a closed Kähler manifold, then we remark that $\mathcal{H}^{p,q}(X^2)$ is finite dimensional. In particular, by (3.57) under the condition $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$, the $(0,2)$-form $\omega \in \mathcal{A}^{0,2}(X^2)$ can be written as follows:

$$\omega = \overline{\partial}\overline{\partial}^*\alpha.$$

Since $\overline{\partial}$ and $\overline{\partial}^*$ define elliptic complexes over closed Kähler surfaces (see for example [22, Chapter IV]):

$$0 \to \mathcal{A}^{0,0}(X^2) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}(X^2) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,2}(X^2) \to 0$$

and

$$0 \to \mathcal{A}^{0,2}(X^2) \xrightarrow{\overline{\partial}^*} \mathcal{A}^{0,1}(X^2) \xrightarrow{\overline{\partial}^*} \mathcal{A}^{0,0}(X^2) \to 0$$

then the operator $\overline{\partial}\overline{\partial}^*$ is elliptic on $(0,2)$-sections over closed Kähler surfaces. In particular, it is Fredholm and moreover, $\overline{\partial}\overline{\partial}^*$ is self-adjoint. Thus, its Fredholm index vanishes:

$$\begin{aligned} \text{index}(\overline{\partial}\overline{\partial}^*) &= \text{dimKer}(\overline{\partial}\overline{\partial}^*) - \text{dimCoker}(\overline{\partial}\overline{\partial}^*) \\ &= \text{dimKer}(\overline{\partial}\overline{\partial}^*) - \text{dimKer}(\overline{\partial}\overline{\partial}^*) = 0. \end{aligned} \tag{3.58}$$

We redefine our operator $T_{A^{0,1}}$ as such:

$$T_\nabla : \Gamma_{W^{2,2}}(\mathcal{A}^{0,2}(X^2)) \to \Gamma_{L^2}(\mathcal{A}^{0,2}(X^2))$$

where ∇ is a $W^{1,2}$ unitary connection over X^2 and $\Gamma_{W^{p,q}}$ is the space of sections of $W^{p,q}$ regularity. We can directly apply Lemma 3.5 to obtain the existence of a global smooth section g so that $\mathrm{Ker} T_{\nabla^g} = 0$. Moreover, by (3.58) and Proposition 3.3 applied to T_{∇^g} it follows that T_{∇^g} is a Fredholm operator of index 0. Hence, $\mathrm{Coker} T_{\nabla^g}$ is empty and the operator T_{∇^g} is invertible.

It follows that we can apply Lemma 3.6 to ∇^g and we can conclude that, similarly to Theorem 3.3, we have proven:

Theorem 3.4. *Let ∇ a $W^{1,2}$ unitary connection over X^2 with $\mathcal{H}^{0,2}_{\overline{\partial}}(X^2) = 0$, satisfying the integrability condition*

$$F^{0,2}_{\nabla} = 0.$$

Then there exists a sequence of smooth unitary connections ∇_k, with $F^{0,2}_{\nabla_k} = 0$ such that

$$\mathrm{dist}_2(\nabla_k, \nabla) \to 0.$$

3.5 Proof of Theorem 1.2

By Theorem 3.4 applied to the given $W^{1,2}$ connection ∇, we obtain the existence of a sequence of smooth connections ∇_k that converge to ∇ in the sense of

$$\mathrm{dist}_2(\nabla_k, \nabla) \to 0.$$

Step 1. There exists $r > 0$ and a finite good cover $\{B^4_r(x_i)\}$ such that

$$\nabla \simeq d + A_i \text{ in } B^4_r(x_i)$$

with

$$\|\nabla A_i\|_{L^2(B^4_r(x_i))} \leq \varepsilon_0,$$

where $\varepsilon_0 > 0$ is given by Theorem 3.2. Since ∇_k converges to ∇ in the sense of dist_2, it follows that in each ball $B^4_r(x_i)$ with $\nabla_k \approx d + A^k_i$ we have $F^{0,2}_{A^k_i} = 0$, $A^k_i \to A_i$ in $W^{1,2}$ and $F_{A^k_i} \to F_{A_i}$ in L^2 as $k \to \infty$.

By Theorem 3.2 there exists $r' < r$ and $\sigma_i \in W^{2,p}(B^4_{r'}(x_i), GL_n(\mathbb{C}))$ for any $p < 2$ such that

$$A^{0,1}_i = -\overline{\partial}\sigma_i \cdot \sigma_i^{-1}.$$

Let $\delta > 0$ be the constant given in Corollary 3.1. There exists $k_0 \geq 0$ so that $\left\|A^k_i - A_i\right\|_{W^{1,2}} \leq \delta$ for all $k \geq k_0$. Corollary 3.1 applied to each A^k_i, $k \geq k_0$

gives the existence of a sequence

$$\sigma_i^k \in W^{2,p}(B^4_{r'/2}(x_i), GL_n(\mathbb{C}))$$

of gauges that holomorphically trivialise A_i^k:

$$\left(A_i^k\right)^{0,1} = -\overline{\partial}\sigma_i^k \cdot \left(\sigma_i^k\right)^{-1}$$

with the estimates

$$\left\|\sigma_i^k - \sigma_i\right\|_{L^p(B^4_{r'/2}(x_i))} \leq C \left\|A_i^k - A_i\right\|_{W^{1,2}(B^4_r(x_i)} \tag{3.59}$$

for some constant $C > 0$ and all $p < 12$. For each $q < 2$ there exists $C_q > 0$ such that

$$\left\|\sigma_i^k - id\right\|_{W^{2,q}(B^4_{r'/2}(x_i))} \leq C_q \left(\left\|A_i^k\right\|_{W^{1,2}(B^4_r(x_i)} + \|A_i\|_{W^{1,2}(B^4_r(x_i)}\right) \tag{3.60}$$

By an abuse of notations, from now on, we will use r' to denote $r'/2$. From (3.59), $\sigma_i^k \to \sigma_i$ in $L^p(B^4_{r'}(x_i))$ for all $p < 12$. Moreover, the uniform bound (3.60) gives that the sequence σ_i^k converges strongly in $W^{2,q}$ for all $q < 2$. Since limits are unique, we obtain that σ_i^k converges strongly to σ_i in $W^{2,q}$ for all $q < 2$. A_i^k is smooth, it follows that each gauge σ_i^k is smooth.

By defining $h_i^k = \overline{\sigma_i^k}^T \cdot \sigma_i^k$, we have that each h_i^k is uniformly bounded in $W^{2,p}(B^4_{r'}, Sym(n))$ and

$$\left(A_i^k\right)^{\sigma_i^k} = \left(h_i^k\right)^{-1} \partial h_i^k \to h_i^{-1}\partial h_i \qquad \text{in } W^{1,p}(B^4_{r'}(x_i)) \text{ for all } p < 2. \tag{3.61}$$

Moreover, the corresponding curvature forms satisfy the following gauge relation:

$$F_{\left(A_i^k\right)^{\sigma_i^k}} = \left(\sigma_i^k\right)^{-1} \left(F_{A_i^k}\right) \sigma_i^k.$$

Combining the estimate (3.59) with the L^2 convergence of the sequence $F_{A_i^k}$, we have

$$F_{\left(A_i^k\right)^{\sigma_i^k}} \to F_{A_i^{\sigma_i}} \qquad \text{in } L^p(B^4_{r'}(x_i)) \text{ for all } p < 2. \tag{3.62}$$

Analogously to the proof of theorem 1.1 in Section 3.3, there exists smooth holomorphic structures $\mathcal{E}_k$ such that

$$\nabla_k^{0,1} = \overline{\partial}_{\mathcal{E}_k}.$$

Thus, (3.61) and (3.62) yield that for all $p < 2$ we obtain the required convergence:

$$\mathrm{dist}_p(\nabla_k, \nabla) \to 0$$

over the holomorphic vector bundle structures $\mathcal{E}_k$ and $\mathcal{E}$. In the next step we prove that this convergence leads to an isomorphism between the two structures.

Step 2. We construct bundle isomorphisms $\mathcal{H}_k$ between the holomorphic bundles $\mathcal{E}_k$ and $\mathcal{E}$ such that $\overline{\partial}_{\mathcal{E}_k} = \mathcal{H}_k^{-1} \circ \overline{\partial}_{\mathcal{E}} \circ \mathcal{H}_k$.

Let i,j such that there exist gauge transition maps $g_{ij} \in W^{2,2}(B_r^4(x_i) \cap B_r^4(x_j), U(n))$ and $g_{ij}^k \in C^\infty(B_r^4(x_i) \cap B_r^4(x_j), U(n))$ satisfying:

$$A_i^{g_{ij}} = A_j \qquad \text{in } B_r^4(x_i) \cap B_r^4(x_j)$$

and

$$\left(A_i^k\right)^{g_{ij}^k} = A_j^k \qquad \text{in } B_r^4(x_i) \cap B_r^4(x_j).$$

Since $g_{ij}^k \to g_{ij}$ in $W^{2,2}$ by construction, from [15] there exist maps $\phi_i^k \in W^{2,2}(B_{r'}^4(x_i), U(n))$ and $\phi_j^k \in W^{2,2}(B_{r'}^4(x_j), U(n))$ such that

$$g_{ij}^k = \left(\phi_i^k\right)^{-1} \cdot g_{ij} \cdot \phi_j^k.$$

Using the notation introduced in Section 3.3, there exists holomorphic transition maps for the structures $\mathcal{E}$ and $\mathcal{E}_k$ defined as:

$$\sigma_{ij} = \sigma_i^{-1} \cdot g_{ij} \cdot \sigma_j \qquad \text{and} \qquad \sigma_{ij}^k = \left(\sigma_i^k\right)^{-1} \cdot g_{ij}^k \cdot \sigma_j^k.$$

Consider the maps $\mathcal{H}_i^k := \sigma_i^{-1} \cdot \phi_i^k \cdot \sigma_i^k$ and $\mathcal{H}_j^k = \sigma_j^{-1} \cdot \phi_j^k \cdot \sigma_j^k$. By construction, we have:

$$\sigma_{ij}^k = \left(\mathcal{H}_i^k\right)^{-1} \cdot \sigma_{ij} \cdot \mathcal{H}_j^k.$$

Thus, $\mathcal{H}_k = \{\mathcal{H}_i^k\}_i$ defines a bundle isomorphism and since σ_{ij}^k are holomorphic, it preserves the holomorphic structure:

$$\overline{\partial}_{\mathcal{E}_k} = \mathcal{H}_k^{-1} \circ \overline{\partial}_{\mathcal{E}} \circ \mathcal{H}_k.$$

This finishes the proof of theorem 1.2.

Chapter 4

$U(n)$ Bundles in Banach Spaces

Using the ideas developed in the previous chapter, we will generalise all our results for d-dimensional closed Kähler manifolds X^d. Let ∇ be a unitary $W^{1,d}$ connection of a hermitian bundle (E, h_0) over X^d. In the last chapter we have benefited from $d = 2$ and consequently from the fact that L^2 is a Hilbert space. In this chapter we will adapt our techniques from before and generalise them for L^d Banach spaces, where $d \geq 3$. Moreover, when translating the theorems we will encounter an additional difficulty which is the fact that $\overline{\partial\partial}^*$ is not elliptic for $d \geq 3$. However, the analogous results still hold true. We recall the theorems we will prove:

Theorem 1.3. *Let ∇ be a unitary $W^{1,d}$ connection of an hermitian bundle (E, h_0) over a closed Kähler manifold X^d. Assume ∇ satisfies the integrability condition*

$$F_\nabla^{0,2} = 0 \tag{4.1}$$

then there exists a <u>smooth</u> holomorphic structure $\mathcal{E}$ on E and a $\bigcap_{q<d} W^{2,q}$ section h of the bundle of positive Hermitian endomorphisms of E such that

$$\nabla = \partial_0 + h^{-1}\partial_0 h + \overline{\partial}_\mathcal{E} \tag{4.2}$$

where $\overline{\partial}_\mathcal{E}$ is the $\overline{\partial}-$operator associated to the holomorphic bundle $\mathcal{E}$ and ∂_0 is the 1-0 *part of the Chern connection associated*[1] *to the holomorphic structure $\mathcal{E}$ and the chosen reference hermitian product h_0.*

and:

Theorem 1.4. *Under the assumptions of Theorem 1.3 and $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$, there exists a sequence of <u>smooth</u> connections ∇_k on smooth holomorphic*

[1] These connections are not necessarily unitary with respect to h_0 anymore.

bundles $\mathcal{E}_k$ satisfying

$$F^{0,2}_{\nabla_k} = 0 \quad ,$$

and converging to ∇ in the sense that for all $p < d$:

$$\operatorname{dist}_p(\nabla_k, \nabla) = \inf_{\sigma \in \mathcal{G}^{1,d}(GL(n,\mathbb{C}))} \int_{X^d} |\nabla_k - \nabla^\sigma|^p \, \omega^d + \int_{X^d} |F_{\nabla_k} - F_{\nabla^\sigma}|^p \, \omega^d \to 0. \tag{4.3}$$

Moreover, there exists a family of isomorphisms $\mathcal{H}_k$ such that

$$\overline{\partial}_{\mathcal{E}_k} = \mathcal{H}_k^{-1} \circ \overline{\partial}_{\mathcal{E}} \circ \mathcal{H}_k.$$

That is, the sequence of connections ∇_k act on equivalent bundles to E. □

4.1 Strategy and Structure of the Chapter

Since the way we will approach the proofs of the theorems will vary in certain points to a significant degree compared to Chapter 3, it is useful to fix the ideas first. Let $d \geq 3$. The invertibility of $\overline{\partial \partial}^*$ as an operator from $W^{2,d}(\Omega^{0,2} X^d)$ to $L^d(\Omega^{0,2} X^d)$ fails in particular because $L^d(\Omega^{0,2} X^d)$ consists of $(0,2)$-forms that are not necessarily $\overline{\partial}$-exact. Thus, our methods from Sections 3.1 and 3.4 do not translate and we will solve the integrability condition (1.1) by using the **extended integrability condition**. Let

$$\overline{\partial}_A \cdot := \overline{\partial} \cdot + [A, \cdot]. \tag{4.4}$$

and $\overline{\partial}^*_A$ its adjoint operator as defined in Proposition C.3. Note that

$$F^{0,2}_A = \overline{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = \overline{\partial}_{A^{0,1}/2} A^{0,1}.$$

For a $(0,2)$-form $\omega \in \Omega^{0,2}$ with ω satisfying the $\overline{\partial}$-Neumann boundary conditions $\omega, \overline{\partial}\omega \in \operatorname{Dom}(\overline{\partial}^*)$ (see Appendix A), we define the **extended integrability condition** as:

$$F^{0,2}_{A + \overline{\partial}^* \omega} + \overline{\partial}^*_{A^{0,1} + \overline{\partial}^* \omega} \overline{\partial} \omega = 0, \tag{4.5}$$

where $\overline{\partial}^*_{A^{0,1} + \overline{\partial}^* \omega}$ is well-defined by Proposition C.3. We will show that under certain assumptions on $A^{0,1}$, we can solve such an equation for ω since the above equation can be expanded as the following elliptic PDE:

$$\left(\overline{\partial \partial}^* + \overline{\partial}^* \overline{\partial} \right) \omega = -[\overline{\partial}^* \omega, A^{0,1}] - *[*\overline{\partial}\omega, \overline{A^{0,1}}^T] - \overline{\partial}^* \omega \wedge \overline{\partial}^* \omega - *[*\overline{\partial}\omega, \overline{\overline{\partial}^* \omega}^T] - F^{0,2}_A.$$

Moreover, we can show that under the $\overline{\partial}$-Neumann boundary conditions $\omega, \overline{\partial}\omega \in \mathrm{Dom}(\overline{\partial}^*)$, we obtain the integrability condition:

$$F^{0,2}_{A+\overline{\partial}^*\omega} = 0.$$

On Kähler manifolds we have that the Kohn-Laplace is related to the Hodge-Laplace in the following way: $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \frac{1}{2}\Delta_d$. However, we have to point out to the reader that even under such a nice relation, we cannot expect an immediate translation of the work in Sections 3.1 and 3.4. Indeed the conditions $\omega, \overline{\partial}\omega \in \mathrm{Dom}(\overline{\partial}^*)$ are the $\overline{\partial}$-Neumann boundary conditions which lead to a problem of lack of coerciveness. Hence, $\Delta_{\overline{\partial}}$ is not an elliptic operator over B^{2d} under $\overline{\partial}$-Neumann boundary conditions by (A.3).

We emphasize the fact that if ω does not satisfy the $\overline{\partial}$-Neumann conditions, then we will use the operator ϑ (see (A.1)) instead of $\overline{\partial}^*$ to emphasize the difference between the two cases. In this case we have $\Delta_{\overline{\partial}} := \overline{\partial}\vartheta + \vartheta\overline{\partial}$ and we note that $\overline{\partial}$ and ϑ are not necessarily L^2 orthogonal.

Sections 3.1 and 3.4 took a "local" approach by perturbing the connection 1-form A in the unit ball B^4 such that it satisfies the integrability condition. Since $\Delta_{\overline{\partial}}$ is not an elliptic operator from $W^{2,d}(\Omega^{0,2}B^{2d})$ to $L^d(\Omega^{0,2}B^{2d})$ under $\overline{\partial}$-Neumann boundary conditions, then a density result obtained through a similar local approach seems out of reach. In order to mitigate for the lack of ellipticity, we will assume the *d-Neumann boundary conditions* for $\omega \in W^{2,d}(\Omega^{0,2}B^{2d})$:

$$i^*_{\partial B^{2d}}\omega = 0 \quad \text{and} \quad i^*_{\partial B^{2d}}(d\omega) = 0, \tag{4.6}$$

where $i_{\partial B^{2d}} : \partial B^{2d} \to B^{2d}$ is the canonical inclusion map. For consistency we will denote the d-Neumann boundary conditions (4.6) by $\omega, d\omega \in \mathrm{Dom}(d^*)$. Thus, by the classical Hodge decomposition, it is known that $\Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ is elliptic under the boundary conditions (4.6) (see [37, Theorem 10.5.1]).

We can use the d-Neumann conditions (4.6) so that we can solve the extended integrability condition (4.5) on the unit ball B^{2d}. Moreover, using the techniques we develop, we are able to conclude that globally on the closed Kähler manifold X^d we obtain the integrability condition from the extended one.

Having fixed these ideas, we can outline the structure of the Chapter. Firstly, we start by proving the existence of holomorphic trivilisations in Section 4.2. This part will use methods that resemble the same techniques which have been used in Section 3.2. Secondly, in Section 4.3 we show density under high-energy assumptions over B^{2d} keeping the extended integrability condition (4.5) under the d-Neumann boundary conditions (4.6). Using this construction we show the extended integrability condition can be globally satisfied on X^d and that this implies the integrability condition.

We prove the key result showing that over closed Kähler manifolds X^d, we have that the extended integrability condition (4.5) implies the integrability condition (1.1).

Proposition 4.1. *Let* $\omega \in W^{2,d}(\Omega^{0,2}X^d)$ *and* $A \in W^{1,d}(\Omega^1 X^d)$ *satisfying the extended integrability condition:*

$$F^{0,2}_{A+\overline{\partial}^*\omega} + \overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega = 0.$$

Then,

$$F^{0,2}_{A+\overline{\partial}^*\omega} = 0 \quad \textit{and} \quad \overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega = 0.$$

Proof of Proposition 4.1. We show that the complex variable analogue of Bianchi's identity $d_A F_A = 0$ holds:

$$\overline{\partial}_{A+\overline{\partial}^*\omega} F^{0,2}_{A+\overline{\partial}^*\omega} = 0.$$

Indeed, by defining $\tilde{A} = A + \overline{\partial}^*\omega$, we have

$$\begin{aligned}
\overline{\partial}_{\tilde{A}}(\overline{\partial}\tilde{A} + \tilde{A}\wedge\tilde{A}) &= \overline{\partial}(\tilde{A}\wedge\tilde{A}) + [\tilde{A},\overline{\partial}\tilde{A}] + [\tilde{A},\tilde{A}\wedge\tilde{A}] \\
&= \overline{\partial}\tilde{A}\wedge\tilde{A} - \tilde{A}\wedge\overline{\partial}\tilde{A} + \tilde{A}\wedge\overline{\partial}\tilde{A} - \overline{\partial}\tilde{A}\wedge\tilde{A} \\
&= 0,
\end{aligned}$$

where we have used the fact that $[\tilde{A},\tilde{A}\wedge\tilde{A}] = 0$. Thus, $F^{0,2}_{A+\overline{\partial}^*\omega} \in \ker \overline{\partial}_{A+\overline{\partial}^*\omega}$. From the extended integrability condition it follows that

$$\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega \in \ker \overline{\partial}_{A+\overline{\partial}^*\omega}.$$

Moreover, we know that $\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega \in \overline{\mathrm{Im}\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}}^{L^d}$. From the Closed Image Theorem [32, Theorem 4.1.16] we know

$$(\ker \overline{\partial}_{A+\overline{\partial}^*\omega})^{\perp} = \overline{\mathrm{Im}\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}}^{L^d}.$$

Thus,

$$\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega \in \ker \overline{\partial}_{A+\overline{\partial}^*\omega} \cap \overline{\mathrm{Im}\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}}^{L^d},$$

and it follows that $\overline{\partial}^*_{A^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega = 0$ and consequently $F^{0,2}_{A+\overline{\partial}^*\omega} = 0$. □

4.2 Existence of holomorphic trivialisations

We setup the framework. Let ∇ be a unitary $W^{1,d}$ connection of the hermitian bundle (E, h_0) over a closed Kähler manifold X^d satisfying $F_\nabla^{0,2} = 0$. We assume that the unit ball B^{2d} is a geodesic ball in X^d and that ∇ trivialises as $\nabla \simeq d + A$ in B^{2d}, where A is a connection 1-form. Moreover, in this section we will work with low $W^{1,d}$ connection norm i.e. A satisfies the smallness condition

$$\|A\|_{W^{1,d}(B^{2d})} \leq \varepsilon_0(X^d, \omega)$$

for some $\varepsilon_0(X^d, \omega) > 0$ depending on the manifold X^d and the Kähler form ω. We will use the smallness assumption throughout this section. Moreover, to fix ideas we will assume that B^{2d} is the flat closed unit ball.

We adapt the methods we have developed in Section 3.2 in order to apply them in our setting. We will rewrite all the results for completeness. We prove that under the integrability condition $F_A^{0,2} = 0$ we obtain the existence of local holomorphic trivialisations assuming low $W^{1,d}$ norm for A as before. We state the result:

Theorem 4.1. *There exists $\varepsilon_0 > 0$ such that if $A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ satisfies $\|A\|_{W^{1,d}(B^{2d})} \leq \varepsilon_0$, and the integrability condition $F_A^{0,2} = 0$, then there exists $r > 0$ and $g, g^{-1} \in W^{2,q}(B_r^{2d}, GL_n(\mathbb{C}))$ for all $q < d$ such that*

$$A^{0,1} = -\overline{\partial} g \cdot g^{-1} \qquad \text{in } B_r^{2d}. \tag{4.7}$$

Moreover, there exists a constant $C_q > 0$ such that the following estimates hold:

$$\|g - id\|_{W^{2,q}(B_r^{2d})} \leq C_q \|A\|_{W^{1,d}(B^{2d})}$$

and

$$\left\|g^{-1} - id\right\|_{W^{2,q}(B_r^{2d})} \leq C_q \|A\|_{W^{1,d}(B^{2d})}.$$

It follows that $A^g = h^{-1}\partial h$ where $h = \overline{g}^T g$.

We skip describing the strategy since it is completely analogous to the Hilbert case in Section 3.2. As before we assume that the ball of radius 2, B_2^{2d} equipped with the canonical complex structure, is holomorphically embedded into $\mathbb{CP}^d$ (simply take the embedding $(z_1, \ldots, z_d) \to [z_1, \ldots, z_d, 1]$).

We show the invertibility of the operator defined by:

$$L_{\tilde{A}}(\omega) = \overline{\partial}\overline{\partial}^* \omega + \overline{\partial}^* \overline{\partial}^* \omega + [\tilde{A}^{0,1}, \overline{\partial}^* \omega] + *[*\overline{\partial}\omega, \overline{A^{0,1}}^T]$$

which is the natural replacement of the operator $\overline{\partial\partial}^* \cdot + [\tilde{A}^{0,1}, \overline{\partial}^* \cdot]$ in order to obtain the extended integrability condition 4.5 in $\mathbb{CP}^d$. We note that over $\mathbb{CP}^d$ we have that trivially $\omega, \overline{\partial}\omega \in \mathrm{Dom}\overline{\partial}^*$ since $\mathbb{CP}^d$ has no boundary.

Proposition 4.2. *There exists $\varepsilon > 0$ such that for any*

$$\tilde{A} \in W^{1,d}(\Omega^1\mathbb{CP}^d \otimes \mathfrak{u}(n))$$

satisfying the bound $\left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \leq \varepsilon$, the operator

$$L_{\tilde{A}} : W^{2,d}(\Omega^2\mathbb{CP}^d \otimes M_n(\mathbb{C})) \to L^d(\Omega^2\mathbb{CP}^d \otimes M_n(\mathbb{C}))$$

defined by

$$L_{\tilde{A}}(\omega) = \overline{\partial\partial}^*\omega + \overline{\partial}^*\overline{\partial}^*\omega + [\tilde{A}^{0,1}, \overline{\partial}^*\omega] + *[*\overline{\partial}\omega, \overline{A^{0,1}}^T] \tag{4.8}$$

is Fredholm and invertible.

Proof of Proposition 4.2. Since $\mathbb{CP}^d$ is a Kähler manifold, it follows that operator $\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial} = \frac{1}{2}\Delta_d$ is elliptic over $\mathbb{CP}^d$ (see [12, p. 93]). Because $\mathbb{CP}^d$ is a compact manifold, then $\mathrm{Ker}\frac{1}{2}\Delta_d$ and $\mathrm{Coker}\frac{1}{2}\Delta_d$ are finite dimensional spaces. By definition it follows that $\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}$ is Fredholm.

Let $\varepsilon > 0$ be as defined in [32, Theorem 4.4.2 (ii), p.185] such that

$$\left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \leq \varepsilon$$

is small in $W^{1,d}$ norm. It follows that the operator $[\tilde{A}^{0,1}, \overline{\partial}^* \cdot] + *[*\overline{\partial}\cdot, \overline{A^{0,1}}^T]$ has small operator norm $W^{2,d}$ to L^d:

$$\begin{aligned}
\left\|\left\|[\tilde{A}^{0,1}, \overline{\partial}^* \cdot] + *[*\overline{\partial}\cdot, \overline{A^{0,1}}^T]\right\|\right\| &= \sup_{\|\omega\|_{W^{2,d}}=1} \left\|[\tilde{A}^{0,1}, \overline{\partial}^*\omega] + *[*\overline{\partial}\omega, \overline{A^{0,1}}^T]\right\|_{L^d(\mathbb{CP}^2)} \\
&\leq \left\|\tilde{A}^{0,1}\right\|_{L^{2d}(\mathbb{CP}^d)} \left\|\overline{\partial}^*\omega\right\|_{L^{2d}(\mathbb{CP}^d)} \\
&\quad + \left\|\tilde{A}^{0,1}\right\|_{L^{2d}(\mathbb{CP}^d)} \left\|\overline{\partial}\omega\right\|_{L^{2d}(\mathbb{CP}^d)} \\
&\leq C \left\|\tilde{A}^{0,1}\right\|_{W^{1,d}(\mathbb{CP}^d)} \|\omega\|_{W^{2,d}(\mathbb{CP}^d)} \\
&\leq C\varepsilon,
\end{aligned}$$

for some constant $C > 0$ coming from the Sobolev embedding $W^{1,d} \hookrightarrow L^{2d}$. Hence, from the continuity of the index maps [32, Theorem 4.4.2, p.185], we have that $L_{\tilde{A}}$ is Fredholm and it has the same index as $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ as an operator mapping $W^{2,d}(\mathbb{CP}^2, M_n(\mathbb{C}))$ to $L^d(\mathbb{CP}^d, M_n(\mathbb{C}))$.

It is well-known that there are no global nonzero holomorphic $(0,2)$-forms on $\mathbb{CP}^d$ [12, p. 118]. The lack of holomorphic $(0,2)$-forms imply that $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ defined from $W^{2,d}(\mathbb{CP}^d)$ to $L^d(\mathbb{CP}^d)$ is an invertible operator on the space of $(0,2)$-forms and consequently has index 0. Thus, it we get that $\operatorname{index}(L_{\tilde{A}}) = \operatorname{index}(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) = 0$.

It remains to show that $L_{\tilde{A}}$ has trivial kernel. Once we have shown this, we can use the zero index of $L_{\tilde{A}}$ in order to conclude that $L_{\tilde{A}}$ is invertible. Assume $\omega \in KerL_{\tilde{A}}$. Hence, ω satisfies

$$\overline{\partial}\overline{\partial}^*\omega + \overline{\partial}^*\overline{\partial}\omega = -[\tilde{A}^{0,1}, \overline{\partial}^*\omega] - *[*\overline{\partial}\omega, \overline{A^{0,1}}^T].$$

By the Fredholm Lemma [32, Lemma 4.3.9] and the Sobolev embedding $W^{1,d} \hookrightarrow L^{2d}$, we obtain

$$\begin{aligned}
\|\omega\|_{W^{2,d}} &\leq C \left\| \overline{\partial}\overline{\partial}^*\omega + \overline{\partial}^*\overline{\partial}\omega \right\|_{L^d} \\
&\leq C \left(\|L_{\tilde{A}}(\omega)\|_{L^d} + \left\| [\tilde{A}^{0,1}, \overline{\partial}^*\omega] \right\|_{L^d} + \left\| *[*\overline{\partial}\omega, \overline{A^{0,1}}^T] \right\|_{L^d} \right) \\
&\leq C \left(\|L_{\tilde{A}}(\omega)\|_{L^d} + \left\| \tilde{A}^{0,1} \right\|_{L^{2d}} \left\| \overline{\partial}^*\omega \right\|_{L^{2d}} + \left\| \tilde{A}^{0,1} \right\|_{L^{2d}} \|\overline{\partial}\omega\|_{L^{2d}} \right) \\
&\leq C \|L_{\tilde{A}}(\omega)\|_{L^d} + C'\varepsilon \|\omega\|_{W^{2,d}}
\end{aligned}$$

for some constants $C, C' > 0$. We can take the term $C'\varepsilon \|\omega\|_{W^{2,d}}$ on the left hand side of the inequality:

$$(1 - C'\varepsilon) \|\omega\|_{W^{2,d}} \leq C \|L_{\tilde{A}}(\omega)\|_{L^d}.$$

Choosing $\varepsilon > 0$ such that $1 - C'\varepsilon > \frac{1}{2}$, then we divide by the positive factor $1 - C'\varepsilon$ and obtain the bound:

$$\|\omega\|_{W^{2,d}} \leq \frac{C}{1 - C'\varepsilon} \|L_{\tilde{A}}(\omega)\|_{L^d}.$$

Because $\omega \in KerL_{\tilde{A}}$, we have that $\omega = 0$. Since ω was arbitrarily chosen from the kernel, it follows that the kernel of $L_{\tilde{A}}$ is trivial: $KerL_{\tilde{A}} = \{0\}$. This finishes the proof. □

Using a similar technique to Proposition 3.2 we will prove the existence of a $\mathbb{CP}^d$ extension of our connection form A, satisfying the integrability condition (1.1). However, we need to take care in order to obtain this condition. We will use Proposition 4.1 in order to achieve the integrability condition. We assume the holomorphic embedding of B^{2d} in $\mathbb{CP}^d$.

