Towards Jaffe and Taubes Conjectures in the Strongly Repulsive Limit.

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Abstract: *We consider the abelian Yang-Mills-Higgs Action on all of* $\mathbb{R}^2$ *for large coupling constants (the strongly repulsive case). The homotopy classes of configurations are defined in a weak sense under the assumption of finite action only. We prove that the homotopy classes which admit a global minimizer for the YMH Action are exactly the* $+1$, -1 *and* 0 *degree classes. Finally we prove that any global minimizer in one of these homotopy classes is the axially symmetric solution (modulo gauge invariance) in the corresponding class.*

I Introduction

In [7] A. Jaffe and C. Taubes considered the abelian Yang-Mills-Higgs Action on all of $\mathbb{R}^2$

$$\mathcal{A}_{\text{YMH}}(A,\Phi)=\int_{\mathbb{R}^2}|\nabla_A\Phi|^2+|F_A|^2+\frac{\lambda}{4}(|\Phi|^2-1)^2 \quad , \tag{I.1}$$

where A is a connection of the trivial hermetian bundle $\mathbb{R}^2\times\mathbb{C}$ (i.e. A is a 1-form on $\mathbb{R}^2$) and Φ is a section of that bundle (i.e. Φ is a map from $\mathbb{R}^2$ into $\mathbb{C}$). F_A denotes the curvature of the connection A (i.e. $F_A=dA$). $\nabla_A\Phi$ denotes the covariant derivative of Φ relative to A (i.e. $\nabla_A\Phi=\nabla\Phi-iA\Phi$) and λ is a positive real parameter known as the coupling constant. Such a pair (A,Φ) is called a *configuration*.

A. Jaffe and C. Taubes gave a complete description of the finite action critical points of $\mathcal{A}_{\text{YMH}}$ in the *non interacting case*, $\lambda=+1$. Technically this is the case where the Euler-Lagrange Equations of (I.1) are reduced to first order equations.

The *vortices* of any critical point (A,Φ) of $\mathcal{A}_{\text{YMH}}$ are the zeros of Φ. They play a major role in the theory. The *degree* of a vortex is the integer representing the topological degree of $\Phi/|\Phi|$ around that point. The name *non interacting case* is justified by the fact that, for that particular value of λ there exists critical points (A,Φ) of $\mathcal{A}_{\rm YMH}$ having arbitrary prescribed number and position of vortices of the same sign (i.e. having degree of the same sign), such that the value $\mathcal{A}_{\rm YMH}(A,\Phi)$ is independent of the position of the vortices. In the present paper we are interested with one of the two *strongly interacting limits* namely $\lambda >> 1$ (the other one beeing $\lambda << 1$). This case is known as *the strong repulsive case*, because in that limit vortices of the same sign produce a high energy and repulse each other.

We will take now $\lambda = \frac{2}{\varepsilon^2}$ and write $\mathcal{A}_\varepsilon$ for $\mathcal{A}_{\rm YMH}$ where ε is a small positive real number. We prove first the existence of homotopy classes for any configuration (A,Φ) under the assumption of finite action only (see proposition II.1). This gives in particular a positive answer to conjecture 1 on page 34 of [7]. The homotopy classes are, roughly speaking, given by the degree of $\Phi/|\Phi|$ at infinity and thus are parametrized by integers $N \in \mathbb{Z}$. This integer (or homotopy class) is of course preserved under the gauge transformation, which preserves the action,

$$(A,\Phi) \longrightarrow (A + d\phi, \Phi \exp(i\phi)) \quad ,$$

where ϕ is any smooth real valued function on $\mathbb{R}^2$.

Our first main result is

Theorem I.1 *For $|N| \geq 2$ and λ sufficiently large there is no minimizer of $\mathcal{A}_{YMH}$ in the N-homotopy class.*

This is a first answer to conjecture 2 on page 58 of [7] where in fact it is expected more generally that there should not exist a stable critical point of $\mathcal{A}$ in the N-homotopy class for $|N| \geq 2$.

Our second main result is

Theorem I.2 *For λ sufficiently large, there exists a minimizer in the $N = \pm 1$ homotopy classes, it is unique (modulo gauge invariance) and it is the ± 1 axially symmetric critical point of $\mathcal{A}_{YMH}$ $(A,\Phi) = (S(r)d\theta, \rho(r)\exp(i\theta))$.* $\blacksquare$

This is also a first answer to the second part of conjecture 2 page 58 of [7], where the same statement is expected but for stable critical point in general instead of minimizers.

Remark I.1 *In fact we prove slightly more than theorem I.2 : we show that there exists a familly of positive constants C_ε, with $\lim_{\varepsilon\rightarrow+\infty} C_\varepsilon = +\infty$ such that any critical point of $\mathcal{A}_\varepsilon$ whose energy differs from the minimal value of $\mathcal{A}_\varepsilon$ in the $\pm1-$homotopy class has to be gauge equivalent to the $\pm1-$axially symmetric solution.*

The axially symmetric solutions and their behavior as $\varepsilon \longrightarrow 0$ are largely described in [1]. Notice that the instability of the axially symmetric solutions of degree different from $0,\pm 1$ is established in [6]

Remark I.2 *For the Yang-Mills-Higgs action on S^2 for the hermetian bundle such that $\frac{1}{2\pi}\int_{S^2} dA = \pm 1$ (i.e. Euler Caracteristic equal to ± 1), similar technics as the ones used to prove theorem I.2 above can yield the uniqueness and the axial symmetry (modulo gauge invariance) of the couple connection-section (A, Φ) minimizing $\mathcal{A}_{YMH}$, still for λ large enough.*

Remark I.3 *The key point of the proof of the axial symmetry of the minimizers in Theorem I.2 relies on the introduction of weighted norms by means of the two fundamental estimates given by Lemma A.3 and Lemma A.4 in order to make the uniqueness argument of P. Mironescu ([8]) in our context work. Similar lemmas are also used to prove a uniqueness result for the non gauge invariant functional (see [9] and [10]).*

We will denote by $\mathcal{A}_\varepsilon^N$ the infimum of the action $\mathcal{A}_{\mathrm{YMH}}$ for the parameter $\lambda = \frac{2}{\varepsilon^2}$ in the N-th homotopy class defined by proposition II.1. Then from Theorem I.1 and Theorem I.2 one can deduce the exact value of $\mathcal{A}_\varepsilon^N$ for ε small.

Corollary I.1 *The following equality holds for ε small enough, depending on N,*

$$\mathcal{A}_\varepsilon^N = |N| \mathcal{A}_\varepsilon(S_\varepsilon(r)d\theta, \rho_\varepsilon(r)\exp(i\theta)) \quad . \tag{I.2}$$

Let us mention that the techniques developped in order to prove Theorems I.1 and I.2 have, as a background in particular the assymptotic analysis of the Ginzburg-Landau Equations on bounded domains and for the large κ limit which was developped in the last 6 years starting from the work of F. Bethuel, H. Brezis and F. Hélein in [3]. See different surveys of it in [12], [13] or in the introduction of [10].

The paper is organized as follows : In part II we prove the existence of homotopy classes under the condition of finite action only, in part III we prove the existence of minimizers in the 0 and ± 1 homotopy classes, in part IV we prove the non-existence of minimizers in the other homotopy classes (Theorem I.1) and in part V we prove the axial symmetry of the minimizers in the ± 1 homotopy classes (Theorem I.2).

Notation. We will often use the following notation : For any two positive quantities A and B, $A \lesssim B$ means that A/B is bounded from above by a positive "universal number" independent on any variable of the paper.

II Weak homotopy classes

In this part we prove the existence of homotopy classes under the assumption of finite action only. We do not restrict ourselves to the case λ large, λ can be any positive number.

Proposition II.1 *Let* (A, Φ) *be a configuration (i.e. a couple connection-section of* $E = \mathbb{R}^2 \times \mathbb{C}$*) such that* $\mathcal{A}(A, \Phi) < +\infty$*. Then there exists* $R > 0$ *such that, for any bounded domain* Ω *of* $\mathbb{R}^2$ *containing* $B_R(0)$ *and satisfying* $|\Phi| \geq 1/2$ *on* $\partial\Omega$ *(such domain exists for* R *as large as we want),*

$$N = deg\left(\frac{\Phi}{|\Phi|}; \partial\Omega\right)$$

is independent of Ω*.*

Definition II.1 *For any configuration* (A, Φ) *verifying* $\mathcal{A}(A, \Phi) < +\infty$*, we define the weak homotopy class of* (A, Φ) *to be the integer* N *given by proposition II.1. It is denoted* $N = deg_\infty(A, \Phi)$*.*

Proof of proposition II.1

Replacing Φ by $T(|\Phi|)\Phi/|\Phi|$, where $T(t) = t$ for $t \leq 1$ and $T(t) \equiv 1$ for $t > 1$, reduces the energy of any given configuration (A, Φ). We may then assume that $|\Phi| \leq 1$. Let $\alpha > 0$ to be fixed sufficiently small later. Let $R \geq 4$ be such that

$$\int_{\mathbb{R}^2 \setminus B_{R/2}(0)} a(A, \Phi) \leq \alpha \quad \text{and} \quad \int_{\partial B_R(0)} a(A, \Phi) \leq \alpha, \tag{II.1}$$

where $a(A, \Phi)$ is the action density $a(A, \Phi) = |\nabla_A \Phi|^2 + |dA|^2 + \frac{1}{2\varepsilon^2}(|\Phi|^2 - 1)^2$. The existence of such an R is a simple consequence of the finiteness of the action on the whole plane. Let χ_R be the characteristic function of $\mathbb{R}^2 \setminus B_R(0)$ ($\chi_R \equiv 1$ on $\mathbb{R}^2 \setminus B_R(0)$, and $\chi_R \equiv 0$ on $B_R(0)$).

