Area Variations under pointwise Lagrangian and Legendrian Constraints

Tristan Rivière

Abstract : *The text below is the proceeding of the talk given by the author on June 14th 2023 in Orkanger (Norway) at the occasion of the 2023 Abel Symposium. It is presenting some of the most recent developments in the study of the area variations under pointwise Lagrangian and Legendrian constraints.*

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I Some element of context

The simplest framework of our investigations in this presentation could be formulated as follows : study area variations among maps u from an oriented 2-manifold Σ into $\mathbb{C}^2$ under the Lagrangian constraint

$$u^*\omega = 0 \ , \tag{I.1}$$

where ω is the standard symplectic form $\omega = dy_1 \wedge dy_2 + dy_3 \wedge dy_4$. More precisely we ask ourselves how to implement variational operations (minimization under various constraints, minmax) for the standard area Lagrangian

$$A(u) := \int_\Sigma |\partial_{x_1} u \wedge \partial_{x_2} u| \ dx_1 \wedge dx_2$$

among maps satisfying the pointwise constraint[1]

$$du_1 \wedge du_2 + du_3 \wedge du_4 = 0 \ .$$

The Lagrangian A is notoriously known in calculus of variations to be delicate to work with (due mostly to the very large invariance group in the domain : the "gauge group" of diffeomorphisms of the surface Σ). In order to "break" this invariance and reduce the gauge group to a finite dimensional one, it is preferable instead to consider the Dirichlet Energy of the map u for a variable metric g of Σ

$$E(u,g) := \frac{1}{2}\int_\Sigma |du|^2_g \ dvol_g \ge A(u) \ . \tag{I.2}$$

with equality if and only if u is conformal with respect to the metric g that is in isothermic or conformal coordinates such that $g = e^{2\lambda}\ [dx_1^2 + dx_2^2]$ there holds

$$|\partial_{x_1} u|^2 = |\partial_{x_2} u|^2 \quad \text{and} \quad \partial_{x_1} u \cdot \partial_{x_2} u = 0 \tag{I.3}$$

[1] In case u is a graph $u(x_1,x_2) = (x_1, x_2, v(x_1,x_2))$ the constraint (I.1) is equivalent to the incompressibility condition

$$\det(\nabla v) = -1 \ .$$

where $\cdot$ denotes the standard scalar product in $\mathbb{C}^2$. Stationary critical points for fixed g have an holomorphic Hopf differential

$$H(u) := \partial_z u \cdot \partial_z u \; dz \otimes dz$$

while making g vary simultaneously in the Moduli space of constant Gauss curvature metrics is known to imply $H(u) = 0$ and equality holds in (I.2). Hence critical points of $E(u, g)$ are critical points of the area.

Assuming u is a conformal critical point of the Dirichlet energy under the Lagrangian constraint, in local conformal coordinates, the Euler Lagrange Equation of our constrained variational problem is given by

$$0 = \frac{d}{dt} E(u + t\psi, g)\Big|_{t=0} = \int_\Sigma du \cdot_g d\psi \; dvol_g \tag{I.4}$$

for any ψ satisfying

$$d\psi_1 \wedge du_2 + d\psi_3 \wedge du_4 + du_1 \wedge d\psi_2 + du_3 \wedge d\psi_4 = 0 \qquad \Longleftrightarrow \qquad d\,[\psi \cdot i du] = 0$$

Hence ψ is an admissible infinitesimal perturbation for the Lagrangian constraint if and only if there exists locally an Hamiltonian function h such that

$$\psi \cdot i du = dh$$

Recall that a 2-plane L is Lagrangian (i.e. the restriction of the symplectic form ω on L is zero) if and only if the multiplication by i realizes an isometry[2] between L and the 2-plane orthogonal to L. Assuming u is locally an conformal Lagrangian immersion we deduce that

$$e^{-\lambda}\,(\partial_{x_1} u, \partial_{x_2} u, i\,\partial_{x_1} u, i\,\partial_{x_2} u) \quad \text{is an orthonormal basis of } \mathbb{C}^2 \qquad \text{where} \qquad e^\lambda = |\partial_{x_1} u| = |\partial_{x_2} u| \ .$$

Hence ψ is an admissible variation if and only if there exists an hamiltonian function h ad two functions a ad b such that

$$\psi = e^{-2\lambda}\, [\partial_{x_1} h\, i\partial_{x_1} u + \partial_{x_2} h\, i\partial_{x_2} u + a\,\partial_{x_1} u + b\,\partial_{x_2} u]$$

ad we deduce the following expression of the Euler Lagrange equation for a conformal Lagrangian immersion

$$\forall\, h \in C^1(\Sigma) \qquad \int_\Sigma (\partial_{x_1} h\; \partial_{x_1} u \cdot i\Delta u + \partial_{x_2} h\; \partial_{x_2} u \cdot i\Delta u)\; |\nabla u|^{-2}\; dx^2 = 0$$

which is also equivalent to

$$\mathrm{div}\left[|\nabla u|^{-2}\, \nabla u \cdot i\Delta u\right] = 0$$

We deduce that the equation is a **3rd order equation**. It has been derived under very strong regularity and non degeneracy assumptions on u and obviously does not make sense for an arbitrary $u \in W^{1,2}(\Sigma, \mathbb{C}^2)$. An alternative formulation which is this time compatible with the sole assumption that u is in $W^{1,2}$ and weakly conformal is derived in [7] and reads

$$\left\{\begin{array}{l} \mathrm{div}\,(\mathfrak{g}\, \nabla u) = 0 \\[2mm] \mathrm{div}\left(\mathfrak{g}^{-1}\, \nabla \mathfrak{g}\right) = 0\ , \end{array}\right. \tag{I.5}$$

[2] Being Lagrangian for a 2-plane is in a sense the opposite of being holomorphic in which case the multiplication by i realizes an isometry from L into itself.

where $\mathfrak{g} = e^{i\beta}$ is an S^1 valued map[3] which is the Lagrange multiplier associated to the Lagrangian pointwise constraint (I.1). We call this system the *Conformal Hamiltonian Stationary System.* The first equation is a structural equation shared by any conformal parametrization of any Lagrangian surface in $\mathbb{C}^2$ while the second one is the Euler Lagrange equation itself. It is saying that the Lagrange multiplier $\mathfrak{g}$ is an S^1 harmonic map. These equations are obviously calling upon the use of classical elliptic theory for solving regularity or existence questions. The difficulty however is that there is a-priori no information on the function space to which $\mathfrak{g}$ belongs to in order to start implementing this theory. In fact one can construct very "pathological" solutions to (I.5) with respect to regularity.