Proposition 4.3. *There exists $\varepsilon > 0$ such that for any*

$$A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$$

satisfying $F_A^{0,2} = 0$ and $\|A\|_{W^{1,d}(B^{2d})} < \varepsilon$, there exists $\tilde{A} \in W^{1,d}(\Omega^1 \mathbb{CP}^d \otimes \mathfrak{u}(n))$ that satisfies the integrability condition

$$F_{\tilde{A}}^{0,2} = 0$$

in $\mathbb{CP}^d$ and $\omega \in W^{2,d}(\Omega^{0,2}\mathbb{CP}^d \otimes M_n(\mathbb{C}))$ such that $\tilde{A}^{0,1} = A^{0,1} + \vartheta\omega$ in B^{2d}. Moreover, ω satisfies the estimate $\|\omega\|_{W^{2,d}(\mathbb{CP}^d)} \leq C\|A\|_{W^{1,d}(B^{2d})}$ for some constant $C > 0$ and

$$\overline{\partial}^*_{\tilde{A}}\overline{\partial}\omega = 0 \quad \text{in } \mathbb{CP}^d.$$

We recall to the reader that according to equation (A.1), we have $\vartheta = -*\partial*$, the formal adjoint of $\overline{\partial}$. In addition, $\overline{\partial}^*_{\tilde{A}}$ is the operator defined as in Proposition C.3:

$$\overline{\partial}^*_{\tilde{A}}\cdot = \overline{\partial}^* \cdot + * [*\overline{\partial}\cdot, \overline{A^{0,1}}^T].$$

Proof of Proposition 4.3. Step 1. Since A is unitary, we can decompose A into its $(0,1)$ and $(1,0)$ parts as such: $A = A^{0,1} - \overline{A^{0,1}}^T$ where

$$A^{0,1} = \sum_{i=1}^{d} \alpha_i d\overline{z_i}$$

and $\alpha_i \in W^{1,d}(B^{2d}, M_n(\mathbb{C}))$ for $i = 1, d$. We extend each α_i to a compactly supported map $\hat{\alpha}_i$ in B_2^{2d}, so that $\hat{\alpha}_i = 0$ in $B_2^{2d} \setminus B_{3/2}^{2d}$. For constructing $\hat{\alpha}_i$, for each $i \leq d$ we solve:

$$\begin{cases} \Delta\phi_i = 0 & \text{in } B_{3/2}^{2d} \setminus B_1^{2d} \\ \phi_i = \alpha_i & \text{on } \partial B_1^{2d} \\ \phi_i = 0 & \text{on } \partial B_{3/2}^{2d} \end{cases}$$

Such solutions exist by [20, Remark 7.2, Chapter 2] and satisfy

$$\|\phi_i\|_{W^{1,d}(B_{3/2}^{2d} \setminus B_1^{2d})} \leq C\|\alpha_i\|_{W^{1-1/d,d}(\partial B_1^{2d})} \leq C'\|\alpha_i\|_{W^{1,d}(B_1^{2d})}$$

for some constants $C, C' > 0$. We can now define the extensions to B_2^{2d}:

$$\hat{\alpha}_i = \begin{cases} \alpha_i & \text{in } B_1^{2d} \\ \phi_i & \text{in } B_{3/2}^{2d} \setminus B_1^{2d} \\ 0 & \text{in } B_2^{2d} \setminus B_{3/2}^{2d}. \end{cases}$$

By construction of ϕ_i, the functions $\hat{\alpha}_i$ are well-defined $W^{1,d}(B_2^{2d})$ Sobolev maps that satisfy the estimate:

$$\|\hat{\alpha}_i\|_{W^{1,d}(B_2^{2d})} \leq C \|\alpha_i\|_{W^{1,d}(B_1^{2d})} .$$

Define the $(0,1)$-form $\hat{A}^{0,1} = \sum_{i=1}^d \hat{\alpha}_i d\overline{z_i}$ and

$$\hat{A} := \hat{A}^{0,1} - \overline{\hat{A}^{0,1}}^T \in W^{1,d}(\Omega^1 B_2^{2d} \otimes \mathfrak{u}(n)).$$

By covering $\mathbb{CP}^d \setminus B_2^{2d}$ with coordinate charts, we can trivially extend $\hat{A}$ by 0 on $\mathbb{CP}^d \setminus B_2^{2d}$. Thus, we have obtained $\hat{A} \in W^{1,d}(\Omega^1 \mathbb{CP}^d \otimes \mathfrak{u}(n))$ and there exists a constant $\hat{C} > 0$ such that $\left\|\hat{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \leq \hat{C} \|A\|_{W^{1,d}(B^{2d})}$.

Step 2. It remains to perturb the form $\hat{A}$ so that we obtain the integrability condition. We first find a solution ω to the extended integrability condition. Note that over $\mathbb{CP}^d$, $\overline{\partial}\omega$ automatically belongs to the space $\text{Dom}(\overline{\partial}^*)$.

$$F^{0,2}_{\hat{A}^{0,1}+\overline{\partial}^*\omega} + \overline{\partial}^*_{\hat{A}^{0,1}+\overline{\partial}^*\omega}\overline{\partial}\omega = 0.$$

This amounts to solving the following PDE globally on the complex projective space $\mathbb{CP}^d$:

$$\begin{aligned} &\overline{\partial}\overline{\partial}^*\omega + \overline{\partial}^*\overline{\partial}\omega + [\hat{A}^{0,1}, \overline{\partial}^*\omega] + * [*\overline{\partial}\omega, \overline{\hat{A}^{0,1}}^T] \\ &\qquad = -\overline{\partial}^*\omega \wedge \overline{\partial}^*\omega - *[*\overline{\partial}\omega, \overline{\overline{\partial}^*\omega}^T] - F^{0,2}_{\hat{A}} \end{aligned} \tag{4.9}$$

where ω is a $(0,2)$ form on $\mathbb{CP}^d$. Using the invertibility of the operator $L_{\hat{A}}$ proven in Proposition 4.2, we can solve equation (4.9) using a fixed point method. We consider the sequence given by:

$$\begin{aligned} L_{\hat{A}}(\omega_1) &= -F^{0,2}_{\hat{A}} \\ L_{\hat{A}}(\omega_2) &= -\overline{\partial}^*\omega_1 \wedge \overline{\partial}^*\omega_1 - *[*\overline{\partial}\omega_1, \overline{\overline{\partial}^*\omega_1}^T] - F^{0,2}_{\hat{A}} \\ &\dots \end{aligned}$$

$$L_{\hat{A}}(\omega_k) = -\overline{\partial}^* \omega_{k-1} \wedge \overline{\partial}^* \omega_{k-1} - *[*\overline{\partial}\omega_{k-1}, \overline{\overline{\partial}^* \omega_{k-1}}^T] - F_{\hat{A}}^{0,2} \tag{4.10}$$
$$\dots$$

By showing that the sequence ω_k converges strongly in $W^{2,d}$, we will obtain a $W^{2,d}$ solution to the required equation (4.9). Since $L_{\hat{A}}$ is invertible as an operator from $W^{2,d}$ to L^d, it is clear that existence holds for each ω_k, $k \geq 1$. We need to show that the sequence $\{\omega_k\}_{k=1}^\infty$ is Cauchy in $W^{2,d}$.

Let $\varepsilon_0 := C \left\| F_{\hat{A}}^{0,2} \right\|_{L^d(\mathbb{CP}^d)}$, where $C > 0$ is the constant appearing in Fredholm inequality:
$$\|\phi\|_{W^{2,d}(\mathbb{CP}^d)} \leq C \left\| L_{\hat{A}}(\phi) \right\|_{L^d(\mathbb{CP}^d)} .$$

Claim. $\{\omega_k\}_{k=1}^\infty$ is a Cauchy sequence in $W^{2,d}(\mathbb{CP}^d)$.

We first show by induction the uniform bound $\|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)} \leq 2\varepsilon_0$. By the Fredholm Lemma [32, Lemma 4.3.9] we have that

$$\|\omega_1\|_{W^{2,d}(\mathbb{CP}^d)} \leq C \left\| L_{\hat{A}}(\omega_1) \right\|_{L^d(\mathbb{CP}^d)} = C \left\| F_{\hat{A}}^{0,2} \right\|_{L^d(\mathbb{CP}^d)} = \varepsilon_0 < 2\varepsilon_0$$

Let $k \geq 1$. By the Sobolev embedding $W^{1,d}(\mathbb{CP}^d) \hookrightarrow L^{2d}(\mathbb{CP}^d)$ there exists a constant $C_1 > 0$ so that

$$\left\| \overline{\partial}^* \omega_k \right\|_{L^{2d}(\mathbb{CP}^d)} \leq C_1 \left\| \overline{\partial}^* \omega_k \right\|_{W^{1,d}(\mathbb{CP}^d)} \leq C_1^2 \|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)} .$$

Similarly for $\overline{\partial}\omega_k$: $\left\| \overline{\partial}\omega_k \right\|_{L^{2d}(\mathbb{CP}^d)} \leq C_1^2 \|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)}$. Thus, from (4.10) the inequalities follow:

$$\begin{aligned}
\|\omega_{k+1}\|_{W^{2,d}(\mathbb{CP}^d)} &\leq C \left\| L_{\hat{A}}(\omega_{k+1}) \right\|_{L^d(\mathbb{CP}^d)} \\
&\leq C \left(\left\| \overline{\partial}^* \omega_k \wedge \overline{\partial}^* \omega_k \right\|_{L^d(\mathbb{CP}^d)} + \left\| *[*\overline{\partial}\omega_{k-1}, \overline{\overline{\partial}^* \omega_{k-1}}^T] \right\|_{L^d(\mathbb{CP}^d)} + \right. \\
&\left. + \left\| F_{\hat{A}}^{0,2} \right\|_{L^d(\mathbb{CP}^d)} \right) \\
&\leq C \|\omega_k\|_{L^{2d}(\mathbb{CP}^d)}^2 + \varepsilon_0 \\
&\leq C \cdot C_1^4 \|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)}^2 + \varepsilon_0
\end{aligned}$$

By the induction hypothesis, we have $\|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)} < 2\varepsilon_0$. Thus,

$$\|\omega_{k+1}\|_{W^{2,d}(\mathbb{CP}^d)} \leq 4(C \cdot C_1^4)\varepsilon_0^2 + \varepsilon_0.$$

Having chosen $\varepsilon > 0$ such that $4(C \cdot C_1^4)\varepsilon < 1$ and

$$\left\|\hat{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \leq \hat{C} \left\|A\right\|_{W^{1,d}(B^{2d})} \leq \varepsilon,$$

it follows that $4(C \cdot C_1^4)\varepsilon_0 < 1$ and we conclude

$$\|\omega_{k+1}\|_{W^{2,d}(\mathbb{CP}^d)} \leq 2\varepsilon_0.$$

By induction, we have proven that we have a uniform bound for the sequence of $(0,2)$-forms $\{\omega_k\}$:

$$\|\omega_k\|_{W^{2,d}(\mathbb{CP}^d)} \leq 2\varepsilon_0.$$

for all $k \geq 1$. It remains to show that $\{\omega_k\}$ is a Cauchy sequence. Let $k \geq 2$. We derive the following bounds from the recurrence relation (4.10) satisfied by the sequence:

$$\begin{aligned}\|\omega_{k+1} - \omega_k\|_{W^{2,d}(\mathbb{CP}^d)} &\leq C \left\|L_{\hat{A}}(\omega_{k+1} - \omega_k)\right\|_{L^d(\mathbb{CP}^d)} \\ &\leq C \cdot \varepsilon_0 \left\|\omega_k - \omega_{k-1}\right\|_{W^{2,d}(\mathbb{CP}^d)}\end{aligned}$$

for some $C > 0$ and $C\varepsilon_0 < 1$. Because $C\varepsilon_0 < 1$, it follows immediately that the sequence is Cauchy and the claim is proven.

Because $\{\omega_k\}_{k=1}^{\infty}$ is a Cauchy sequence in the Banach space $W^{2,d}(\Omega^{0,2}\mathbb{CP}^d \otimes M_n(\mathbb{C}))$, it has a limit ω and converges strongly in $W^{2,d}$ to it. Moreover, from the $W^{2,d}$ convergence, we have that the uniform bound is satisfied by the limiting form ω, indeed $\|\omega\|_{W^{2,d}} < 2\varepsilon_0 = 2C\left\|F_{\hat{A}}\right\|_{L^d}$. By construction of $\hat{A}$, it is clear that there exists a constant $C' > 0$ so that

$$\|F_{\hat{A}}\|_{L^d(\mathbb{CP}^d)} \leq C' \left\|A\right\|_{W^{1,d}(B^{2d})}.$$

This leads to the required estimate on ω, $\|\omega\|_{W^{2,d}(\mathbb{CP}^d)} \leq C\left\|A\right\|_{W^{1,d}(B^{2d})}$, where $C > 0$ is some constant.

By construction, ω satisfies the extended integrability condition (4.5) and from Proposition 4.1, it follows that

$$F^{0,2}_{\hat{A}+\overline{\partial}^*\omega} = 0 \quad \text{and} \quad \overline{\partial}^*_{\hat{A}+\overline{\partial}^*\omega}\overline{\partial}\omega = 0 \quad \text{in } \mathbb{CP}^d.$$

We conclude by defining $\tilde{A} = \hat{A} + \overline{\partial}^*\omega - \overline{\overline{\partial}^*\omega}^T$ to be a skew-Hermitian 1-form, satisfying $F^{0,2}_{\tilde{A}} = 0$ in $\mathbb{CP}^d$ and $\tilde{A}^{0,1} = \hat{A}^{0,1} + \vartheta\omega = A^{0,1} + \vartheta\omega$ in B^{2d}.

This finishes the proof of Proposition 4.3. $\square$

We are ready to prove the existence of a holomorphic trivialisation of $\tilde{A}$ over $\mathbb{CP}^d$.

Lemma 4.1. *There exists* $\varepsilon > 0$ *such that for any form* $\tilde{A} \in W^{1,d}(\Omega^1\mathbb{CP}^d \otimes \mathfrak{u}(n))$ *satisfying the integrability condition (1.1) and* $\left\|\tilde{A}\right\|_{W^{1,d}} < \varepsilon$*, then there exist gauges* $\tilde{g}, \tilde{g}^{-1} \in W^{2,q}(\mathbb{CP}^d, GL_n(\mathbb{C}))$ *for all* $q < d$ *such that*

$$\tilde{A}^{0,1} = -\overline{\partial}\tilde{g} \cdot \tilde{g}^{-1}.$$

Furthermore for each $q < d$ *there exists a constant* $C_q > 0$ *such that*

$$\|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^d)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \tag{4.11}$$

and

$$\left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^d)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}. \tag{4.12}$$

Proof of Lemma 4.1. We divide the proof into three steps. Using a fixed point argument the first two steps show the existence of a map $\tilde{g}$ satisfying $\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g}$. Step 3 shows the existence of $\tilde{g}^{-1}$ and proves the estimates (4.11) and (4.12).

Step 1. We consider the linear operator:

$$T : W^{1,d}(\mathbb{CP}^d, M_n(\mathbb{C})) \to W^{1,d}(\mathbb{CP}^d, M_n(\mathbb{C}))$$

given by

$$T(\tilde{g}) = -\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) + id, \tag{4.13}$$

where id is the constant identity matrix and N is the inverse operator of $\Delta_{\overline{\partial}}$ as defined in (A.3). Consequently up to a constant N is the inverse operator of Δ_d on $\mathbb{CP}^d$, since $\mathbb{CP}^d$ is Kähler. We verify that the operator T is well-defined. On $\mathbb{CP}^d$ we can use the fact that $\overline{\partial}$ is an elliptic operator and hence we get the following estimate:

$$\left\|\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{W^{1,d}(\mathbb{CP}^d)} \leq C \left\|\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g})\right\|_{L^d(\mathbb{CP}^d)} \tag{4.14}$$

for some constant C. Since $\mathbb{CP}^d$ is Kähler, we have $\frac{1}{2}\Delta_d = \Delta_{\overline{\partial}}$ on $\mathbb{CP}^d$ and the $\overline{\partial}$-Hodge decomposition (A.5) follows from the d-Hodge decomposition for 1-forms. The lack of global holomorphic $(0,1)$-forms on $\mathbb{CP}^d$ (see [12, p. 118]) gives us the following decomposition:

$$\tilde{A}^{0,1}\tilde{g} = \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) + \overline{\partial}\overline{\partial}^*N(\tilde{A}^{0,1}\tilde{g}) = \frac{1}{2}\Delta_d N(\tilde{A}^{0,1}\tilde{g}) \tag{4.15}$$

From the elliptic regularity of Δ_d [37, Theorem 10.5.1] we have the estimate:

$$\left\| N(\tilde{A}^{0,1}g) \right\|_{W^{2,d}} \leq \left\| A^{0,1}g \right\|_{L^d},$$

from which it follows that

$$\left\| \overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) \right\|_{L^d} \leq \left\| \tilde{A}^{0,1}\tilde{g} \right\|_{L^d}. \tag{4.16}$$

Putting the inequalities (4.14) and (4.16) together, we obtain:

$$\left\| \overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) \right\|_{W^{1,d}(\mathbb{CP}^d)} \leq C \left\| \tilde{A}^{0,1}\tilde{g} \right\|_{L^d(\mathbb{CP}^d)} \leq C \left\| \tilde{A}^{0,1} \right\|_{L^{2d}(\mathbb{CP}^d)} \|\tilde{g}\|_{L^{2d}(\mathbb{CP}^d)}. \tag{4.17}$$

Furthermore, using the Sobolev embedding in $2d$-dimensions $W^{1,d} \hookrightarrow L^{2d}$, there exists a constant C' so that

$$\left\| \overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}) \right\|_{W^{1,d}} \leq C' \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}} \|\tilde{g}\|_{W^{1,d}}. \tag{4.18}$$

Thus, the operator T is well-defined, mapping $W^{1,d}$ to $W^{1,d}$.

We can now show that T has a unique fixed point. Consider $\tilde{g}_1, \tilde{g}_2 \in W^{1,d}(\mathbb{CP}^d, M_n(\mathbb{C}))$. Then

$$\|T(\tilde{g}_1) - T(\tilde{g}_2)\|_{W^{1,d}(\mathbb{CP}^d)} = \left\| \overline{\partial}^* N(\tilde{A}^{0,1}(\tilde{g}_1 - \tilde{g}_2)) \right\|_{W^{1,d}(\mathbb{CP}^d)}.$$

From (4.17), we obtain

$$\left\| \overline{\partial}^* N(\tilde{A}^{0,1}(\tilde{g}_1 - \tilde{g}_2)) \right\|_{W^{1,d}(\mathbb{CP}^d)} \leq C' \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} \|\tilde{g}_1 - \tilde{g}_2\|_{W^{1,d}(\mathbb{CP}^d)}$$

and we can choose $\varepsilon > 0$ such that the bound $\left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} < \varepsilon$ is small such that the factor $C' \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)}$ is strictly smaller than 1. Hence, T is a contraction operator and there exists a unique fixed point $\tilde{g} \in W^{1,d}(\mathbb{CP}^d, M_n(\mathbb{C}))$, $T(\tilde{g}) = \tilde{g}$. Thus, we have

$$\overline{\partial}\tilde{g} = -\overline{\partial}\overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}). \tag{4.19}$$

Step 2. We can now show that the equation (4.19) above coupled with the integrability condition $F^{0,2}_{\tilde{A}^{0,1}} = 0$ satisfied by $\tilde{A}^{0,1}$, imply that $\tilde{g}$ solves the required PDE: $\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g}$. The $\overline{\partial}$-Hodge decomposition (4.15) gives

$$\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g} + \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}). \tag{4.20}$$

Since the operators N and $\overline{\partial}$ compute, $N\overline{\partial} = N\overline{\partial}$ (see [8, Theorem 4.4.1 (3)]), we can further compute the term $\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})$:

$$\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) = \overline{\partial}^* N\overline{\partial}(\tilde{A}^{0,1}\tilde{g}) = \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}\tilde{g}).$$

Using (4.20), then we have

$$\begin{aligned}\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) &= \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}\tilde{g}) \\ &= \overline{\partial}^* N(\overline{\partial}\tilde{A}^{0,1}\tilde{g} + \tilde{A}^{0,1} \wedge \tilde{A}^{0,1}\tilde{g} - \tilde{A}^{0,1} \wedge \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})).\end{aligned}$$

Since $\tilde{A}$ satisfies the integrability condition $F^{0,2}_{\tilde{A}} = 0$, we have the recurrence relation:

$$\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g}) = -\overline{\partial}^* N(\tilde{A}^{0,1} \wedge \overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})) := \mathcal{L}\left(\overline{\partial}^*\overline{\partial}N(\tilde{A}^{0,1}\tilde{g})\right) \tag{4.21}$$

where we defined

$$\mathcal{L} : L^d(\Omega^1\mathbb{CP}^d \otimes M_n(\mathbb{C})) \to L^d(\Omega^1\mathbb{CP}^d \otimes M_n(\mathbb{C}))$$
$$V \mapsto -\overline{\partial}^* N(\tilde{A}^{0,1} \wedge V).$$

We need to establish whether $\mathcal{L}$ is a well-defined operator and find its fixed points in order to analyse equation (4.21).

Since N is the inverse of an elliptic operator, we have:

$$\begin{aligned}\left\|N(\tilde{A}^{0,1} \wedge V)\right\|_{W^{2,2d/(d+1)}(\mathbb{CP}^d)} &\leq C\left\|\tilde{A}^{0,1} \wedge V\right\|_{L^{2d/(d+1)}(\mathbb{CP}^d)} \\ &\leq C\left\|\tilde{A}^{0,1}\right\|_{L^{2d}(\mathbb{CP}^d)} \|V\|_{L^d(\mathbb{CP}^d)}\end{aligned}$$

for some constant $C > 0$. Moreover, by the Sobolev embedding $W^{1,2d/(d+1)} \hookrightarrow L^d$ it follows that

$$\|\mathcal{L}(V)\|_{L^d(\mathbb{CP}^d)} \leq C\,\|\mathcal{L}(V)\|_{W^{1,2d/(d+1)}(\mathbb{CP}^d)}\,.$$

and consequently,

$$\|\mathcal{L}(V)\|_{L^d(\mathbb{CP}^d)} \leq C \left\| \tilde{A}^{0,1} \wedge V \right\|_{L^{2d/(d+1)}(\mathbb{CP}^d)} \leq C \left\| \tilde{A}^{0,1} \right\|_{L^{2d}(\mathbb{CP}^d)} \|V\|_{L^d(\mathbb{CP}^d)} .$$

Similarly as before, this means that $\mathcal{L}$ is a well-defined contraction operator and has a unique fixed point. In particular, 0 is its fixed point. We know from equation (4.21) that $\overline{\partial}^* \overline{\partial} N(\tilde{A}^{0,1}\tilde{g})$ is also a fixed point for $\mathcal{L}$ and thus, we have that the term $\overline{\partial}^* \overline{\partial} N(\tilde{A}^{0,1}\tilde{g})$ vanishes. In particular, from (4.20) the equation is solved:

$$\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g}.$$

Step 3. We prove that $\tilde{g} \in W^{2,q}(\mathbb{CP}^2, GL_n(\mathbb{C}))$ for all $q < d$ and satisfies the required estimate (4.11). Afterwards, we show that $\tilde{g}^{-1}$ satisfies (4.12). Let $q < d$. We know that $\tilde{g}$ is a $W^{1,d}$ map and satisfies:

$$\tilde{g} - id = \overline{\partial}^* N(\tilde{A}^{0,1}\tilde{g}).$$

Since $\tilde{g}$ is a fixed point of T (4.13), then it satisfies the estimate (4.18), which means:

$$\begin{aligned}\|\tilde{g} - id\|_{W^{1,d}(\mathbb{CP}^d)} &\leq C \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} \|\tilde{g}\|_{W^{1,d}} \\ &\leq C \left(\left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} \|\tilde{g} - id\|_{W^{1,d}} + \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} \right).\end{aligned}$$

Because $\left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} < \varepsilon$, where ε is small, then there exists a constant $C > 0$ such that

$$\|\tilde{g} - id\|_{W^{1,d}(\mathbb{CP}^d)} \leq C \left\| \tilde{A}^{0,1} \right\|_{W^{1,d}(\mathbb{CP}^d)} .$$

Since $\tilde{g}$ satisfies this estimate, we can bootstrap using Lemma D.1 and Remark D.1(i), from which the required estimate (4.11) follows:

$$\|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^d)} \leq C_q \left\| \tilde{A} \right\|_{W^{1,d}(\mathbb{CP}^d)} , \tag{4.22}$$

for some constant $C_q > 0$.

We need to show that $\tilde{g}$ is in $GL_n(\mathbb{C})$ over $\mathbb{CP}^d$ and that its inverse satisfies a similar estimate as (4.22). Arguing in a similar way to Step 1, 2 and the way we obtained the regularity estimates for $\tilde{g}$ in (4.22), we can show that there exists $\tilde{u} \in W^{2,q}(\mathbb{CP}^d, GL_n(\mathbb{C}))$ for any $q < d$ such that

$$\overline{\partial}\tilde{u} = \tilde{u}\tilde{A}^{0,1}$$

and

$$\|\tilde{u} - id\|_{W^{2,q}(\mathbb{CP}^d)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \tag{4.23}$$

for some constant $C_q > 0$. In particular, we have that $\overline{\partial}(\tilde{u}\tilde{g}) = 0$. Hence, $\tilde{u}\tilde{g} =: \tilde{h}$ is holomorphic. However, since the only holomorphic functions on $\mathbb{CP}^d$ are the constant ones [12, p. 118], then $\tilde{h}$ is a constant.

We can pick $(2d-1)/2 < q_0 < d$ so that we obtain the Sobolev embedding $W^{2,q_0} \hookrightarrow L^\infty$ on any $2d-1$ dimensional hypersurface. Moreover, by the Sobolev products results in [31, Section 4.8.2, Theorem 1], there exists $q_1 \in (q_0, 2)$ such that from $\tilde{u}, \tilde{g} \in W^{2,q_1}(\mathbb{CP}^d, M_n(\mathbb{C}))$ we have $\tilde{u}\tilde{g} \in W^{2,q_0}$ and $\|\tilde{u}\tilde{g} - id\|_{W^{2,q_0}(\mathbb{CP}^d)} \leq C_{q_0} \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}$ for some constant $C_{q_0} > 0$. By Fubini, there exists a radius $r > 0$ and $z_0 \in \mathbb{CP}^d$ such that

$$\|\tilde{u}\tilde{g} - id\|_{W^{2,q_0}(\partial B_r^{2d}(z_0))} < 2C'_{q_0} \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}$$

where $B_r^{2d}(z_0)$ is a ball in $\mathbb{CP}^d$. Thus, by the embedding of W^{2,q_0} into L^∞ in $2d-1$ dimensions, there exists a constant $C''_{q_0} > 0$ so that

$$\|\tilde{u}\tilde{g} - id\|_{L^\infty(\partial B_r^{2d}(z_0))} \leq C''_{q_0} \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}. \tag{4.24}$$

We can choose a possibly smaller $\varepsilon > 0$ than we have done for the estimate $\left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} < \varepsilon$ such that we obtain that $\tilde{h} = \tilde{u}\tilde{g} \in GL_n(\mathbb{C})$ over $\partial B_r^{2d}(z_0)$. Because $\tilde{h}$ is a constant, then $\tilde{h} \in GL_n(\mathbb{C})$ over $\mathbb{CP}^d$ and satisfies the estimate:

$$\left\|\tilde{h} - id\right\|_{L^\infty(\mathbb{CP}^d)} \leq C \left\|\tilde{A}\right\|_{W^{1,d}(B^{2d})},$$

for some constant $C > 0$.

Hence, we can define $\tilde{g}^{-1} := \tilde{h}^{-1}\tilde{u}$. Since $\tilde{g}^{-1}\tilde{g} = id$ by construction, we obtain that $\tilde{g}$ takes values in $GL_n(\mathbb{C})$. Moreover, from the fact that $\tilde{h}^{-1}$ is a constant and from the estimate (4.23) it follows that $\tilde{g}^{-1} \in W^{2,q}(\mathbb{CP}^d, GL_n(\mathbb{C}))$ for all $q < d$, and by the estimates on $\tilde{u}$, we obtain that for each $q < d$ there exists a constant $C_q > 0$ such that

$$\left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^d)} \leq C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}.$$

This concludes the proof of Lemma 4.1. □

Before proving the existence of a local holomorphic trivialisation for our initial $W^{1,d}$ form, we show a stronger version of existence similar to Lemma 3.3. We consider forms of small norm in $W^{1,p}$, $p > 2d-1$. This will be a useful result for our final theorem.