For any $x_0 \in [0, 1]^2$ we consider the integer lattice having x_0 as a node

$$\mathcal{L}_{x_0} = x_0 + \mathbb{Z} \times \mathbb{R} + \mathbb{R} \times \mathbb{Z} \quad .$$

Fubini's theorem implies

$$\int_{x_0 \in [0,1]^2} dx_0 \int_{\mathcal{L}_{x_0}} \chi_R \; a(A, \Phi) \; d\mathcal{H}^1 \leq 2 \int_{\mathbb{R}^2 \setminus B_R(0)} a(A, \Phi) \leq 2\alpha \tag{II.2}$$

Thus there exists $x_0 \in [0, 1]^2$ such that

$$\int_{\mathcal{L}_{x_0}} \chi_R \; a(A, \Phi) \; d\mathcal{H}^1 \leq 2\alpha \quad . \tag{II.3}$$

Take such x_0. We claim that for α chosen sufficiently small we have $|\Phi| \geq 1/2$ on $\mathcal{L}_{x_0} \cap \mathbb{R}^2 \setminus B_R(0)$.

Let $y \in \mathcal{L}_{x_0} \cap \mathbb{R}^2 \setminus B_R(0)$ and assume $|\Phi|(y) < 1/2$. Let $I = \{y + s\vec{i} \; : \; s \in [0, 1]\}$. We can assume $I \subset \mathcal{L}_{x_0} \cap \mathbb{R}^2 \setminus B_R(0)$ (otherwise just replace the first basic vector $\vec{i}$ by $-\vec{i}$). Since $\int_I |\nabla|\Phi||^2 \leq \int_I |\nabla_A \Phi|^2 \leq 2\alpha$, for $y' \in I$ we have

$$||\Phi|(y) - |\Phi|(y')| \le \int_{[y,y']} |\nabla|\Phi|| \le \sqrt{2}\alpha^{1/2} \quad .$$

Thus for $\sqrt{2}\alpha^{1/2} \le 1/4$ we have $|\Phi| \le 3/4 \quad$ on $\quad I$ and this implies

$$2\alpha \ge \frac{\lambda}{4}\int_I (1-|\Phi|^2)^2 \ge \frac{\lambda}{4}\left(\frac{7}{16}\right)^2 \quad ,$$

and we get a contradiction for $\alpha \le \frac{\lambda}{8}\left(\frac{7}{16}\right)^2$. Observe that exactly the same way of arguing yields $|\Phi| \ge 1/2$ on $\partial B_{R(0)}$, if we choose $R \ge 1$ (as we may and α small enough. Thus, taking α sufficiently small, we ensure

$$|\Phi| \ge \frac{1}{2} \quad \text{on} \quad \left(\mathcal{L}_{x_0} \cap \mathbb{R}^2 \setminus B_R(0)\right) \cup \partial B_R(0) \quad . \tag{II.4}$$

Let Ω be a bounded domain of $\mathbb{R}^2$ such that $B_R(0) \subset \Omega$ and $|\Phi| \ge 1/2$ on $\partial\Omega$. Let $(\Omega_j)_{j\in J}$ be the connected components of $(\Omega \setminus B_R(0)) \setminus \mathcal{L}_{x_0}$. On $\partial\Omega_j$, $|\Phi| \ge 1/2$. Assume there exists a $j \in J$ such that $\deg\left(\frac{\Phi}{|\Phi|}; \partial\Omega_j\right) \ne 0$. Let B_2 be a ball of radius 2 in $\Omega \setminus B_{R/2}(0)$ containing Ω_j (note Ω_j is necessaryly included in one of the squares, connected components of $\mathbb{R}^2 \setminus \mathcal{L}_{x_0}$). Let ξ be the solution of

$$\left\{\begin{array}{l} \Delta\xi = *dA \quad \text{in } B_2 \\ \\ \xi = 0 \quad \text{on } \partial B_2 \end{array}\right.$$

($*dA$ is the function $\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$). We have $\|\xi\|_{W^{2,2}} \lesssim \int_{B^2} |dA|^2 \lesssim \alpha$. Choose a gauge such that $A = \left(-\frac{\partial\xi}{\partial x_2}, \frac{\partial\xi}{\partial x_1}\right)$ on B_2.
Thus $\|A\|^2_{W^{1,2}(B_2)} \lesssim \alpha$ and since $\int_{B_2} a(A,\Phi) \le \alpha$ we get

$$\int_{B_2} |\nabla\Phi|^2 \lesssim \alpha. \tag{II.5}$$

(recall that $|\Phi| \le 1$). Indeed we have $\|A\|^2_{L^4(B_2)} \lesssim \|A\|^2_{W^{1,2}(B_2)} \lesssim \alpha$ and we bound

$$\int_{B_2} |\Phi|^2|A|^2 \lesssim \left(\int_{B_2} 1 + (1-|\Phi|^2)^2\right)^{\frac{1}{2}} \left(\int_{B_2} |A|^4\right)^{1/2} \lesssim \alpha + \alpha^{3/2} \quad , \tag{II.6}$$

where we may assume that $\alpha \le 1$. Since $\deg\left(\frac{\Phi}{|\Phi|}; \partial\Omega_j\right) \ne 0$ and since $|\Phi| \ge 1/2$ on $\partial\Omega_j$, we have $\Phi(\Omega_j) \supset D_{1/2}(0)$ where $D_{1/2}(0)$ is the disk of center 0 and radius 1/2. Thus

$$\frac{\pi}{4} \le |D_{1/2}(0)| \le \int_{\Omega_j} |\Phi^* dx_1 \wedge dx_2| \le \frac{1}{2}\int_{\Omega_j} |\nabla\Phi|^2 \quad . \tag{II.7}$$

This contradicts (II.5) for α chosen small enough. Thus $\deg\left(\frac{\Phi}{|\Phi|};\partial\Omega_j\right) = 0$ for any $j \in J$ and this implies $\deg\left(\frac{\Phi}{|\Phi|};\partial\Omega\right) = \deg\left(\frac{\Phi}{|\Phi|};\partial B_R(0)\right)$ which is the desired result. ■

III Existence of a global minimizer in the ± 1 homotopy classes for large λ.

As mentioned in the introduction we take $\lambda = \frac{2}{\varepsilon^2}$ with ε small. We will need the following Lemma. Denote by $a_\varepsilon(A,\Phi)$ the action density $a_\varepsilon(A,\Phi) = |\nabla_A\Phi|^2 + |dA|^2 + \frac{1}{2\varepsilon^2}(|\Phi|^2-1)^2$ and set. $\mathcal{A}_\varepsilon(A,\Phi) = \int_{\mathbb{R}^2} a_\varepsilon(A,\Phi)$. Denote also

$$\mathcal{A}_\varepsilon^N = \inf\left\{\mathcal{A}_\varepsilon(A,\Phi) \quad ; \quad \deg_\infty(A,\Phi) = N\right\} \quad .$$

We have

Lemma III.1 *For $|N| \geq 1$*

$$\lim_{\varepsilon\to 0}\frac{\mathcal{A}_\varepsilon^N}{2\pi|N|\log\frac{1}{\varepsilon}} = 1 \tag{III.1}$$

Proof of Lemma III.1

As in [4], using suitable configurations one proves easily that there exists $C > 0$ independent of ε such that

$$\mathcal{A}_\varepsilon^N < 2\pi|N|\log\frac{1}{\varepsilon} + C \tag{III.2}$$

C depends of course on N. Take any (A,Φ) such that

$$\begin{cases} \mathcal{A}_\varepsilon(A,\Phi) \leq 2\pi|N|\log\dfrac{1}{\varepsilon} + C \\ \\ \text{and} \quad \deg_\infty(A,\Phi) = N \quad . \end{cases}$$