Theorem I.1. *[7] For any $1 \le p < 2$ there exists $(u, \mathfrak{g}) \in W^{1,p}(D^2, \mathbb{C}^2) \times W^{1,p}(D^2, S^1)$ solution to the Conformal Hamiltonian Stationary System*

$$\left\{ \begin{array}{l} div(\mathfrak{g}\,\nabla u) = 0 \quad in \ \ \mathcal{D}'(D^2) \\[2ex] |\partial_{x_1} u|^2 = |\partial_{x_2} u|^2 \qquad a.\ e.\ in\ D^2 \\[2ex] \partial_{x_1} u \cdot \partial_{x_2} u = 0 \qquad a.\ e.\ in\ D^2 \\[2ex] \mathrm{div}\left[\mathfrak{g}^{-1}\,\nabla \mathfrak{g}\right] = 0 \quad in\ \mathcal{D}'(D^2) \end{array} \right. \tag{I.6}$$

and u as well as $\mathfrak{g}$ are nowhere continuous. □

This result is suggesting that questions regarding existence and regularity cannot be settled at the strictly "PDE level" whose formulation even is problematic. Indeed in order to give a distributional meaning to (I.5) one needs at least an assumption like $\mathfrak{g}$ is in $W^{1,1}$ or $H^{1/2}$. This is certainly an interesting elliptic PDE problem by itself worth to be studied but somehow artificial though since these assumptions are not given a-priori by any variational operation we are aiming at doing.

Having exclusively a functional analysis approach to the system (I.5) and to the variation of the area under Lagrangian constraints is insufficient for answering natural analysis questions regarding this problem. In the next subsection we consider this variational problem from a more geometric perspective in order to collect useful informations.

I.1 Hamiltonian and Lagrangian Stationary Immersions

A Kähler surface is a complex two dimensional manifold (M^2, J) with a compatible[4] symplectic form ω. Lagrangian sub-manifolds are the ones on which the restriction of the symplectic form vanishes. In particular, Lagrangian sub-manifolds are the ones for which the complex structure realizes an isometry between the tangent and the normal spaces at every points.

The study of area variations under Lagrangian constraints has been initiated in a series of work by Yong-Geun Oh ([15], [16]). He proposed to minimize the area under the Lagrangian constraint within

[3]The uni-valued map $e^{i\beta}$ plays the rôle of a Lagrange multiplier issued from the incompressibility constraint similarly as the rôle played by the "pressure" in the variational formulation of Euler equations. The parallel between the Lagrangian and the incompressibility constraint is maybe more obvious while considering the particular case of a graph

$$u(x_1, x_2) := (x_1, x_2, w(x_1, x_2))$$

where $w\ :\ \Sigma \rightarrow\ \mathbb{R}^2$. We have

$$u^*\omega = 0 \quad \Longleftrightarrow \quad \det(\nabla w) = -1$$

[4]A symplectic form is said to be compatible with a complex or with an almost complex structure J if $g(X, Y) := \omega(X, JY)$ defines a riemannian metric.

a Lagrangian homology class with the objective to obtain in this way very special representative of these classes. He called the sub-manifolds critical points of the area within Lagrangian sub-manifolds *hamiltonian stationary* for sub-manifolds which are critical points under "local" Lagrangian deformations[5] only which are given by hamiltonian vector-fields of the form

$$J\,\nabla h\ . \tag{I.7}$$

where h is an arbitrary compactly supported function in M^2. A Lagrangian immersion u is then *Hamiltonian stationary* (or H-minimal) if

$$\forall h\in C^1(M)\qquad \int_\Sigma J\,\nabla^\Sigma h\cdot\vec{H}_u\ dvol_{g_u}=0\ , \tag{I.8}$$

where $\vec{H}$ is the mean curvature vector of the immersion u. This implies that the contraction of the mean curvature vector $\vec{H}_u$ with the Kähler form

$$\omega\llcorner\vec{H}_u=\alpha_{\vec{H}}$$

defines an <u>harmonic</u> one form of the sub-manifold Σ. Such surfaces are also called *Hamiltonian minimal* or simply $H-minimal$ surfaces. These pioneered works on the subject contain very interesting and now classical conjectures in the field about possible volume minimizing candidates under hamiltonian deformations within $\mathbb{C}^2$ or $\mathbb{C}P^2$.

Y.-G. Oh computed in particular the Euler Lagrange equations associated to this variational problem. In the special case of Kähler-Einstein manifolds (satisfying $Ric(g)=\lambda\,g$), it has been discovered by Dazor [6] that the mean curvature of a Lagrangian surface Σ is given by

$$\vec{H}:=J\,\nabla^\Sigma\beta\ ,$$

where β is a locally well defined function on Σ and ∇^Σ is the gradient of β for the induced metric on Σ. In the particular case when the manifold is Ricci flat (which implies that M^2 is Calabi Yau), if Ω is a normalized global holomorphic section of the canonical bundle $K:=\wedge^{2,0}M$ the function β is given by

$$e^{i\beta}\iota_\Sigma^\ast\Omega=dvol_g$$

where ι_Σ is the canonical embedding of Σ in M^2. Hence β is a globally well defined function in $\mathbb{R}/2\pi\mathbb{Z}$ and is called *Lagrangian angle*. For general Kähler-Einstein manifolds β is still a well defined function in $\mathbb{R}/2\pi\mathbb{Z}$ in case the induced canonical connection of the restriction to Σ of the canonical bundle $\iota_\Sigma^\ast K$ of M^2 (which is flat) has no monodromy (see [29]). In the $M^2:=\mathbb{C}^2$ case that we were considering in the first section, for any Lagrangian immersion u from Σ into $\mathbb{C}^2$, the map $\mathfrak{g}:=e^{i\beta}$ satisfies

$$\iota_\Sigma\,[dz_1\wedge dz_2]=\mathfrak{g}^{-1}\ |\partial_{x_1}u\wedge\partial_{x_2}u|\ \ dx_1\wedge dx_2\ .$$