Lemma 4.2. *Let $p > 2d-1$. There exists $\varepsilon > 0$ such that for any $\omega \in W^{1,p}(\Omega^{0,1}B^{2d} \otimes M_n(\mathbb{C}))$ satisfying $F_\omega^{0,2} = 0$ and $\|\omega\|_{W^{1,p}(B^{2d})} \leq \varepsilon$, there exists $r \in (1/2, 1)$ and gauges $u, u^{-1} \in W^{2,p}(B_r^{2d}, GL_n(\mathbb{C}))$ so that*

$$\omega = -\overline{\partial} u \cdot u^{-1} \qquad \text{in } B_r^{2d},$$

with estimates

$$\begin{gathered} \|u - id\|_{W^{2,p}(B_r^{2d})} \leq C \|\omega\|_{W^{1,p}(B^{2d})} \\ \text{and} \\ \left\|u^{-1} - id\right\|_{W^{2,p}(B_r^{2d})} \leq C \|\omega\|_{W^{1,p}(B^{2d})}. \end{gathered} \tag{4.25}$$

Remark 4.1. *We note that in proving this result, we will use a more direct approach than in Lemma 3.3. The techniques can be easily interchanged. We do not have to worry about working in a Hilbert setting, since this result is not formulated in it in the case B^4 either.*

Proof of Lemma 4.2. Step 1. Let $q = 2dp/(2d-p)$. We show the existence of a gauge $u \in GL_n(\mathbb{C})$ that "almost" solves our equation modulo a perturbation term. Indeed, in Step 2 we can show that the perturbation term vanishes and consequentially u is the solution. Again, let N be the inverse operator of $\Delta_{\overline{\partial}}$.

We define the operator

$$\mathcal{H} : L^\infty(B^{2d}, M_n(\mathbb{C})) \to L^\infty(B^{2d}, M_n(\mathbb{C}))$$

given by

$$\mathcal{H}(u) = id + \overline{\partial}^* N(-\omega \cdot u), \tag{4.26}$$

where id is the constant identity matrix and N is the inverse of the operator $\Delta_{\overline{\partial}}$ as defined in (A.3).

Claim. $\mathcal{H}$ is well-defined.

Using the fact that $\overline{\partial}^* N$ maps L^q to $W^{1/2,q}$ in B^{2d} [5, Theorem 4], then

$$\|\mathcal{H}(u) - id\|_{W^{1/2,q}(B^{2d})} \leq C \|-\omega \cdot u\|_{L^q(B^{2d})} \leq C \|\omega\|_{L^q(B^{2d})} \|u\|_{L^\infty(B^{2d})},$$

where $C > 0$ is a constant. By the Sobolev embedding $W^{1,p} \hookrightarrow L^q$ on B^{2d}, then

$$\|\mathcal{H}(u) - id\|_{W^{1/2,q}(B^{2d})} \leq C \|\omega\|_{W^{1,p}(B^{2d})} \|u\|_{L^\infty(B^{2d})},$$

for a constant $C > 0$. Moreover, since $1/q - 1/(4d) < 0$, then $W^{1/2,q}$ embeds into L^∞ over B^{2d} and we get

$$\|\mathcal{H}(u) - id\|_{L^\infty(B^{2d})} \leq C \|\omega\|_{W^{1,p}(B^{2d})} \|u\|_{L^\infty(B^{2d})} \leq C\varepsilon \|u\|_{L^\infty(B^{2d})}.$$

Thus, by picking $\varepsilon > 0$ such that $C\varepsilon < 1$, then $\mathcal{H}$ is well-defined and has a fixed point $u \in L^\infty(B^{2d}, M_n(\mathbb{C}))$. This fixed point "almost" solves the required equation. We will show in the next step that the error we obtain vanishes in light of the integrability condition $F_\omega^{0,2} = 0$.

Step 2. Having obtained the fixed point u for the operator $\mathcal{H}$, we show that u satisfies $\overline{\partial} u = -\omega \cdot u$. Since we have proven that $u - id = \overline{\partial}^* N(-\omega \cdot u)$, we get by [5, Theorem 1] that

$$\overline{\partial} u = \overline{\partial}\overline{\partial}^* N(-\omega \cdot u) \in L^q. \tag{4.27}$$

We can apply the $\overline{\partial}$-Hodge decomposition (A.5) to get:

$$-\omega \cdot u = \overline{\partial}\overline{\partial}^* N(-\omega \cdot u) + \overline{\partial}^*\overline{\partial} N(-\omega \cdot u) \tag{4.28}$$

and by using $F_\omega^{0,2} = 0$, we expand the last term to get

$$\overline{\partial}^*\overline{\partial} N(-\omega \cdot u) = \overline{\partial}^* N(\omega \wedge \overline{\partial}^*\overline{\partial} N(-\omega \cdot u)),$$

which is similar to (4.21). We want to show that this recurrence equation implies that $\overline{\partial}^*\overline{\partial} N(-\omega \cdot u) = 0$.

$$\begin{aligned} \left\|\overline{\partial}^*\overline{\partial} N(-\omega \cdot u))\right\|_{L^\infty(B^{2d})} &\leq C \left\|\overline{\partial}^*\overline{\partial} N(-\omega \cdot u)\right\|_{W^{1/2,q}(B^{2d})} \\ &\leq C \left\|\omega \wedge \overline{\partial}^*\overline{\partial} N(-\omega \cdot u)\right\|_{L^q(B^{2d})} \end{aligned}$$

$$\leq C \|\omega\|_{L^q(B^{2d})} \left\| \overline{\partial}^* \overline{\partial} N(-\omega \cdot u) \right\|_{L^\infty(B^{2d})}$$
$$\leq CC_1\varepsilon \left\| \overline{\partial}^* \overline{\partial} N(-\omega \cdot u) \right\|_{L^\infty(B^{2d})},$$

where $C > 0$ is the Sobolev constant given by the Sobolev embedding $W^{1/2,q} \hookrightarrow L^\infty$. For $1 - C \cdot C_1\varepsilon > 0$ there is a contradiction unless

$$\overline{\partial}^* \overline{\partial} N(-\omega \cdot u) = 0.$$

From (4.27) and (4.28), we conclude that the $\overline{\partial}$-equation is solved:

$$\overline{\partial} u = -\omega \cdot u \qquad \text{in } B^{2d}.$$

Step 3. Since the procedure is identical to Step 3 in the proof of Lemma 3.3, we refer the reader to it in order to show that $u \in GL_n(\mathbb{C})$ and satisfies the required estimates (4.25).

This finishes the proof of Lemma 4.2. □

Having the results above at our disposal, we are ready to proceed with showing the existence of local holomorphic trivialisations of $\nabla \approx d + A$ in B_r^{2d} for some $r > 0$:

Theorem 4.1. *There exists $\varepsilon_0 > 0$ such that if $A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ satisfies $\|A\|_{W^{1,d}(B^{2d})} \leq \varepsilon_0$, and the integrability condition $F_A^{0,2} = 0$, then there exists $r > 0$ and gauges $g, g^{-1} \in W^{2,q}(B_r^{2d}, GL_n(\mathbb{C}))$ for all $q < d$ such that*

$$A^{0,1} = -\overline{\partial} g \cdot g^{-1} \qquad \text{in } B_r^{2d}. \tag{4.29}$$

Moreover, there exists a constant $C_q > 0$ such that the following estimates hold:

$$\|g - id\|_{W^{2,q}(B_r^{2d})} \leq C_q \|A\|_{W^{1,d}(B^{2d})}$$
$$\text{and} \tag{4.30}$$
$$\left\|g^{-1} - id\right\|_{W^{2,q}(B_r^{2d})} \leq C_q \|A\|_{W^{1,d}(B^{2d})}.$$

It follows that $A^g = h^{-1}\partial h$ where $h = \overline{g}^T g$.

Proof of Theorem 4.1. From Proposition 4.3, there exists a 1-form

$$\tilde{A} \in W^{1,d}(\Omega^1 \mathbb{CP}^d \otimes \mathfrak{u}(n))$$

satisfying the integrability condition $F^{0,2}_{\tilde{A}^{0,1}} = 0$ so that $\tilde{A}^{0,1} = A^{0,1} + \vartheta\omega$ in B^{2d}, where

$$\omega \in W^{2,d}(\Omega^{0,2}\mathbb{CP}^d \otimes M_n(\mathbb{C}))$$

with estimate $\|\omega\|_{W^{2,d}(\mathbb{CP}^d)} \le \|A\|_{W^{1,d}(B^{2d})}$. This implies that

$$\left\|\tilde{A}^{0,1}\right\|_{W^{1,d}(\mathbb{CP}^d)} \le C\,\|A\|_{W^{1,d}(B^{2d})} \tag{4.31}$$

for some constant $C > 0$.

Lemma 4.1 applied to the form $\tilde{A}$ gives the existence of a gauge

$$\tilde{g} \in W^{2,q}(\mathbb{CP}^d, GL_n(\mathbb{C}))$$

for all $q < d$ so that

$$\overline{\partial}\tilde{g} = -\tilde{A}^{0,1}\tilde{g} \qquad \text{in } \mathbb{CP}^d$$

and for each $q < d$ there exists $C_q > 0$ such that

$$\begin{gathered} \|\tilde{g} - id\|_{W^{2,q}(\mathbb{CP}^d)} \le C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)} \\ \text{and} \\ \left\|\tilde{g}^{-1} - id\right\|_{W^{2,q}(\mathbb{CP}^d)} \le C_q \left\|\tilde{A}\right\|_{W^{1,d}(\mathbb{CP}^d)}. \end{gathered} \tag{4.32}$$

On the unit ball B^{2d} we can rewrite $(A^{0,1})^{\tilde{g}}$ as such:

$$\left(A^{0,1}\right)^{\tilde{g}} = \tilde{g}^{-1}\overline{\partial}\tilde{g} + \tilde{g}^{-1}A^{0,1}\tilde{g} = \tilde{g}^{-1}\overline{\partial}\tilde{g} + \tilde{g}^{-1}\tilde{A}^{0,1}\tilde{g} - \tilde{g}^{-1}\vartheta\omega\tilde{g} = -\tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g}.$$

In order to find a gauge g for $A^{0,1}$ that gives a holomorphical trivialisation, it remains to find a gauge change u that cancels the perturbation term $-\tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g}$:

$$\overline{\partial}u = \tilde{g}^{-1}\left(\vartheta\omega\right)\tilde{g}\cdot u. \tag{4.33}$$

We claim that the composition of gauges $\tilde{g}\cdot u$ satisfies the statement.

Since the Sobolev embedding $W^{2,q} \hookrightarrow L^{dq/(d-q)}$ holds for any $q < d$, it implies that $\tilde{g}, \tilde{g}^{-1} \in \bigcap_{q<\infty} L^q$. The fact that A and $\tilde{A}$ satisfy the integrability condition on B^{2d}: $F^{0,2}_A = 0$ and $F^{0,2}_{\tilde{A}} = 0$, implies that $\omega \in W^{2,d}(\Omega^{0,2}B^{2d})$ satisfies the following PDE:

$$\overline{\partial}\vartheta\omega = -[A^{0,1}, \vartheta\omega] - \vartheta\omega \wedge \vartheta\omega.$$

Moreover, by Proposition 4.3, we also know that

$$\vartheta\overline{\partial}\omega = - * [*\overline{\partial}\omega, \overline{A^{0,1}}^T] - *[*\overline{\partial}\omega, \overline{\vartheta\omega}^T].$$

Thus, using the fact that $\vartheta\overline{\partial} + \overline{\partial}\vartheta = \frac{1}{2}\Delta_d$, it follows that

$$\frac{1}{2}\Delta_d\omega + [A^{0,1}, \vartheta\omega] + *[*\overline{\partial}\omega, \overline{A^{0,1}}^T] + [\vartheta\omega, \vartheta\omega] + *[*\overline{\partial}\omega, \overline{\vartheta\omega}^T] = 0.$$

Proposition D.2 applied to this PDE allows to bootstrap the regularity of ω inside B^{2d}. Indeed, we have an improved inner regularity $\omega \in W^{2,q}_{loc}(B^{2d}, M_n(\mathbb{C}))$ for any $q < 2d$. Sobolev embeddings yield:

$$\vartheta\omega \in \bigcap_{q<2d} W^{1,q}_{loc}(B^{2d}, M_n(\mathbb{C})) \hookrightarrow \bigcap_{q<\infty} L^q_{loc}.$$

Putting together the regularity of $\vartheta\omega$, $\tilde{g}$ and $\tilde{g}^{-1}$ we can obtain the regularity of $\tilde{g}^{-1}(\vartheta\omega)\tilde{g}$:

$$\tilde{g}^{-1}(\vartheta\omega)\tilde{g} \in \bigcap_{q<2d} W^{1,q}_{loc} \hookrightarrow \bigcap_{q<\infty} L^q_{loc}. \tag{4.34}$$

Fix $p > 2d-1$ and $\delta > 0$ small. There exists $r_0 \in (0,1)$ so that

$$\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,p}(B^{2d}_{r_0})} < \delta.$$

This $(0,1)$-form also solves $F^{0,2}_{\tilde{g}^{-1}(\vartheta\omega)\tilde{g}} = 0$ in $B^{2d}_{r_0}$. Hence, we apply Lemma 4.2 to $\tilde{g}^{-1}(\vartheta\omega)\tilde{g}$ in $B^{2d}_{r_0}$ (by rescaling) to get the existence of $r \in (r_0/2, r_0)$ and $u \in W^{2,p}(B^{2d}_r, GL_n(\mathbb{C}))$ that solves the $\overline{\partial}$-equation above (4.33):

$$\overline{\partial}u = \tilde{g}^{-1}(\vartheta\omega)\tilde{g}\cdot u \qquad \text{in } B^{2d}_r.$$

and satisfies the estimates

$$\|u - id\|_{W^{2,p}(B^{2d}_r)} \leq C\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,d}(B^{2d}_r)} \leq C\|A\|_{W^{1,d}(B^{2d})} \tag{4.35}$$

and

$$\left\|u^{-1} - id\right\|_{W^{2,p}(B^{2d}_r)} \leq C\left\|\tilde{g}^{-1}(\vartheta\omega)\tilde{g}\right\|_{W^{1,d}(B^{2d}_r)} \leq C\|A\|_{W^{1,d}(B^{2d})}, \tag{4.36}$$

for some constant $C > 0$.

Define $g := \tilde{g}u$ in B^{2d}_r. By construction of g, the required $\overline{\partial}$-equation is solved:

$$\overline{\partial}g = -A^{0,1}g \qquad \text{in } B_r^{2d} \tag{4.37}$$

We show that g and g^{-1} satisfy the required estimates (4.30). Let $q < d$ arbitrary. The triangle inequality applied on the norm $W^{2,q}$ gives:

$$\|g - id\|_{W^{2,q}(B_r^{2d})} \leq \|(\tilde{g} - id)(u - id)\|_{W^{2,q}(B_r^{2d})} + \|\tilde{g} - id\|_{W^{2,q}(B_r^{2d})} + \|u - id\|_{W^{2,q}(B_r^{2d})}.$$

Using the Sobolev product estimates of [31, Section 4.8.2, Theorem 1], from the regularity of $\tilde{g} - id \in \bigcap_{q<d} W^{2,q}(B_r^{2d})$ and $u - id \in W^{2,p}(B_r^{2d})$, it follows that

$$(\tilde{g} - id)(u - id) \in \bigcap_{q<d} W^{2,q}(B_r^{2d})$$

with

$$\|(\tilde{g} - id)(u - id)\|_{W^{2,q}(B_r^{2d})} \leq C\,\|\tilde{g} - id\|_{W^{2,q_1}(B_r^{2d})} \cdot \|u - id\|_{W^{2,p}(B_r^{2d})},$$

for some $q_1 \in (q, d)$ and constant $C > 0$. Hence, from (4.31), (4.32) and (4.35) it immediately follows that there exists a constant $C_q > 0$ such that:

$$\|g - id\|_{W^{2,q}(B_r^{2d})} \leq C_q\,\|A\|_{W^{1,d}(B^{2d})}.$$

By arguing in a completely analogous way to how we obtained the regularity of g, we obtain

$$\left\|g^{-1} - id\right\|_{W^{2,q}(B_r^{2d})} \leq C_q\,\|A\|_{W^{1,d}(B^{2d})}.$$

It remains to show the existence of h satisfying $A^g = h^{-1}\partial h$. We apply g to A in B_r^{2d} to get:

$$A^g = g^{-1}(\overline{\partial}g + \partial g) + g^{-1}A^{0,1}g - g^{-1}\overline{A^{0,1}}^T g = g^{-1}\partial g - g^{-1}\overline{A^{0,1}}^T g.$$

Since (4.37) holds, then $\partial\overline{g}^T = -\overline{g}^T\overline{A^{0,1}}^T$. Hence, $\left(\overline{g}^T\right)^{-1}\partial\overline{g}^T = -\overline{A^{0,1}}^T$. By plugging this into the equation above, we get

$$A^g = g^{-1}\partial g + g^{-1}\left(\overline{g}^T\right)^{-1}\partial\overline{g}^T g = (\overline{g}^T g)^{-1}\partial(\overline{g}^T g).$$

We conclude the proof of Theorem 4.1 by defining $h := \overline{g}^T g$, and $h \in W^{2,q}(B_r^{2d}, i\mathfrak{u}(n))$ for any $q < d$. □

Taking into account Remark 3.2, we end the chapter by proving a stability result for holomorphic trivialisations similar to Corollary 3.1.

Corollary 4.1. *Let $A_1 \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ and $r < 1$ so that $g_1 \in W^{2,q}(B^{2d}_r, GL_n(\mathbb{C}))$ satisfies theorem 4.1. There exists $\delta > 0$ such that for all $A_2 \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ with $F^{0,2}_{A_2} = 0$ satisfying*

$$\|A_1 - A_2\|_{W^{1,d}(B^{2d})} \leq \delta,$$

there exists a radius $r_0 \in (r/2, r)$ depending only on A_1 and a gauge

$$g_2 \in \bigcap_{q<d} W^{2,q}(B^{2d}_{r_0}, GL_n(\mathbb{C}))$$

that trivialises A_2 in the sense that:

$$A_2 = -\overline{\partial} g_2 \cdot g_2^{-1} \qquad \text{in } B^{2d}_{r_0}$$

The following estimates hold: for all $q < d$ there exists $C_q > 0$ such that

$$\|g_2 - id\|_{W^{2,q}(B^{2d}_{r_0})} \leq C_q \left(\|A_1\|_{W^{1,d}(B^{2d})} + \|A_2\|_{W^{1,d}(B^{2d})} \right)$$

and there exists $C > 0$ such that

$$\|g_1 - g_2\|_{L^p(B^{2d}_{r_0})} \leq C \|A_1 - A_2\|_{W^{1,d}(B^{2d})}$$

for all $p < d(2d+2)$.

Proof of Corollary 3.1. Choose $\delta > 0$ such that A_2 is a small perturbation of A_1. By Remark 3.2 and Theorem 4.1 applied to the forms A_1 and A_2 we obtain the existence of $r > 0$ and gauges $g_1, g_2 \in W^{2,q}(B^{2d}_r, GL_n(\mathbb{C}))$ for all $q < d$ so that

$$\overline{\partial} g_1 = -A_1^{0,1} \cdot g_1 \quad \text{and} \quad \overline{\partial} g_2 = -A_2^{0,1} \cdot g_2 \text{ in } B^{2d}_r \tag{4.38}$$

and there exists a constant $C_q > 0$ such that

$$\begin{aligned} \|g_1 - id\|_{W^{2,q}(B^{2d}_r)} &\leq C_q \|A_1\|_{W^{1,d}(B^{2d})}, \\ \|g_2 - id\|_{W^{2,q}(B^{2d}_r)} &\leq C_q \|A_2\|_{W^{1,d}(B^{2d})} \\ &\leq C_q \left(\|A_1\|_{W^{1,d}(B^{2d})} + \|A_1 - A_2\|_{W^{1,d}(B^{2d})} \right) \\ \text{and} & \end{aligned} \tag{4.39}$$

$$\begin{aligned}\left\|g_2^{-1} - id\right\|_{W^{2,q}(B_r^{2d})} &\leq C_q \left\|A_2\right\|_{W^{1,d}(B^{2d})} \\ &\leq C_q \left(\left\|A_1\right\|_{W^{1,d}(B^{2d})} + \left\|A_1 - A_2\right\|_{W^{1,d}(B^{2d})}\right)\end{aligned}$$

By (4.38), g_1 and g_2 holomorphically trivialise A_1 and A_2 respectively, we can relate the *transition gauge* $g_2^{-1}g_1$ with the difference 1-form $A_2 - A_1$ through the following $\overline{\partial}$-equation:

$$\overline{\partial}(g_2^{-1}g_1) = g_2^{-1}(A_2 - A_1)^{0,1}g_2 \cdot (g_2^{-1}g_1). \tag{4.40}$$

We first estimate $g_2^{-1}g_1 - id$ using the inequalities (4.39) and then use equation (4.40) to show that $g_2^{-1}g_1 - id$ is only bounded by the norm of $A_2 - A_1$. Fix $q < d$. The triangle inequality gives:

$$\begin{aligned}\left\|g_2^{-1}g_1 - id\right\|_{W^{2,q}(B_r^{2d})} \leq{}& \left\|(g_2^{-1} - id)(g_1 - id)\right\|_{W^{2,q}(B_r^{2d})} \\ &+ \left\|g_2^{-1} - id\right\|_{W^{2,q}(B_r^{2d})} + \left\|g_1 - id\right\|_{W^{2,q}(B_r^{2d})}\end{aligned}$$

Hence, by the results of [31, Section 4.8.2, Theorem 1] applied to the product $(g_2^{-1} - id)(g_1 - id)$ and estimates (4.39), there exists a constant $C_q > 0$ so that

$$\begin{aligned}\left\|g_2^{-1}g_1 - id\right\|_{W^{2,q}(B_r^{2d})} &\leq C_q(\left\|A_1\right\|_{W^{1,d}} + \left\|A_2\right\|_{W^{1,d}}) \\ &\leq 2C_q(\left\|A_1\right\|_{W^{1,d}} + \left\|A_1 - A_2\right\|_{W^{1,d}}).\end{aligned} \tag{4.41}$$

for all $q < d$. We can use equation (4.40) in order to find an a-posteriori estimate of $g_2^{-1}g_1 - id$ involving *only* the 1-form $A_2 - A_1$. Let $s < 2d$. By the regularity of $\overline{\partial}$ in L^s (see [18, Theorem 1 (b)]) there exists a holomorphic map h and a constant $C_s > 0$ such that

$$\begin{aligned}\left\|g_2^{-1}g_1 - h\right\|_{L^{(2d+2)s/((2d+2)-s)}(B_r^{2d})} &\leq C_s \left\|\overline{\partial}(g_2^{-1}g_1)\right\|_{L^s(B_r^{2d})} \\ &\leq C_s \left\|g_2^{-1}(A_2 - A_1)^{0,1}g_2\right\|_{L^{sp/(p-s)}(B_r^{2d})} \left\|g_2^{-1}g_1\right\|_{L^p(B_r^{2d})},\end{aligned}$$

where $p \in (s, \infty)$ arbitrary. Hence, it follows that there exists $C > 0$ depending on A_1 such that

$$\begin{aligned}\left\|g_2^{-1}g_1 - h\right\|_{L^{(2d+2)s/((2d+2)-s)}(B_r^{2d})} \leq{}& C\left\|A_1 - A_2\right\|_{W^{1,d}(B^{2d})} \left\|g_2^{-1}g_1 - id\right\|_{L^p(B_r^{2d})} \\ &+ C\left\|A_1 - A_2\right\|_{W^{1,d}(B^{2d})}.\end{aligned}$$

There exists $q < d$ such that $W^{2,q} \hookrightarrow L^p$. Since $g_2^{-1}g_1 - id$ is bounded in

$W^{2,q}$ by (4.41), then it is also bounded in L^p. Hence,

$$\left\| g_2^{-1} g_1 - h \right\|_{L^{(2d+2)s/((2d+2)-s)}(B_r^{2d})} \leq C \left\| A_1 - A_2 \right\|_{W^{1,d}(B^{2d})} .$$

Since this holds for any $s < 2d$, there exists a constant $C > 0$ such that

$$\left\| g_2^{-1} g_1 - h \right\|_{L^p(B_r^{2d})} \leq C \left\| A_1 - A_2 \right\|_{W^{1,d}(B^{2d})} ,$$

for all $p < d(2d+2)$. Having this inequality at our disposal, we can turn to estimate $g_1 - g_2 \cdot h$. Let $p < d(2d+2)$, then:

$$\left\| g_1 - g_2 \cdot h \right\|_{L^p(B_r^{2d})} = \left\| (g_2 - id)(h - g_2^{-1} g_1) + h - g_1^{-1} g_2 \right\|_{L^p(B_r^{2d})} .$$

For $v \in (p, d(2d+2))$, we get:

$$\begin{aligned} \left\| g_1 - g_2 \cdot h \right\|_{L^p(B_r^{2d})} \leq & \left\| g_2 - id \right\|_{L^{vp/(v-p)}(B_r^{2d})} \left\| g_2^{-1} g_1 - h \right\|_{L^v(B_r^{2d})} \\ & + \left\| g_2^{-1} g_1 - h \right\|_{L^p(B_r^{2d})} . \end{aligned}$$

Thus, there exists a constant $C_{v,p} > 0$ depending on v, p and A_1 such that

$$\left\| g_1 - g_2 \cdot h \right\|_{L^p(B_r^{2d})} \leq C_{v,p} \left\| A_1 - A_2 \right\|_{W^{1,d}(B^{2d})} .$$

Moreover, $g_2 \cdot h$ solves the equation:

$$\overline{\partial}(g_2 \cdot h) = A_2^{0,1}(g_2 \cdot h) \tag{4.42}$$

in a distributional sense. It remains to show that the $g_2 \cdot h$ is bounded in $W^{2,q}$ for all $q < d$ by the norms of A_1 and A_2 in a possible slightly smaller ball. Let $r_0 \in (r/2, r)$, then there exists a constant $C > 0$ such that

$$\left\| g_2 \cdot h - id \right\|_{W^{1,d}(B_{r_0}^{2d})} \leq C \left(\left\| \overline{\partial} g_2 \cdot h \right\|_{L^2(B_r^{2d})} + \left\| g_2 \cdot h - id \right\|_{L^d(B_r^{2d})} \right) .$$

Consequently, by using the $\overline{\partial}$-equation (4.42) satisfied by $g_2 \cdot h$, it follows that:

$$\begin{aligned} \left\| g_2 \cdot h - id \right\|_{W^{1,d}(B_{r_0}^{2d})} \leq C \Big(& \left\| A_2 \right\|_{L^{2d}(B_r^{2d})} \left\| g_2 \cdot h \right\|_{L^{2d}(B_r^{2d})} \\ & + \left\| g_2 \cdot h - g_1 \right\|_{L^d(B_r^{2d})} + \left\| id - g_1 \right\|_{L^d(B_r^{2d})} \Big) . \end{aligned}$$

Having shown that $g_2 \cdot h \in L^p$ for all $p < d(2d+2)$, we obtain

$$\|g_2 \cdot h - id\|_{L^{2d}(B^{2d}_{r_0})} \leq C \left(\|A_1\|_{W^{1,d}(B^{2d})} + \|A_2\|_{W^{1,d}(B^{2d})} \right).$$

Hence, given that $g_2 \cdot h \in W^{1,d}$ and $g_2 \cdot h - id$ is bounded by A_1 and A_2, we get from Lemma D.1 and Remark D.1(ii) the estimate: for any $q < d$ there exists a constant $C_q > 0$ such that:

$$\|g_2 \cdot h - id\|_{W^{2,q}(B^{2d}_{r_0})} \leq C_q \left(\|A_1\|_{W^{1,d}(B^{2d})} + \|A_2\|_{W^{1,d}(B^{2d})} \right).$$

By redefining g_2 as $g_2 \cdot h$, we have proven our stability result. □

4.3 Density under high energy

For proving theorem 1.4 we have to show that for any smooth approximating sequence of ∇, there exists a perturbation of the sequence such that the integrability condition $F^{0,2}_{\nabla} = 0$ is satisfied throughout. In the previous chapter, we have heavily used the fact that $\overline{\partial}\overline{\partial}^*$ is an elliptic operator on Kähler surfaces. However, this is not the case for Kähler manifolds as we have explained in section 4.1. Thus, we aim at solving the extended integrability condition (4.5) locally and consequently showing that globally this implies the integrability condition.

Over B^{2d} we assume without loss of generality that $\nabla \approx d + A$. We start by studying the invertibility of the operator

$$L_A : W^{2,d}(\Omega^{0,2}B^{2d}) \cap \mathrm{Dom}(L_A) \to L^d(\Omega^{0,2}B^{2d})$$

defined by

$$L_A\omega := \overline{\partial}\vartheta\omega + \vartheta\overline{\partial}\omega + [A^{0,1}, \vartheta\omega] + *[*\overline{\partial}\omega, \overline{A^{0,1}}^T]. \tag{4.43}$$

where $\mathrm{Dom}(L_A) = \{\omega : \ \omega_N = 0, \ (d\omega)_N = 0\}$ and by ω_N we understand the components of the form ω involving dr, i.e. $\mathrm{Dom}(L_A) = \{\omega : \ \omega, d\omega \in \mathrm{Dom}(d^*)\}$ by (4.6). In particular, since $\Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$, the boundary conditions $\omega_N = 0, \ (d\omega)_N = 0$ make the operator L_A elliptic. Globally we do not need these conditions since we are working over a closed Kähler manifold.

We show that for perturbations g of A with $\|g - id\|_{L^\infty}$ small, L_{A^g} is an invertible operator that maps $W^{2,d}(\Omega^{0,2}B^{2d} \otimes \mathfrak{u}(n))$ to $L^d(\Omega^{0,2}B^{2d} \otimes \mathfrak{u}(n))$.

In order to be able to study the invertibility of L_A, we first need a set of general results derived from the theory of Fredholm Operators. Secondly, we prove that under linear perturbations of A, we can find $\mathfrak{u}(n)$-valued maps U such that $L_{A+\beta\overline{\partial}U}$ is invertible for all $0<\beta<\beta_0$, for some $\beta_0>0$. Finally, we generalise these ideas in Section 4.3.3 and find gauge changes g of A such that L_{A^g} is invertible.

The first result of this Chapter is the analogous of Proposition 3.3.

Proposition 4.4. *The operator*

$$L_A : W^{2,d}(\Omega^2 B^{2d}\otimes\mathfrak{u}(n))\cap \mathrm{Dom}(L_A)\to L^d(B^{2d},\mathfrak{u}(n))$$

is Fredholm and has index zero.