By the argument mentioned in the first lines of the proof of Proposition II.1 we may allways assume that $|\Phi| \leq 1$. For $\eta > 0$ to be chosen sufficiently small later as in the proof of the existence of weak homotopy classes we introduce the lattice for $x_0 \in [0,\eta]^2$

$$\mathcal{L}_{x_0}^\eta = x_0 + \eta\mathbb{Z}\times\mathbb{R} + \mathbb{R}\times\eta\mathbb{Z} \quad .$$

Fubini's theorem implies $\int_{x_0\in[0,\eta]^2} dx_0 \int_{\mathcal{L}_{x_0}^\eta} a_\varepsilon(A,\Phi)\ d\mathcal{H}^1 \lesssim \eta\ \mathcal{A}_\varepsilon(A,\Phi)$. Thus there exists $x_0 \in [0,\eta]^2$ such that

$$\int_{\mathcal{L}^\eta_{x_0}} a_\varepsilon(A,\Phi)\ d\mathcal{H}^1 \lesssim \frac{1}{\eta}\mathcal{A}_\varepsilon(A,\Phi) \lesssim \frac{|N|}{\eta}\log\frac{1}{\varepsilon} \quad , \tag{III.3}$$

assuming at this stage that ε is smaller than some universal positive number. We claim that for ε sufficiently small, depending on η of course, we have

$$|\Phi| \geq \frac{1}{2} \qquad \text{on} \qquad \mathcal{L}^\eta_{x_0} \quad . \tag{III.4}$$

Indeed, assume there exists $y \in \mathcal{L}^\eta_{x_0}$ such that $|\Phi|(y) < 1/2$. Let $I_\tau = \{y + t\vec{i}\ \tau > t > 0\} \subset \mathcal{L}^\eta_{x_0}$. From (III.3) and the fact that $|\nabla|\Phi||^2 \leq a_\varepsilon(A,\Phi)$ we deduce that

$$\forall z \in I_\tau \qquad ||\Phi|(y) - |\Phi|(z)|^2 \lesssim \tau\frac{|N|}{\eta}\log\frac{1}{\varepsilon} \quad . \tag{III.5}$$

Choose $\tau = \frac{\eta}{C|N|\log\frac{1}{\varepsilon}}$, for some universal constant C, (III.5) then implies that $|\Phi| \leq \frac{3}{4}$ on I_τ, thus

$$\int_{\mathcal{L}^\eta_{x_0}} a_\varepsilon(A,\Phi) \geq \int_{I_\tau} \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2 \geq \frac{C}{\varepsilon^2}|I_\tau| = \frac{C\ \eta}{\varepsilon^2\log\frac{1}{\varepsilon}} \quad . \tag{III.6}$$

(III.6) contradicts (III.3) for ε sufficiently small and (III.4) holds.

Consider a square Λ_j (connected component of $\mathbb{R}^2\setminus\mathcal{L}^\eta_{x_0}$) such that $\deg\left(\frac{\Phi}{|\Phi|};\partial\Lambda_j\right) = d_j \neq 0$. Let $B_{2\eta}$ be a ball containing Λ_j. On $B_{2\eta}$ we consider the solution ξ of

$$\begin{cases} \Delta\xi = *dA \quad \text{in } B_{2\eta} \\ \\ \xi = 0 \quad \text{on } \partial B_{2\eta} \quad . \end{cases}$$

We have $\int_{B_{2\eta}} \eta^2|\nabla^2\xi|^2 + |\nabla\xi|^2 \lesssim \eta^2\int_{B_{2\eta}}|dA|^2 \lesssim \eta^2\mathcal{A}_\varepsilon(A,\Phi)$. Take the gauge $A = \left(-\frac{\partial\xi}{\partial x_2};\frac{\partial\xi}{\partial x_1}\right)$ on $B_{2\eta}$, we then have

$$\int_{\Lambda_j} \eta^2\|\nabla A\|^2 + |A|^2 \lesssim \eta^2|N|log\frac{1}{\varepsilon} \quad . \tag{III.7}$$

This implies

$$\left(\int_{\Lambda_j}|A|^4\right)^{\frac{1}{2}} \lesssim \eta N\log\frac{1}{\varepsilon} \quad . \tag{III.8}$$

Thus we have

$$\begin{aligned} \int_{\Lambda_j}|\Phi|^2|A|^2 &\lesssim \left(\int_{\Lambda_j}(1+(1-|\Phi|^2)^2)\right)^{\frac{1}{2}}\left(\int_{\Lambda_j}|A|^4\right)^{\frac{1}{2}} \\ &\lesssim (\eta^2+\varepsilon^2\log\frac{1}{\varepsilon})^{\frac{1}{2}}\eta\log\frac{1}{\varepsilon} \lesssim \eta^2\log\frac{1}{\varepsilon} + \varepsilon\eta(\log\frac{1}{\varepsilon})^{3/2} \lesssim \eta^2\log\frac{1}{\varepsilon} \quad . \end{aligned} \tag{III.9}$$

where we assumed that ε is smaller than a positive constant depending on η. Using the fact that, for any $\delta > 0$, $|\nabla\Phi - iA\Phi|^2 \geq ||\nabla\Phi| - |A||\Phi||^2 \geq (1-\delta)|\nabla\Phi|^2 + (1-\frac{1}{\delta})|A|^2|\Phi|^2$, from (III.9) we deduce that

$$\int_{\Lambda_j} a_\varepsilon(A,\Phi) \geq (1-\delta)\int_{\Lambda_j}\left[|\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2\right] - C|N|\frac{\eta^2}{\delta}\log\frac{1}{\varepsilon} \quad \text{(III.10)}$$

for any $1/2 > \delta > 0$. Moreover since $\|\xi\|^2_{H^{\frac{3}{2}}(\partial\Lambda_i)} \lesssim \int_{B_{2\eta}} |dA|^2$, we have

$$\left(\int_{\partial\Lambda_j} |A|^4\right)^{\frac{1}{2}} \lesssim \eta^{\frac{1}{2}}|N|\log\frac{1}{\varepsilon} \quad \text{(III.11)}$$

This implies

$$\begin{aligned} &\textstyle\int_{\partial\Lambda_j} |\Phi|^2|A|^2 \lesssim \left(\int_{\partial\Lambda_j}(1+(1-|\Phi|^2)^2)\right)^{\frac{1}{2}}\left(\int_{\partial\Lambda_j}|A|^4\right)^{\frac{1}{2}} \\ &\lesssim \eta\log\frac{1}{\varepsilon} + \varepsilon\log\frac{1}{\varepsilon} \lesssim \eta\log\frac{1}{\varepsilon} \quad , \end{aligned} \quad \text{(III.12)}$$

where we have used (III.3 and assumed that ε is small enough.

Combining (III.12) and (III.3) we get

$$\int_{\partial\Lambda_j} |\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2 \leq \frac{C}{\eta}\log\frac{1}{\varepsilon} \quad . \quad \text{(III.13)}$$

Denote by $\Lambda_{j,s}$ the image square of Λ_j by the homothety with center the center of Λ_j and of rate $1+s$, and denote by $\tilde{\Phi}$ the resulting map on $\Lambda_{j,\delta}$, equal to Φ on Λ_j and to its radial extension between $\Lambda_{j,\delta}$ and Λ_j with modulus interpolated to 1 and $\tilde{\Phi}$ is radially constant in $\Lambda_{j,2\delta}\setminus\Lambda_{j,\delta}$. Precisely

$$\begin{aligned} &\tilde{\Phi} = \Phi \qquad \text{on } \Lambda_j \\ &\tilde{\Phi}(x) = (t_x + (1-t_x)|\Phi|(\tilde{x}))\,\tfrac{\Phi}{|\Phi|}(\tilde{x}) \qquad \text{on } \Lambda_{j,\delta}\setminus\Lambda_j \\ &\tilde{\Phi}(x) = \tfrac{\Phi}{|\Phi|}(\tilde{x}) \qquad \text{on } \Lambda_{j,2\delta}\setminus\Lambda_{j,\delta} \quad , \end{aligned}$$

where x_j is the center of the square Λ_j, $\{\tilde{x}\} = [x_j;x]\cap\partial\Lambda_j$, $[x_j;x]$ the segment joining x and x_j and t_x is a smooth function in $\Lambda_{j,\delta}\setminus\Lambda_j$ satisfying $t_x = 1$ on $\partial\Lambda_{j,\delta}$, $t_x = 0$ on $\partial\Lambda_j$, $0 \leq t_x \leq 1$ and finally $|\nabla t_x| \lesssim \frac{1}{\delta\eta}$. Clearly we have

$$\begin{aligned} &\int_{\Lambda_{j,2\delta}} |\nabla\tilde{\Phi}|^2 + \frac{1}{2\varepsilon^2}(1-|\tilde{\Phi}|^2)^2 \leq \int_{\Lambda_j} |\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2 \\ &+C\delta\eta\int_{\partial\Lambda_j}\frac{(1-|\Phi|)^2}{\delta^2\eta^2} + C\delta\eta\int_{\partial\Lambda_j}|\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2 \quad . \end{aligned} \quad \text{(III.14)}$$