In the case where u is Hamiltonian stationary (i.e. u satisfies (I.4)), the map $\mathfrak{g}$ is the one in (I.5).

Observe now that for general Kähler-Einstein Target M^2 the Hamiltonian stationarity condition (I.8) is becoming

$$\forall h\in C^1(M)\qquad \int_\Sigma \nabla^\Sigma h\cdot\nabla^\Sigma\beta\ dvol_\Sigma=0\ . \tag{I.9}$$

This is equivalent to the fact that locally

$$\Delta^\Sigma\beta=0\ , \tag{I.10}$$

[5] A slightly more restrictive condition is allowing for more deformations. A Lagrangian immersion is called *Lagrangian stationary* when it is a critical point of the area for area deformations of the form JY where Y is a vector-field associated to a closed form which is not necessarily exact.

which is nothing but the second equation in (I.5) in the $M^2 = \mathbb{C}^2$ case.

In the particular case when β is a globally univalued function then, for a compact surface Σ, the harmonic function equation (I.10) is implying $\beta = \beta_0$ and the Euler-Lagrange equation is simply the minimal surface equation $\vec{H} = 0$. Such a sub-manifold is called *minimal lagrangian* (This holds in particular if the immersion is *Lagrangian stationary* and not only *Hamiltonian stationary*. If this is the case the infinitesimal perturbation $J\nabla^\Sigma\beta$ is admissible and one gets $\nabla^\Sigma\beta \equiv 0$.) . If the Kähler-Einstein manifolds is Ricci flat then any *minimal lagrangian* sub-manifold is calibrated by $e^{i\beta_0}\,\Omega$ and and realizes in this way an absolute minimizer[6] in it's homology class (see [8]). While it is a very interesting object of research with numerous applications in geometry (and in high energy physics[7]) in general there are no reasons why $d\beta$ should be equal to zero and the general hamiltonian stationary equation in Kähler-Einstein Manifold is the 3rd order equation[8]. It is interesting to observe that, while having $d\beta$ being harmonic we have a "finite dimensional perturbation" of $\beta = \beta_0$ but nevertheless H-minimal surfaces have very different features. For instance there exists closed $H-$minimal surfaces in $\mathbb{C}^2$ (the Clifford torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ is an example of such a surface[9]) while the existence of closed minimal surfaces is prevented by the maximum principle.

To end this section on the interplay between the Lagrangian constraint and the area variations, we would like to briefly mention the *Lagrangian Mean Curvature Flow* without really giving justice to the importance of this geometric analysis object. It has been discovered by Knut Smoczyk [25] that the classical Mean Curvature Flow is preserving the Lagrangian constraint in Kähler Einstein manifolds. Since this discovery, numerous works have been devoted to the mean curvature flow applied to Lagrangian surfaces in particular in a Calabi-Yau manifold ([12], [13], [14]...) where the Lagrangian angle function exists and for which the Lagrangian Mean Curvature flow takes the elegant formulation

$$\frac{\partial\beta}{\partial t} + \Delta^\Sigma\beta = 0\ .$$

These works are partly motivated by the Thomas Yau Conjecture about the existence of a "stability condition"[10] for the characterisation of Hamiltonian Isotopy classes by a minimal Lagrangian representatives (see [27], [28]).

II The Schoen Wolfson existence and regularity result

In a breakthrough paper [23], Rick Schoen and Jon Wolfson performed the very first variational construction of *minimizing stationary Lagrangian surfaces* in lagrangian homology classes[11] of Kähler Einstein surfaces . Their original motivation was to realize any Lagrangian homology class of a Kähler Einstein surface by a sum of Lagrangian minimal branched immersions. In case the Kähler Einstein surface would be Ricci Flat (i.e. Calabi Yau) this is implying that these surfaces are calibrated and these homology

[6]The local stability of arbitrary minimal lagrangian sub-manifolds generalizes to the situation of general Kähler-Einstein manifolds of non positive Ricci. This being said we will be interested in the other case in the present work that is $Ric(g) > 0$.

[7]Minimal Lagrangian submanifolds of Calabi Yau 3 and 4-folds appear in string theory under the name of "supersymmetric cycle" and they play a key role in the Strominger Yau Zaslow theory of mirror symmetry.

[8]The equation is second order with respect to the multivalued Lagrangian angle function β but the expression of $d\beta$ involves 2 derivatives with respect to the Lagrangian immersion.

[9]One of the Oh conjecture says that this torus should be minimizing in it's Hamiltonian isotopy class : The class of tori which can be obtained through hamiltonian deformations that is deformations which infinitesimally are given by (I.7) (see [15]).

[10]A classical stability condition in algebraic geometry is the one for the so called Hitchin-Kobayachi correspondence between holomorphic structures over a Kähler Manifold and solutions to the Hermitian Yang-Mills equation.

[11]Lagrangian homology classes are elements in $H_2(M, \mathbb{Z})$ that can be realized by integral Lagrangian cycles. It is proved in [23] that this is simply equivalent to

$$[\omega](\alpha) = 0\ .$$

classes would have then been realised by very rigid objects called *Special Lagrangian cycles*. This project bumped into the discovery by Schoen and Wolfson of a new type of singularities. The two authors proved that area minimizing lagrangian stationary surfaces are branched immersions with possible isolated "conical" singularities. Precisely the following result holds.