Proof of Proposition 4.4. Firstly, L_A is elliptic, because it is a lower order perturbation of $\Delta_{\overline{\partial}}=\frac{1}{2}\Delta_d$ under well-posed boundary conditions, thus it is Fredholm.

Next, note that $\Delta_{\overline{\partial}}$ is an elliptic operator of Fredholm index zero from $W^{2,d}(\Omega^{0,2}B^{2d}\otimes\mathfrak{u}(n))\cap\mathrm{Dom}(L_A)$ into $L^d(\Omega^2 B^{2d}\otimes\mathfrak{u}(n))$.

Let A_k be a sequence of smooth 1-forms converging strongly in $W^{1,d}$ to A. Then the bracket operator

$$\omega\mapsto [A_k^{0,1},\vartheta\omega]+*[*\overline{\partial}\omega,\overline{A_k^{0,1}}^T]$$

is compact $W^{2,d}(\Omega^{0,2}(B^{2d}))$ to $L^d(\Omega^{0,2}(B^{2d}))$. Indeed, A_k is bounded in L^∞ and hence:

$$\left\|[A_k^{0,1},\vartheta\omega]+*[*\overline{\partial}\omega,\overline{A_k^{0,1}}^T]\right\|_{L^d(B^{2d})}\le C\left\|A_k\right\|_{L^\infty(B^{2d})}\Big(\left\|\vartheta\omega\right\|_{W^{1,d}(B^{2d})}+\left\|\overline{\partial}\omega\right\|_{W^{1,d}(B^{2d})}\Big),$$

for some constant $C>0$, where we have used the fact that $W^{1,d}$ is compactly embedded in L^d in $2d$-dimensions, by Rellich-Kondrachov [2] and therefore the operators

$$\omega\mapsto [A_k^{0,1},\vartheta\omega]+*[*\overline{\partial}\omega,\overline{A_k^{0,1}}^T]$$

are compact $W^{2,d}$ to L^d for all k. Hence, using the compactness of these operators and the fact that $\Delta_{\overline{\partial}}$ is Fredholm, by [32, Theorem 4.4.2, p.185]) we have

$$\mathrm{index}L_{A_k}=\mathrm{index}(\Delta_{\overline{\partial}}).$$

By the Hodge decomposition, we know that Δ_d has index zero over $\mathrm{Dom}(L_A)$. Thus, $\Delta_{\overline{\partial}}$ has index zero as well and we have

$$\mathrm{index} L_{A_k} = \mathrm{index}(\Delta_{\overline{\partial}}) = 0.$$

Moreover, for a fixed $\varepsilon > 0$ given by [32, Theorem 4.4.2, p.185], there exists $k_0 > 0$ such that for all $k \geq k_0$:

$$\left|\!\left|\!\left| [A_k^{0,1} - A^{0,1}, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{A_k^{0,1} - A^{0,1}}^T] \right|\!\right|\!\right| \leq \varepsilon$$

since A_k converges strongly to A in $W^{1,d}$.
Thus, by applying [32, Theorem 4.4.2, p.185] to the perturbation operator $[A_k^{0,1} - A^{0,1}, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{A_k^{0,1} - A^{0,1}}^T]$ and L_A, we obtain that

$$\begin{aligned}\mathrm{index} L_A &= \mathrm{index}\left(L_A + [A_k^{0,1} - A^{0,1}, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{A_k^{0,1} - A^{0,1}}^T]\right) \\ &= \mathrm{index} L_{A_k} = 0\end{aligned}$$

This proves the statement. $\square$

In the following section we will discuss the invertibility of general Fredholm operators. The results we prove here will be crucial to our density arguments.

4.3.1 Invertibility of Fredholm Operators

Recall that a bounded linear operator $L : X \to Y$ between Banach spaces is Fredholm if it has closed range and its kernel and cokernel are finite-dimensional. If $L : X \to Y$ is Fredholm and $P : X \to Y$ is another bounded linear operator, we can write $X = \mathrm{Ker}(L) \oplus X'$ and $Y = Y' \oplus \mathrm{Ran}(L)$, as direct sum decompositions of X, Y. Then L, P and $L + P$ can be written in the form

$$L = \begin{pmatrix} 0 & 0 \\ 0 & L_4 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \quad L + P := \begin{pmatrix} P_1 & P_2 \\ P_3 & \widetilde{L}_4 \end{pmatrix}, \quad (4.44)$$

where $L_4 : X' \to \mathrm{Ran}(L)$ is a bounded invertible operator, $\widetilde{L}_4 := L_4 + P_4$, and where via the expression

$$\widetilde{L}_4^{-1} = (L_4 + P_4)^{-1} = \sum_{k=0}^{\infty} (-L_4^{-1} P_4)^k L_4^{-1}, \quad \text{for } \|L_4^{-1} P_4\| < 1 \tag{4.45}$$

we see that if $\|P_4\| < \|L_4^{-1}\|^{-1}$ then $\widetilde{L}_4$ is invertible with bounded inverse. In particular, we have

$$\left|\left|\left| \widetilde{L}_4^{-1} \right|\right|\right| \leq |||L_4^{-1}||| \frac{1}{1 - |||L_4^{-1}||| \, |||P_4|||} \tag{4.46}$$

In this case we can use the Schur complement of $\widetilde{L}_4$ and write

$$L + P = \begin{pmatrix} P_1 & P_2 \\ P_3 & \widetilde{L}_4 \end{pmatrix} \tag{4.47}$$

$$= \begin{pmatrix} I_{Y'} & P_2 \widetilde{L}_4^{-1} \\ 0 & I_{\mathrm{Ran}(L)} \end{pmatrix} \begin{pmatrix} P_1 - P_2 \widetilde{L}_4^{-1} P_3 & 0 \\ 0 & \widetilde{L}_4 \end{pmatrix} \begin{pmatrix} I_{\mathrm{Ker}(L)} & 0 \\ \widetilde{L}_4^{-1} P_3 & I_{X'} \end{pmatrix}.$$

We note that the first and last operators in (4.47) are bounded and invertible, acting respectively on Y and on X, thus the index of $L + P$ is equal to the index of the middle operator in (4.47). Considering the middle operator in (4.47), since it is block-diagonal and $L_4 + P_4$ is invertible, its kernel and co-kernel dimensions are determined by the upper-left block

$$\widetilde{L}_1 := P_1 - P_2 \widetilde{L}_4^{-1} P_3 : \quad \mathrm{Ker}(L) \to Y', \tag{4.48}$$

which is a bounded linear operator between finite-dimensional spaces.

Note that denoting $x_0 \in \mathrm{Ker} L$ and $x_1 \in X'$ according to the decomposition $X = \mathrm{Ker} L \oplus X'$, we have

$$\mathrm{Ker}(L + P) \ni x_0 + x_1 \Leftrightarrow \begin{cases} P_1 x_0 + P_2 x_1 = 0, \\ P_3 x_0 + \widetilde{L}_4 x_1 = 0, \end{cases} \Leftrightarrow \begin{cases} \widetilde{L}_1 x_0 = 0, \\ x_1 = -\widetilde{L}_4^{-1} P_3 x_0. \end{cases} . \tag{4.49}$$

Note that the map written in block form as

$$J_{L,P} := (I_{\mathrm{Ker}(L)}, -\widetilde{L}_4^{-1}P_3) \ : \ \mathrm{Ker}(L) \to \mathrm{Ker}(L) \oplus X' \tag{4.50}$$

is injective, and thus an isomorphism between $\mathrm{Ker}(L)$ and its range, which is

$$K_{L,P} := \{x_0 + x_1 : x_0 \in \mathrm{Ker}(L), x_1 = -\widetilde{L}_4^{-1}P_3x_0\}. \tag{4.51}$$

Also note that $\mathrm{Ker}(L+P) \subset K_{L,P} \subset X$, and that the map $J_{L,P}$ depends continuously on P.

Lemma 4.3. *Let $L, P : X \to Y$ be bounded linear operators between Banach spaces such that L is Fredholm and that decomposition* (4.44) *holds, with furthermore $\|P_4\| < \|L_4^{-1}\|^{-1}$. Let the operator $\widetilde{L}_1$ be defined as in* (4.48) *using* (4.45)*. Then the following hold:*

1. *The map $P \mapsto \widetilde{L}_1$ is continuous with respect to the operator norms induced by the Banach space norms on $X, Y, \mathrm{Ker}(L), Y'$.*
2. *There holds $\mathrm{Ker}(L+P) \simeq \mathrm{Ker}(\widetilde{L}_1)$ and $\mathrm{CoKer}(L+P) \simeq \mathrm{CoKer}(\widetilde{L}_1)$.*
3. *There holds $\mathrm{dimKer}(L+P) \leq \mathrm{dimKer}(L)$ and $\mathrm{dimCoKer}(L+P) \leq \mathrm{dimCoKer}(L)$.*

Sketch of proof: Points 1 and 2 follow from the decomposition (4.44) and formula (4.45). Point 3 follows from point 2 and (4.48). □

In view of Lemma 4.3, the following result gives us a necessary and sufficient condition for $\mathrm{dimKer}(L+P) = \mathrm{dimKer}(L)$:

Proposition 4.5. *Let $L, P : X \to Y$ be bounded linear operators between Banach spaces, such that $P|_{\mathrm{Ker}(L)}$ is injective and $\|P_4\| < \|L_4^{-1}\|^{-1}$ with notation as in* (4.44)*. Then $\mathrm{dimKer}(L+P) = \mathrm{dimKer}(L)$ if and only if*

$$P_1 = P_2\widetilde{L}_4^{-1}P_3 = 0. \tag{4.52}$$

Proof of Proposition 4.5. By (4.47) we have the following decomposition induced by the direct sum $Y = Y' \oplus \mathrm{Ran}(L)$:

$$P|_{\mathrm{Ker}(L)} = \begin{pmatrix} P_1 \\ \\ P_3 \end{pmatrix}. \tag{4.53}$$

First assume that $\dim\mathrm{Ker}(L+P) = \dim\mathrm{Ker}(L)$. By the isomorphism (4.50) we obtain that $\dim\mathrm{Ker}(L+P) = \dim K_{L,P}$ and since $\mathrm{Ker}(L+P) \subset \dim K_{L,P}$, we obtain $\mathrm{Ker}(L+P) = K_{L,P}$. Comparing the rightmost system in (4.51) to (4.49), we deduce $\widetilde{L}_1 = 0$, and via (4.48) we find

$$P_1 = P_2 \widetilde{L}_4^{-1} P_3. \tag{4.54}$$

Inserting this in (4.53), we get

$$P|_{\mathrm{Ker}(L)} = \begin{pmatrix} P_2 \widetilde{L}_4^{-1} \\ \\ I_{\mathrm{Ran}(L)} \end{pmatrix} P_3. \tag{4.55}$$

Since by hypothesis $P|_{\mathrm{Ker}(L)}$ is injective, (4.55) implies that

$$P_3 : \mathrm{Ker}(L) \to \mathrm{Ran}(L)$$

is also injective. Hence,

$$\dim\mathrm{Ran}(P_3) = \dim\mathrm{Ker}(L). \tag{4.56}$$

On the other hand, since the decomposition (4.53) comes from a direct sum expression of Y, we have $\dim\mathrm{Ker}(L) \geq \dim\mathrm{Ran}(P_1) + \dim\mathrm{Ran}(P_3)$. This together with (4.56) implies that $\dim\mathrm{Ran}(P_1) = 0$, and thus $P_1 = 0$. This, together with (4.54), concludes the proof of (4.52).

Conversely, assume that (4.52) holds. In particular by (4.48) $\widetilde{L}_1 = 0$. Thus, from (4.49) we obtain that $\dim\mathrm{Ker}(L+P) = \dim\mathrm{Ker}(L)$. □

Crucially, we are now able to show that for perturbation operators P that are injective on the kernel of L we can decrease the dimension of the $\mathrm{Ker}L$.

Proposition 4.6. *Let $L, P : X \to Y$ be bounded linear operators between Banach spaces, such that $P|_{\mathrm{Ker}(L)}$ is injective. Then there exists $\beta > 0$ such that*

$$\dim\mathrm{Ker}(L+\beta P) < \dim\mathrm{Ker}(L). \tag{4.57}$$

Proof of Proposition 4.6. Assume that $\dim\mathrm{Ker}(L+\beta P) = \dim\mathrm{Ker}(L)$ for all $\beta \in (0,1)$. Since $P|_{\mathrm{Ker}(L)}$ is injective, so is $\beta P|_{\mathrm{Ker}(L)}$. We use notation (5.5) and define $\beta_0 := \|L_4\|\|\beta_4\|$, so that $\|\beta P_4\| \leq \|L_4\|^{-1}$ for all $\beta \in (0,\beta_0)$. Then we can apply Proposition 4.5 to L and βP in order to get $\beta P_1 = 0$ and $\beta P_2 = 0$ on $\mathrm{Im}(L_4 + \beta P_4)^{-1} \beta P_3$ for all $\beta \in (0, \beta_0)$.

Let $0 \neq e \in KerL$. Since $P_1 = 0$, all elements in Ker$L$ are mapped into Ran$L$ by P_3. Thus, there exists $\omega_e \in X'$ such that $Pe = L\omega_e$ and for $\beta \in (0, \beta_0)$, we also have

$$(\beta P)e = L(\beta\omega_e).$$

This implies the following relation:

$$(\beta P)e = L(\beta\omega_e) \iff (\beta P_3)e = L_4(\beta\omega_e).$$

The above equivalence together with the fact that $\beta P_2 = 0$ on the space $\mathrm{Im}(L_4 + \beta P_4)^{-1}\beta P_3$, give the following equation:

$$\begin{aligned}
0 &= (L_4 + \beta P_4)^{-1}\beta P_3 e = (L_4 + \beta P_4)^{-1} L(\beta\omega_e) \\
&= (L_4 + \beta P_4)^{-1} L_4(\beta\omega_e) = \sum_{k=0}^{\infty} (-1)^k \beta^k (L_4^{-1} P_4)^k L_4^{-1} L_4 \beta\omega_e \\
&= \sum_{k=0}^{\infty} (-1)^k \beta^k (L_4^{-1} P_4)^k \beta\omega_e \\
&= \beta\omega_e + o(\beta^2).
\end{aligned}$$

Since this holds for all $\beta \in (0, \beta_0)$, we find $\omega_e = 0$, and thus $e = 0$. We have obtained a contradiction. Hence, there exists $\beta > 0$ such that $\mathrm{dimKer}(L + \beta P) < \mathrm{dimKer}(L)$. □

4.3.2 Linear perturbation

Having Proposition 4.6 at our disposal, we can now construct linear perturbations P of L_A (4.43) such that

$$\mathrm{dimKer}(L_A + P) < \mathrm{dimKer}(L_A). \tag{4.58}$$

In particular, we find a small perturbation of the form

$$P = P_U = [\overline{\partial} U, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{\overline{\partial} U}^T] \tag{4.59}$$

that satisfies (4.58), where U is a smooth $\mathfrak{u}(n)$-valued map over B^{2d}.

The following proof resembles Proposition 3.4, however now we want to show the existence of perturbation U such that P is injective on KerL_A.

Proposition 4.7. *There exists $P = P_U$ with the same domain and range as L_A and of the form* (4.59) *such that U is a smooth, $\mathfrak{u}(n)$-valued map and P is injective on* $\mathrm{Ker}L_A$.

Proof of Proposition 4.7. Since $\mathrm{Ker}L_A$ is finite dimensional, let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of it.

Claim 1. *For each $v \in \mathrm{Ker}L_A \setminus \{0\}$, there exists $U_v \in C^\infty(B^n, \mathfrak{g})$ such that $P_{U_v}v \neq 0$.*

Let $v \in \mathrm{Ker}L_A \setminus \{0\}$. We show that for each such v, we can find U_v such that $P_{U_v}v \neq 0$. Assume by contradiction that $P_U v = 0$ for all smooth $M_n(\mathbb{C})$-valued maps U on B^{2d}. Define the linear operator

$$H(\omega) := [\omega, \vartheta v] + *[*\overline{\partial} v, \overline{\omega}^T] : C^\infty(\Omega^{0,1}B^{2d} \otimes M_n(\mathbb{C})) \to C^\infty(\Omega^{0,2}B^{2d} \otimes M_n(\mathbb{C})).$$

Observe that $H(\omega(x)) = H(\omega)(x)$, in which $\omega(x)$ is identified to the constant form equal to $\omega(x)$. Moreover, we have that

$$0 = P_U v = [\overline{\partial} U, \vartheta v] + *[*\overline{\partial} v, \overline{\overline{\partial} U}^T] = H(\overline{\partial} U)$$

for all smooth maps U on B^{2d}. Applying Proposition C.1 to H, we obtain that $H = 0$. By density of smooth 1-forms into $W^{1,d}$ 1-forms, it follows in particular that

$$H(A) = [A^{0,1}, \vartheta v] + *[*\overline{\partial} v, \overline{A^{0,1}}^T] = 0.$$

Putting this together with the fact that $v \in \mathrm{Ker}L_A$, we obtain:

$$0 = L_A v = \Delta_{\overline{\partial}} v.$$

Since $v \in \mathrm{Ker}\Delta_{\overline{\partial}}$, then $v = 0$. This is a contradiction, since we have picked $v \neq 0$. Hence, there exists $U_v \in C^\infty(B^{2d}, M_n(\mathbb{C}))$ so that $P_{U_v}v \neq 0$, as claimed.

Next, we show that such a U_v can be chosen to be $\mathfrak{u}(n)$-valued. Indeed, since $U_v \in M_n(\mathbb{C})$, there exists a decomposition in terms of its Hermitian and anti-Hermitian part:

$$U_v = U_1 + U_2,$$

where $U_1 \in C^\infty(B^{2d}, \mathfrak{u}(n))$ and $U_2 \in C^\infty(B^{2d}, i\mathfrak{u}(n))$. Assume that $P_{U_1}v = 0$, otherwise we redefine $U_v := U_1$. Under this assumption, by linearity it then necessarily follows that

$$P_{U_2}v \neq 0.$$

If this condition holds, then by multiplying with i,

$$iP_{U_2}v = P_{iU_2}v \neq 0.$$

Moreover, $iU_2 \in C^\infty(B^{2d}, \mathfrak{u}(n))$ and in this case we redefine $U_v := iU_2$. Hence, there exists

$$U_v \in C^\infty(B^{2d}, \mathfrak{u}(n)) \text{ so that } P_{U_v}v \neq 0, \text{ for any } v \in KerL_A, v \neq 0. \quad (4.60)$$

Claim 2. *There exists U smooth $\mathfrak{u}(n)$-valued map over B^{2d} such that P_U is injective on* $\mathrm{Ker}L_A$.
We formulate the following inductive hypothesis:

$$\mathcal{I}(k): \quad \begin{array}{l} \text{there exists } U^k \in C^\infty(B^{2d}, \mathfrak{u}(n)) \text{ supported in } V^k \subsetneq B^{2d} \text{ such that} \\ \\ \text{the forms } P_{U^k}e_1, \dots P_{U_k}e_k \text{ are linearly independent.} \end{array}$$

We show by induction that $\mathcal{I}(k)$ holds for $k \leq N$, from which it follows that P_{U^N} is injective on $KerL_A$.

By Claim 1, there exists U_1 such that $P_{U_1}e_1 \neq 0$. Without loss of generality, by multiplying with a compactly supported function ρ_1, we can localise U_1 in $V_1 \subsetneq B^{2d}$. Hence $\mathcal{I}(1)$ holds. We now assume that for $k < N$, $\mathcal{I}(k)$ holds, and we prove $\mathcal{I}(k+1)$.

If $P_{U^k}e_1, \dots, P_{U^k}e_{k+1}$ already are linearly independent, then we just set $U^{k+1} = U^k$. Otherwise there exists $\lambda_1, \dots, \lambda_{k+1}$ not all 0 such that $\sum_{i=1}^{k+1} \lambda_i P_{U^k}e_i = 0$. In this case $\lambda_{k+1} \neq 0$.

By Claim 1 there exists U_{k+1} such that $P_{U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0$. We can choose a neighbourhood V_{k+1} and $\tilde{V}^k \subseteq V^k$ disjoint from V_{k+1} such that $\{P_{U^k}e_j\}_{j=1}^k$ is linearly independent in $\tilde{V}^k$ and

$$P_{\rho_{k+1}U_{k+1}} \sum_{i=1}^{k+1} \lambda_i e_i \neq 0 \qquad \text{in } V_{k+1}$$

In particular, we can define functions ρ_{k+1} compactly supported in V_{k+1}, ρ_k compactly supported in $\tilde{V}^k$.

We now define $U^{k+1} := \rho_k U^k + \rho_{k+1}U_{k+1}$, and show that $P_{U^{k+1}}e_1, \dots, P_{U^{k+1}}e_{k+1}$ now are linearly independent. Assume there exists $\beta_1, \dots, \beta_{k+1}$ such that

$$\sum_{j=1}^{k+1} \beta_j P_{\rho_k U^k + \rho_{k+1} U_{k+1}} e_j = \sum_{j=1}^{k+1} \beta_j P_{U^{k+1}} e_j = 0. \tag{4.61}$$

In the neighbourhood $\tilde{V}^k$, we have that

$$\sum_{j=1}^{k+1} \beta_j P_{U^k} e_j = 0.$$

Then $(\beta_1, \ldots, \beta_{k+1}) = c(\lambda_1, \ldots, \lambda_{k+1})$ for some constant c. Hence, in V_{k+1}, we have that

$$c \sum_{j=1}^{k+1} \lambda_j P_{U_{k+1}} e_j = 0.$$

By the choice of U_{k+1}, we obtain that $c = 0$. Hence $\beta_1 = \ldots = \beta_{k+1} = 0$. To conclude, define $V^{k+1} = \tilde{V}^k \cup V_{k+1}$. This proves the induction.

Hence, we have obtained $U = U^N$ such that $\{P_U e_j\}_{j=1}^N$ are linearly independent, where e_j the orthonormal basis of KerL_A we have picked initially. It follows that P_U is injective on KerL_A. □

The above construction of $P = P_U$ allows us to apply Proposition 4.6 to the operator L_A as follows:

Proposition 4.8. *Let L_A be defined as in* (4.43) *and assume that L_A is not injective. There exists $P = P_U$ as in Proposition 4.7, such that furthermore*

$$\text{dimKer}(L_A + P) < \text{dimKer}(L_A). \tag{4.62}$$

Proof of Proposition 4.8. By Proposition 4.7, there exists a 1-form $\widetilde{U}$ such that the operator

$$\widetilde{P} = [\overline{\partial}\widetilde{U}, \vartheta \cdot] + *[*\overline{\partial} \cdot, \overline{\overline{\partial}\widetilde{U}}^T]$$

is injective on KerL_A. After rescaling $\widetilde{U}$, we can assume that $\widetilde{P}$ is small in operator norm. Applying Proposition 4.6 to L_A and $\widetilde{P}$, we obtain P that is of the form $[\overline{\partial}U, \vartheta \cdot] + *[*\overline{\partial} \cdot, \overline{\overline{\partial}U}^T]$ and satisfies (4.62). This concludes the statement. □

Having now the result that shows that we can always find a perturbation that decreases the dimension of the kernel of L_A, we want to iterate the procedure such that we obtain a perturbation P that makes the kernel of $L_A + P$ trivial.

In light of the fact that L_A is an operator of index zero, we can conclude that $L_A + P$ is invertible.

Theorem 4.2. *Let L_A be defined as in* (4.43) *and assume that L_A is not injective. There exists P with same domain and range spaces as L_A, of the form*

$$P(\eta) := [\overline{\partial} U, \vartheta\eta] + *[*\overline{\partial}\eta, \overline{\overline{\partial} U}^T]$$

and small in operator norm such that $L_A + P$ is invertible.

Proof of Theorem 4.2. By applying Proposition 4.8 to L_A, we obtain a perturbation $P_1 = [\overline{\partial} U_1, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{\overline{\partial} U_1}^T]$ such that

$$\dim\mathrm{Ker}(L_A + P_1) < \dim\mathrm{Ker}(L_A).$$

The operator $L_A + P_1 = L_{A+\overline{\partial} U_1}$ is in addition Fredholm of the same index as L_A, since P_1 is compact. Similarly we can find a perturbation operator P_2 for $L_{A+\overline{\partial} U_1} = L_A + P_1$. This leads us to the idea of iterating the procedure. Since $\ell = \dim\mathrm{Ker}(L_A)$ is finite, after at most n perturbations P_k, $k \leq \ell$, we obtain that

$$\dim\mathrm{Ker}(L_A + P_1 + \ldots + P_\ell) = 0.$$

Thus, by denoting $P := P_1 + \ldots + P_\ell$, we obtain a linear perturbation that makes the kernel of $L_A + P$ trivial. Moreover, because L_A is a Fredholm operator of index zero and P is a compact operator, it follows that $L_A + P$ is an operator of index zero and trivial kernel. Hence, $L_A + P$ is invertible.

Moreover, since each P_k is of the form $[\overline{\partial} U_k, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{\overline{\partial} U_k}^T]$ for each $k \leq \ell$, then

$$P = [\overline{\partial} U, \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{\overline{\partial} U}^T]$$

where $U = U_1 + \ldots + U_\ell$. This concludes the statement. □

4.3.3 Gauge perturbation

As in the Hilbert case in Chapter 3, we now consider perturbations of A of the form $A^g = g^{-1}dg + g^{-1}Ag$ with g a $U(n)$-valued gauge transformation. Using the approach we took in the previous section and relying on Proposition 4.6, we find a smooth $\mathfrak{u}(n)$-valued function U such that for $g = \exp U \in U(n)$, the operator L_{A^g} is invertible.

Consider the family of gauges $g_t = \exp tU$, $t = [0,1]$. We can analytically expand A^{g_t} in $t \in [0,1]$ to obtain

$$A^{g_t} - A = t(dU + [U, A]) + O(t^2).$$

In particular, we have

$$\left(A^{0,1}\right)^{g_t} - A^{0,1} = t(\overline{\partial} U + [U, A^{0,1}]) + O(t^2). \tag{4.63}$$

Moreover, it will be useful to define the bounded linear operators

$$G_{t,U} : W^{2,d}(\Omega^{0,2} B^{0,2} \otimes \mathfrak{M}_n(\mathbb{C})) \to L^d(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C}))$$

given by

$$\begin{aligned} G_{t,U} &:= [\left(A^{0,1}\right)^{g_t} - A^{0,1} - t(\overline{\partial} U + [U, A^{0,1}]), \vartheta \cdot] \\ &\quad + *[*\overline{\partial} \cdot, \overline{(A^{0,1})^{g_t} - A^{0,1} - t(\overline{\partial} U + [U, A^{0,1}])}^T] \end{aligned} \tag{4.64}$$

and

$$P_U : W^{2,d}(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C})) \to L^d(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C}))$$

given by

$$P_U = [\overline{\partial} U + [U, A^{0,1}, \vartheta \cdot] + *[*\overline{\partial} \cdot, \overline{\overline{\partial} U + [U, A^{0,1}}^T].$$

Observe that

$$L_{A^{g_t}} = L_A + [\left(A^{0,1}\right)^{g_t} - A^{0,1}, \vartheta \cdot] + *[*\overline{\partial} \cdot, \overline{(A^{0,1})^{g_t} - A^{0,1}}^T]$$

and we can write $G_{t,U}$ as $G_{t,U} = L_{A^{g_t}} - L_A + tP_U$. It follows that

$$|||G_{t,U}||| \leq C \left\| A^{g_t} - A \right\|_{L^d(B^{2d})} + t|||P_U|||, \tag{4.65}$$

for some constant $C > 0$. Moreover, there exists t_0 such that for all $t \leq t_0$, we have that the operator

$$[\left(A^{0,1}\right)^{g_t} - A^{0,1}, \vartheta \cdot] + *[*\overline{\partial} \cdot, \overline{(A^{0,1})^{g_t} - A^{0,1}}^T]$$

is small in operator norm mapping $W^{2,d}(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C}))$ to $L^d(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C}))$ and by [32, Theorem 4.4.2]:

$$\text{index} L_{(A^{0,1})^{g_t}} = \text{index} L_A = 0. \tag{4.66}$$

We show that it is enough to find a perturbation P_U that is injective on the kernel L_A in order to decrease the kernel size. We start with a proposition which is a non-linear analogue to Proposition 4.7:

Proposition 4.9. *There exists $P = P_U$ with the same domain and range as L_A and of the form*

$$P_U(\eta) := [\overline{\partial}U + [U, A^{0,1}], \vartheta\eta] + *[*\overline{\partial}\eta, \overline{\overline{\partial}U + [U, A^{0,1}]}^T]$$

for U $\mathfrak{u}(n)$-valued smooth, such that P is injective on $\mathrm{Ker}L_A$.

Proof of Proposition 4.9. **Claim**. *For each $v \in \mathrm{Ker}L_A$,$v \neq 0$, there exists*

$$U_v \in C^\infty(B^{2d}, \mathfrak{u}(n))$$

such that $P_{U_v}v \neq 0$.

Let $v \in KerT_0$, $v \neq 0$. We show that for each such v, we can find U_v such that $P_{U_v}v \neq 0$. Assume by contradiction that $P_U v = 0$ for all smooth $M_n(\mathbb{C})$-valued maps U on B^{2d}. Define the linear operators

$$H_0 : C^\infty(B^{2d}, M_n(\mathbb{C})) \to C^\infty(\Omega^{0,2}B^{2d} \otimes M_n(\mathbb{C}))$$

given by

$$H_0(\omega) := [[\omega, A^{0,1}], \vartheta v] + *[*\overline{\partial}v, \overline{[\omega, A^{0,1}]}^T]$$

and

$$H_1 : C^\infty(\Omega^{0,1}B^{2d} \otimes M_n(\mathbb{C})) \to C^\infty(\Omega^{0,2}B^{2d} \otimes M_n(\mathbb{C})).$$

given by

$$H_1(\omega) := [\omega, \vartheta v] + *[*\overline{\partial}v, \overline{\omega}^T].$$

We have $H_0(\omega(x)) = H_0(\omega)(x)$, where $\omega(x)$ is identified with a constant form. Moreover,

$$0 = P_U v = [\overline{\partial}U + [U, A^{0,1}], \vartheta v] + *[*\overline{\partial}v, \overline{\overline{\partial}U + [U, A^{0,1}]}^T] = H_1(\overline{\partial}U) + H_0(U)$$

for all smooth maps U on B^{2d}. Applying Proposition C.2 to H_1 and H_0, we obtain that $H_1 \circ \overline{\partial} = 0$ and $H_0 = 0$. In particular, we have obtained that for all U smooth functions on B^{2d},

$$H_1(\overline{\partial}U) = [\overline{\partial}U, \vartheta v] + *[*\overline{\partial}v, \overline{\overline{\partial}U}^T] = 0.$$

This contradicts Proposition 4.7. Hence, there exists $U_v \in C^\infty(B^{2d}, M_n(\mathbb{C}))$ so that $P_{U_v}v \neq 0$.