Using (III.13) for ε smaller than a positive constant depending eventually on δ and η we get

$$\int_{\Lambda_{j,2\delta}} |\nabla\tilde{\Phi}|^2 + \frac{1}{2\varepsilon^2}(1-|\tilde{\Phi}|^2)^2 \leq \int_{\Lambda_j} |\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2 + C\delta\log\frac{1}{\varepsilon}\ , \quad \text{(III.15)}$$

where C depends only on N.
$\tilde{\Phi}$ is a map into $\mathbb{C}$ such that $|\tilde{\Phi}| \geq 1$ on $\Lambda_{j,2\delta}\setminus\Lambda_{j,\delta}$, having degree d_j on $\partial\Lambda_j$. Observe that $|d_j| \lesssim \int_{\partial\Lambda_j}|\nabla\Phi| \lesssim (\log\frac{1}{\varepsilon})^{1/2}$. Applying Lemma A.1 for $u=\tilde{\Phi}$, $\Omega=\Lambda_{j,2\delta}$ and $\mu=\delta\eta$, we get

$$\int_{\Lambda_{j,2\delta}} |\nabla\tilde{\Phi}|^2 + \frac{1}{2\varepsilon^2}(1-|\tilde{\Phi}|^2)^2 \geq 2\pi|d_j|\log\frac{1}{\varepsilon} - o_\varepsilon(\log\frac{1}{\varepsilon}) \quad , \qquad \text{(III.16)}$$

where $o(\log\frac{1}{\varepsilon})$ may depend on δ and η. Combining (III.10), (III.15) and (III.16) one gets

$$\int_{\Lambda_j} a_\varepsilon(A,\Phi) \geq 2\pi|d_j|(1-\delta)\left(1-C\delta-C\frac{\eta^2}{\delta}\right)\log\frac{1}{\varepsilon} - o(\log\frac{1}{\varepsilon}) \quad , \quad \text{(III.17)}$$

where again $o(\log\frac{1}{\varepsilon})$ may depend on η and δ. Summing over all the js such that $d_j\neq 0$ (there can be only finitely many such a j because of proposition II.1), since $\sum d_j = N$, taking $\delta\to 0$ and $\eta << \delta^{\frac{1}{2}}$ and using (III.2) we get the desired result and Lemma III.1 is proved. ∎

Proof of theorem I.2 (existence part). We prove now the existence of a global minimizer of the action in the ± 1 homotopy classes for large λ.

Take a minimizing sequence (A_n,Φ_n) of $\mathcal{A}_\varepsilon$ in the homotopy class given by $N=1$, i.e.

$$\begin{cases} \mathcal{A}_\varepsilon(A_n,\Phi_n) \longrightarrow \mathcal{A}^1_\varepsilon & \text{for } n\to+\infty \\ \\ \deg_\infty(A_n,\Phi_n) = +1 & . \end{cases}$$

Observe that the truncation of the modulus of Φ_n at 1 decreases the energy, thus we may always assume that $|\Phi_n|\leq 1$. As in the proof of Lemma III.1, there exists $\varepsilon_0>0$ such that for any $\varepsilon<\varepsilon_0$ we can choose $x_0^n\in[0,1]^2$ such that $|\Phi|\geq 1/2$ on $\mathcal{L}_{x_0^n}$. We keep denoting by (A_n,Φ_n) the previous configurations translated by x_0^n (which is still a minimizing sequence) and the previous lattice on which $|\Phi_n|\geq 1/2$ is now $\mathcal{L}_0$.

Let $(\Lambda_i)_{i\in I}$ be the connected components of $\mathbb{R}^2\setminus\mathcal{L}_0$. Denote $d_i^n = \deg\left(\frac{\Phi_n}{|\Phi_n|};\partial\Lambda_i\right)$. Since $\int_{\mathcal{L}_{x_0^n}} a_\varepsilon(A_n,\Phi_n)$ is bounded independently of n (not of ε of course), one verifies easily that the d_i^n are bounded independently of $i\in I$ and $n\in\mathbb{N}$.

Moreover, since $\mathcal{A}_\varepsilon(A_n,\Phi_n)$ is uniformly bounded independently of n, using arguments like in the proof of Lemma III.1, one easily sees that, for ε sufficiently small, the number of $i\in I$ such that $d_i\neq 0$ is also bounded independently of n. Let $J^n=\left\{j\in I \quad \text{s.t.} \quad d_j^n\neq 0\right\}$. Denote by $(x_j)_{j\in J}$ the centers of the Λ_j. Since Card J^n is uniformly bounded, one can extract a subsequence from (A_n,Φ_n) such that, after suitable renumering, $J^n=J$ is independent of n and for all $j,k\in J$

$$\left\{\begin{array}{l} \text{either} \quad x_j^n-x_k^n \quad \text{converges} \ , \\ \\ \text{or} \quad |x_j^n-x_k^n|\to\infty \quad . \end{array}\right.$$

This defines an equivalence relation on J saying that $j\simeq k$ iff $x_j^n-x_k^n$ converges. Since x_j^n and x_k^n are centers of lattice squares one can again extract a subsequence such that $j\simeq k \Longleftrightarrow x_j^n-x_k^n$ is independent of n. To each equivalence class $\mathcal{C}$ and each n one can associate the barycenter $z_{\mathcal{C}}^n$ of the x_j^n with $j\in\mathcal{C}$ and there exists $R_n\to+\infty$ such that $j\in\mathcal{C} \quad \Longleftrightarrow \quad x_j^n\in B_{R_n}(z_{\mathcal{C}}^n)$. Extracting a further subsequence, if necessary, we may also assume that $d_i^n=d_i$ is independent of n for all i and we set $D_{\mathcal{C}}=\sum_{j\in\mathcal{C}}d_j$. We claim that for each $\mathcal{C}$ there exists $(A_{\mathcal{C}},\Phi_{\mathcal{C}})$ such that

$$\sum_{\mathcal{C}}\int_{\mathbb{R}^2} a_\varepsilon(A_{\mathcal{C}},\Phi_{\mathcal{C}})\le\mathcal{A}_\varepsilon^1 \quad \text{and} \quad \deg_\infty(A_{\mathcal{C}},\Phi_{\mathcal{C}})=D_{\mathcal{C}} \tag{III.18}$$

Let $\mathcal{C}$ be a class and translate $z_{\mathcal{C}}^n$ such that $z_{\mathcal{C}}^n=0$. Let $R>0$ and let $\tilde{\xi}_n$ be the solution of

$$\left\{\begin{array}{ll} \Delta\tilde{\xi}_n=*d\tilde{A}_n & \text{in } B_R(0) \\ \\ \tilde{\xi}_n=0 & \text{on } \partial B_R(0) \quad , \end{array}\right.$$

where $d\tilde{A}_n(z)=dA_n(z+z_{\mathcal{C}}^n)$. Modulo a change of gauge we may take $\tilde{A}_n=\left(-\frac{\partial\tilde{\xi}_n}{\partial x_2};\frac{\partial\tilde{\xi}_n}{\partial x_1}\right)$ and $\tilde{\Phi}_n$ denotes the representative of the section in this trivialisation. Since $*d\tilde{A}_n$ is uniformly bounded in $L^2(B_R(0))$, one can extract a subsequence such that it converges weakly in L^2 to some h. Thus $\tilde{A}_n$ converges weakly in $H^1(B_R(0))$ to some $\tilde{A}$ and since $\int_{B_R(0)}|\nabla_{\tilde{A}_n}\tilde{\Phi}_n|^2$ is bounded, one can extract a subsequence such that $\tilde{\Phi}_n$ converges also weakly in H^1 to some $\tilde{\Phi}$, moreover

$$\int_{B_R(0)}|\nabla_{\tilde{A}}\tilde{\Phi}|^2+|d\tilde{A}|^2+\frac{1}{2\varepsilon^2}(1-|\tilde{\Phi}|^2)^2\le\underline{\lim}\int_{B_R(0)}a_\varepsilon(\tilde{A}_n,\tilde{\Phi}_n) \quad .$$