Theorem II.1. *[23] Let (M,ω,J) be a Kähler Einstein surface. Every Lagrangian homology class in $\alpha\in H_2(M,\mathbb{Z})$ can be represented by the Lagrangian Lipschitz image by a map u of a closed surface Σ. The map u is a branched Lagrangian smooth Hamiltonian stationary (i.e. $H-$minimal) immersion away from isolated singularities $(a_k)_{k\in K}$ at which the associated Lagrangian angle function $\mathfrak{g}$ satisfies*

$$*\,d\left[\mathfrak{g}^{-1}\,d\mathfrak{g}\right]=\sum_{k\in K}d_k\,\delta_{a_k}\qquad in\ \mathcal{D}'(\Sigma)\ ,$$

where $d_k=\pm 1$ and K is a finite set that can be empty. In particular

$$\sum_{k\in K}d_k=2^{-1}\,c_1(M)(\alpha)$$

where $c_1(M)$ is the first Chern Class of M. If $c_1(M)(\alpha)=0$ and $K=\emptyset$ then u is a minimal Lagrangian branched immersion and if in addition to $K=\emptyset$ we havee that M^2 is Calabi Yau ($c_1(M)=0$) the current $u_*[\Sigma]$ is Special Lagrangian. □

The proof of theorem II.1 is based on the following regularity result for $W^{1,2}-$Lagrangian minimizing maps.

Theorem II.2. *Let $u\in W^{1,2}(D^2,\mathbb{C}^2)$, satisfying the following three conditions*

i) Weakly Lagrangian :

$$u^*\omega=0$$

ii) Weakly Conformal :

$$|\partial_{x_1}u|^2=|\partial_{x_2}u|^2\qquad and\qquad \partial_{x_1}u\cdot\partial_{x_2}u=0$$

iii) Locally minimizing :

$$\forall\ B_r(x)\subset D^2\quad \forall v\in W^{1,2}(D^2,\mathbb{C}^2)\qquad s.\ t.\qquad v^*\omega=0\quad and\quad v=u\ on\ \partial B_r(x)$$

$$then\qquad \int_{B_r(x)}|\nabla v|^2\ dx^2\geq\int_{B_r(x)}|\nabla u|^2\ dx^2$$

Then u is a possibly branched smooth H-minimal immersion away from isolated conical point singularities. There exists a map $\mathfrak{g}\bigcap_{p<2}W^{1,p}(D^2,\mathbb{C}^2)$ such that

$$\left\{\begin{array}{l} div\,(\mathfrak{g}\,\nabla u)=0\\[4mm] div\left(\mathfrak{g}^{-1}\,\nabla\mathfrak{g}\right)=0\ ,\end{array}\right.$$

and there exists isolated points $(a_k)_{k\in K}$ in the disc D^2 such that[12]

$$curl(\mathfrak{g}^{-1}\,\nabla\mathfrak{g})=2\pi\,i\sum_{k\in K}d_k\,\delta_{a_k}\qquad in\ \mathcal{D}'(D^2)\ .$$

□

[12]The fact that the points a_k are isolated is implying that for any open subset $\omega\subset D^2$ with $\overline{\omega}\subset D^2$, ω must contain at most finitely many a_k and, while K is a-priori not necessarily finite, the sum $\sum_{k\in K}d_k\,\delta_{a_k}$ nevertheless makes sense in $\mathcal{D}'(D^2)$.

III The Schoen Wolfson Conical Singularities

In fact these conical singularities are proved to exist for some Lagrangian homology classes α (see [11] and [30]) in some Kähler-Einstein surfaces even if $c_1(M)(\alpha) = 0$. This last fact is a surprise and is one of the reasons why the variational analysis of Lagrangian surfaces, even in the lowest co-dimension 2, is particularly involved.

The explicit form of the conical singularities is fully given in [23]. It can be observed that, unlike for the branched points, at these singularities the Gauss map[13] is not continuous. This stands in contrast with classical stationary surfaces for the area (or minimal surfaces) in $\mathbb{C}^2$ which can degenerate only at branched points at which the Gauss map is still continuous. These isolated point singularities discovered by Schoen and Wolfson are asymptotically given by cones parametrized by maps of the form

$$u_{p,q} : D^2 \to \mathbb{C}^2, \quad (r,\theta) \mapsto \frac{r^{\sqrt{pq}}}{\sqrt{p+q}} \begin{pmatrix} \sqrt{q}e^{ip\theta} \\ i\sqrt{p}e^{-iq\theta} \end{pmatrix}. \tag{III.1}$$

called nowadays Schoen–Wolfson cones. A careful analysis of this expression shows that the pullback by the Gauss map, taking values into the Lagrangian Grassmann manifold $\Lambda(2)$, of the generator $H^1(\Lambda(2),\mathbb{Z})$ realizes a non trivial element in $H^1(D^2 \setminus \{0\},\mathbb{Z})$ called *Maslov index*[14] (see for instance [3]). Some of these cones, the ones for which $p - q = \pm 1$, are shown in [23] to be stable with respect to area variations under the pointwise Lagrangian constraint (I.1). While we have seen that the presence of Schoen-Wolfson conical singularities cannot be excluded even in the case even if $c_1(M)(\alpha) = 0$ where the sum of their Maslov indices is zero, a question naturally emerged :

is there any restriction on the location of the Schoen–Wolfson conical singularities?