The rest of the proof (the existence of hermitian U_v such that $P_{U_v}v \neq 0$ and the existence of U such that P_U is injective on $\mathrm{Ker}L_A$) follows exactly as in Proposition 4.7. $\square$

Proposition 4.10. *Let* L_A *be defined as in* (4.43) *and assume that* L_A *is not injective. There exists a unitary smooth gauge perturbation* g *such that*

$$\mathrm{dimKer}(L_{A^g}) < \mathrm{dimKer}(L_A) \tag{4.67}$$

and $\mathrm{index}L_{A^g} = \mathrm{index}L_A = 0$.

Proof of Proposition 4.10. By Proposition 4.9, there exists U smooth hermitian such that the operator

$$P_U = [\overline{\partial}U + [U, A^{0,1}], \vartheta\cdot] + *[*\overline{\partial}\cdot, \overline{\overline{\partial}U + [U, A^{0,1}]}^T]$$

is injective on $\mathrm{Ker}L_A$. Moreover, there exists t_0 such that for $t \leq t_0$, we can assume that $P_{tU} = tP_U$ is small in operator norm $|||\cdot|||$ such that Proposition 4.6 applies to L_A and tP. Then $L_A + tP_U$ is Fredholm and

$$\mathrm{dimKer}(L_A + tP_U) < \mathrm{dimKer}(L_A).$$

With the same notation as in Section 4.3.1 applied to the operator L_A and perturbation tP_U, we can rewrite $L_A + tP_U$ as:

$$L_A + tP_U = \begin{pmatrix} 0 & 0 \\ 0 & H_4 \end{pmatrix},$$

where $H_4 = (L_A + tP_U)_4$ is a bounded invertible operator depending on L_4 and P_U. Note that it will not be sufficient to only include P_4 in the estimates, but we need the operator $\pi_{\mathrm{Ran}L_A + tP_U} P_U|_{(\mathrm{Ker}L_A + tP_U)^\perp}$, where $\pi_{RanL_A+tP_U}$ is the projection onto the range of the operator $L_A + tP_U$. Denote this operator $\pi_{\mathrm{Ran}L_A + tP_U} P_U|_{(\mathrm{Ker}L_A + tP_U)^\perp}$ by $\widetilde{P}_4$. Thus, $H_4 = L_4 + \widetilde{P}_4$. We have by (4.46) the estimate applied to H_4:

$$|||(L_A + tP_U)_4^{-1}||| \leq |||L_4^{-1}||| \frac{1}{1 - t|||L_4^{-1}||| \, \left|\left|\left|\widetilde{P}_4\right|\right|\right|}$$

and hence for some $t_1 \in (0, t_0)$ we obtain for all $t \leq t_1$:

$$\left|\!\left|\!\left|(L_A + tP_U)_4^{-1}\right|\!\right|\!\right|^{-1} \geq \frac{1}{|||L_4^{-1}|||}\left(1 - t\left|\!\left|\!\left|L_4^{-1}\right|\!\right|\!\right|\left|\!\left|\!\left|\widetilde{P}_4\right|\!\right|\!\right|\right) \geq \frac{1}{2|||L_4^{-1}|||}.$$

Since $\exp tU$ is continuous in $t > 0$ and converging to 0 uniformly, then

$$\left\|A^{\exp tU} - A\right\|_{L^n(B^n)} \to 0,$$

as $t \to 0$. Choose $t_2 \in (0, t_1)$ such that

$$\left\|A^{\exp tU} - A\right\|_{L^n(B^n)} \leq \frac{1}{4|||L_4^{-1}|||} \quad \text{and} \quad t|||P_U||| \leq \frac{1}{4|||L_4^{-1}|||}$$

for all $t \leq t_2$. Thus, by the operator norm estimate (4.65) we obtain

$$|||G_{t,U}||| \leq \frac{1}{2|||L_4^{-1}|||} \leq \left|\!\left|\!\left|(L_A + tP_U)_4^{-1}\right|\!\right|\!\right|^{-1}$$

for all $t \leq t_2$. Hence, Lemma 4.3 applied to the operator $L_A + tP_U$ and perturbation $G_{t,U}$ together with the definition of the operator $G_{t,U}$ (4.64) that

$$\text{dimKer}L_{A^g} = \text{dimKer}(L_A + tP_U + G_{t,U}) \leq \text{dimKer}(L_A + tP_U) < \text{dimKer}(L_A),$$

for all $t \leq t_2$. Thus (4.67) holds if we define $g := \exp tU$ for $t \leq t_2$ and from the construction of U, g is unitary and smooth. By (4.66), we can also choose $t > 0$ slightly smaller so that we also guarantee that $\text{index}L_{A^g} = \text{index}L_A = 0$. □

With this result in hand, we are ready to prove the main theorem of this section - that we can find a gauge perturbation g such that L_{A^g} is invertible.

Theorem 4.3. *Let L_A be as in* (4.43)*. There exists a unitary smooth gauge perturbation g such that L_{A^g} is Fredholm and invertible.*

Proof of Theorem 4.3. We know that L_A is Fredholm and by choosing $g = \exp(tU)$, for $t > 0$ sufficiently small, we obtain (4.66): $\text{index}L_A = \text{index}L_{A^g} = 0$. It suffices to prove that L_{A^g} is injective. Let

$$\ell := \text{dimKer}(L_A).$$

If $\ell = 0$ then it suffices to take $g = id$, so let us assume now that $\ell > 0$.

By applying Proposition 4.10 to L_A, we obtain a gauge perturbation g_1 such that

$$\text{dimKer}(L_{A^{g_1}}) < \text{dimKer}(L_A).$$

The operator $L_{A^{g_1}}$ is in addition Fredholm of the same index as L_A. Similarly we can find a gauge perturbation operator g_2 for $L_{A^{g_1}}$, and so on. By iterating the procedure for at most ℓ times and denoting $g := g_1 \cdot \ldots \cdot g_\ell$, we obtain a gauge perturbation that makes the kernel of L_{A^g} trivial. This concludes the proof. □

4.3.4 Local density result in the high energy case

Having built the framework that allows us to find a unitary gauge transformation g such that L_{A^g} is invertible, we can proceed with a very similar procedure as in section 3.4.3 in order to show a local density result that preserves the extended integrability condition (4.5) throughout the sequence. More specifically, we find a sequence of unitary smooth 1-forms A_k converging to A and such that A_k satisfies the extended integrability condition (4.5). Locally on B^{2d} we cannot conclude the integrability condition. This will be done globally in the next section.

We state an analogous result to Proposition 3.8 and refer the reader to its proof. Since the arguments are identical, we will not prove it.

Proposition 4.11. *Let $A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ and a sequence of smooth 1-forms $\tilde{A}_k \to A$ in $W^{1,d}$. Then there exists a gauge $g \in C^\infty(B^{2d}, U(n))$ and $k_0 \in \mathbb{N}$ such that*

(i)

$$Ker L_{A^g} = \{0\} \qquad \text{and} \qquad Ker L_{\tilde{A}_k^g} = \{0\}$$

for all $k \geq k_0$. In particular, the operators $L_{\tilde{A}_k^g}$, $k \geq k_0$ and L_{A^g} are all invertible.

(ii)

$$\sup_{k \geq k_0} \left|\left|\left| L^{-1}_{\tilde{A}_k^g} \right|\right|\right| \leq 2 |||L^{-1}_{A^g}|||.$$

Remark 4.2. *Unlike the proof of Proposition 3.8, the existence of g is now given by theorem 4.3.*

Under the assumption that the norm of $F_A^{0,2}$ is sufficiently small, the next lemma allows us to perturb the 1-form A such that the extended integrability condition (4.5) is satisfied.

Lemma 4.4. *There exists $C', C > 0$ such that for every $A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ such that L_A is invertible and $\left\|F_A^{0,2}\right\|_{L^d} < \frac{C'}{|||L_A^{-1}|||}$, there exists $\omega \in W^{2,d} \cap \mathrm{Dom}(L_A)(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C}))$ such that the extended integrability condition holds:*

$$F_{A+\vartheta\omega}^{0,2} + \vartheta_{A^{0,1}+\vartheta\omega}\overline{\partial}\omega = 0.$$

and

$$\|\omega\|_{W^{2,d}(B^{2d})} \leq C\left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^d(B^{2d})}. \tag{4.68}$$

Proof of Lemma 4.4. Observe the fact that the extended integrability condition can be obtained by solving

$$L_A(\omega) + [\vartheta\omega, \vartheta\omega] + *[*\overline{\partial}\omega, \overline{\vartheta\omega}^T] = -F_A^{0,2},$$

for $\omega \in W^{2,d}(\Omega^{0,2} B^{2d}) \cap \mathrm{Dom}(L_A)$. In order to achieve this, we argue via a fixed point argument using the fact that L_A is invertible. We construct a sequence of $(0,2)$-forms $\omega_0, \cdot, \omega_k, \cdot$ such that:

$$\begin{aligned}
L_A\omega_0 &= -F_A^{0,2} \\
L_A\omega_1 &= -\left([\vartheta\omega_0, \vartheta\omega_0] + *[*\overline{\partial}\omega_0, \overline{\vartheta\omega_0}^T]\right) - F_A^{0,2} \\
L_A\omega_2 &= -\left([\vartheta\omega_1, \vartheta\omega_1] + *[*\overline{\partial}\omega_1, \overline{\vartheta\omega_1}^T]\right) - F_A^{0,2} \\
&\cdots \\
L_A\omega_k &= -\left([\vartheta\omega_{k-1}, \vartheta\omega_{k-1}] + *[*\overline{\partial}\omega_{k-1}, \overline{\vartheta\omega_{k-1}}^T]\right) - F_A^{0,2} \\
&\cdots
\end{aligned}$$

where $\omega_k \in \mathrm{Dom}(L_A)$ for each $k \in \mathbb{N}$. Thus, the boundary conditions are

$$(\omega_k)_N = 0 \quad \text{and} \quad (d\omega_k)_N = 0 \quad \text{on } \partial B^{2d}.$$

Claim 1. *For each $k \in \mathbb{N}$ there holds*

$$\|\omega_k\|_{W^{2,d}(B^{2d})} \leq 2|||L_A||| \left\|F_A^{0,2}\right\|_{L^d(B^{2d})}. \tag{4.69}$$

Since we have assumed that L_A is invertible as a map $W^{2,d} \cap \mathrm{Dom}(L_A) \to L^d$, then we have the identity:

$$\omega_0 = L_A^{-1} L_A \omega_0.$$

Hence, from the definition of the norm of operators, it follows that:

$$\begin{aligned}\|\omega_0\|_{W^{2,d}} &\leq |||L_A^{-1}||| \, \|L_A\omega_0\|_{L^d(B^{2d})} \\ &= |||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})} \\ &< 2|||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})}\end{aligned}$$

Let $k > 0$. By the Sobolev embedding of $W^{1,d} \hookrightarrow L^{2d}$ there exists a constant $C_1 > 0$ so that

$$\|\vartheta\omega_k\|_{L^{2d}(B^{2d})} \leq C_1 \|\vartheta\omega_k\|_{W^{1,d}(B^{2d})} \leq C_1 \|\omega_k\|_{W^{2,d}(B^{2d})}$$

and similarly

$$\|\overline{\partial}\omega_k\|_{L^{2d}(B^{2d})} \leq C_1 \|\omega_k\|_{W^{2,d}(B^{2d})} .$$

Then ω_k satisfies the following estimate:

$$\begin{aligned}\|\omega_k\|_{W^{2,d}(B^{2d})} &\leq |||L_A^{-1}||| \, \|L_A\omega_k\|_{L^d(B^{2d})} \\ &\leq |||L_A^{-1}||| \, \|[\vartheta\omega_{k-1}, \vartheta\omega_{k-1}]\|_{L^d(B^{2d})} \\ &\quad + |||L_A^{-1}||| \, \left\| *[*\overline{\partial}\omega_{k-1}, \overline{\vartheta\omega_{k-1}}^T] \right\|_{L^d(B^{2d})} \\ &\quad + |||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})} \\ &\leq 2C_1|||L_A^{-1}||| \, \|\vartheta\omega_{k-1}\|^2_{L^{2d}(B^{2d})} + |||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})} \\ &\leq 2C_1^2|||L_A^{-1}||| \, \|\omega_{k-1}\|^2_{W^{2,d}} + |||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})}\end{aligned}$$

By inductive hypothesis we have $\|\omega_{k-1}\|_{W^{2,d}} < 2|||L_A^{-1}||| \, \|F_A\|_{L^d(B^{2d})}$. Thus,

$$\|\omega_k\|_{W^{2,d}} \leq 4C_1^2|||L_A^{-1}|||^3 \, \|F_A\|^2_{L^d(B^{2d})} + |||L_A^{-1}||| \, \|F_A^{0,2}\|_{L^d(B^{2d})} .$$

If $1/C' > 4C_1^2$ then

$$\|F_A^{0,2}\|_{L^d(B^{2d})} \leq \frac{C'}{|||L_A^{-1}|||^2} \leq \frac{1}{4C_1^2|||L_A^{-1}|||^2},$$

and this allows to prove (4.69) and conclude the proof of Claim 1.

Claim 2. $\{\omega_k\}_{k=0}^{\infty}$ forms a Cauchy sequence.

Let $k > 0$. It follows that

$$\begin{aligned}
\|\omega_{k+1} - \omega_k\|_{W^{2,d}} \leq & \left|\left|\left| L_A^{-1} \right|\right|\right| \|L_A(\omega_{k+1} - \omega_k)\|_{L^d(B^{2d})} \\
\leq & \left|\left|\left| L_A^{-1} \right|\right|\right| \|[\vartheta(\omega_k - \omega_{k-1}), \vartheta\omega_k]\|_{L^d(B^{2d})} \\
& + \left|\left|\left| L_A^{-1} \right|\right|\right| \|\vartheta\omega_{k-1} \wedge \vartheta(\omega_k - \omega_{k-1})\|_{L^d(B^{2d})} \\
& + \left|\left|\left| L_A^{-1} \right|\right|\right| \left\| *[*(\overline{\partial}\omega_k - \overline{\partial}\omega_{k-1}), \overline{\overline{\partial}^* \omega_k}^T] \right\|_{L^d(B^{2d})} \\
& + \left|\left|\left| L_A^{-1} \right|\right|\right| \left\| *[*\overline{\partial}\omega_k, \overline{\vartheta\omega_k - \vartheta\omega_{k-1}}^T] \right\|_{L^d(B^{2d})} \\
\leq & 8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2 \|F_A\|_{L^d(B^{2d})} \|\omega_k - \omega_{k-1}\|_{W^{2,d}}
\end{aligned}$$

If we choose $C' > 0$ such that $1/C' > 8C_1$ then

$$\left\|F_A^{0,2}\right\|_{L^d(B^{2d})} \leq \frac{C'}{\left|\left|\left| L_A^{-1} \right|\right|\right|^2} < \frac{1}{8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2}.$$

Thus, the factor $8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2 \left\|F_A^{0,2}\right\|_{L^d(B^{2d})}$ is strictly less than 1 and hence, we can conclude that the sequence $\{\omega_k\}_{k=0}^{\infty}$ is Cauchy, proving the claim.

As $W^{2,d}$ is a Banach space and $\{\omega_k\}_{k=0}^{\infty}$ is a Cauchy sequence, there exists ω_∞ such that $\omega_k \to \omega_\infty$ as $k \to \infty$ in $W^{2,d}$. By the strong convergence of ω_k and (4.69), we obtain that $\|\omega_\infty\|_{W^{2,d}(B^{2d})} \leq 2\left|\left|\left| L_A^{-1} \right|\right|\right| \|F_A\|_{L^d(B^{2d})}$ and

$$L_A(\omega_\infty) + [\vartheta\omega, \vartheta\omega] + *[*\overline{\partial}\omega, \overline{\vartheta\omega}^T] = -F_A^{0,2}.$$

Thus, we have obtained the extended integrability condition:

$$F_{A+\vartheta\omega}^{0,2} + \vartheta_{A^{0,1}+\vartheta\omega}\overline{\partial}\omega = 0.$$

□

Using the above arguments, we can prove the local density result:

Theorem 4.4. *Let $A \in W^{1,d}(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$, with $F_A^{0,2} = 0$. There exists a sequence of smooth 1-forms $A_k \in C^\infty(\Omega^1 B^{2d} \otimes \mathfrak{u}(n))$ and $\omega_k \in C^\infty(\Omega^{0,2} B^{2d} \otimes M_n(\mathbb{C})$ such that the extended integrability condition is satisfied for all k:*

$$F_{A_k}^{0,2} + \vartheta_{A_k^{0,1}}\overline{\partial}\omega_k = 0$$

and $A_k \to A$ *in* $W^{1,d}$, $\omega_k \to 0$ *in* $W^{2,d}$.

Proof of Theorem 4.4. By Theorem 4.3 there exists a gauge change $g \in C^\infty(B^{2d}, U(n))$ so that L_{A^g} is invertible $W^{2,d} \cap \mathrm{Dom}(L_A)$ to L^d. We can obtain a sequence A_k of smooth $\mathfrak{u}(n)$-valued 1-forms such that $\left\| F^{0,2}_{\tilde{A}_k} \right\|_{L^d(B^{2d})} \to 0$ and $\tilde{A}_k \to A$ in $W^{1,d}$ as $k \to \infty$. By Proposition 4.11(i) there exists k_0 so that for all $k \geq k_0$, $L_{\tilde{A}^g_k}$ is invertible. Since g is a smooth gauge change independent of k, it also follows that $\tilde{A}^g_k \to A^g$ in $W^{1,d}$ as $k \to \infty$.

Moreover, by Proposition 4.11(ii) we know that

$$\sup_{k \geq k_0} \left|\left|\left| L^{-1}_{\tilde{A}^g_k} \right|\right|\right| \leq 2 \left|\left|\left| L^{-1}_{A^g} \right|\right|\right|.$$

Because $L_{A^g} L^{-1}_{A^g} = id$, then $\left|\left|\left| L_{A^g} L^{-1}_{A^g} \right|\right|\right| = 1$. It follows that

$$1 \leq \left|\left|\left| L_{A^g} \right|\right|\right| \left|\left|\left| L^{-1}_{A^g} \right|\right|\right|$$

and hence, $\left|\left|\left| L^{-1}_{A^g} \right|\right|\right|^{-1} \leq \left|\left|\left| L_{A^g} \right|\right|\right|$. Using this inequality, we can choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \frac{1}{C'} \left|\left|\left| L^{-1}_{\tilde{A}^g_k} \right|\right|\right|^{-2} \leq \frac{1}{C'} \left|\left|\left| L_{\tilde{A}^g_k} \right|\right|\right|^{2},$$

uniformly in k, where C' is the constant given by Lemma 4.4. Hence, we can pick k_1 large enough such that $\left\| F_{\tilde{A}^g_k} \right\|_{L^d(B^{2d})} < \varepsilon_0$ for all $k \geq k_1$. For each $k \geq k_1$, Lemma 4.4 gives the existence of 2-forms ω_k that satisfy

$$\|\omega_k\|_{W^{2,d}(B^{2d})} \leq C \left|\left|\left| L^{-1}_{\tilde{A}^g_k} \right|\right|\right| \left\| F_{\tilde{A}^g_k} \right\|_{L^d(B^{2d})} \leq 2C \left|\left|\left| L^{-1}_{A^g} \right|\right|\right| \left\| F_{\tilde{A}_k} \right\|_{L^d(B^{2d})}$$

and the extended integrability condition

$$F^{0,2}_{\tilde{A}^g_k + \vartheta\omega_k} + \vartheta_{(\tilde{A}^g_k)^{0,1} + \vartheta\omega_k} \overline{\partial}\omega_k = 0 \tag{4.70}$$

Since $\left\| F_{\tilde{A}_k} \right\|_{L^d(B^{2d})}$ converges strongly to 0, the estimates on the 2-forms ω_k give $\omega_k \to 0$ in $W^{2,d}$ as $k \to \infty$. Thus, we obtain the strong convergence

$$\tilde{A}^g_k + \vartheta\omega_k - \overline{\vartheta\omega_k}^T \to A^g \text{ in } W^{1,d}.$$

The connection forms

$$A_k := \left(\tilde{A}^g_k + \vartheta\omega_k - \overline{\vartheta\omega_k}^T \right)^{g^{-1}} \in C^\infty(\Omega^{0,1} B^{2d} \otimes \mathfrak{u}(n))$$

are uniformly bounded in $W^{1,d}$ and $A_k^g \to A^g$ in $W^{1,d}$ automatically. In particular, we obtain

$$\|A_k^g - A^g\|_{L^d} \to 0 \iff \left\|g^{-1}(A_k - A)\,g\right\|_{L^d} \to 0.$$

Moreover, since g is a smooth gauge change, we have the following estimates:

$$\begin{aligned}\|A_k - A\|_{L^d} &= \left\|gg^{-1}(A_k - A)\,gg^{-1}\right\|_{L^d}\\ &\le \|g\|_{L^\infty}\left\|g^{-1}(A_k - A)\,g\right\|_{L^d}\left\|g^{-1}\right\|_{L^\infty}.\end{aligned}$$

Hence, $\|A_k - A\|_{L^d} \to 0$. Similarly, we can prove the strong convergence of their gradients, and hence $A_k \to A$ in $W^{1,d}$. In addition, the smooth sequence A_k satisfies the extended integrability condition:

$$F^{0,2}_{A_k} + \vartheta_{A_k^{0,1}}\overline{\partial}\omega_k = 0$$

by construction of ω_k. □

4.3.5 Global density result

Using the results in the previous section, we show that they can be immediately generalised and applied to the whole closed Kähler manifold X^d. We conclude that the extended integrability condition (4.5) over X^d implies the integrability condition (1.1) by Proposition 4.1. This observation is be possible since X^d does not have any boundary. We will work under the assumption that the $(0,2)$ Dolbeault cohomology group vanishes $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$. This is equivalent to saying that $H^2(X^d, \mathcal{O}) = 0$, where $\mathcal{O}$ is the sheaf of holomorphic functions [12, Dolbeault Theorem, p. 45]. We use similar arguments as in Section 3.4.4.

We work on sections of the vector bundle (E, h_0) over X^d. The $\overline{\partial}$ operator over X^d is well-defined and acts on the space of E-valued (p,q)-forms:

$$\overline{\partial} : \mathcal{A}^{p,q}(X^d) \to \mathcal{A}^{p,q+1}(X^d).$$

Its corresponding dual operator, $\overline{\partial}^*$, is defined as a map:

$$\overline{\partial}^* : \mathcal{A}^{p,q}(X^d) \to \mathcal{A}^{p,q-1}(X^d).$$

On the space $\mathcal{A}^{p,q}(X^d)$ the $\overline{\partial}$-Hodge theorem gives the L^d decomposition:

$$\mathcal{A}^{p,q}(X^d) = \overline{\partial}\mathcal{A}^{p,q-1}(X^d) + \overline{\partial}^*\mathcal{A}^{p,q+1}(X^d) + \mathcal{H}^{p,q}(X^d), \tag{4.71}$$

where $\mathcal{H}^{p,q}(X^d)$ is the space of holomorphic (p,q)-sections. Since X^d is a closed Kähler manifold, then we remark that $\mathcal{H}^{p,q}(X^d)$ is finite dimensional. In particular, by (4.71) any $(0,2)$-form $\omega \in \mathcal{A}^{0,2}(X^d)$ over X^d can be decomposed as follows:

$$\omega = \overline{\partial}\overline{\partial}^*\alpha + +\overline{\partial}^*\overline{\partial}\alpha,$$

under the assumption that $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$.

Recall that since $\overline{\partial}$ and $\overline{\partial}^*$ define elliptic complexes over closed Kähler surfaces (see for example [22, Chapter IV]) then the operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is elliptic on $(0,2)$-sections over closed Kähler manifolds. In particular it is Fredholm and moreover, $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is self-adjoint. Thus, its Fredholm index vanishes.

We globalise our operator L_A from the previous sections as such:

$$L_\nabla : \Gamma_{W^{2,d}}(\mathcal{A}^{0,2}(X^d)) \to \Gamma_{L^d}(\mathcal{A}^{0,2}(X^d))$$

where ∇ is a $W^{1,d}$ unitary connection over X^d and $\Gamma_{W^{p,q}}$ is the space of sections with $W^{p,q}$ regularity. We can directly apply theorem 4.3 to obtain the existence of a smooth section g so that L_∇ is Fredholm and invertible.

Remark 4.3. *Since X^d is a closed Kähler manifold, we trivially have that $\omega, \overline{\partial}\omega \in \mathrm{Dom}\overline{\partial}^*$. Thus, ϑ and $\overline{\partial}^*$ coincide.*

It follows that we can apply Lemma 4.4 to ∇ and obtain similarly to Theorem 4.4:

Lemma 4.5. *Let ∇ a $W^{1,d}$ unitary connection over X^d with $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$, satisfying the integrability condition*

$$F^{0,2}_\nabla = 0.$$

Then there exists a sequence of smooth unitary connections ∇_k, and smooth $(0,2)$ sections ω_k satisfying the extended integrability condition

$$F^{0,2}_{\nabla_k} + \overline{\partial}^*_{\nabla_k^{0,1}}\overline{\partial}\omega_k = 0$$

such that

$$\mathrm{dist}_d(\nabla_k, \nabla) \to 0 \quad \textit{and} \quad \omega_k \to 0 \ \textit{in}\ \Gamma_{W^{1,d}}.$$

It remains to show that this lemma implies that the integrability condition (1.1) is satisfied for each connection ∇_k.

Theorem 4.5. *Let ∇ a $W^{1,d}$ unitary connection over X^d with $\mathcal{H}^{0,2}_{\overline{\partial}}(X^d) = 0$, satisfying the integrability condition*

$$F^{0,2}_{\nabla} = 0.$$

Then there exists a sequence of smooth unitary connections ∇_k satisfying the integrability condition

$$F^{0,2}_{\nabla_k} = 0$$

such that

$$\mathrm{dist}_d(\nabla_k, \nabla) \to 0.$$

Proof of Theorem 4.5. From Lemma 4.5 there exists a sequence of smooth unitary connections ∇_k, and smooth $(0,2)$ sections ω_k satisfying the extended integrability condition

$$F^{0,2}_{\nabla_k} + \overline{\partial}^*_{\nabla^{0,1}_k} \overline{\partial}\omega_k = 0$$

such that

$$\mathrm{dist}_d(\nabla_k, \nabla) \to 0 \quad \text{and} \quad \omega_k \to 0 \text{ in } \Gamma_{W^{1,d}}.$$

Note that since X^d does not have any boundary then $\overline{\partial}\omega_k \in \mathrm{Dom}(\overline{\partial}^*)$. Hence, by applying the earlier Proposition 4.1 to ∇_k, we obtain the integrability condition:

$$F^{0,2}_{\nabla_k} = 0.$$

□

4.4 Proof of Theorems 1.3 and 1.4

We have obtained complete analogous results of Theorem 3.2 and Theorem 3.4 in the setting of Banach spaces see: Theorem 4.1 and Theorem 4.5. Then the proofs of Theorem 1.3 and Theorem 1.4 are completely analogous to the proofs of Theorem 1.1 and Theorem 1.2. Thus, we refer the reader to Sections 3.3 and 3.5.

Chapter 5

Weak Flat Connections

Through our work we have worked on several techniques that deal the issues of the invertibility of Fredholm operators. We have realised that these can tackle other problems as well. In this Chapter we prove one of them.

Let B^n be the n-dimensional unit ball. For a compact Lie group G and its associated Lie algebra $\mathfrak{g}$, theorem 1.5 holds and it can be easily proven with results from the literature (see below):

Theorem 1.5. *A is a $\mathfrak{g}$-valued 1-form in $W^{1,n/2}$ over the ball B^n, and we assume that $F_A = dA + A \wedge A = 0$. Then there exist a sequence of smooth 1-forms A_k with $k \in \mathbb{N}$ such that*

$$A_k \to A \qquad \text{in } W^{1,n/2},$$

furthermore satisfying

$$F_{A_k} = 0 \quad \text{for all } k \in \mathbb{N}.$$

This result can be proven using the same approach as in [4, Lemma 3]. Indeed, from the flatness condition $F_A = 0$ and the compactness of the gauge group G, it is shown that $A = u^{-1}du$ for u a G-valued gauge transformation. Since u can be bootstrapped to $W^{2,n/2}$ regularity, by [6] or [33] there exists a sequence of smooth G-valued maps u_k such that $u_k \to u$ in $W^{2,n/2}$. Thus, we can define $A_k := u_k^{-1}du_k$.

Note that the technique used in [4, Lemma 3] requires only L^2 regularity of A in n-dimensions in order to get the existence of $u \in W^{1,2}$. Another way of obtaining u is using the Uhlenbeck gauge extraction method for $W^{1,n/2}$ forms [41]. This gives us a gauge $u \in W^{2,n/2}$ such that for a constant $C > 0$

we have $\|A^u\|_{W^{1,n/2}} \le C\,\|F_A\|_{L^{n/2}}$ and $d^*A^u = 0$. However, since $F_A = 0$, then $A^u = 0$ and consequently $A = u^{-1}du$.

All the above techniques rely on the compactness of the group G in order to be able to show the density result. In this chapter we propose a more general approach to this result that does not rely on the compactness of the Lie group. Moreover, we will conclude the chapter with an open problem. The strategy we will use relies heavily on the techniques presented in Section 4.3.

Hence, we will assume from now on that G is a Lie group not necessarily compact inducing a matrix-valued Lie algebra $\mathfrak{g} \subset M_r(\mathbb{R})$. The reader can notice that, unlike in Chapter 4, we can prove the result on the bounded domain B^n. This is possible due to the ellipticity of the operator d.