Following a standard terminology in functional analysis for gauge theory we say that a sequence of configurations connection-section (A_n,Φ_n) converges in some function space F to a limiting configuration (A,Φ) if in a given trivialisation (U_α,ψ_α) of the complex line bundle over the base, there exists a change of gauge $(A_n^\alpha+d\phi_n^\alpha,\Phi_n^\alpha\, e^{i\phi_n^\alpha})$ such that this sequence of pairs form-function converges to (A^α,Φ^α) in the function space F. Clearly we have proved that modulo extraction of a subsequence (A_n,Φ_n) converges in the above mentioned sense to a limiting configuration (A,Φ) weakly in $H^1(B_R(0))$. Since this holds for any $R>0$ one can apply a diagonalisation procedure to extract a subsequence such that we get a $(A_{\mathcal{C}},\Phi_{\mathcal{C}})$ defined on all of $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2}|\nabla_{A_{\mathcal{C}}}\Phi_{\mathcal{C}}|^2+|dA_{\mathcal{C}}|^2+\frac{1}{2\varepsilon^2}(1-|\Phi_{\mathcal{C}}|^2)^2\leq\underline{\lim}\int_{B_{R_n}(z_{\mathcal{C}}^n)}a_\varepsilon(A_n,\Phi_n)\qquad\text{(III.19)}$$

and $(\tilde{A}_n,\tilde{\Phi}_n)\rightharpoonup(\tilde{A},\tilde{\Phi})$ in $H^1(K)$ for any compact $K\subset\mathbb{R}$, where $(\tilde{A}_n,\tilde{\Phi}_n)$ and $(\tilde{A},\tilde{\Phi})$ are the coulomb gauges of the configurations on a fixed ball $B_R(0)$ containing K (i.e. $d^*\tilde{A}_n=0$ on $B_R(0)$ and $\tilde{A}_n.\nu=0$ on $\partial B_R(0)$). We claim that

$$\deg_\infty(\tilde{A}_{\mathcal{C}},\tilde{\Phi}_{\mathcal{C}})=D_{\mathcal{C}}\quad .\qquad\text{(III.20)}$$

We now translate such that 0 becomes a closest vertex of $\mathcal{L}_0$ to $z_{\mathcal{C}}^n$ and we still denote the translated configurations by (A_n,Φ_n). Then for any $j\in\mathcal{C}$ the translated x_j^n are now at a fixed distance from 0. Let C be a **fixed** square included in $\mathcal{L}_0$ containing all the translated x_j^n's. We have

$$\deg\left(\frac{\Phi_n}{|\Phi_n|};\partial C\right)=D_{\mathcal{C}}\qquad\text{(III.21)}$$

Take an $R>0$ such that $C\subset B_R(0)$ and use tilde to express the Coulomb gauges of the configurations on this ball. Since $(\tilde{A}_n,\tilde{\Phi}_n)\rightharpoonup(\tilde{A},\tilde{\Phi})$ in $H^1(C)$, we have $\tilde{\Phi}_n\rightharpoonup\tilde{\Phi}$ and $\tilde{A}_n\rightharpoonup\tilde{A}$ in $H^{1/2}(\partial C)$ which implies by compactness of the embedding $H^{1/2}\to L^2$

$$\tilde{\Phi}_n\to\tilde{\Phi}\quad\text{ and }\quad\tilde{A}_n\to\tilde{A}\quad\text{ strongly in }L^2(\partial C)\quad .\qquad\text{(III.22)}$$

Moreover,

$$\int_{\partial C}a_\varepsilon(A_n,\Phi_n)\leq\int_{\mathcal{L}}a_\varepsilon(A_n,\Phi_n)\qquad\text{(III.23)}$$

is bounded independently of n. Since $|\Phi_n|\leq1$ and since $\tilde{A}_n$ is bounded in $L^2(\partial C)$ we deduce that

$$\|\tilde{\Phi}_n\|_{H^1(\partial C)}\leq C_0\quad ,\qquad\text{(III.24)}$$

where C_0 is independent on n. Thus we may extract a subsequence (still denoted Φ_n) such that Φ_n converges weakly in $H^1(\partial C)$ and because of (III.22) the limit can only be $\tilde{\Phi}$. Weak convergence in H^1 implies uniform convergence in 1 dimension, thus the degree passes to the limit and (III.20) holds.

Clearly from (III.19), since we are dealing with a minimizing sequence in the $+1$ homotopy class, we deduce that

$$\int_{\mathbb{R}^2} |\nabla_{A_{\mathcal{C}}}\Phi_{\mathcal{C}}|^2 + |dA_{\mathcal{C}}|^2 + \frac{1}{2\varepsilon^2}(1-|\Phi_{\mathcal{C}}|^2)^2 \leq C\log\frac{1}{\varepsilon} \quad , \tag{III.25}$$

where C is independent of ε (ε is taken small enough). So applying Lemma A.2 we deduce that $D_{\mathcal{C}}$ is bounded independently of ε.

Combining (III.20) and (III.19) we get (III.18) and combining (III.18) and Lemma III.1 we get that one of the $D_{\mathcal{C}}$ is $+1$ and the other ones are 0. We then have proved the existence of a minimizer for the action in the $+1$-homotopy class for ε small enough. ∎

IV The non existence of global Minimizers of the YMH Action in the N-homotopy classes for $|N| \geq 2$ and large λ.

IV.1 A brief review of the assymptotic analysis of N-minimizers of $\mathcal{A}$ as $\lambda \to +\infty$.

In this part we recall the description given mostly in [4] (see also [12] for more direct arguments) for minimizers of $\mathcal{A}$ on bounded domain as $\lambda \to +\infty$, which can be extended to the eventual N-minimizers of $\mathcal{A}$ on all of $\mathbb{R}^2$ following arguments like in [11]. The reader can also find a presentation of most of the estimates of the present subsection in [10] chapter 12.

Assume then $(A^\varepsilon, \Phi^\varepsilon)$ is a minimizer of $\mathcal{A}_\varepsilon$ in the N-homotopy class (we will omit the superscript ε except when it is really necessary), then the following equations hold

$$\left\{\begin{array}{l} \nabla_A^*\nabla_A\Phi = \dfrac{\Phi}{\varepsilon^2}(1-|\Phi|^2) \\[4mm] -d^*dA = (i\Phi, \nabla_A\Phi) \quad . \end{array}\right. \tag{IV.1}$$

Here $(\cdot,\cdot)$ denotes the Euclidian inner product on $\mathbb{C} = \mathbb{R}^2$ that we will also sometimes simply denote $\cdot,\cdot$. Combining (IV.1) and the global upper-bound (III.2), we get, like in the proof of proposition II.6 of [4]

$$|\nabla h| \leq |\nabla_A\Phi| \leq \frac{C}{\varepsilon} \quad \text{in } \mathbb{C} \quad . \tag{IV.2}$$

where h denotes $*dA$. It is usefuhl also to remind at this stage to the reader the pointwise identity that holds for any configurations, minimizing or not,

$$|\nabla_A\Phi|^2 = |\Phi|^{-2}|(i\Phi, \nabla_A\Phi)|^2 + |\nabla|\Phi||^2 = |\Phi|^2|A - d(arg\,\Phi)|^2 + |\nabla|\Phi||^2$$

Changing (A,Φ) into $(A, T^1(|\Phi|)\Phi/|\Phi|)$, where $T^1(s)=s$ for $s\le 1$ and $T^1(s)=1$ for $s\ge 1$, reduces the energy but preserves the homotopy class. Thus, the minimizing property of (A,Φ) implies we have

$$|\Phi|\le 1 \qquad \text{in } \mathbb{C}. \tag{IV.3}$$

On any ball $B_r(x_0)$ the following Pohozaev identity holds (see lemma 12.2 of [10])

$$\begin{aligned}\int_{B_r(x_0)} \frac{1}{\varepsilon^2}(1-|\Phi|^2)^2 \quad -2|dA|^2 &= r\int_{\partial B_r(x_0)} \frac{1}{2\varepsilon^2}(1-|\Phi|^2)^2-|dA|^2\\ &+r\int_{\partial B_r(x_0)}|\nabla_A\Phi.\tau|^2-|\nabla_A\Phi.\nu|^2 \quad ,\end{aligned} \tag{IV.4}$$

where τ and ν are respectively the tangential and normal unit vector fields on $\partial B_r(x_0)$ with (ν,τ) positively oriented in $\mathbb{R}^2$.