We are not able at this stage to give a satisfying answer to this question but an attempt has been made for constructing surfaces (mostly discs) with isolated Schoen–Wolfson conical singularities for which the location will be strongly correlated to the boundary data. Precisely we have the following result

Theorem III.1. *Let $(d_k)_{k=1\cdots N}$ such that $d_k = \pm 1$. Let a_k be N distinct points in D^2 and let G be the Green's function solution to*

$$\left\{ \begin{array}{ll} \Delta G & = 2\pi \displaystyle\sum_{k=1}^{N} d_k\, \delta_{a_k} \quad \text{in } \mathcal{D}'(D^2) \\[2mm] G & = 0 \quad \text{on} \quad \partial D^2 \ . \end{array} \right. \tag{III.2}$$

Assume each of the connected components of $G \neq 0$ is a disc containing exactly one a_k. Let $\psi \in C^1(\partial D^2,\mathbb{C})$. Then there exists a solution

$$(u,\mathfrak{g}) \in \bigcap_{p<2} W^{1,p}(D^2,\mathbb{C}^2) \times W^{1,p}(D^2,S^1)$$

smooth away from the a_k of the Hamiltonian Stationary Equation

$$\left\{ \begin{array}{l} div(\mathfrak{g}\,\nabla u) = 0 \quad in \ \ D^2 \\[2mm] u_1 + i\,u_2 = \phi \quad \ in\ \partial D^2 \\[2mm] \text{div}\left[\mathfrak{g}^{-1}\,\nabla \mathfrak{g}\right] = 0 \quad in\ D^2 \\[2mm] \mathfrak{g}^{-1}\,\partial_r \mathfrak{g} = 0 \quad \ in\ \partial D^2 \ . \end{array} \right. \tag{III.3}$$

[13]The Gauss map is the map which assigns at every point the oriented tangent 2-plane. This is a map from Σ into the 3 dimensional Lagrangian Grassmanian $\Lambda(2) \simeq U(2)/O(2)$, sub-manifold of the Grassmann manifold $G_2(\mathbb{R}^4) \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$

[14]The Maslov Index of the cone III.1 is $p - q$. This is exactly the degree of the Lagrangian angle map around the origin.

with

$$\mathfrak{g}^{-1}\nabla\mathfrak{g} = i\,\nabla^\perp G\ .$$

where $\nabla^\perp G := (-\partial_{x_2}G, \partial_{x_1}G)$. *Moreover* u *is conformal on* D^2 *and on each connected component* ω *of* $G \neq 0$ *and there exists* $\tilde{u} \in W^{1,2}(\omega, \mathbb{C}^2)$ *and* $A^\omega \in \mathbb{C}$ *such that*

$$u = \tilde{u} + A^\omega\, g\ . \tag{III.4}$$

This solution (u, g) *is unique in this class moreover* $\tilde{u} \in C^{1,\sqrt{2}-1}(D^2)$. *There exists a sub-space of co-dimension at most* N *of* $\phi \in H^{1/2}(\partial D^2, \mathbb{C})$ *such that* u *has finite area (i.e.* $\forall\omega\ A^\omega = 0$ *and* $\tilde{u} = u$*) and realizes an Hamiltonian Stationary Conformal Immersion with isolated Schoen–Wolfson conical Singularities of Maslov degree* d_k *at the* a_k. □

Remark III.1. *In the cases* $N = 1, 2$ *we are able to prove that the subspace of* $\phi \in H^{1/2}(\partial D^2, \mathbb{C})$ *such that* u *has finite area has exactly co-dimension 2.* □

Here is a list of natural open problems that come out of the statement of theorem III.1.

Some open questions

- Can Theorem III.1 be generalised to construct Hamiltonian stationary Lagrangian discs with prescribed Schoen–Wolfson singularities in Calabi-Yau surfaces?
- It would be interesting to investigate the same question for closed surfaces: can one construct closed Hamiltonian stationary surfaces (in $\mathbb{C}^2$, in Calabi-Yau manifolds or even in Kähler–Einstein manifolds) with isolated Schoen–Wolfson singularities? Closed Hamiltonian stationary surfaces have been constructed, among others, by I. Castro and F. Urbano [5], F. Hélein and P. Romon [10, 9], H. Anciaux [2], but in all cases they construct branched immersions without cone singularities.
- Does there exist complete Hamiltonian Stationary planes in $\mathbb{C}^2$ with isolated singularities beside the Schoen–Wolfson cones (III.1)? If yes, are they stable with respect to Hamiltonian deformations?
- Is the sufficient condition on G given by theorem III.1 necessary for the existence of finite area Hamiltonian Stationary discs in $\mathbb{C}^2$ with ± 1 Schoen–Wolfson cones? Is there some notion of renormalized energy in the style of the Ginzburg–Landau renormalized energy governing the location of the singular Schoen Wolfson cones ?
- Is there a counterpart to theorem III.1 for higher multiplicities ($|d_k| > 1$)?
- Is it possible to construct Hamiltonian stationary Lagrangian discs with locally finite area but infinitely many Schoen–Wolson singularities accumulating at the boundary?

IV The Monotonicity Formula and the need to "lift" to Legendrian immersions

The starting point for performing Schoen Wolfson analysis is the search of a monotonicity formula. Conserved quantities and monotonicity formula, which are integrated versions of conservation laws, are fundamental notions in the calculus of variations. These identities are reflecting the existence of groups of symmetries of the underlying Lagrangian most of the time in locally isotropic spaces. The most illustrative example is maybe the monotonicity formula for minimal immersion of a surfaces in a Euclidian space $\mathbb{R}^n$. This identity says the following: let u be a minimal immersion of a surface Σ without boundary into $\mathbb{R}^n$, then the following **conservation law** holds

$$\forall\, r > 0 \qquad \frac{1}{r^2}\int_{\rho(u)<r} dvol_{u^*g_{\mathbb{R}^n}} = \int_{\rho(u)<r} \frac{|(\nabla\rho)^\perp|^2}{\rho(u)^2}\, dvol_{u^*g_{\mathbb{R}^n}} + \theta_0 \tag{IV.1}$$

where ρ is the distance function to the origin of the space, $dvol_\Sigma$ denotes the volume form on Σ induced by the immersion Φ , $(\nabla^\Sigma\rho)^\perp$ is the projection to the normal co-dimension 2 plane to the immersion of the gradient of ρ and

$$\theta_0 := \pi\, \mathrm{Card}\left(u^{-1}(0)\right)$$

where $\mathrm{Card}\left(u^{-1}(0)\right)$ is the number of pre-images of the origin 0 by the immersion u (see for instance [24]). In particular we obtain

$$\frac{d}{dr}\left[\frac{1}{r^2}\int_{|u|<r} dvol_{u^*g_{\mathbb{R}^n}}\right]\ge 0\ . \tag{IV.2}$$

This formula which holds as well for much weaker mathematical objects critical point of the area than minimal immersions (such as *stationary varifolds* for instance) is the starting point for the analysis of the variation of the area in euclidian or even Riemannian spaces.