5.1 Strategy

We describe the strategy of proving Theorem 1.5. The first idea is to mollify A in order to obtain a sequence of smooth 1-forms $\tilde{A}_k$. Note that $F_{\tilde{A}_k}$ does not necessarily vanish, and instead we find a perturbation by $d^*\omega_k$ such that

$$F_{\tilde{A}_k + d^*\omega_k} = 0,$$

where ω_k is a sequence of 2-forms. In particular, due to Cartan formula $F_A = dA + \frac{1}{2}[A, A]$ and to the formula $[\omega, \tau] = \omega \wedge \tau + \tau \wedge \omega$ valid for 1-forms ω, τ, the above translates into the following PDE for ω_k:

$$dd^*\omega_k + [\tilde{A}_k, d^*\omega_k] + d^*\omega_k \wedge d^*\omega_k = -F_{\tilde{A}_k}. \tag{5.1}$$

The leading-order operator dd^* in the above PDE is not elliptic. In particular, it is not Fredholm. We note however that in general there holds the Bianchi identity $d_A F_A = 0$, where

$$d_A\beta := d\beta + [A, \beta] \quad \text{for all } \mathfrak{g}\text{-valued } p\text{-forms } \beta.$$

In order to complete the symbol of (5.1), note that the linear part of (5.1) reads $d_{\tilde{A}_k} d^*\omega_k$ and thus we can complete it in a canonical way to an elliptic (self-adjoint with respect to the L^2-scalar product) operator by adding the term $dd^*_{\tilde{A}_k}$.

We are now helped in the computations by the fact that so-called "Bianchi identity" $d_A F_A = d_A(dA + \frac{1}{2}[A, A]) = 0$ holds for any A. This follows directly from the formula

$$[A, B] = A \wedge B + (-1)^{k+p} B \wedge A \quad \text{if } A \text{ is a } p\text{-form and } B \text{ is a } q\text{-form.} \tag{5.2}$$

The Bianchi identity implies $\langle d_A^* \tau, F_A \rangle = \langle \tau, d_A F_A \rangle = 0$ for any 3-form τ, where by Proposition C.4 we have

$$d_A^* \tau := d^* \tau + *[*\tau, A^T].$$

Then we find the **extended flatness condition**

$$F_A + d_A^* \tau = 0. \tag{5.3}$$

Under appropriate boundary condition on τ this is then actually equivalent to $F_A = d_A^* \tau = 0$.

Applying the above with $A \mapsto \tilde{A}_k + d^* \omega_k$ and $\tau \mapsto d\omega_k$, we find the equation

$$dd^*\omega_k + d^*d\omega_k + [\tilde{A}_k, d^*\omega_k] + *[*d\omega_k, \tilde{A}_k^T] + d^*\omega_k \wedge d^*\omega_k + *[*d\omega_k, (d^*\omega_k)^T] = -F_{\tilde{A}_k}. \tag{5.4}$$

Now the linear part of equation (5.4) is formally self-adjoint and we pass to investigate its Fredholmness. To do that we define the operator

$$L_A : W^{2,n/2} \cap \mathrm{Dom}(L_A)(\Omega^2 B^n \otimes \mathfrak{g}) \to L^{n/2}(\Omega^2 B^n \otimes \mathfrak{g})$$

by

$$L_A := dd^* + d^*d + [d^*, A] + *[*d, A^T], \tag{5.5}$$

where $\mathrm{Dom}(L_A) = \{\omega : \omega_N = 0,\ (d\omega)_N = 0 \text{ on } \partial B^n\}$, where ω_N are forms with components involving dr. In particular over the space $\mathrm{Dom}(L_A)$ we have $\mathrm{Ker}(dd^* + d^*d) = \{0\}$.

Remark 5.1. *We will show that on the space* $W^{2,n/2}(\Omega^2 B^n \otimes \mathfrak{g}) \cap \mathrm{Dom}(L_A)$ *we obtain that* $\mathrm{index} L_A = 0$ *and*

$$\mathrm{index} L_A = \mathrm{index}(d^*d + d^*d) = 0.$$

Going back to our initial problem, (5.4) reduces to:

$$L_{\tilde{A}_k} + d^*\omega_k \wedge d^*\omega_k + *[*d\omega_k, (d^*\omega_k)^T] = -F_{\tilde{A}_k},$$

where, as we have shown above, $L_{\tilde{A}_k}$ is a 0-index operator. This equation cannot be solved using a fixed point argument due to lack of invertibility of L_A. However, using the Fredholm-ness of the operator and the fact that it has 0 index, we show that under smooth gauge perturbations g, the operator L_{A^g} is invertible. Thus, using all these pieces of information, we will solve

$$L_{\tilde{A}_k^g} + d^*\omega_k \wedge d^*\omega_k + *[*d\omega_k, (d^*\omega_k)^T] = -F_{\tilde{A}_k^g},$$

and from this we will prove the theorem.

5.2 Invertibility of L_A

Since L_A is a Fredholm operator, we can immediately assume the results of Section 4.3.1. In particular, Proposition 4.6 enables us to decrease the kernel of L_A by applying perturbations to it. Since the strategy is analogous, then it is enough to adapt our arguments from Section 4.3.3. We only replace Propositions 4.9 and 4.10, and then Theorem 4.3 will follow. We set up the perturbation problem in our setting.

Consider the family of G-valued gauges $g_t = \exp tU$, $t = [0, 1]$. We can analytically expand A^{g_t} in $t \in [0, 1]$ to obtain

$$A^{g_t} - A = t(dU + [U, A]) + O(t^2). \tag{5.6}$$

Moreover, it will be useful to define the bounded linear operators

$$G_{t,U} : W^{2,n/2}(\Omega^2 B^n \otimes \mathfrak{g}) \to L^{n/2}(\Omega^2 B^n \otimes \mathfrak{g})$$

given by

$$\begin{aligned} G_{t,U} :=\, & [A^{g_t} - A - t(dU + [U, A]), d^*] \\ & + *[*d, (A^{g_t} - A - t(dU + [U, A]))^T], \end{aligned} \tag{5.7}$$

and

$$P_U : W^{2,n/2}(\Omega^2 B^n \otimes \mathfrak{g}) \to L^{n/2}(\Omega^2 B^n \otimes \mathfrak{g})$$

given by

$$P_U = [dU + [U, A], d^*] + *[*d, (dU + [U, A])^T].$$

Observe that

$$L_{A^{g_t}} = L_A + [A^{g_t} - A, d^*] + *[*d, (A^{g_t} - A)^T]$$

and we can write $G_{t,U}$ as $G_{t,U} = L_{A^{g_t}} - L_A + tP_U$. It follows that

$$|||G_{t,U}||| \leq C \, \|A^{g_t} - A\|_{L^n(B^n)} + t|||P_U|||, \tag{5.8}$$

for some constant $C > 0$. Moreover, there exists t_0 such that for all $t \leq t_0$, we have the operator

$$[A^{g_t} - A, d^*] + *[*d, (A^{g_t} - A)^T]$$

is small in operator norm mapping $W^{2,n/2}(\Omega^2 B^n \otimes \mathfrak{g})$ to $L^{n/2}(\Omega^2 B^n \otimes \mathfrak{g})$ such that (see [32]):

$$\text{index} L_{A^{g_t}} = \text{index} L_A = 0. \tag{5.9}$$

As we have seen before it is enough to find a perturbation P_U that is injective on the kernel L_A in order to decrease the kernel size. Thus, we prove the analogous result of Proposition 4.9 in our setting.

Proposition 5.1. *There exists $P = P_U$ with the same domain and range as L_A and of the form $P_U(\eta) := [d^*\eta, dU + [U, A]] + *[*d\eta, (dU + [U, A])^T]$ for U smooth, such that P is injective on* $\text{Ker} L_A$.

Proof of Proposition 5.1. Since L_A is Fredholm, $\text{Ker} L_A$ is a finite dimensional space. Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of $\text{Ker} L_A$.

Claim 1. *For each $v \in \text{Ker} L_A$, $v \neq 0$, there exists $U_v \in C^\infty(B^n, \mathfrak{g})$ such that $P_{U_v} v \neq 0$.*

Let $v \in KerL_A$, $v \neq 0$. We show that for each such v, we can find U_v such that $P_{U_v} v \neq 0$. Assume by contradiction that $P_U v = 0$ for all smooth $\mathfrak{g}$-valued maps U on B^n. Define the linear operators

$$H_0 : C^\infty(B^n, \mathfrak{g}) \to C^\infty(\Omega^2 B^n \otimes \mathfrak{g})$$

given by

$$H_0(\omega) := [[\omega, A], d^*v] + *[*dv, ([\omega, A])^T]$$

and

$$H_1 : C^\infty(\Omega^1 B^n \otimes \mathfrak{g}) \to C^\infty(\Omega^2 B^n \otimes \mathfrak{g}).$$

given by

$$H_1(\omega) := [\omega, d^*v] + *[*dv, \omega^T]$$

We have $H_0(\omega(x)) = H_0(\omega)(x)$, where $\omega(x)$ is identified with a constant form. Moreover,

$$0 = P_U v = [d^*v, dU + [U, A]] + *[*dv, (dU + [U, A])^T] = H_1(dU) + H_0(U)$$

for all smooth maps U on B^n. Applying Proposition C.2 to H_1 and H_0, we obtain that $H_1 \circ d = 0$ and $H_0 = 0$. In particular, taking into account that $(dU)^T = d(U^T)$, we have obtained that for all U smooth maps on B^n,

$$H_1(dU) = [dU, d^*v] + *[*dv, (dU)^T] = 0.$$

Define the linear operator

$$H(\omega) := (H_1 \circ d)(\omega) : C^\infty(\Omega^1 B^n \otimes \mathfrak{g}) \to C^\infty(\Omega^2 B^n \otimes \mathfrak{g}).$$

Thus, we have

$$0 = [dU, d^*v] + *[*dv, d(U^T)] = H(dU)$$

for all smooth maps U on B^n. Applying Proposition C.1 to H, we obtain that $H = 0$. By density of smooth 1-forms into $W^{1,n/2}$ 1-forms, it follows in particular that

$$H(A) = [A, d^*v] + *[*dv, A^T] = 0.$$

Putting this together with the fact that $v \in \mathrm{Ker} L_A$, we obtain:

$$0 = L_A v = (dd^* + d^*d)v.$$

Since the kernel of L_A is empty when $v \in \mathrm{Dom}(L_A)$, then $v = 0$. This is a contradiction with the fact that $v \neq 0$. Hence, there exists $U_v \in C^\infty(B^n, \mathfrak{g})$ so that $P_{U_v} v \neq 0$, as claimed.

Claim 2. *There exists U smooth $\mathfrak{g}$-valued map such that P_U is injective on* $\mathrm{Ker} L_A$.

The proof of this claim follows identically as in Proposition 4.7. □

Proposition 5.2. *Let L_A be defined as in* (5.5) *and assume that L_A is not injective. There exists a smooth gauge perturbation g such that*

$$\mathrm{dimKer}(L_{A^g}) < \mathrm{dimKer}(L_A) \tag{5.10}$$

and $\mathrm{index} L_{A^g} = \mathrm{index} L_A = 0$.

Proof of Proposition 5.2. By Proposition 5.1, there exists a 1-form U such that the operator $P_U = [d^*, dU + [U, A]] + *[*d, (dU + [U, A])^T]$ is injective on $\mathrm{Ker} L_A$. Moreover, there exists t_0 such that for $t \leq t_0$, we can assume that $P_{tU} = tP_U$ is small in operator norm such that Proposition 4.6 applies to L_A

and tP. Then $L_A + tP_U$ is Fredholm and

$$\dim\mathrm{Ker}(L_A + tP_U) < \dim\mathrm{Ker}(L_A).$$

The rest of the proof follows exactly as in Proposition 4.10 and by the operator norm estimate (5.8) we obtain

$$|||G_{t,U}||| \leq \frac{1}{2|||L_4^{-1}|||} \leq \left|\left|\left|(L_A + tP_U)_4^{-1}\right|\right|\right|^{-1}$$

for all $t \leq t_2$, with the notation as in Section 4.3.1. Hence, Lemma 4.3 applied to the operator $L_A + tP_U$ and perturbation $G_{t,U}$ together with the definition of the operator $G_{t,U}$ (5.7) that

$$\dim\mathrm{Ker}L_{A^g} = \dim\mathrm{Ker}(L_A+tP_U+G_{t,U}) \leq \dim\mathrm{Ker}(L_A+tP_U) < \dim\mathrm{Ker}(L_A),$$

for all $t \leq t_2$. Thus (5.10) holds, if we define $g := \exp tU$ for $t \leq t_2$. By (5.9), we can also choose $t > 0$ slightly smaller so that we also guarantee that $\mathrm{index}L_{A^g} = \mathrm{index}L_A = 0$. □

5.3 Density for non-compact groups G

Having obtained the results above results, we are now able to show theorem 1.5 under non-compactness of the Lie group G.

We start the section by showing that if we satisfy the extended flatness condition, under the boundary conditions imposed by the space $\mathrm{Dom}(L_A)$ we recover the flatness condition.

Proposition 5.3. *Let $\omega \in W^{2,n/2} \cap \mathrm{Dom}(L_A)(\Omega^2 B^n \otimes \mathfrak{g})$ be a solution to the PDE:*

$$L_A(\omega) + d^*\omega \wedge d^*\omega + *[*d\omega, (d^*\omega)^T] = -F_A, \tag{5.11}$$

which is equivalent to ω satisfying the extended flatness condition:

$$F_{A+d^*\omega} + d^*_{A+d^*\omega} d\omega = 0.$$

Then

$$F_{A+d^*\omega} = 0 \qquad \text{and} \qquad d^*_{A+d^*\omega} d\omega = 0.$$

Remark 5.2. *Since* $\omega \in \mathrm{Dom}(L_A)$ *then* $\omega \in \mathrm{Dom}(d^*)$ *and* $d\omega \in \mathrm{Dom}(d^*)$. *Thus, the terms* $d^*_{A+d^*\omega}d\omega$ *and* $F_{A+d^*\omega}$ *are well-defined.*

Proof of Proposition 5.3. We expand L_A in the PDE (5.11):

$$dd^*\omega + d^*d\omega + [A, d^*\omega] + *[*d\omega, A^T] + d^*\omega \wedge d^*\omega + *[*d\omega, (d^*\omega)^T] = -F_A.$$

By rearranging the terms, we obtain:

$$\underbrace{F_A + dd^*\omega + [A, d^*\omega] + d^*\omega \wedge d^*\omega}_{=F_{A+d^*\omega}} + \underbrace{d^*d\omega + *[*d\omega, T] + *[*d\omega, (d^*\omega)^T]}_{=d^*_{A+d^*\omega}d\omega} = 0.$$

Hence, (5.3) translates to the extended flatness condition:

$$F_{A+d^*\omega} + d^*_{A+d^*\omega}d\omega = 0. \tag{5.12}$$

Note that by the second Bianchi identity $d_AF_A = 0$ we have $F_{A+d^*\omega} \in \mathrm{Ker}d_{A+d^*\omega}$. Moreover, the equation above gives us that $d^*_{A+d^*\omega}d\omega \in \mathrm{Ker}d_{A+d^*\omega}$. Clearly we also have

$$d^*_{A+d^*\omega}d\omega \in \overline{\mathrm{Im}d^*_{A+d^*\omega}}^{L^{n/2}} = \left(\mathrm{Ker}d_{A+d^*\omega}\right)^{\perp},$$

where the set equality holds by the Closed Image Theorem. Thus, because

$$d^*_{A+d^*\omega}d\omega \in \mathrm{Ker}d_{A+d^*\omega} \quad \text{and} \quad d^*_{A+d^*\omega}d\omega \in \left(\mathrm{Ker}d_{A+d^*\omega}\right)^{\perp},$$

we obtain $d^*_{A+d^*\omega}d\omega = 0$. Moreover, since (5.12) holds, then we also get that $F_{A+d^*\omega} = 0$. This concludes the proof. □

We restate without proof the analogue of Proposition 3.6 in our setting:

Proposition 5.4. *Let* $A \in W^{1,n/2}(\Omega^1 B^n \otimes \mathfrak{g})$ *and a sequence of smooth* 1-*forms* $\tilde{A}_k \to A$ *in* $W^{1,n/2}$. *Then there exists a gauge* $g \in C^\infty(B^n, G)$ *and* $k_0 \in \mathbb{N}$ *such that*

(i)

$$KerL_{A^g} = \{0\} \qquad \text{and} \qquad KerL_{\tilde{A}^g_k} = \{0\}$$

for all $k \geq k_0$. *In particular, the operators* $L_{\tilde{A}^g_k}$ *and* L_{A^g} *are all invertible.*

(ii)

$$\sup_{k\geq k_0} \left|\left|\left| L^{-1}_{\tilde{A}^g_k} \right|\right|\right| \leq 2 |||L^{-1}_{A^g}|||.$$

Similarly as Lemma 4.4, we show that for small enough curvature form F_A, we can perturb A such that we satisfy the extended flatness condition (5.3).

Lemma 5.1. *There exists $C', C > 0$ such that for every $A \in L^n(\Omega^1 B^n \otimes \mathfrak{g})$ such that L_A is invertible and $\|F_A\|_{L^{n/2}} < \frac{C'}{|||L_A^{-1}|||}$, there exists $\omega \in W^{2,n/2} \cap$* $\mathrm{Dom}(L_A)(\Omega^2 B^n \otimes \mathfrak{g})$ *such that the extended flatness condition is satisfied:*

$$F_{A+d^*\omega} + d^*_{A+d^*\omega} d\omega = 0$$

and

$$\|\omega\|_{W^{2,n/2}(B^n)} \leq C \left|\left|\left| L_A^{-1} \right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}. \tag{5.13}$$

Proof of Lemma 5.1. We construct the following sequence of solutions:

$$\begin{aligned}
L_A\omega_0 &= -F_A \\
L_A\omega_1 &= -d^*\omega_0 \wedge d^*\omega_0 - *[*d\omega_0, (d^*\omega_0)^T] - F_A \\
L_A\omega_2 &= -d^*\omega_1 \wedge d^*\omega_1 - *[*d\omega_1, (d^*\omega_1)^T] - F_A \\
&\dots \\
L_A\omega_k &= -d^*\omega_{k-1} \wedge d^*\omega_{k-1} - *[*d\omega_{k-1}, (d^*\omega_{k-1})^T] - F_A \\
&\dots
\end{aligned}$$

where each $\omega_k \in \mathrm{Dom}(L_A)$, i.e. it satisfies the boundary conditions:

$$(\omega_k)_N = 0 \quad \text{and} \quad (d\omega_k)_N = 0 \quad \text{on } \partial B^n.$$

Claim 1. For each $k \in \mathbb{N}$ there holds

$$\|\omega_k\|_{W^{2,n/2}} \leq 2|||L_A||| \, \|F_A\|_{L^{n/2}(B^n)}. \tag{5.14}$$

Since L_A is invertible, then we have the identity:

$$\omega_0 = L_A^{-1} L_A \omega_0.$$

Hence, from the definition of the norm of operators, it follows that:

$$\|\omega_0\|_{W^{2,n/2}} \leq \left|\left|\left| L_A^{-1} \right|\right|\right| \|L_A\omega_0\|_{L^{n/2}(B^n)}$$

$$= \left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}$$
$$< 2\left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}$$

Let $k > 0$. By the Sobolev embedding of $W^{1,n/2} \hookrightarrow L^n$ there exists a constant $C_1 > 0$ so that

$$\|d^*\omega_k\|_{L^n(B^n)} \leq C_1 \|d^*\omega_k\|_{W^{1,n/2}(B^n)} \leq C_1 \|\omega_k\|_{W^{2,n/2}} .$$

Then we have:

$$\begin{aligned}
\|\omega_k\|_{W^{2,n/2}} &\leq \left|\left|\left|L_A^{-1}\right|\right|\right| \|L_A\omega_k\|_{L^{n/2}(B^n)} \\
&\leq \left|\left|\left|L_A^{-1}\right|\right|\right| \|d^*\omega_{k-1} \wedge d^*\omega_{k-1}\|_{L^{n/2}(B^n)} \\
&\quad + \left|\left|\left|L_A^{-1}\right|\right|\right| \left\|*[*d\omega_{k-1}, (d^*\omega_{k-1})^T]\right\|_{L^{n/2}(B^n)} + \left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)} \\
&\leq 2C_1\left|\left|\left|L_A^{-1}\right|\right|\right| \|d^*\omega_{k-1}\|^2_{L^n(B^n)} + \left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)} \\
&\leq 2C_1^2\left|\left|\left|L_A^{-1}\right|\right|\right| \|\omega_{k-1}\|^2_{W^{2,n/2}} + \left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}
\end{aligned}$$

By inductive hypothesis we have $\|\omega_{k-1}\|_{W^{2,n/2}} < 2\left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}$. Thus,

$$\|\omega_k\|_{W^{2,n/2}} \leq 4C_1^2\left|\left|\left|L_A^{-1}\right|\right|\right|^3 \|F_A\|^2_{L^{n/2}(B^n)} + \left|\left|\left|L_A^{-1}\right|\right|\right| \|F_A\|_{L^{n/2}(B^n)} .$$

If $1/C' > 4C_1^2$ then

$$\|F_A\|_{L^{n/2}(B^n)} \leq \frac{C'}{\left|\left|\left|L_A^{-1}\right|\right|\right|^2} \leq \frac{1}{4C_1^2\left|\left|\left|L_A^{-1}\right|\right|\right|^2},$$

and this allows to prove (5.14) and conclude the proof of Claim 1.

Claim 2. $\{\omega_k\}_{k=0}^{\infty}$ forms a Cauchy sequence.

Let $k > 0$. It follows that

$$\begin{aligned}
\|\omega_{k+1} - \omega_k\|_{W^{2,n/2}} &\leq \left|\left|\left|L_A^{-1}\right|\right|\right| \|L_A(\omega_{k+1} - \omega_k)\|_{L^{n/2}(B^n)} \\
&\leq \left|\left|\left|L_A^{-1}\right|\right|\right| \|d^*(\omega_k - \omega_{k-1}) \wedge d^*\omega_k\|_{L^{n/2}(B^n)} \\
&+ \left|\left|\left|L_A^{-1}\right|\right|\right| \|d^*\omega_{k-1} \wedge d^*(\omega_k - \omega_{k-1})\|_{L^{n/2}(B^n)} \\
&+ \left|\left|\left|L_A^{-1}\right|\right|\right| \left\|*[*(d\omega_k - d\omega_{k-1}), (d^*\omega_k)^T]\right\|_{L^{n/2}(B^n)} \\
&+ \left|\left|\left|L_A^{-1}\right|\right|\right| \left\|*[*d\omega_k, (d^*\omega_k - d^*\omega_{k-1})^T]\right\|_{L^{n/2}(B^n)}
\end{aligned}$$

$$\le 8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2 \|F_A\|_{L^{n/2}(B^n)} \|\omega_k - \omega_{k-1}\|_{W^{2,n/2}}$$

If $C' > 0$ is such that $1/C' > 8C_1$ then

$$\|F_A\|_{L^{n/2}(B^n)} \le \frac{C'}{\left|\left|\left| L_A^{-1} \right|\right|\right|^2} < \frac{1}{8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2}.$$

Thus, the factor $8C_1 \left|\left|\left| L_A^{-1} \right|\right|\right|^2 \|F_A\|_{L^{n/2}(B^n)}$ is strictly less than 1 and hence, we can conclude that the sequence $\{\omega_k\}_{k=0}^{\infty}$ is Cauchy, proving the claim.

As $W^{2,n/2} \cap \mathrm{Dom}(L_A)$ is a Banach space and $\{\omega_k\}_{k=0}^{\infty}$ is a Cauchy sequence, there exists ω_∞ such that $\omega_k \to \omega_\infty$ as $k \to \infty$ in $W^{2,n/2} \cap \mathrm{Dom}(L_A)$. By strong convergence and (5.14), we obtain that

$$\|\omega_\infty\|_{W^{2,n/2}} \le 2 \left|\left|\left| L_A^{-1} \right|\right|\right| \|F_A\|_{L^{n/2}(B^n)}$$

and

$$L_A(\omega_\infty) + d^*\omega_\infty \wedge d^*\omega_\infty + *[*d\omega_\infty, (d^*\omega_\infty)^T] = -F_A.$$

Thus, by defining $\omega := \omega_\infty$ we recover the extended integrability condition:

$$F_{A+d^*\omega} + d^*_{A+d^*\omega} d\omega = 0.$$

□

Having all the necessary results in hand, we can now proceed to proving theorem 1.5.

Proof of Theorem 1.5. We can apply Theorem 4.3 to L_A since we have obtained Proposition 5.2. Then there exists a gauge change $g \in C^\infty(B^n, G)$ so that L_{A^g} is invertible $W^{2,n/2} \cap \mathrm{Dom}(L_A)$ to $L^{n/2}$. Moreover, there exists a sequence A_k of smooth $\mathfrak{g}$-valued 1-forms such that $\left\|F_{\tilde{A}_k}\right\|_{L^{n/2}(B^n)} \to 0$ and $\tilde{A}_k \to A$ in $W^{1,n/2}$ as $k \to \infty$. By Proposition 5.4(i) there exists k_0 so that for all $k \ge k_0$, $L_{\tilde{A}_k^g}$ is invertible. Since g is a smooth gauge change independent of k, it also follows that $\tilde{A}_k^g \to A^g$ in $W^{1,n/2}$ as $k \to \infty$.

Moreover, by Proposition 5.4(ii) we know that

$$\sup_{k \ge k_0} \left|\left|\left| L^{-1}_{\tilde{A}_k^g} \right|\right|\right| \le 2 \left|\left|\left| L^{-1}_{A^g} \right|\right|\right|.$$

Because $L_{A^g} L^{-1}_{A^g} = id$, then $\left|\left|\left| L_{A^g} L^{-1}_{A^g} \right|\right|\right| = 1$. It follows that

$$1 \leq |||L_{A^g}||| \, |||L_{A^g}^{-1}|||$$

and hence, $|||L_{A^g}^{-1}|||^{-1} \leq |||L_{A^g}|||$. Using this inequality, we can choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \frac{1}{C'}|||L_{\tilde{A}_k^g}^{-1}|||^{-2} \leq \frac{1}{C'}|||L_{\tilde{A}_k^g}|||^2,$$

uniformly in k, where C' is the constant given by Lemma 5.1. Hence, we can pick k_1 large enough such that $\left\|F_{\tilde{A}_k^g}\right\|_{L^{n/2}(B^n)} < \varepsilon_0$ for all $k \geq k_1$. For each $k \geq k_1$, Lemma 5.1 gives the existence of 2-forms ω_k that satisfy

$$\|\omega_k\|_{W^{2,n/2}(B^n)} \leq C|||L_{\tilde{A}_k^g}^{-1}||| \, \left\|F_{\tilde{A}_k^g}\right\|_{L^{n/2}(B^n)} \leq 2C|||L_{A^g}^{-1}||| \, \left\|F_{\tilde{A}_k}\right\|_{L^{n/2}(B^n)}$$

and

$$L_{\tilde{A}_k^g}(\omega_k) + d^*\omega_k \wedge d^*\omega_k + *[*d\omega_k, (d^*\omega_k)^T] = -F_{\tilde{A}_k^g}. \tag{5.15}$$

By Proposition 5.11, implies that $F_{\tilde{A}_k^g + d^*\omega_k} = 0$. Since $\left\|F_{\tilde{A}_k}\right\|_{L^{n/2}(B^n)}$ converges strongly to 0, the estimates on the 2-forms ω_k give $\omega_k \to 0$ in $W^{2,n/2}$ as $k \to \infty$. Thus, we obtain the strong convergence

$$\tilde{A}_k^g + d^*\omega_k \to A^g \text{ in } W^{1,n/2}.$$

The connection forms

$$A_k := \left(\tilde{A}_k^g + d^*\omega_k\right)^{g^{-1}} \in C^\infty$$

are uniformly bounded in $W^{1,n/2}$ and $A_k^g \to A^g$ in $W^{1,n/2}$ automatically. In particular, we obtain

$$\|A_k^g - A^g\|_{L^{n/2}} \to 0 \iff \left\|g^{-1}(A_k - A)\, g\right\|_{L^{n/2}} \to 0.$$

Moreover, since g is a smooth gauge change, we have the following estimates:

$$\begin{aligned} \|A_k - A\|_{L^{n/2}} &= \left\|gg^{-1}(A_k - A)\, gg^{-1}\right\|_{L^{n/2}} \\ &\leq \|g\|_{L^\infty} \left\|g^{-1}(A_k - A)\, g\right\|_{L^{n/2}} \left\|g^{-1}\right\|_{L^\infty}. \end{aligned}$$

Hence, $\|A_k - A\|_{L^{n/2}} \to 0$. Similarly, we can prove the strong convergence of their gradients, and hence $A_k \to A$ in $W^{1,n/2}$. In addition, the smooth sequence A_k satisfies the flatness condition $F_{A_k} = 0$ by construction of ω_k. □

Appendix A

Results in Several Complex Variables

We briefly recall some of the results we will be using from the theory of several complex variables. Valuable reads include [36], [8] and [9].