Local estimates like (66) and (67) in [12] here also hold and as in [12], combining η-compactness lemma, local estimates and the Pohozaev identity (IV.4) we deduce that there exist $\mu>0$ and $n_0\in\mathbb{N}$ independent on ε and finitely many points in $\mathbb{R}^2$ $x_1...x_{n^\varepsilon}$ such that

$$\begin{aligned}&\{y \quad \text{s. t.} \quad |\Phi|<1/2\}\subset\cup_{i=1}^{n^\varepsilon}B_{\mu\varepsilon(x_i)}\\ &n^\varepsilon\le n_0 \quad \text{moreover} \quad B_{\mu\varepsilon}(x_i)\cap B_{\mu\varepsilon}(x_j)\neq\emptyset\Longrightarrow i=j \quad .\end{aligned} \tag{IV.5}$$

Following a sequence of minimizers $(A^\varepsilon,\Phi^\varepsilon)$ as $\varepsilon\to 0$ one can find $\mu>0$ such that

$$\frac{|x_i-x_j|}{\varepsilon}\longrightarrow+\infty \qquad \text{for } i\neq j \quad . \tag{IV.6}$$

and one can ensure

$$\deg\left(\frac{\Phi}{|\Phi|};\partial B_{\mu\varepsilon}(x_i)\right)=d_i\neq 0 \quad , \tag{IV.7}$$

(and because of (IV.2) the d_i's are bounded independently of ε). Indeed let $R_i\to\infty$ such that $x_j\in B_{R_i\varepsilon}(x_i)\Rightarrow j=i$, reasoning like in [4] for proving proposition II.6, we can take a gauge on $B_{R_i\varepsilon}(x_i)$ such that $d^*A=0$, $\|A\|_{L^\infty(B_{R_i\varepsilon})}\to 0$ and $\|\nabla\Phi\|_\infty\le\frac{C}{\varepsilon}$. Changing the scale $1\to\frac{1}{\varepsilon}$ we would get a limiting map $\tilde\Phi$ on all of $\mathbb{R}^2$ such that

$$\left\{\begin{array}{l}\Delta\tilde\Phi+\Phi(1-|\tilde\Phi|^2)=0 \quad , \quad \displaystyle\int_{\mathbb{R}^2}(1-|\tilde\Phi|^2)^2<+\infty\\[2ex] |\tilde\Phi|\ge\dfrac{1}{2} \quad \text{on } \mathbb{R}^2\setminus B_\mu(0) \quad \text{and} \quad \exists y\in B^2_\mu(0) \quad \text{s. t. } |\tilde\Phi|<\dfrac{1}{2} \quad .\end{array}\right.$$

From [5] we then deduce that $d_i \neq 0$. See a more complete presentation of these arguments above in lemma 10.3 of [10] for the corresponding non gauge invariant functional and in the first part of the proof of proposition 12.1 of [10] for the present problem.

Introducing the function k solution of

$$-\Delta k + k = \text{div}\left(\frac{\nabla^\perp h}{|\nabla^\perp h|^\beta}\right) \qquad \text{in } \mathcal{D}'(\mathbb{R}^2) \quad ,$$

where $\nabla^\perp h = (\frac{\partial h}{\partial x_2}; -\frac{\partial h}{\partial x_1})$, for some $0 < \beta < 1$ (recall $h = *dA$) and arguing like in the proof of lemma 5.1 of [12] we deduce that

$$\forall 1 < p < 2 \quad \|dA\|_{W^{1,p}(\mathbb{R}^2)} \leq C \quad \text{independent of } \varepsilon \quad . \tag{IV.8}$$

It is here much easier since we are in 2D and the proof of lemma 5.1 of [12] is given for the 3D corresponding problem. Since $h = *dA$ verifies the nice elliptic equation

$$-\text{div}\left(\frac{1}{|\Phi|^2}\nabla h\right) + h = 0 \qquad \text{on } \mathbb{R}^2 \setminus \cup B_{\mu\varepsilon}(x_i) \quad , \tag{IV.9}$$

we have for any η fixed

$$\int_{\mathbb{R}^2\setminus\cup B_\eta(x_i)} |\nabla h|^2 + |h|^2 \leq C_\eta \quad \text{independent of } \varepsilon \quad . \tag{IV.10}$$

Thus, using the second equation of (IV.1) we deduce

$$\int_{\mathbb{R}^2\setminus\cup B_\eta(x_i)} |(i\Phi, \nabla_A\Phi)|^2 \leq C_\eta \quad . \tag{IV.11}$$

Since for each ε we have $\int_{\mathbb{C}} a_\varepsilon(A, \Phi) < +\infty$ we can choose a sequence of radii r_n such that $r_n \to +\infty$ and $r_n \int_{\partial B_{r_n}(0)} a_\varepsilon(A, \Phi) \to 0$. By means of the Pohozaev identity on the balls $B_{r_n}(0)$, we deduce that

$$\int_{\mathbb{R}^2} |dA|^2 = \int_{\mathbb{R}^2} \frac{(1-|\Phi|^2)^2}{2\varepsilon^2} \quad , \tag{IV.12}$$

and from (IV.8) we have that $\int_{\mathbb{R}^2} \frac{(1-|\Phi|^2)^2}{2\varepsilon^2}$ is bounded independently of ε. Argueing like in [4] for the proof of (V.41), we get, replacing Ω by $\mathbb{C}$, for any $p < 2$

$$\int_{\mathbb{R}^2} |\nabla|\Phi||^p \leq C \qquad \text{independent on } \varepsilon \quad . \tag{IV.13}$$

We claim now that, for any $\eta > 0$, the following estimates hold :

$$\||\nabla^k h| + \varepsilon^{-2}|\nabla^k(|\Phi| - 1)|\|_{L^\infty(\mathbb{R}^2\setminus\cup B_{2\eta}(x_i))} \leq C_{\eta,k} \quad , \tag{IV.14}$$

where $C_{\eta,k}$ only depends on η, k and N but not on ε and not on the location of the x_i.

We argue by contradiction and we may then find a k_0, a sequence $\varepsilon_n \to 0$ and a sequence of points z_n such that

$$\begin{aligned} &\mathrm{dist}(z_n, (x_i^{\varepsilon_n})) \geq 2\eta \\ &|\nabla^{k_0} h|(z_n) + \varepsilon^{-2}|\nabla^k(|\Phi| - 1)|(z_n) \to +\infty \quad . \end{aligned} \tag{IV.15}$$

We translate the configuration by z_n and we keep denoting it $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$. Argueing now word by word like in the proof of proposition VI.1 in [4] replacing K by $B_\eta(0)$ we get that, modulo extraction of a subsequence, h^{ε_n} converges strongly in $W^{1,2}(B_\eta(0))$, that $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$ converges also strongly in the same space and that $\int_{B_\eta(0)} \frac{(1-|\Phi^{\varepsilon_n}|^2)^2}{\varepsilon^2} \to 0$. We are now in position to follow almost word by word [2], replacing the equations 51 and 52 of [2] by the following two equations derived from (IV.1)

$$\begin{aligned} &-\mathrm{div}\left(\frac{1}{|\Phi|^2}\nabla h\right) + h = 0 \qquad \text{on } \mathbb{R}^2 \setminus \cup B_\eta(x_i) \\ &-\Delta|\Phi| + \frac{|\nabla h|^2}{|\Phi|^3} = \frac{1}{\varepsilon^2}|\Phi|(1 - |\Phi|^2) \end{aligned}$$

and we get that $\|\nabla^k h^{\varepsilon_n}\|_{L^\infty(B_\eta(0))} + \varepsilon^{-2}\|\nabla^k(|\Phi^{\varepsilon_n}| - 1)\|_{L^\infty(B_\eta(0))}$ remains bounded independently of n for any $k \in \mathbb{N}$. This contradicts (IV.15) and (IV.14) is proved.

Take N such that $N \geq 1$ and assume there exists a sequence $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$ of N-minimizers of $\mathcal{A}_{\varepsilon_n}$ for some $\varepsilon_n \to 0$ (from now on we will often omit the superscript ε_n). All the estimates in IV.1 hold. One can extract a subsequence such that n^{ε_n} given by (IV.5) is fixed, denoted by n_0 and there exists $C > 0$ for which the following holds

$$\forall i \neq j \quad \text{either } x_i^{\varepsilon_n} - x_j^{\varepsilon_n} \longrightarrow 0 \quad \text{or} \quad |x_i^{\varepsilon_n} - x_j^{\varepsilon_n}| \geq C \quad \text{as } \varepsilon_n \to 0 \tag{IV.16}$$

Let $0 < \eta < \frac{C}{4}$ chosen sufficiently small later but independent of ε_n. In $\{1...n\}$ we can establish the equivalence relation such that $i \simeq j$ if $x_i^{\varepsilon_n} - x_j^{\varepsilon_n} \longrightarrow 0$, take $z_{\mathcal{C}}$ to be the barycenter of the $x_i^{\varepsilon_n}$ for all the i's in a given class $\mathcal{C}$. We may extract again a subsequence such that the number of classes is fixed and denote the barycenters by $z_1...z_p$, then for ε_n small enough we have

$$\left\{ \begin{array}{l} \{x \text{ s. t. } |\Phi|(x) < 1/2 \quad \} \subset \cup_{k=1}^p B_\eta(z_k) \\ \\ |z_i - z_j| \geq \dfrac{C}{2} \geq 2\eta \quad \text{ for } i \neq j \quad . \end{array} \right. \tag{IV.17}$$

Let $D_k = \deg\left(\frac{\Phi}{|\Phi|}; \partial B_\eta(z_k)\right)$. Since the d_i given by (IV.7) are bounded independently of ε_n and since the number of $x_j^{\varepsilon_n}$ is fixed, passing to a subsequence again we may assume that the D_k are independent of ε_n. we claim that

$$\forall k = 1...p \quad , \quad D_k > 0 \quad . \tag{IV.18}$$

Assume first $D_k = 0$, using comparison argument and the eta-compactness Lemma (see lemma 24 in [12]), we obtain $|\Phi| \geq 1/2$ on $B_\eta(z_k)$, so we can forget this $B_\eta(z_k)$.