At this point Schoen and Wolfson made the following crucial observation that such a monotonicity formula is not true in $\mathbb{C}^2$ for $H-$minimal surfaces in general (even minimizing !). They give an explicit simple counter-example in [22] (section 4). It can be described as follows consider the sequence of maps from the cylinders $\Sigma := S^1\times[-1/\varepsilon,+1/\varepsilon]$ into $\mathbb{C}^2$ given by

$$u_\varepsilon(e^{i\theta},t) := (\varepsilon\,\cos\theta,\varepsilon\,\sin\theta,\varepsilon\, t,0)$$

It is obviously a conformal Lagrangian immersion for the canonical metric $g = d\theta^2 + dt^2$ on $\Sigma := S^1\times[-1/\varepsilon,+1/\varepsilon]$, moreover it satisfies

$$\frac{\partial}{\partial\theta}\left[e^{-i\theta}\,\frac{\partial u}{\partial\theta}\right]+\frac{\partial}{\partial t}\left[e^{-i\theta}\,\frac{\partial u}{\partial t}\right]=0$$

Hence u is $H-$minimal with Lagrangian angle function $e^{i\beta} = e^{-i\,\theta}$ which satisfies the S^1-harmonic equation

$$-\left[\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial t^2}\right](e^{-i\theta}) = e^{-i\,\theta}\ \left[\left|\frac{\partial e^{-i\theta}}{\partial\theta}\right|^2+\left|\frac{\partial e^{-i\theta}}{\partial t}\right|^2\right]\ .$$

We have

$$\int_{|u|<1} dvol_{u^*g_{\mathbb{C}^2}} = O(\varepsilon)\quad\text{ and }\quad (2\,\varepsilon)^{-2}\int_{|u|<2\,\varepsilon} dvol_{u^*g_{\mathbb{C}^2}}\ge\pi\,\sqrt{3}\ .$$

This contradicts obviously the monotonicity (IV.2).

The existence of such a counter-example could be explained by the fact that the generator of dilation[15] in $\mathbb{C}^2$ is not hamiltonian :

$$y_1\,\partial_{y_1}+y_2\,\partial_{y_2}+y_3\,\partial_{y_3}+y_4\,\partial_{y_4}\ne i\nabla h$$

The major observation made by Schoen and Wolfson is that by adding the Legendrian coordinate (if it locally exists) this problem can be overcome. If u is Lagrangian in $\mathbb{C}^2$ there holds

$$d(u_1\,du_2-u_2\,du_1+u_3\,du_4-u_4\,du_3)=0$$

and one introduces the local Legendrian coordinate

$$d\varphi_u := u_1\,du_2-u_2\,du_1+u_3\,du_4-u_4\,du_3\ .$$

In $\mathbb{R}^5$ one introduce the form

$$\alpha := -d\varphi + y_1\,dy_2-y_2\,dy_1+y_3\,dy_4-y_4\,dy_3$$

[15] The existence of monotonicity formula, which is a conservation law, is intimately linked to the existence of infinitesimal isotropy of the space (see [1]) and technically in general is obtained by considering the variation of the surface with respect to the generator of dilations “exploring” this isotropy.

If u is Lagrangian the map $v := (\varphi_u, u_1, u_2, u_3, u_4)$ is cancelling the form α which is non integrable $\alpha \wedge d\alpha \wedge d\alpha \neq 0$. The map v is called *Legendrian.* Infinitesimal variations among Legendrian maps in $\mathbb{R}^5$ are given by Hamiltonian vector fields in the contact space $(\mathbb{R}^5, \alpha)$ of the form

$$\vec{X} := J_H\, \nabla^H h - 2\, h\, \partial_\varphi \ . \tag{IV.3}$$

where H is the 4-plane given by the kernel of α, ∇^H is the projection on H of the gradient in $\mathbb{R}^5$ with respect to the metric such that π_*, the differential of the canonical projection $\pi(\varphi, y_1, y_2, y_3, y_4) = (y_1, y_2, y_3, y_4)$ from H into $\mathbb{C}^2$, realizes an isometry and the length of ∂_φ is 1. $\mathbb{R}^5$ equipped with this metric is defining the Heisenberg group $\mathbb{H}^2$. Finally J_H is the complex structure on H such that it projects by π_* onto the standard complex structure of $\mathbb{C}^2$. The authors in this work introduce the notion of $H-$minimal Legendrian for the immersion which are critical point of the area with respect to all infinitesimal variations generated by Hamiltonian vector fields in $\mathbb{R}^5$ of the form (IV.3). One then observe

$$u = \pi \circ v \text{ is H-Minimal in } \mathbb{C}^2 \quad \Longleftrightarrow \quad v \text{ is H-Minimal Legendrian in } (\mathbb{R}^5, \alpha) \ .$$

Testing $h = -\varphi$ produces the generator of dilations in $\mathbb{C}^2$ plus dilations in the 5th Legendrian direction but with a different scale.

$$-\, i\, \nabla^H \varphi + 2\, \varphi\, \partial_\varphi = y_1\, \partial_{y_2} - y_2\, \partial_{y_1} + y_3\, \partial_{y_4} - y_4\, \partial_{y_3} + 2\, \varphi\, \partial_\varphi \ .$$

This fact is an encouragement for looking for a monotonicity formula for H-Minimmal Legendrian surfaces. However, as a matter of fact, it cannot be deduced from a simple insertion of this generator of the dilation in the stationarity condition. The derivation of a monotonicity formula for H-Minimal Legendrian immersions in $(\mathbb{R}^5, \alpha)$ proposed in [23], which is maybe one of the main achievement of this work, is quite indirect and involved (going through the resolution of a wave equation, the use of Bessel functions...etc). Another drawback is that the formula is semi-explicit.