Definition A.1. *Let D be a bounded domain in $\mathbb{C}^n$. We say D has boundary of class C^k if for every $p \in \partial D$ and U neighbourhood of p, there exists a C^k function $r : U \to \mathbb{R}$ such that $U \cap D = \{z \in U |\ r(z) < 0\}$, $U \cap \partial D = \{z \in U |\ r(z) = 0\}$ and $\nabla r(z) \neq 0$ on $U \cap \partial D$. Then r is called a C^k* ***local defining function*** *for D. If $\overline{D} \subset U$, then r is a* ***global defining function****.*

Moreover, we need to define what pseudoconvexity is:

Definition A.2. *Let D be a bounded domain in $\mathbb{C}^n$ and r a C^2 defining function. D is* ***pseudoconvex*** *at $p \in \partial D$ if the Levi form*

$$L_p(r,t) = \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} t_j \bar{t}_k \geq 0$$

for all $t \in T_p^{1,0}(\partial D)$. D is ***strictly pseudoconvex*** *at p if $L_p(r,t) > 0$ whenever $t \neq 0$. If D is (strictly) pseudoconvex for all $p \in \partial D$ then D is (strictly) pseudoconvex.*

Over a bounded domain $D \subset \mathbb{C}^n$, we define the operator

$$\overline{\partial} : \Omega^{p,q} D \to \Omega^{p,q+1} D$$

acting on forms $\omega = \sum_{IJ} \omega_{IJ} dz_I \wedge d\overline{z}_J$ by

$$\overline{\partial}\omega = \sum_{kIJ} \partial_{\overline{z}_k} \omega_{IJ} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J,$$

where I is an index set of size p and J is an index set of size q. Let $\phi \in \Omega^{p,q}D$ and $\psi \in \Omega^{p,q-1}D$ a compactly supported form. Then, we compute the dual operator ϑ of $\overline{\partial}$ with respect to the L^2 scalar product:

$$\int_D \operatorname{tr}\left(\phi \wedge *\overline{\overline{\partial}\psi}^T\right) = (\phi, \overline{\partial}\psi)$$
$$= \left(\sum_{IJ} \phi_{IJ} dz_I \wedge d\overline{z}_J, \sum_{kIJ} \partial_{\overline{z}_k} \psi_{IJ} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J)\right).$$

where $*$ is the Hodge star operator mapping $\Omega^{p,q}D$ to $\Omega^{n-q,n-p}$. Using the fact that $(d\overline{z_i}, d\overline{z_j}) = 2\delta_{ij}$, then by integrating by parts and rearranging the terms we get

$$(\phi, \overline{\partial}\psi) = 2^{p+q} \sum_{IH} \left(\sum_{kJ} \partial_{z_k}\phi_{IJ}, \psi_{IH}\right),$$

for H index set of size $q-1$, J index set of size q and I index set of size p. Hence, by defining

$$\vartheta\phi = 2 \sum_{kIJH} \partial_{z_k}\phi_{IJ} dz_I \wedge d\overline{z}_H,$$

we obtain the required duality

$$(\phi, \overline{\partial}\psi) = (\vartheta\phi, \psi).$$

In geometric applications, it is useful to work on local orthonormal basis $\tau_1 \cdot \tau_n$ spanning the complex tangent space at a point P, $T_P^{1,0}D$. Thus, we can replace the frame $\{dz_i\}$ with $\{\tau_i\}$ and define L_i the vector fields dual to τ_i. Note that τ_i are not necessarily $\overline{\partial}$-closed and error terms or lower order will appear when applying the $\overline{\partial}$ operator on forms. We have:

$$\overline{\partial}\phi = \sum_{kIJ} \overline{L}_k \omega_{IJ} \overline{\tau}_k \wedge \tau_I \wedge \overline{\tau}_J + \text{ terms of order zero,}$$

and

$$\vartheta\phi = \sum_{kIJH} L_k \phi_{IJ} \tau_I \wedge \tau_H + \text{ terms of order zero.}$$

Moreover, we can also obtain the following expression of ϑ in terms of the operator ∂:

$$\vartheta = - * \partial *, \tag{A.1}$$

see [9, Proposition 5.1.1]. If ϕ and ψ do not necessarily have compact support, then

$$(\vartheta\phi, \psi) = (\phi, \overline{\partial}\psi) + \int_{\partial D} \langle \sigma(\vartheta, dr)\phi, \psi \rangle dS,$$

where ϕ is a (p, q)-form and ψ a $(p, q-1)$-form and we are using the notation in the literature $\sigma(\vartheta, dr)\phi$ to denote

$$\sigma(\vartheta, dr)\phi = \overline{*\overline{\partial}r \wedge *\overline{\phi}^T}^T. \tag{A.2}$$

More explicitly, $\sigma(\vartheta, dr)\phi$ is the form whose components are in the $\overline{\partial}r$ frame of ϕ. We denote the components of ϕ that are in the $\overline{\partial}r$ frame, by ϕ_N. Thus, if $\sigma(\vartheta, dr)\phi = 0$, then $\vartheta = \overline{\partial}^*$, where $\overline{\partial}^*$ is the Hilbert adjoint operator. In this case we say that $\phi \in \mathrm{Dom}(\overline{\partial}^*)$, i.e. ϕ is in the domain of $\overline{\partial}^*$.

It is also useful to remark the fact that if a $(0, q)$-form α vanishes on the boundary, it follows that α vanishes component wise on the boundary. This is because the frame $\overline{\partial}r$ does not vanish on the boundary, unlike dr which does vanish! In terms of the notation above, $\phi = 0$ on ∂D is equivalent to $\sigma(\vartheta, dr)\phi = 0$ and $\sigma(\overline{\partial}, dr)\phi = 0$ on ∂D, where $\sigma(\overline{\partial}, dr)\cdot = \overline{\partial}r \wedge \cdot$ is the adjoint operator of $\sigma(\vartheta, dr)$.

Before we state a few results regarding regularity and Hodge type decomposition, we say that ϕ is in the domain of the Kohn-Laplace $\Delta_{\overline{\partial}}$ (also denoted in the literature by $\Box$) $\phi \in \mathrm{Dom}(\Delta_{\overline{\partial}})$ if and only if $\phi \in \mathrm{Dom}(\overline{\partial}^*)$ and $\overline{\partial}\phi \in \mathrm{Dom}(\overline{\partial})$. These conditions are called the **$\overline{\partial}$-Neumann boundary conditions** over the domain D, i.e. they amount to saying that $\phi_N = 0$ and $(\overline{\partial}\phi)_N = 0$ over ∂D.

Moreover, it was shown for Hilbert spaces [9] and Sobolev spaces [5] that over pseudoconvex domains D there exists the inverse operator

$$N : W^{s,p}(\Omega^{p,q}D) \to W^{s+1,p}(\Omega^{p,q}D) \tag{A.3}$$

of the operator $\Delta_{\overline{\partial}}$ and moreover $N\phi \in \mathrm{Dom}(\Delta_{\overline{\partial}})$, $\Delta_{\overline{\partial}}N\phi = \phi$,

$$\|N\phi\|_{W^{s+1,p}} \leq C\,\|\phi\|_{W^{s,p}} \qquad \text{for any } \phi \in \Omega^{p,q}D.$$

We also have the estimate:

$$\left\|\overline{\partial}N\phi\right\|_{W^{s+1/2,p}} + \left\|\overline{\partial}^*N\phi\right\|_{W^{s+1/2,p}} \leq C\,\|\phi\|_{W^{s,p}}.$$

This shows that over domains with boundary the operators $\Delta_{\overline{\partial}}$, $\overline{\partial}$ and consequently $\overline{\partial}^*$ are not elliptic, but they do satisfy the above sub-elliptic estimates.

We also obtain the $\overline{\partial}$-Hodge decomposition for any form $\phi \in \Omega^{p,q}$:

$$\phi = \overline{\partial}\overline{\partial}^* N\phi + \overline{\partial}^*\overline{\partial}N\phi + h, \tag{A.4}$$

where h is a holomorphic form, i.e. it satisfies: $\overline{\partial}h = 0$ and $\overline{\partial}^* h = 0$. For domains D with vanishing (p,q) Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(D) = 0$ or for domains $D \subseteq \mathbb{C}^n$ for some n we have

$$\phi = \overline{\partial}\overline{\partial}^* N\phi + \overline{\partial}^*\overline{\partial}N\phi. \tag{A.5}$$

We refer the reader to [8] and [12]. In particular, over Stein manifolds (manifolds that holomorphically embedd into $\mathbb{C}^n$, for some n), there are no holomorphic forms in the $\overline{\partial}$-Hodge decomposition.

We recall the **Integral Representation Theorem** which was proven in [24] for $(0,q)$-forms and initially in [14] for $(0,1)$-forms. Before doing so, we will have to define a few key operators. Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with defining function r such that

$$\sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial x_i \partial x_j} t_i t_j \geq c|t|^2$$

for some $c \in \mathbb{R}$ and $t \in \mathbb{R}^{2n}$.

The Bochner-Martinelli-Koppelman kernel (see [8, Theorem 11.1.2]) is given by:

$$K(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\sum_{i=1}^n (\overline{\zeta}_i - \overline{z}_i) d\zeta_i}{|\zeta - z|^2} \wedge \left(\frac{\sum_{i=1}^n (d\overline{\zeta}_i - d\overline{z}_i) \wedge d\zeta_i}{|\zeta - z|^2} \right)^{n-1}. \tag{A.6}$$

We define the kernel K_q as being the form of $(0,q)$ degree in z and $(n, n-q-1)$ degree in ζ. Moreover, the boundary kernels K^∂, K^∂_{00} are given by

$$\begin{aligned} K^\partial =& \frac{1}{(2\pi i)^n} \frac{\sum_{i=1}^n (\overline{\zeta}_i - \overline{z}_i) d\zeta_i}{|\zeta - z|^2} \wedge \frac{\sum_{i=1}^n \partial_{\zeta_i} r(\zeta) d\zeta_i}{\sum_{i=1}^n \partial_{\zeta_i} r(\zeta)(\zeta_i - z_i)} \\ & \wedge \sum_{k_1 + k_2 = n-2} \left(\frac{\sum_{i=1}^n (\overline{\zeta}_i - \overline{z}_i) d\zeta_i}{|\zeta - z|^2} \right)^{k_1} \wedge \left(\frac{\sum_{i=1}^n \partial_{\overline{\zeta}_i} \partial_{\zeta_i} r(\zeta) d\overline{\zeta}_i \wedge d\zeta_i}{\sum_{i=1}^n \partial_{\zeta_i} r(\zeta)(\zeta_i - z_i)} \right)^{k_2} \end{aligned}$$

and

$$K_{00}^{\partial} = \frac{1}{(2\pi i)^n} \frac{\sum_{i=1}^n (\overline{\zeta}_i - \overline{z}_i) d\zeta_i}{|\zeta - z|^2} \wedge \left(\frac{\sum_{i=1}^n \partial_{\overline{\zeta}_i} \partial_{\zeta_i} r(\zeta) d\overline{\zeta}_i \wedge d\zeta_i}{\sum_{i=1}^n \partial_{\zeta_i} r(\zeta)(\zeta_i - z_i)} \right)^{n-1}.$$

For each $q \geq 1$ we can, thus, define

$$T_q : C^\infty(\Omega^{0,q} D) \to C^\infty(\Omega^{0,q-1} D)$$

given by

$$T_1(\alpha) = \int_D K_0 \wedge \alpha - \int_{\partial D} K_0^{\partial} \wedge \alpha \qquad \text{when } q = 1 \tag{A.7}$$

and

$$T_q(\alpha) = \int_D K_{q-1} \wedge \alpha \qquad \text{when } q > 1, \tag{A.8}$$

where by K_0^{∂} we understand the form of $(0,0)$ degree in z.

We now formulate the representation theorem (this can be found in [24, Section 3] and [8, Theorem 11.2.7]):

Theorem A.1. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, $0 \leq q \leq n$ and $\alpha \in C^\infty(\Omega^{0,q} D, \mathbb{C})$. Then we have the following representations:*

$$\begin{aligned} \alpha(z) &= \int_{\partial D} K_{00}^{\partial}(\zeta, z)\alpha(\zeta) + T_1(\overline{\partial}\alpha) && \textit{when } q = 0 \\ \alpha(z) &= \overline{\partial}(T_q \alpha) + T_{q+1}(\overline{\partial}\alpha) && \textit{when } q > 0. \end{aligned}$$

Moreover, for results concerning regularity of operators T_q, we recommend to the reader [19] (optimal L^p results for $(0,1)$-forms) and [24] (L^p and Hölder regularity results for $(0,q)$-forms).

The operators T_q are key to proving Lemma 3.3, which is used to prove theorem 1.1.

A.1 Results in the unit ball B^4

Since the operators $\overline{\partial}$, ϑ have been stated in their full generality, in this section we will be more concrete and deal with the case when the domain D is chosen as B^4, embedded into $\mathbb{C}^2$. At the end of this section we prove regularity results for the operators T_1 and T_2 we have defined above and can also be found in [24].

In particular, on B^4 we define

$$r(z) = |z|^2 - 1.$$

Then r is a global defining function for B^4. Thus, B^4 is an example of a strictly pseudoconvex domain. Indeed, if the pick the defining function above, we have that the Levi form

$$L_p(r, t) = |t|^2 > 0$$

for all $p \in B^4$, $t \in T_p^{1,0}(\partial B^4)$, $t \neq 0$. On B^4 we take the canonical complex structure J. At each point $p \in \partial B^4$, we can find an orthonormal $(0, 1)$ fields $L_{\overline{\tau}}$ and $L_{\overline{\partial} r}$ that span the complexified tangential space $T_p^{\mathbb{C}}(\partial B^4)$. We will explictly compute them.

Let e^*, Je^*, dr, Jdr define the Hopf frame. These define the orthonormal basis for $(0, 1)$-forms. Namely,

$$\overline{\tau} = e^* + iJe^* \tag{A.9}$$

and

$$\overline{\partial} r = dr + iJdr. \tag{A.10}$$

In terms of $\overline{z}_1$ and $\overline{z}_2$, they satisfy:

$$\begin{aligned}
dr &= \frac{1}{2r}(\overline{z_1}dz_1 + z_1 d\overline{z_1} + z_2 d\overline{z_2} + \overline{z_2}dz_2) = \frac{1}{2}(\partial r + \overline{\partial} r) \\
Jdr &= \frac{1}{2ir}(-\overline{z_1}dz_1 + z_1 d\overline{z_1} + z_2 d\overline{z_2} - \overline{z_2}dz_2) = \frac{1}{2}(\overline{\partial} r - \partial r) \\
e^* &= \frac{1}{2r}(z_2 dz_1 - z_1 dz_2 + \overline{z_2}d\overline{z_1} - \overline{z_1}d\overline{z_2}) = \frac{1}{2}(\tau + \overline{\tau}) \\
Je^* &= \frac{1}{2ir}(\overline{z_2}d\overline{z_1} - \overline{z_1}d\overline{z_2} - z_2 dz_1 + z_1 dz_2) = \frac{1}{2r}(\overline{\tau} - \tau).
\end{aligned}$$

We obtain the explicit formulation of $\overline{\tau}$ and $\overline{\partial} r$:

$$\begin{aligned}
\overline{\tau} &= \tfrac{1}{r}\left(\overline{z_2}d\overline{z_1} - \overline{z_1}d\overline{z_2}\right) \\
\overline{\partial} r &= \tfrac{1}{r}\left(z_1 d\overline{z_1} + z_2 d\overline{z_2}\right)
\end{aligned} \tag{A.11}$$

Moreover, the vector fields $L_{\overline{\tau}}$ and $L_{\overline{\partial}r}$ can be computed as follows:

$$L_{\overline{\tau}} = \frac{1}{2}\left(\partial_{e^*} - i\partial_{Je^*}\right) = \frac{1}{r}\left(z_2\partial_{\overline{z}_1} - z_1\partial_{\overline{z}_2}\right)$$
$$L_{\overline{\partial}r} = \frac{1}{2}\left(\partial_r - i\partial_{Jdr}\right) = \frac{1}{r}\left(\overline{z}_1\partial_{\overline{z}_1} + \overline{z}_2\partial_{\overline{z}_2}\right).$$

We prove a regularity result for $(0,2)$-forms in the domain B^4, which comes in-handy in our results, for example in the proof of Lemma 3.3. We are unaware of such a result being available in the literature.

Proposition A.1. *The operator* T_2 (A.8) *maps* $L^p(\Omega^{0,2}B^4)$ *into* $W^{1,p}(\Omega^{0,1}B^4)$ *whenever* $p > 1$.

Proof of Proposition A.1. From the formula (A.6), we obtain

$$K_1(\zeta, z) = -\frac{1}{4\pi^2}\frac{\sum_{i=1}^2(\overline{\zeta}_i - \overline{z}_i)d\zeta_i}{|\zeta - z|^2} \wedge \frac{\sum_{i=1}^2 -d\overline{z}_i \wedge d\zeta_i}{|\zeta - z|^2}.$$

We expand the equation above to get:

$$K_1(\zeta, z) = -\frac{1}{4\pi^2|\zeta - z|^4}\left((\overline{\zeta}_1 - \overline{z}_1)d\zeta_1 \wedge d\overline{z_2} \wedge d\zeta_2 + (\overline{\zeta}_2 - \overline{z}_2)d\zeta_2 \wedge d\overline{z_1} \wedge d\zeta_1\right).$$

Let $\alpha = f d\overline{z}_1 \wedge d\overline{z}_2 \in L^p(\Omega^{0,2}B^4)$. From the formula of T_2 it follows that:

$$T_2(\alpha) = \left(-\frac{1}{4\pi^2}\int_{B^4}\frac{1}{|\zeta - z|^4}(\overline{\zeta}_2 - \overline{z}_2) f d\zeta_1 \wedge d\overline{\zeta}_1 \wedge d\zeta_2 \wedge d\overline{\zeta}_2\right) d\overline{z_1}$$
$$+ \left(-\frac{1}{4\pi^2}\int_{B^4}\frac{1}{|\zeta - z|^4}(\overline{\zeta}_1 - \overline{z}_1) f d\zeta_1 \wedge d\overline{\zeta}_1 \wedge d\zeta_2 \wedge d\overline{\zeta}_2\right) d\overline{z_2}.$$

Since each component of K_1 is a quasi-potential in the sense of [21, Definition 3.7.1], then we can apply [21, Theorem 3.7.1] component wise to $T_2(\alpha)$ to get the required result:

$$\|T_2(\alpha)\|_{W^{1,p}(B^4)} \leq \|\alpha\|_{L^p(B^4)}.$$

□

In addition, the following result builds upon the sharp estimates of the Henkin operator (T_1 in our notation) found by [19]. In particular, we show that for estimating $T_1\alpha$, where α is a $(0,1)$ form, we can relax the condition $\overline{\partial}\alpha = 0$. The estimates we find are not sharp.

Proposition A.2. *Let $p > 6$ and $q > 6$ such that $W^{1,p}(B^4) \hookrightarrow L^q(B^4)$ and $\alpha \in L^q(\Omega^{0,1}B^4)$ satisfying $\overline{\partial}\alpha \in L^p(B^4)$. Then there exists a constant $C > 0$ depending on p and q such that:*

$$\|T_1\alpha\|_{L^\infty(B^4)} + \left\|\overline{\partial}T_1\alpha\right\|_{L^q(B^4)} \leq C\left(\|\alpha\|_{L^q(B^4)} + \left\|\overline{\partial}\alpha\right\|_{L^p(B^4)}\right).$$

Proof of Proposition A.2. We refer to the proofs presented in [19]. We recall:

$$T_1(\alpha) = \int_{B^4} K_0 \wedge \alpha - \int_{\partial B^4} K_0^\partial \wedge \alpha. \tag{A.12}$$

In [19, Section 5] it is shown that the first term has "good" regularity. In particular it follows that $\int_{B^4} K_0 \wedge \alpha$ belongs to a Lipschitz space when $\alpha \in L^q$ for $q > 6$. We focus our attention to the second term in (A.12), which is problematic. By Stokes Theorem we obtain:

$$\int_{\partial B^4} K_0^\partial \wedge \alpha = \int_{B^4} \overline{\partial}K_0^\partial \wedge \alpha - \int_{B^4} K_0^\partial \wedge \overline{\partial}\alpha. \tag{A.13}$$

Note that $\overline{\partial}\alpha \neq 0$ in general. The first integral in (A.13) is estimated in [19, Section 5,6] and yields the regularity result:

$$\left\|\int_{B^4} \overline{\partial}K_0^\partial \wedge \alpha\right\|_{L^\infty(B^4)} \leq C\,\|\alpha\|_{L^q(B^4)}.$$

for some constant $C > 0$. It remains to deal with the term: $\int_{B^4} K_0^\partial \wedge \overline{\partial}\alpha$. However, since $\overline{\partial}K_0^\partial$ is more singular than K_0^∂, since $\overline{\partial}\alpha \in L^p$, we have the estimate:

$$\left\|\int_{B^4} K_0^\partial \wedge \overline{\partial}\alpha\right\|_{L^\infty(B^4)} \leq C\left\|\overline{\partial}\alpha\right\|_{L^p(B^4)}.$$

Hence, we have that there exists a constant $C > 0$ such that

$$\|T_1\alpha\|_{L^\infty(B^4)} \leq C\left(\|\alpha\|_{L^q(B^4)} + \left\|\overline{\partial}\alpha\right\|_{L^p(B^4)}\right). \tag{A.14}$$

Since $\overline{\partial}\alpha$ is well-defined, by density of smooth forms, we obtain by Theorem A.1 the following equation:

$$\alpha = \overline{\partial}T_1(\alpha) + T_2(\overline{\partial}\alpha),$$

and

$$\left\|\overline{\partial}T_1(\alpha)\right\|_{L^q(B^4)} \leq \|\alpha\|_{L^q(B^4)} + \left\|T_2(\overline{\partial}\alpha)\right\|_{L^q(B^4)}.$$

By Proposition A.1, we have that $T_2(\overline{\partial}\alpha) \in W^{1,p} \hookrightarrow L^q$. In particular, there exist constants $C, C' > 0$ such that:

$$\begin{aligned}\left\|\overline{\partial} T_1(\alpha)\right\|_{L^q(B^4)} &\leq C\left(\|\alpha\|_{L^q(B^4)} + \left\|T_2(\overline{\partial}\alpha)\right\|_{W^{1,p}(B^4)}\right) \\ &\leq C'\left(\|\alpha\|_{L^q(B^4)} + \left\|\overline{\partial}\alpha\right\|_{L^p(B^4)}\right). \end{aligned} \tag{A.15}$$

Hence, by putting (A.14) and (A.15) together we get:

$$\|T_1\alpha\|_{L^\infty(B^4)} + \left\|\overline{\partial} T_1\alpha\right\|_{L^q(B^4)} \leq C\left(\|\alpha\|_{L^q(B^4)} + \left\|\overline{\partial}\alpha\right\|_{L^p(B^4)}\right).$$

□

Appendix B

Extension Procedures

This appendix section introduces a short result concerning extensions of $(0, q)$-forms.

Proposition B.1. *Let U, V be bounded smooth domains in $\mathbb{C}^n$ with $U \cap V \neq \emptyset$ and $\alpha \in W^{2,p}(\Omega^{0,q} U \cap V)$. Then there exists a $W^{2,p}$ extension $\tilde{\alpha}$ of α in U such that $\tilde{\alpha} = \alpha$ in $U \cap V$.*

This extension will be used in Chapter 2 for the gluing procedure.

Proof of Proposition B.1. It is sufficient to extend α component wise in $\mathbb{C}^n$ and then restrict each component to U. Hence, the problem reduces to harmonic extensions. We solve the PDEs:

$$\begin{cases} \Delta^2 \omega_I &= 0 \quad \text{in } \mathbb{C}^n \setminus U \cap V \\ \partial_r \omega_I &= \partial_r \alpha_I \quad \text{on } \partial(U \cap V) \\ \omega_I &= \alpha_I \quad \text{on } \partial(U \cap V), \end{cases}$$

where $\alpha = \sum_I \alpha_I d\bar{z}_I$, I index of size q. Such a solution exists and elliptic estimates give

$$\|\omega_I\|_{W^{2,p}(\mathbb{C}^n)} \leq C \|\alpha\|_{W^{2,p}(U \cap V)},$$

for some constant $C > 0$. Moreover $\omega_I = \alpha_I$ on $U \cap V$ by construction. Hence, by defining $\tilde{\alpha} = \sum_I \omega_I d\bar{z}_I$ and restricting it to U, we obtain our result. □

Appendix C

Linear Operators

We prove the following results that are be used in several chapters.

Proposition C.1. *Let D be a bounded domain in $\mathbb{R}^n$, $\mathfrak{g}$ a Lie algebra and for $k \in \mathbb{N}$, let*

$$H : C^\infty(\Omega^1 D \otimes \mathfrak{g}) \to C^\infty(\Omega^k D \otimes \mathfrak{g})$$

be a linear operator such that $H(A)(x) = H(A(x))$ for all $x \in D$ and $A \in C^\infty(\Omega^1 D \otimes \mathfrak{g})$. If for all $U \in C^\infty(D, \mathfrak{g})$ we have

$$H(dU) = 0 \tag{C.1}$$

then $H = 0$.

Before we prove this statement we remark that the condition $H(A)(x) = H(A(x))$ in general prevents H from being a differential operator acting on forms A. Otherwise the statement cannot be true. For example take $H = d$, $H(dU) = 0$ - since $d^2 = 0$, but $d \neq 0$.

Proof of Proposition C.1. Fix $A \in C^\infty(\Omega^1 D \otimes \mathfrak{g})$, then we can write it as $A = \sum_{i=1}^n a_i dx_i$. Fix $x_0 \in D$ arbitrary and define the function $V(x) := \sum_{i=1}^n a_i(x_0)x_i$. Then $dV = A(x_0)$. Moreover, by (C.1) we have

$$H(dV) = H(A(x_0)) = H(A)(x_0) = 0.$$

Hence, because x_0 is arbitrary, we have that $H(A) = 0$ and since A was an arbitrarily chosen smooth 1-form, then $H = 0$. □

Next, we prove a more general statement than the one above:

Proposition C.2. *Let D be a bounded domain in $\mathbb{R}^n$, $\mathfrak{g}$ a Lie algebra and for $k \in \mathbb{N}$, let*

$$H_0 : C^\infty(D, \mathfrak{g}) \to C^\infty(\Omega^k D \otimes \mathfrak{g})$$

and

$$H_1 : C^\infty(\Omega^1 D \otimes \mathfrak{g}) \to C^\infty(\Omega^k D \otimes \mathfrak{g})$$

be linear operators such that $H_0(U)(x) = H_0(U(x))$ for all $x \in D$. If for all $U \in C^\infty(D, \mathfrak{g})$ we have

$$H_0(U) + H_1(dU) = 0 \tag{C.2}$$

then $H_0 = 0$ and $H_1 \circ d = 0$.

Proof of Proposition C.2. Fix $U \in C^\infty(D, \mathfrak{g})$ and $x \in D$. Define $V_x := U(x)$ a constant function. Then by (C.2), we have

$$H_0(V_x) = H_0(U(x)) = H_0(U)(x) = 0.$$

Since x is arbitrary in D, then we have that $H_0(U) = 0$. Since U is an arbitrarily chosen smooth function, then $H_0 = 0$. Hence, from this and (C.2), we also obtain $H_1 \circ d = 0$. This concludes the proof. $\square$

The following results find the expression of the adjoints of $\overline{\partial}_A$ and d_A. These expressions will be of critical use in Chapters 4 and 5.

Proposition C.3. *Let $\mathfrak{g} \subseteq M_r(\mathbb{C})$ a Lie algebra, A be a $\mathfrak{g}$-valued 1-form and $\overline{\partial}_A : C^\infty(\Omega^{p,q} B^{2d} \otimes \mathfrak{g}) \to C^\infty(\Omega^{p,q+1} B^{2d} \otimes \mathfrak{g})$, be the operator given by:*

$$\overline{\partial}_A \omega = \overline{\partial}\omega + [A^{0,1}, \omega].$$

Then the formal adjoint $\overline{\partial}_A^*$ to d_A with respect to the L^2 inner product over (p,q)-forms induced by the Frobenius inner product $(A, B) \mapsto \operatorname{tr}(A\overline{B}^T)$ on $\mathfrak{g}$, is defined on $C^\infty(\Omega^{p,q+1} B^{2d} \otimes \mathfrak{g}) \cap \operatorname{Dom}(\overline{\partial}^*)$ by

$$\overline{\partial}_A^* \tau = \overline{\partial}^* \tau + *[*\tau, \overline{A^{0,1}}^T].$$

Remark C.1. *$\overline{\partial}_A^*$ defines the Hilbert adjoint of $\overline{\partial}_A$. We can similarly find ϑ_A, the formal adjoint of $\overline{\partial}_A$. Indeed, assuming compactly supported forms, we analogously find*

$$\vartheta_A^* \tau = \vartheta \tau + *[*\tau, \overline{A^{0,1}}^T].$$

Proof of Proposition C.3. Let $k \geq 0$, $\omega \in \Omega^{p,q}$ and $\tau \in \Omega^{p,q+1} \cap \mathrm{Dom}(\overline{\partial}^*)$, then:

$$\langle \overline{\partial}_A \omega, \tau \rangle = \langle \overline{\partial}\omega + [A^{0,1}, \omega], \tau \rangle = \langle \omega, \overline{\partial}^* \tau \rangle + \langle [A^{0,1}, \omega], \tau \rangle.$$

Recall that $[A^{0,1}, \omega] \overset{def}{=} A^{0,1} \wedge \omega + (-1)^{p+q+1} \omega \wedge A^{0,1}$ when ω is a (p,q)-form and A a 1-form. Thus,

$$\langle A^{0,1} \wedge \omega, \tau \rangle \overset{def}{=} \int_{B^{2d}} \mathrm{tr}(A^{0,1} \wedge \omega \wedge *\overline{\tau}^T) = -\int_{B^{2d}} \mathrm{tr}(\omega \wedge *\overline{\tau}^T \wedge A^{0,1}),$$

Where we used the fact that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$ and the antisymmetry of the wedge product. Using the fact that

$$* * \eta = (-1)^{(p+q)^2} \eta \quad \text{ if } \eta \text{ is a } (p,q)\text{-form},$$

we obtain by linearity of the trace and noting that $*\overline{\tau}^T \wedge A^{0,1}$ is a $(d-p, d-q)$-form:

$$\begin{aligned}
\langle A^{0,1} \wedge \omega, \tau \rangle &= -(-1)^{(p+q)^2} \int_{B^{2d}} \mathrm{tr}\left(\omega \wedge * * (*\overline{\tau}^T \wedge A^{0,1})\right) \\
&= -(-1)^{(p+q)^2} \int_{B^{2d}} \mathrm{tr}\left(\omega \wedge \overline{*\overline{*(*\overline{\tau}^T \wedge A^{0,1})}^T}^T\right) \\
&= -(-1)^{(p+q)^2} \int_{B^{2d}} \mathrm{tr}\left(\omega \wedge *\overline{(*\overline{(*\overline{\tau}^T \wedge A^{0,1})}^T)}^T\right) \\
&= -(-1)^{(p+q)^2} \langle \omega, *\overline{(*\overline{\tau}^T \wedge A^{0,1})}^T \rangle.
\end{aligned}$$

Using the fact that for matrices we have $(AB)^T = B^T A^T$ and the antisymmetry of the wedge product, we obtain:

$$\begin{aligned}
\langle A^{0,1} \wedge \omega, \tau \rangle &= -(-1)^{(p+q)^2} (-1)^{2d-p-q-1} \langle \omega, *(\overline{A^{0,1}}^T \wedge *\tau) \rangle \\
&= (-1)^{(p+q)^2+p+q} \langle \omega, *(\overline{A^{0,1}}^T \wedge *\tau) \rangle.
\end{aligned}$$

Interchanging now the role of $*\overline{\tau}^T \wedge A^{0,1}$ with $A^{0,1} \wedge *\overline{\tau}^T$, we get as above

$$\begin{aligned}
\langle (-1)^{p+q+1} \omega \wedge A^{0,1}, \tau \rangle &= (-1)^{p+q+1} \int_{B^{2d}} \mathrm{tr}\left(\omega \wedge A^{0,1} \wedge *\overline{\tau}^T\right) \\
&= (-1)^{(p+q)^2+p+q+1} \langle \omega, *\overline{(A^{0,1} \wedge *\overline{\tau}^T)}^T \rangle \\
&= (-1)^{(p+q)^2} \langle \omega, *(\tau \wedge *\overline{A^{0,1}}^T) \rangle.
\end{aligned}$$

Note now that $*\tau$ is a $(d-q-1, d-p)$-form thus

$$[*\tau, \overline{A^{0,1}}^T] = *\tau \wedge \overline{A^{0,1}}^T + (-1)^{p+q}\overline{A^{0,1}}^T \wedge *\tau,$$

and by the above computations we get

$$\begin{aligned}\langle [A^{0,1}, \omega], \tau\rangle &= (-1)^{(p+q)^2+p+q}\langle \omega, *[*\tau, \overline{A^{0,1}}^T]\rangle \\ &= (-1)^{(p+q)(p+q+1)}\langle \omega, *[*\tau, \overline{A^{0,1}}^T]\rangle.\end{aligned}$$

Using the fact that $(p+q)(p+q+1)$ is a product of two consecutive integers, we obtain

$$\langle [A^{0,1}, \omega], \tau\rangle = \langle \omega, *[*\tau, \overline{A^{0,1}}^T]\rangle.$$

This allows to obtain the desired formula for the adjoint operator to $\overline{\partial}_A$:

$$\langle \omega, \overline{\partial}_A^* \tau\rangle = \langle \overline{\partial}_A \omega, \tau\rangle = \langle \omega, \overline{\partial}^* \tau + *[*\tau, \overline{A^{0,1}}^T]\rangle,$$

which concludes our proof. □

Proposition C.4. *Let $\mathfrak{g} \subseteq M_r(\mathbb{R})$ a Lie algebra, A be a $\mathfrak{g}$-valued 1-form and $d_A : C^\infty(\Omega^k B^n \otimes \mathfrak{g}) \to C^\infty(\Omega^k B^n \otimes \mathfrak{g})$, be the operator given by:*

$$d_A\omega = d\omega + [A, \omega].$$

Then the formal adjoint d_A^* to d_A with respect to the L^2 inner product over k-forms induced by the Frobenius inner product $(A, B) \mapsto \mathrm{tr}(AB^T)$ on $\mathfrak{g}$, is defined as

$$d_A^* : C^\infty(\Omega^{k+1} B^n \otimes \mathfrak{g}) \cap \mathrm{Dom}(d^*) \to C^\infty(\Omega^k B^n \otimes \mathfrak{g})$$

by

$$d_A^*\tau = d^*\tau + (-1)^{k(n-k+1)} * [*\tau, A^T].$$

In particular if $k = 2$, then $d_A^*\tau = d^*\tau + *[*\tau, A^T]$.