Pick a z_k, for ε_n small enough all the $B_{\mu\varepsilon_n}(x_i^{\varepsilon_n})$ in $B_\eta(z_k)$ are in fact in $B_{\eta/2}(z_k)$. Using the mean value formula we may find a $\nu \in (\eta/2, \eta)$ such that

$$\int_{\partial B_\nu(z_k)} a_{\varepsilon_n}(A^{\varepsilon_n}, \Phi^{\varepsilon_n}) \lesssim \frac{N}{\eta} \log \frac{1}{\varepsilon_n} \quad .$$

We can proceed then exactly like for establishing (III.17) to deduce that for any $\alpha > 0$

$$\int_{B_\nu(z_k)} a_\varepsilon(A, \Phi) \geq 2\pi |D_k| (1-\alpha) \log \frac{1}{\varepsilon} + o(\log \frac{1}{\varepsilon}) \quad . \tag{IV.19}$$

$o(\log \frac{1}{\varepsilon})$ is a function that may depend on α. Summing over k, if one of the D_k is negative, since $\sum D_k = N$ we would contradict lemma III.1.

We claim that for any $\delta > 0$, there exists $\eta_0 > 0$ such that for $\eta < \eta_0$, if ε is sufficiently small

$$(1+\delta) D_k \log \frac{1}{\eta} \geq h(x) \geq (1-\delta) D_k \log \frac{1}{\eta} \quad \text{on } \partial B_\eta(z_k) \quad . \tag{IV.20}$$

Take k_0 and translate $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$ such that $z_{k_0} = 0$. One can always extract a subsequence as $\varepsilon_n \to 0$ such that

$$\text{for each } k \text{ either} \quad z_k^{\varepsilon_n} \to \overline{z}_k \in \mathbb{R}^2 \quad \text{or } |z_k| \to +\infty \quad ,$$

thus clearly, from [4] proposition VII.2, $h \to h_\star$ in $W^{1,p}_{loc}(\mathbb{R}^2)$, $p < 2$, where $h_\star$ is a solution of

$$-\Delta h_\star + h_\star = 2\pi \sum_{k \in K} D_k \delta_{\overline{z}_k} \quad ,$$

where K is the subset of $\{1...p\}$ such that $z_k^{\varepsilon_n}$ converges. K contains of course 0. From (IV.14) we deduce that

$$\|h - h_\star\|_{L^\infty(\partial B_\eta(0))} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0$$

Since $h_\star$ verifies (IV.20) (for any δ there exists η_0 such that for all $\eta < \eta_0$ (IV.20) holds for $h_\star$), this is also the case for h provided ε is small enough depending on the given pair δ, η. This information will be used later on in the proof.

We claim that there exist at least two disjoint $B_\eta(z_k)$. Assume there is only one and take $z_k = 0$. Then h tends to $h_\star$, solution of

$$-\Delta h_\star + h_\star = 2\pi N\delta_0$$

Take on B_1 Coulomb gauges in order to be able to ensure that $\int_{B_1}|A|^2$ is uniformly bounded (see part III). Using theorem 4 and 5 of [5] (observe that there is a missprint in the statement of theorem 5 of [5] and that a 2π is missing - compare the statement of the theorem with the last inequality in the proof) we deduce that

$$\int_{B_1}|\nabla\Phi^{\varepsilon_n}|^2 \geq 2\pi\sum_i d_i^2\log\frac{1}{\varepsilon_n} - 2\pi\sum_{k\neq l} d_k d_l \log|x_k - x_l| - C \quad , \qquad \text{(IV.21)}$$

since $\frac{|x_k^{\varepsilon_n} - x_l^{\varepsilon_n}|}{\varepsilon_n} \rightarrow +\infty$ we have $\sum_{k\neq l} d_k d_l \log|x_k - x_l| = o(\log\frac{1}{\varepsilon_n})$. Because of (III.1) we must have $\sum_i |d_i|^2 = |D_k| = D_k = \sum_i d_i$ and this implies $d_i = +1$ for all i. Thus

$$-\sum_{k\neq l} d_k d_l \log|x_k - x_l| \longrightarrow +\infty \quad ,$$

and this contradicts (III.1). In fact this argument implies

$$\forall k \qquad D_k = +1 \quad . \qquad \text{(IV.22)}$$

Thus there exists exactly N z_k and $|z_k - z_l|$ is bounded from below by a positive constant $l^\star$, for $k \neq l$, as ε tends to 0.

We shall prove now that there exists exactly one cancellation point in each of the $B_{l^\star}(z_k)$.

Consider z_1. We may assume, modulo a translation, that $z_1 = 0$. Since 0 is the barycenter of the $x_i^{\varepsilon_n}$ that get close together in a fixed neighborhood of 0 as $\varepsilon_n \rightarrow 0$, We have that all the $x_i^{\varepsilon_n}$ in $B_{l^\star}(0)$ tend to 0. As we explained above we can extract a subsequence such that, in $B_{l^\star}(0)$, h^{ε_n} converges in $W^{1,p}\cap C^l_{loc}(B_{l^\star}(0)\setminus\{0\})$, for any $l\in\mathbb{N}$, to a solution $h_\star$ of

$$-\Delta h_\star + h_\star = 2\pi\delta_0$$

Therefore, for any $\delta > 0$, there exists $\eta_0 > 0$ such that, for any given $\eta_0 > \eta > 0$, there exists n large enough, such that,

$$\frac{1+\delta}{\eta} \geq \frac{\partial h^{\varepsilon_n}}{\partial r}(x) \geq \frac{1-\delta}{\eta} \qquad \text{(IV.23)}$$

for $x\in\partial B_\eta(0)$. We have also

$$\int_{B_\eta}|h|^2 \leq C\eta\left(\int_{B_\eta}|h|^4\right)^{\frac{1}{2}} \leq C\eta\ \|h\|^2_{W^{1,4/3}(B_1)} \qquad \text{(IV.24)}$$

Combining (IV.4), (IV.23) and (IV.24) we get that for any δ, there exists $\eta > 0$ and $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$,

$$\int_{B_\eta(0)} \frac{(1-|\Phi|^2)^2}{\varepsilon_n^2} \leq 2\pi + \delta \tag{IV.25}$$

Consider now the gauge A on $B_\eta(0)$ given by $A = \nabla^\perp \xi$ where ξ solves $\Delta\xi = h^{\varepsilon_n}$ on $B_\eta(0)$ and $\xi = 0$ on $\partial B_1(0)$. Since $\|h\|_{W^{1,p}}$ remains bounded as $\varepsilon_n \to 0$, we have, for any $p < 2$, that $\|A\|_{L^\infty(B_1)}$ remains also bounded. Since $\operatorname{div} A = 0$, the configuration (A, Φ) solves in B_η

$$\Delta\Phi + \Phi\frac{(1-|\Phi|^2)}{\varepsilon^2} - |A|^2\Phi - 2iA.\nabla\Phi = 0 \quad . \tag{IV.26}$$

Since $|\nabla_A \Phi| \leq \frac{C}{\varepsilon}$ on B_1 and since $\|A\|_\infty$ is uniformly bounded, this implies

$$|\nabla\Phi| \leq \frac{C}{\varepsilon} \quad . \tag{IV.27}$$

Assume that $|\Phi(0)| = 0$. We proceed to a dilation $1 \to \frac{1}{\varepsilon}$: that is $\tilde{\Phi}(z) = \Phi(\varepsilon z)$. We have $|\nabla\tilde{\Phi}| \leq C$ on $B_{\frac{1}{\varepsilon}}$ and it verifies

$$\Delta\tilde{\Phi} + \tilde{\Phi}(1-|\tilde{\Phi}|^2) - \varepsilon^2|A|^2(\varepsilon z)\tilde{\Phi} - 2i\varepsilon^2 A(\varepsilon z)\frac{\nabla\tilde{\Phi}}{\varepsilon} = 0 \tag{IV.28}$$