Recently the author in [20] obtained a new monotonicity formula by inserting in the Legendrian stationarity condition an explicit hamiltonian : $h := \chi(\mathfrak{r})\, \arctan(2\, \varphi/|y|^2)$, where $\mathfrak{r}$ is the Folland-Korányi gauge given by $\mathfrak{r}^4 = |y|^4 + 2\, \varphi^2$ and χ is some well chosen cut-off function. Because of it's explicit nature the formula is a way more handful and is the starting point of the next section. Precisely the following result holds.

Theorem IV.1. [Almost Monotonicity] *There exists a universal constant $C > 0$ such that for any smooth H-minimal Legendrian proper immersion v of an oriented surface Σ without boundary into $\mathbb{H}^2$, we have $\forall\, r < 1$*

$$C^{-1} \left[\theta_0 + \int_{\mathfrak{r}(v) < r/2} \frac{|(\nabla^\Sigma \mathfrak{r})^\perp|^2}{\mathfrak{r}^2(v)}\ dvol_{v^* g_{\mathbb{H}^2}} \right] \le \frac{1}{r^2} \int_{\mathfrak{r}(v) < r} dvol_{v^* g_{\mathbb{H}^2}} \le C \int_{1/2 < \mathfrak{r}(v) < 2} dvol_{v^* g_{\mathbb{H}^2}} \ , \tag{IV.4}$$

where

$$\theta_0 := 2\pi\ Card\, v^{-1}(0) \ ,$$

where $\mathfrak{r} := (\rho^4 + 4\, \varphi^2)^{1/4}$ is the Folland-Korányi gauge and where $(\nabla^\Sigma \mathfrak{r})^\perp$ denotes the projection of the gradient of the Folland-Korányi gauge onto the orthogonal 2-plane to the tangent space of the immersion v within the horizontal plane H. □

V Minmax Operation under Legendrian constraints

While the work of Schoen and Wolfson is mostly addressing existence and regularity questions for Lagrangian/legendrian minimizing maps, there are multiple reasons for considering Lagrangian/legendrian surfaces critical points of the area with higher *Hamiltonian Morse indices*[16] .

[16] The Hamiltonian Morse Index of an Hamiltonian Stationary Surface is the maximal dimension of the subspace of Hamiltonian deformations on which the restriction of the second derivative of the Area is strictly negative (see for instance [4])

Let (N^5, g) be a 5 dimensional oriented riemannian manifold and α a non degenerate $1-$form on N^5 (i.e. (N^5, α) is contact) satisfying

$$\alpha \wedge d\alpha \wedge d\alpha > 0 \ ,$$

The triple (N^5, g, α) is called a Sasakian structure if the cone $(N^5 \times \mathbb{R}_+, k := dt^2 + t^2 g)$ with the non degenerate symplectic form $\Omega := 2^{-1}\, d(t^2\, \alpha)$ is Kähler. Let J be the compatible complex structure $(\Omega(\cdot, J\cdot) = g)$. The tangent vector field along N^5 given by $\vec{R} := J(t\partial_t)$ is unit on N^5 and is orthogonal to the "horizontal hyperplanes" given by $H = \mathrm{Ker}(\alpha)$. It is called *Reeb Vector-field* of the distribution H. This distribution of plane is invariant under the action of J in $N^5 \times \mathbb{R}_+$ and we shall denote J_H it's restriction on H. Such a structure is called Sasakian structure (see [26]). Classical examples are the Heisenberg group $\mathbb{H}^2$, the unit sphere S^5 for the Reeb vector-field tangent to the fibers of the Hopf fibration into $\mathbb{C}P^2$ or the Stiefel manifold $V_2(\mathbb{R}^4) \simeq S^3 \times S^2$ of orthonormal 2-frames in $\mathbb{R}^4$ for the Reeb vector-field given by fibers of the tautological projection onto the Grassman manifold $G_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$.

A Sasakian structure being given, we introduce the Sobolev space of Legendrian $W^{2,4}$ immersions of a closed oriented surface Σ in N^5

$$\mathfrak{E}^{2,4}_{\Sigma,Leg}(\Sigma, N^5) := \left\{ v \in W^{2,4}(\Sigma, N^5) \quad ; \quad |dv \dot{\wedge} dv| > 0 \ \text{ and } \ v^*\alpha = 0 \quad \text{ on } \Sigma \right\} \ .$$

It is proved in[21] that $\mathfrak{M} := \mathfrak{E}^{2,4}_{\Sigma,Leg}(\Sigma, N^5)$ has the structure of Banach manifold and posses a compatible Finsler structure for which the associated *Palais distance* is complete.

An admissible family $\mathcal{A}$ in $\mathfrak{M}$ is a set of subsets of $\mathfrak{M}$ which is invariant under isotopies in $\mathfrak{M}$. The min-max value or the "width" associated to such a family is the number given by

$$\beta := \inf_{A \in \mathcal{A}} \sup_{v \in A} \int_\Sigma dvol_v \ .$$

The main result in [21] is asserting that any such min-max is achieved by a continuous conformal Legendrian map from a closed riemann surface S into N^5 equipped with an integer multiplicity bounded in L^∞ and satisfying a weak version of the $H-$minimal equation. Such a surface is called *Legendrian Hamiltonian stationary parametrized integer varifold.* Precisely we have.