Remark C.2. *By $\mathrm{Dom}(d^*)$ we mean the forms τ whose normal component (involving the frame dr) vanishes over the boundary of the domain, in this case the boundary of B^n.*

Proof of Proposition C.4. Let $k \geq 0$, $\omega \in \Omega^k$ and $\tau \in \Omega^{k+1} \cap \mathrm{Dom}(d^*)$.

Since $\tau \in \mathrm{Dom}(d^*)$, by definition $\tau_N = 0$ (i.e. the normal component vanishes over the boundary of B^n) and using integration by parts we have:

$$\langle d_A\omega, \tau\rangle = \langle d\omega + [A,\omega], \tau\rangle = \langle \omega, d^*\tau\rangle + \langle [A,\omega], \tau\rangle.$$

Recall that $[A,\omega] \overset{def}{=} A\wedge\omega + (-1)^{k+1}\omega\wedge A$ when ω is a k form and A a 1-form, and

$$\langle A\wedge\omega, \tau\rangle \overset{def}{=} \int_{B^n} \mathrm{tr}(A\wedge\omega\wedge *\tau^T) = (-1)^{n-1}\int_{B^n} \mathrm{tr}(\omega\wedge *\tau^T\wedge A),$$

Where we used the fact that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$ and the antisymmetry of the wedge product. Using the fact that

$$**\,\eta = (-1)^{k(n-k)}\eta \quad \text{if } \eta \text{ is a } k\text{-form},$$

we obtain by linearity of the trace and noting that $*\tau\wedge A$ is a $(n-k)$-form:

$$\begin{aligned}
\langle A\wedge\omega, \tau\rangle &= (-1)^{n-1}(-1)^{k(n-k)}\int_{B^n} \mathrm{tr}(\omega\wedge **\,(*\tau^T\wedge A)) \\
&= (-1)^{n-1}(-1)^{k(n-k)}\int_{B^n} \mathrm{tr}(\omega\wedge *(*(*\tau^T\wedge A)^T)^T) \\
&= (-1)^{k(n-k)+n-1}\langle \omega, *(*\tau^T\wedge A)^T\rangle.
\end{aligned}$$

Using the fact that for matrices we have $(AB)^T = B^T A^T$ and the antisymmetry of the wedge product, we obtain:

$$\begin{aligned}
\langle A\wedge\omega, \tau\rangle &= (-1)^{k(n-k)+n-1}(-1)^{n-k-1}\langle \omega, *(A^T\wedge *\tau)\rangle \\
&= (-1)^{k(n-k)+k}\langle \omega, *(A^T\wedge *\tau)\rangle.
\end{aligned}$$

Interchanging now the role of $*\tau\wedge A$ with $A\wedge *\tau$, we get as above

$$\begin{aligned}
\langle (-1)^{k+1}\omega\wedge A, \tau\rangle &= (-1)^{k+1}\int_{B^n} \mathrm{tr}(\omega\wedge A\wedge *\tau^T) \\
&= (-1)^{k(n-k)+k+1}\langle \omega, *(A\wedge *\tau^T)^T\rangle \\
&= (-1)^{k(n-k)+n}\langle \omega, *(\tau\wedge *A^T)\rangle.
\end{aligned}$$

Note now that $*\tau$ is a $(n-k-1)$-form thus,

$$[*\tau, A^T] = *\tau\wedge A^T + (-1)^{n-k}A^T\wedge *\tau,$$

By the above computations we get

$$\langle [A,\omega],\tau\rangle = (-1)^{k(n-k)+k}\langle \omega, *[*\tau, A^T]\rangle = (-1)^{k(n-k+1)}\langle \omega, *[*\tau, A^T]\rangle.$$

This allows to obtain the desired formula for the adjoint operator to d_A:

$$\langle \omega, d_A^*\tau\rangle = \langle d_A\omega,\tau\rangle = \langle \omega, d^*\tau + (-1)^{k(n-k+1)} * [*\tau, A^T]\rangle,$$

which concludes our proof. □

Appendix D

Estimates

In this section of the Appendix, we will prove a few results that help us bootstrap certain $\overline{\partial}$-equations. These results have been heavily used in order to prove the regularity of sections which yield a holomorphic structure over a given closed Kähler surface or manifold.

Lemma D.1. *Let D a domain holomorphically embedded in $\mathbb{CP}^d$ and $\omega \in W^{1,d}(\Omega^{0,1}D \otimes M_n(\mathbb{C}))$ such that ω satisfies the integrability condition* (1.1). *Let $g \in L^{2d}(D, M_n(\mathbb{C}))$ be a distributional solution of the equation:*

$$\overline{\partial} g = -\omega \cdot g$$

satisfying the estimate $\|g - id\|_{L^{2d}(D)} \leq C \|\omega\|_{W^{1,d}(D)}$. Then there exists a subdomain $D_0 \subseteq D$, and for each $q \in (1, d)$, a constant $C_q > 0$ such that:

$$\|g - id\|_{W^{2,d}(D_0)} \leq C_q \|\omega\|_{W^{1,d}(D)} .$$

Proof of Lemma D.1. We start with a bootstrapping procedure. Firstly, let D_0 be a slightly smaller subdomain of D such that we obtain the existence of a constant $C > 0$ and the following inequality holds:

$$\begin{aligned} \|g - id\|_{W^{1,d}(D_0)} &\leq C \left(\|\overline{\partial} g\|_{L^d(D)} + \|g - id\|_{L^d(D)} \right) \\ &\leq C \left(\|\omega\|_{L^{2d}(D)} \|g\|_{L^{2d}(D)} + \|g - id\|_{L^d(D)} \right) . \end{aligned}$$

By using the embedding of $W^{1,d}$ into L^{2d} in $2d$-dimensions, it follows that for some constant $C > 0$, we have:

$$\|g - id\|_{W^{1,d}(D_0)} \leq C \Big(\|\omega\|_{W^{1,d}(D)} \|g - id\|_{L^{2d}(D)}$$

$$+ \|\omega\|_{W^{1,d}(D)} + \|g - id\|_{L^d(D)}\Big) .$$

Hence, by $\|g - id\|_{L^{2d}(D)} \leq C \|\omega\|_{W^{1,d}(D)}$, we obtain:

$$\|g - id\|_{W^{1,d}(D_0)} \leq C \|\omega\|_{W^{1,d}(D)} .$$

Once we have obtained the $W^{1,d}$ estimate on D_0, we can proceed to bootstrapping to $W^{2,q}$ regularity. We can apply ϑ to the $\overline{\partial}$-equation to obtain the elliptic PDE:

$$\vartheta\overline{\partial}g = -\vartheta(\omega \cdot g) = -\vartheta\omega \cdot g - *(*\omega \wedge \partial g),$$

which holds in a distributional sense. Since $\vartheta\overline{\partial}$ is equal to the Hodge Laplacian d^*d acting on functions, we can apply Proposition D.1 and obtain that $g \in W^{2,q}_{loc}(D_0, M_n(\mathbb{C}))$ for all $q < d$ and for each $q < d$ there exists a constant $C_q > 0$ such that

$$\|g\|_{W^{2,q}_{loc}(D_0)} \leq C_q \|\omega\|_{W^{1,d}(D_0)} . \tag{D.1}$$

Without loss of generality, we can assume $g \in W^{2,q}(D_0, M_n(\mathbb{C}))$ for all $q < d$, otherwise we pick a slightly smaller domain than D_0. It remains to show the required bound. Since $\vartheta\overline{\partial}$ is elliptic we have the a-priori estimate:

$$\|g - id\|_{W^{2,q}(D_0)} \leq C \left(\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} + \|g - id\|_{L^q(D_0)} \right), \tag{D.2}$$

for any $q \in (1, d)$. We estimate $\vartheta\overline{\partial}g$:

$$\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} \leq \|\nabla\omega\|_{L^d(D_0)} \|g\|_{L^{dq/(d-q)}(D_0)} + \|\omega\|_{L^{2d}(D_0)} \|\nabla g\|_{L^{2dq/(2d-q)}(D_0)} .$$

Since $W^{1,d}$ embeds into L^{2d}, it follows that

$$\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} \leq C \|\omega\|_{W^{1,d}(D)} \left(\|g\|_{L^{dq/(d-q)}(D_0)} + \|\nabla g\|_{L^{2dq/(2d-q)}(D_0)} \right),$$

for some constant $C > 0$. Moreover, $W^{2,q}$ embeds into $L^{dq/(d-q)}$ and $W^{1,q}$ embeds into $L^{2dq/(2d-q)}$. Using these embeddings, there exists a constant $C_q > 0$ and $C > 0$ such that:

$$\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} \leq C_q \|\omega\|_{W^{1,d}(D)} \|g\|_{W^{2,q}(D_0)} + C \|\omega\|_{W^{1,d}(D)} .$$

By the bound (D.1), it follows that there exists $C_q > 0$:

$$\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} \leq C_q \|\omega\|_{W^{1,d}(D)} \|\omega\|_{W^{1,d}(D_0)} + C \|\omega\|_{W^{1,d}(D)} .$$

Then for some constant $C_q > 0$, we have

$$\left\|\vartheta\overline{\partial}g\right\|_{L^q(D_0)} \leq C_q \left\|\omega\right\|_{W^{1,d}(D)}.$$

Putting this together with the fact that

$$\|g - id\|_{L^q(D)} \leq \|g - id\|_{L^{2d}(D)} \leq C_q \|\omega\|_{W^{1,d}(D)}$$

and (D.2), there exists a constant $C_q > 0$ such that

$$\|g - id\|_{W^{2,q}(D_0)} \leq C_q \|\omega\|_{W^{1,d}(D)}.$$

Since $q \in (1, d)$ is arbitrary, we have proven the result. □

Remark D.1.

(i) In the statement above, if $D = \mathbb{CP}^d$, then $D_0 = D = \mathbb{CP}^d$. This is the case because $\overline{\partial}$ is elliptic on $\mathbb{CP}^d$.

(ii) Assume instead of $\|g - id\|_{L^{2d}(D)} \leq C \|\omega\|_{W^{1,d}(D)}$, the slightly perturb inequality:

$$\|g - id\|_{L^{2d}(D)} \leq C \|\omega\|_{W^{1,d}(D)} + C_0,$$

where $C_0 > 0$ is a small constant. We can conclude from the proof of the statement that all arguments pass through and we can reach the natural conclusion:

$$\|g - id\|_{W^{2,q}(D_0)} \leq C \left(\|\omega\|_{W^{1,d}(D)} + C_0\right) \qquad \text{for all } q < d.$$

(iii) If we higher regularity of ω, we can obtain similar estimates using classical elliptic regularity results. Let $p > d$, a $(0,1)$-form $\omega \in W^{1,p}(\Omega^{0,1}D \otimes M_n(\mathbb{C}))$, satisfying a smallness condition and the integrability condition (1.1). Moreover,

$$\|g - id\|_{L^\infty(D)} \leq C \|\omega\|_{W^{1,p}(D)}.$$

Then we can bootstrap the equation solved by g to show that $g \in W^{2,p}_{loc}$ with the expected estimate:

$$\|g - id\|_{W^{2,p}_{loc}(D)} \leq C_p \|\omega\|_{W^{1,p}(D)}.$$

This means that there exists a domain $D_0 \subseteq D$ such that

$$\|g - id\|_{W^{2,p}(D_0)} \leq C_p \|\omega\|_{W^{1,p}(D)} .$$

We can start proving two bootstrap procedures for two types of PDE-s. The general technique is to use show the boundedness of Morrey norms in order to bootstrap beyond the critical embedding level. We use the ideas from [30].

Proposition D.1. *Let $N \in \mathbb{N}^*$, $A \in W^{1,d}(B^{2d}, \mathbb{C}^N)$ and $f_A \in C^\infty(\mathbb{C}^N, \mathbb{C}^N)$ such that there exists $C > 0$ satisfying:*

$$|f_A(\xi)| \leq C|\xi||\nabla A| + |A||\nabla \xi|$$

and $u \in W^{1,d}(B^{2d}, \mathbb{R}^N)$ solving the equation:

$$\Delta u = f_A(u)$$

in a distributional sense, then $u \in W^{2,p}_{loc}(B^{2d}, \mathbb{C})$ for any $p < d$ there exists $C_p > 0$ such that $\|u\|_{W^{2,p}_{loc}(B^{2d})} \leq C_p \|A\|_{W^{1,d}(B^{2d})}$.

Proof of Proposition D.1. Dimension 4 is critical in this case because $W^{2,2d/3} \hookrightarrow L^{2d}$ and we cannot directly bootstrap. In order to improve on the regularity of u, we will use the Adams-Morrey embedding.

Claim. $\exists \gamma > 0$ such that

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<1/4} \rho^{-\gamma} \int_{B^{2d}_\rho(x_0)} |u|^{2d} + |\nabla u|^d dx^{2d} < \infty$$

Let $\varepsilon > 0$ to be fixed later. There exists $\rho_0 > 0$ such that:

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<\rho_0} \|A\|_{W^{1,d}(B^{2d}_\rho(x_0))} < \varepsilon.$$

We can always find such ε and ρ_0 since $\rho \mapsto \int_{B^{2d}_\rho(x_0)}$ is continuous. Fix $x_0 \in B^{2d}_{1/2}(0)$ and $\rho < \rho_0$ arbitrary. To prove this claim we first consider:

$$\begin{aligned} \Delta\varphi &= f_A(u) \quad \text{in } B^{2d}_\rho(x_0) \\ \varphi &= 0 \qquad\quad \text{on } \partial B^{2d}_\rho(x_0) \end{aligned}$$

Let $v := u - \varphi$. Then $\Delta v = 0$ and it is easy to see that $\Delta |v|^{2d} \geq 0,$ and $\Delta |\nabla v|^d \geq 0$ in $B_\rho(x_0)$. Applying the divergence theorem, we get that $\forall r < \rho$:

$$\int_{\partial B_r^{2d}(x_0)} \frac{\partial |v|^{2d}}{\partial r} \geq 0, \text{ and } \int_{\partial B_r^{2d}(x_0)} \frac{\partial |\nabla v|^d}{\partial r} \geq 0.$$

These inequalities imply that:

$$\frac{d}{dr}\left[\frac{1}{r^{2d}}\int_{B_r^{2d}(x_0)} |v|^{2d} dx^{2d}\right] \geq 0 \text{ and } \frac{d}{dr}\left[\frac{1}{r^{2d}}\int_{B_r^{2d}(x_0)} |\nabla v|^d dx^{2d}\right] \geq 0.$$

Since these derivatives are non-negative, it follows that the functions

$$r \mapsto \frac{1}{r^{2d}}\int_{B_r^{2d}(x_0)} |v|^{2d} dx^{2d} \quad \text{and } r \mapsto \frac{1}{r^{2d}}\int_{B_r^{2d}(x_0)} |\nabla v|^d dx^{2d}$$

are increasing in r. In particular:

$$\int_{B_{\rho/4}^{2d}(x_0)} |v|^{2d} dx^{2d} \leq 4^{-2d} \int_{B_\rho^{2d}(x_0)} |v|^{2d} dx^{2d}$$

and

$$\int_{B_{\rho/4}^{2d}(x_0)} |\nabla u|^d dx^{2d} \leq 4^{-2d} \int_{B_\rho^{2d}(x_0)} |\nabla u|^d dx^{2d}.$$

Using these decays, we can bound $\int_{B_{\rho/4}^{2d}(x_0)} |u|^{2d} dx^{2d}$ and $\int_{B_{\rho/4}^{2d}(x_0)} |\nabla u|^d dx^{2d}$ as such:

$$\begin{aligned} \int_{B_{\rho/4}^{2d}(x_0)} |u|^{2d} dx^{2d} &\leq 2^{2d-1} \int_{B_{\rho/4}^{2d}(x_0)} |v|^{2d} + |\varphi|^{2d} dx^{2d} \\ &\leq 2^{-2d-1} \int_{B_\rho^{2d}(x_0)} |v|^{2d} dx^{2d} + 2^{2d-1} \int_{B_\rho^{2d}(x_0)} |\varphi|^{2d} dx^{2d} \\ &\leq 2^{-2} \int_{B_\rho^{2d}(x_0)} |u|^{2d} dx^{2d} + 2^{2d} \int_{B_\rho^{2d}(x_0)} |\varphi|^{2d} dx^{2d} \end{aligned} \tag{D.3}$$

Similarly, for ∇u we get the bound:

$$\int_{B_{\rho/4}^{2d}(x_0)} |\nabla u|^d dx^d \leq 2^{-2d-2} \int_{B_\rho^{2d}(x_0)} |\nabla u|^d dx^{2d} + 2^d \int_{B_\rho^{2d}(x_0)} |\nabla \varphi|^d dx^{2d}. \tag{D.4}$$

There exists a constant $C > 0$ so that we can bound $\Delta\varphi$ in the $L^{2d/3}$ norm:

$$\begin{aligned} ||\Delta\varphi||_{L^{2d/3}(B^{2d}_\rho(x_0))} &= ||f_A(u)||_{L^{2d/3}(B^{2d}_\rho(x_0))} \\ &\leq C\Big(\|A\|_{L^{2d}(B^{2d}_\rho(x_0))}\|\nabla u\|_{L^d(B^{2d}_\rho(x_0))} \\ &\quad + \|\nabla A\|_{L^d(B^{2d}_\rho(x_0))}\|u\|_{L^{2d}(B^{2d}_\rho(x_0))}\Big). \end{aligned}$$

Since φ vanishes on the boundary, by Calderon-Zygmund inequality [35], it follows that

$$\begin{aligned} \|\varphi\|_{W^{2,2d/3}(B^{2d}_\rho(x_0))} \leq C\Big(&\|A\|_{L^{2d}(B^{2d}_\rho(x_0))}\|\nabla u\|_{L^d(B^{2d}_\rho(x_0))} \\ &+ \|\nabla A\|_{L^d(B^{2d}_\rho(x_0))}\|u\|_{L^{2d}(B^{2d}_\rho(x_0))}\Big), \end{aligned}$$

for some constant $C > 0$. Moreover, the Sobolev embedding $W^{1,d} \hookrightarrow L^{2d}$ gives:

$$\begin{aligned} \|\varphi\|_{W^{2,2d/3}(B^{2d}_\rho(x_0))} \leq C\|A\|_{W^{1,d}(B^{2d}_\rho(x_0))}\Big(&\|\nabla u\|_{L^d(B^{2d}_\rho(x_0))} \\ &+ \|u\|_{L^{2d}(B^{2d}_\rho(x_0))}\Big), \end{aligned} \tag{D.5}$$

Thus, combining (D.5) with the inequalities (D.3) and (D.4), we obtain the decay:

$$\begin{aligned} &\int_{B^{2d}_{\rho/4}(x_0)} |u|^{2d} + |\nabla u|^d dx^{2d} \\ &\qquad \leq \Big(2^{-2} + C_0\|A\|_{W^{1,d}(B^{2d}_\rho(x_0))}\Big)\int_{B^{2d}_\rho(x_0)} |u|^{2d} + |\nabla u|^d dx^{2d} \end{aligned}$$

for some constant $C_0 > 0$. We can choose $\varepsilon > 0$ so that $C_0\varepsilon^{2d} \leq 2^{-2}$ to get:

$$\int_{B^{2d}_{\rho/4}(x_0)} |u|^{2d} + |\nabla u|^d dx^{2d} \leq 2^{-1}\int_{B^{2d}_\rho(x_0)} |u|^{2d} + |\nabla u|^d dx^{2d}. \tag{D.6}$$

This estimate gives the required existence of $\gamma > 0$, and proves the claim.

It remains to prove the main regularity result using the claim. From the equation satisfied by u and the decay inequality (D.6), we obtain the bound:

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<1/4} \rho^{-\gamma} \int_{B^{2d}_\rho(x_0)} |\Delta u|^{2d/3} dx^{2d} < \infty$$

By Adams-Morrey embedding, we get a bound on $||I_1 \Delta u||_{L^p(B^{2d}_{1/2}(0))}$, $p > d$ where I_1 is the Riesz potential (see [1]). We obtain $\nabla u \in L^p_{loc}(B^{2d}, \mathbb{C})$ for $p > d$. Hence, the PDE becomes sub-critical and we can bootstrap to get $u \in W^{2,p}_{loc}(B^{2d}, \mathbb{C})$ for any $p < d$. □

Proposition D.2. *Let $A \in L^{2d}(B^{2d}, \mathbb{C})$ and $f_A \in C^\infty(B^{2d}, \mathbb{C})$ such that there exists $C > 0$ satisfying:*

$$|f_A(\xi)| \leq C|\xi|^2 + |A||\xi|$$

and $u \in W^{2,d}(B^{2d}, \mathbb{R}^N)$ satisfying

$$\Delta u = f_A(\nabla u)$$

then $u \in W^{2,p}_{loc}(B^{2d}, \mathbb{C})$ for any $p < 2d$ and $\|u\|_{W^{2,p}_{loc}} \leq C_p \|A\|_{L^{2d}(B^{2d})}$ where C_p is a constant.

Proof of Proposition D.2. Dimension 4 is critical in this case because $\nabla u \in W^{1,d} \hookrightarrow L^{2d}$ and we cannot directly bootstrap. In order to improve on the regularity of u, we will use the Adams-Morrey embedding.

Claim. $\exists \gamma > 0$ such that

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<1/4} \rho^{-\gamma} \int_{B^{2d}_\rho(x_0)} |\nabla u|^{2d} dx^{2d} < \infty$$

Let $\varepsilon > 0$ to be fixed later. There exists $\rho_0 > 0$ such that:

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<\rho_0} \|A\|_{L^{2d}(B^{2d}_\rho(x_0))} < \varepsilon$$

We can always find such ε and ρ_0 since $\rho \mapsto \int_{B^{2d}_\rho(x_0)}$ is continuous. Fix $x_0 \in B^{2d}_{1/2}(0)$ and $\rho < \rho_0$ arbitrary. To prove this claim we first consider :

$$\begin{aligned}\Delta\varphi &= f_A(\nabla u) && \text{in } B^{2d}_\rho(x_0)\\ \varphi &= 0 && \text{on } \partial B^{2d}_\rho(x_0)\end{aligned}$$

Let $v := u - \varphi$. Then $\Delta v = 0$ and it is easy to see that $\Delta|\nabla v|^{2d} \geq 0$ in $B^{2d}_r(x_0)$, for some $r < \rho$. Applying the divergence theorem, we get that $\forall r < \rho$:

$$\int_{\partial B^{2d}_r(x_0)} \frac{\partial|\nabla v|^{2d}}{\partial r} \geq 0$$

This implies that

$$\frac{d}{dr}\left[\frac{1}{r^{2d}}\int_{B^{2d}_r(x_0)} |\nabla v|^{2d} dx^{2d}\right] \geq 0$$

and consequently the function $r \mapsto \dfrac{1}{r^{2d}}\int_{B^{2d}_r(x_0)} |\nabla v|^{2d} dx^{2d}$ is increasing. In particular,

$$\int_{B^{2d}_{\rho/4}(x_0)} |\nabla v|^{2d} dx^{2d} \leq 4^{-2d}\int_{B^{2d}_\rho(x_0)} |\nabla v|^{2d} dx^{2d}$$

Using this decay, we can obtain a bound for $\int_{B^{2d}_{\rho/4}(x_0)} |\nabla u|^{2d} dx^{2d}$:

$$\begin{aligned}\int_{B^{2d}_{\rho/4}(x_0)} |\nabla u|^{2d} dx^{2d} &\leq 2^{2d-1}\int_{B^{2d}_{\rho/4}(x_0)} |\nabla v|^{2d} + |\nabla\varphi|^{2d} dx^{2d}\\ &\leq 2^{-2d-1}\int_{B^{2d}_\rho(x_0)} |\nabla v|^{2d} dx^{2d} + 2^{2d-1}\int_{B^{2d}_\rho(x_0)} |\nabla\varphi|^{2d} dx^{2d}\\ &\leq 2^{-2}\int_{B^{2d}_\rho(x_0)} |\nabla u|^{2d} dx^{2d} + 2^{2d}\int_{B^{2d}_\rho(x_0)} |\nabla\varphi|^{2d} dx^{2d}\end{aligned}$$

Moreover, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}||\Delta\varphi||^{2d}_{L^d(B^{2d}_\rho(x_0))} &\leq C_1||f_A(\nabla u)||^{2d}_{L^d(B^{2d}_\rho(x_0))}\\ &\leq C_2\left(\|A\|_{L^{2d}(B^{2d}_\rho(x_0))}\|\nabla u\|_{L^{2d}(B^{2d}_\rho(x_0))} + \|\nabla u\|^2_{L^{2d}(B^{2d}_\rho(x_0))}\right)^{2d}\\ &\leq 2^{2d-1}C_2\Big(\|A\|^{2d}_{L^{2d}(B^{2d}_\rho(x_0))}\|\nabla u\|^{2d}_{L^{2d}(B^{2d}_\rho(x_0))}\\ &\qquad + \|\nabla u\|^{4d}_{L^{2d}(B^{2d}_\rho(x_0))}\Big)\end{aligned}$$

Since φ vanishes on the boundary of $B^{2d}_\rho(x_0)$, then by elliptic estimates, we have for some constant $C > 0$ the inequality:

$$\|\varphi\|^{2d}_{W^{2,d}(B^{2d}_\rho(x_0))} \leq C||\Delta\varphi||^{2d}_{L^d(B^{2d}_\rho(x_0))},$$

from which we deduce that $\|\nabla\varphi\|^{2d}_{L^{2d}(B^{2d}_\rho(x_0))} \leq C||\Delta\varphi||^{2d}_{L^d(B^{2d}_\rho(x_0))}$.
Putting this inequality together with the bound on $\Delta\varphi$, we get the following decay:

$$\int_{B^{2d}_{\rho/4}(x_0)} |\nabla u|^{2d} dx^{2d} \leq \left(2^{-2} + C_0||A||^{2d}_{L^{2d}(B^{2d}_\rho(x_0))}\right) \int_{B^{2d}_\rho(x_0)} |\nabla u|^{2d} dx^{2d}$$

We can choose $\varepsilon > 0$ so that $C_0\varepsilon^{2d} \leq 2^{-2}$ and obtain:

$$\int_{B^{2d}_{\rho/4}(x_0)} |\nabla u|^{2d} dx^{2d} \leq 2^{-1} \int_{B^{2d}_\rho(x_0)} |\nabla u|^{2d} dx^{2d}.$$

This decay implies the existence of $\gamma > 0$ and proves the claim.

Thus, there exists $\gamma > 0$ such that

$$\sup_{x_0 \in B^{2d}_{1/2}(0),\ 0<\rho<1/4} \rho^{-\gamma} \int_{B^{2d}_\rho(x_0)} |\Delta u|^d dx^{2d} < \infty$$

By Adams-Morrey embedding, we get a bound on $||I_1\Delta u||_{L^p}$, $p > d$ where I_1 is the Riesz potential (see [1]). Thus, it follows that $\nabla u \in L^p_{loc}(B^{2d}, \mathbb{C})$ for $p > 2d$. Since the PDE becomes sub-critical, we can bootstrap to get $u \in W^{2,p}_{loc}(B^{2d}, \mathbb{C})$ for any $p < 2d$. □

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