Theorem V.1. *[21] Let A be an admissible family for the space of $W^{2,4}$ Legendrian immersions of a closed oriented surface Σ into a closed Sasakian manifold (N^5, g, α). Assume that the associated width is strictly positive*

$$\beta := \inf_{A \in \mathcal{A}} \sup_{v \in A} \int_\Sigma dvol_v \ > 0 \ .$$

Then there exists a closed Riemann surface[17] *(S, σ) with genus$(S) \le$ genus(Σ) a map $v \in C^0(S, N^5) \cap W^{1,2}(S, N^5)$ and $N \in L^\infty(S, \mathbb{N}^*)$ such that*

i) Weakly Legendrian

$$v^*\alpha = 0 \ ,$$

ii) Weakly Conformal : in local conformal coordinates for S

$$|\partial_{x_1} v|^2 = |\partial_{x_2} v|^2 \quad \text{and} \quad \partial_{x_1} v \cdot \partial_{x_2} v = 0$$

iii) Realization of the Width

$$\int_S N\, |\partial_{x_1} v \wedge \partial_{x_2} v|\ dx_1 \wedge dx_2 = \frac{1}{2} \int_S N\, |\nabla v|^2_\sigma\ dvol_\sigma = \beta$$

[17] We equip the surface with a compatible metric σ. All the statement below only depend on the conformal class of σ but not on the choice of σ within the coformal class.

iv) Weakly Hamiltonian Stationarity

$$\forall\, h \in C^3(S) \quad , \quad \forall f \in C^1(S) \quad \textit{for a.e. } \lambda \in \mathbb{R} \quad \textit{s.t.} \quad v(f^{-1}\{\lambda\}) \cap Supp(h) = \emptyset$$

$$\int_{f^{-1}((\lambda,+\infty))} N\ \left\langle dv, d\left[\vec{X}_h \circ v\right]\right\rangle\ dvol_\sigma = 0\ , \tag{V.1}$$

where $\vec{X}_h$ *is the Hamiltonian vector-field associated to* h *and given by*

$$\vec{X}_h := J_H(\nabla^H h) - 2\, h\, \vec{R}\ , \tag{V.2}$$

where ∇^H *is the projection onto* H *of the gradient in* (N^5, g).

v) Almost Monotonicity. There exist two constants $c_* > 0$ *and* $C > 1$ *depending only on* (N^5, g, α) *such that*

$$\textit{for } \nu \textit{ a.e. } x \in S \quad \textit{and} \quad \forall\ s > t > 0$$

$$c_* < t^{-2} \int_{v^{-1}(\mathfrak{B}_t(v(x)))} N\, dvol_v < C\ s^{-2} \int_{v^{-1}(\mathfrak{B}_s(v(x)))} N\, dvol_v \tag{V.3}$$

$\nu := 2^{-1}\, N\, |\nabla v|^2_\sigma\ dvol_\sigma$ *and* $\mathfrak{B}_t(\vec{p})$ *denote the ball of radius* $t > 0$ *and center* $\vec{p}$ *for the Carnot Caratheodory distance on* (N^5, g, α). □

In [19] the corresponding result to theorem V.1 was proven without the pointwise Legendrian constraint. Solutions to (V.1) for arbitrary $\vec{X}$ (non necessarily hamiltonian) were called *parametrised stationary integer varifolds*. In an analogous way we shall be calling any triple (S, v, N) solving (V.1) for any Hamiltonian vector-field of the form (V.2) and the almost monotonicity (V.3) *Legendrian parametrised hamiltonian stationary integer varifolds*.

The hypothesis that the stationarity identity (V.1) holds for a.e. $\lambda \in \mathbb{R}$ as long as the image by v of the level set $f^{-1}\{\lambda\}$ is avoiding the support of the testing vector-field $\vec{X}$ is called "localisation hypothesis". Combining the nullity of the variation of the area for perturbation $\vec{X}$ in the target with the "localisation hypothesis" is a substitute to the Euler Lagrange equation (I.5) which has no satisfying weak formulation as we discussed in the foreword.

Unlike the general stationary case without pointwise constraint, the author does not know how to derive the almost monotonicity property (V.3) from the stationarity condition (with the localisation property) (V.1). Hence hypothesis v) is apriori not redundant.The almost monotonicity property is obtained from the smoothing approach we have adopted.

The strategy adopted to establish theorem V.1 is the penalisation approach introduced by the author in [19] under the name "viscosity method" (This method has then been implemented for free boundary surfaces in [17]). Precisely we study the smoothed minmax operation

$$\beta(\varepsilon) := \inf_{A \in \mathcal{A}}\ \sup_{v \in A} E_\varepsilon(v) := \int_\Sigma dvol_v + \varepsilon^4 \int_\Sigma (1 + |\vec{\mathbb{I}}_v|^2)^2\ dvol_v$$

where $\vec{\mathbb{I}}_v$ is the second fundamental form of v. We construct almost critical points v_k of E_{ε_k} for a sequence $\varepsilon_k \to 0$ such that

$$E_{\varepsilon_k}(v_k) - \beta \longrightarrow 0 \quad \text{and} \quad \varepsilon_k^4 \int_\Sigma (1 + |\vec{\mathbb{I}}_{v_k}|^2)^2\ dvol_{v_k} = o\left(\frac{1}{\log \varepsilon_k^{-1}}\right)\ .$$

The main difficulty consists then in passing to the limit for some well chosen subsequence of v_k in order to obtain the triple (S, v, N) satisfying the conclusions of theorem V.1. While we follow mainly the same scheme as the one introduced in [19], working with the Legendrian constraint however is making the proof of theorem V.1 obviously much more challenging.

In [18] it is proved that any *parametrised stationary varifolds* is a smooth branched minimal immersion equipped with a smooth multiplicity. We make the following conjecture.

Open Problem Prove that 2-dimensional Legendrian parametrised Hamiltonian stationary integer varifolds are branched Legendrian immersions and smooth away from isolated Schoen Wolfson conical singularities. □

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