Thus, because of (V.7) and (IV.28) we deduce that $\tilde{\Phi}$ converges, modulo extraction of a subsequence, in $C^1_{\text{loc}}(\mathbb{R}^2)$ to a solution of

$$\begin{cases} \Delta\hat{\Phi} + \hat{\Phi}(1-|\hat{\Phi}|^2) = 0 \quad \text{on } \mathbb{R}^2 \quad \text{ and } \quad \hat{\Phi}(0) = 0 \\ \int_{\mathbb{R}^2}(1-|\hat{\Phi}|^2)^2 < +\infty \end{cases}$$

From [5] we know that for such a solution we have

$$\int_{\mathbb{R}^2}(1-|\hat{\Phi}|^2)^2 \in 2\pi\mathbb{Z}_+^*$$

where the given integer is the degree of $\hat{\Phi}$ on circles centered at the origin with sufficiently large radius. Denote $\deg_\infty\hat{\Phi}$ this degree. Combining this fact with (IV.25) we obtain that $\deg_\infty\hat{\Phi} = 1$. Using now [8], we know that this solution is unique and vanishes only at 0 where $\operatorname{rank}\nabla\hat{\Phi} = 2$. Since, for any given radius R

$$\|\tilde{\Phi} - \hat{\Phi}\|_{L^\infty(B_R(0))} + \|\nabla(\tilde{\Phi} - \hat{\Phi})\|_{L^\infty(B_R(0))} \longrightarrow 0$$

then, for ε_n small enough, Φ vanishes only at 0 in $B_{R\varepsilon_n}(0)$. Moreover on $B_{R\varepsilon_n}(0)$ we have

$$\det\nabla\Phi(z) \geq \frac{C}{\varepsilon_n^2} \quad \text{on} \quad B_{R\varepsilon_n} \quad \text{and } |\Phi|(z) \geq c\frac{|z|}{\varepsilon_n} \tag{IV.29}$$

Granting this information for $R = \mu$ where μ is given by (IV.6), we deduce that, if Φ vanishes somewhere else in $B_\eta(0)$, this has to happen in another ball $B_{\mu\varepsilon_n}(x_n^j)$ where $|x_n^j - 0| >> \varepsilon_n$ as $n \to +\infty$. Dilating now about such an eventual zero for Φ we will get the same conclusions as above (while we dilated about 0) and a new 2π contribution to the potential part of the energy $\int_{B_\eta(0)} \frac{(1-|\Phi|^2)^2}{\varepsilon_n^2}$ that would become larger than any constant less than 4π for η small enough and n large enough. This would contradict (IV.25). We have then finaly established the following lemma.

Lemma IV.1 *Let $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$ be a sequence of $N-$minimizers of $\mathcal{A}_{\varepsilon_n}$ for some sequence $\varepsilon_n \to 0$. Then one can extract a subsequence, still denoted ε_n, such that there exists N points $\{z_1^{\varepsilon_n} \cdots z_N^{\varepsilon_n}\}$ verifying*

i) $\forall\ i \neq j\ |z_i^{\varepsilon_n} - z_j^{\varepsilon_n}| \geq C > 0$ *where C is independent of n.*

ii) $|\Phi^{\varepsilon_n}|^{-1}(\{0\}) = \{z_1^{\varepsilon_n} \cdots z_N^{\varepsilon_n}\}$.

iii) $\forall l \in \mathbb{N}\ |\nabla^l(1 - |\Phi^{\varepsilon_n}|)| \leq C\frac{\varepsilon^2}{dist(x,\{z_1 \cdots z_N\})^{2+l}}$.

iv) There exists c, a universal constant, such that, for any $k = 1 \cdots N$

$$det\,\nabla\Phi^{\varepsilon_n}(z) \geq \frac{c}{\varepsilon_n^2} \qquad in\ B_{\varepsilon_n}(z_k) \quad ,$$

$$|\Phi^{\varepsilon^n}| \geq c\frac{|z|}{\varepsilon_n} \qquad in\ B_{\varepsilon_n}(z_k) \quad .$$

IV.2 The non existence result.

In this part we prove theorem I.1. The proof is based on the following decomposition of h^{ε_n}.

Lemma IV.2 *Let $|\Phi^{\varepsilon_n}|$ given by lemma IV.2. For any $k = 1 \cdots N$ there exists a unique solution $h_k^{\varepsilon_n}$ in $W^{1,2}(\mathbb{R}^2)$ solving*

i)

$$\frac{\nabla h_k}{|\Phi^{\varepsilon_n}|^2} \in L^p(\mathbb{R}^2)\ for\ some\ p > 1$$

ii)

$$-div\left(\frac{\nabla h_k}{|\Phi^{\varepsilon_n}|^2}\right) + h_k = 2\pi\ \delta_{z_k} \qquad in\ \mathcal{D}'(\mathbb{R}^2)$$

moreover $h_k > 0$ on $\mathbb{R}^2$.

Proof of lemma IV.2 : Since the result of lemma IV.2 does not involve estimates but is purely a qualitative result, in order to simplify the presentation, we give the proof of lemma IV.2 for $|\Phi^{\varepsilon_n}|$ replaced by the function $\overline{r} := \min(|x|, 1)$ (which has the same behavior at z_i as $|\Phi^{\varepsilon_n}|$) and with a Dirac mass centered at the origin : we look for a solution of

$$-div\left(\frac{\nabla h}{\overline{r}^2}\right) + h = 2\pi\ \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2) \tag{IV.30}$$

Let f^δ be an approximation of the Dirac mass at the origin : $f^\delta(x) := \frac{1}{\delta^2}\psi(x/\delta)$ with $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}_+)$ and $\int_{\mathbb{R}^2}\psi = 1$. Consider the solution h^δ given by lemma A.5 of

$$-div\left(\frac{\nabla h^\delta}{\overline{r}^2}\right) + h^\delta = 2\pi\ f^\delta \qquad \text{in } \mathcal{D}'(\mathbb{R}^2) \quad . \tag{IV.31}$$

Multiplying the equation by h^δ and integrating by parts we get

$$\int_{\mathbb{R}^2} \frac{|\nabla h^\delta|^2}{\overline{r}^2} + |h^\delta|^2 \le \frac{1}{\delta}\left(\int_{B_\delta} |h^\delta|^2\right)^{\frac{1}{2}} \quad . \tag{IV.32}$$

Therefore h^δ is in $W^{1,2}$. A standard application of the maximum principle yields $h^\delta > 0$. Combining this two facts, since

$$\int_{\mathbb{R}^2} h^\delta = \lim_{R_i\to+\infty}\int_{B_{R_i}} f^\delta + \int_{\partial B_{R_i}} \frac{\partial h^\delta}{\partial r} \quad ,$$

for any $R_i \to +\infty$. Choosing R_i such that

$$\left|\int_{\partial B_{R_i}} \frac{\partial h^\delta}{\partial r}\right| \le R_i^{\frac{1}{2}}\left|\int_{\partial B_{R_i}} |\nabla h^\delta|^2\right|^{\frac{1}{2}} \le C\left|\int_{B_{2R_i}\setminus B_{R_i/2}} |\nabla h^\delta|^2\right|^{\frac{1}{2}} \to 0 \ ,$$

we deduce that

$$\int_{\mathbb{R}^2} |h^\delta| \le C \qquad \text{indep. of } \delta \tag{IV.33}$$

Multiplying the equation (IV.31) by $\overline{r}^2$ we obtain

$$-\Delta h^\delta + 2\frac{\nabla\overline{r}}{\overline{r}}\cdot\nabla h^\delta + \overline{r}^2 h^\delta = \overline{r}^2 f^\delta \quad ,$$

and since $r^2 f^\delta = \frac{r^2}{\delta^2}\psi(x/\delta)$ is bounded indepentently of δ we may apply lemma A.4 for ρ replaced by $\overline{r}$ to deduce that there exists $\nu_0 < 1$ such that for any $\nu_0 < \nu < 1$

$$\int_{\mathbb{R}^2} \overline{r}^{2-2\nu}\frac{|\nabla h^\delta|^2}{\overline{r}^2} \qquad \text{indep. of } \delta \quad . \tag{IV.34}$$

Therefore we have that $\int_{B_1} r^{-2\nu}|\nabla h^\delta|^2 \le C$ for any $\nu < 1$ where C is independent on δ. For any $\rho > 0$ we consider the equation on $B_{2\rho} \setminus B_\rho$, we dilate everything by $1/\rho$ and a standard bootstrap of the inequality (IV.34) yields

$$\forall x \in B_1(0) \setminus \{0\} \qquad |\nabla h^\delta|(x) \le \frac{C}{|x|^{1-\nu}}$$

and we deduce, using the L^1 bound (IV.33) for h^δ that

$$\|h^\delta\|_{C^{0,\mu}(B_1)} \qquad \text{is bounded independently of } \delta \tag{IV.35}$$

as δ tends to 0.

Multiply now the equation by $r^{-2\nu}[h^\delta - h^\delta(0)]$ for ν chosen small enough later on. We integrate by parts this product on $B_1 \setminus B_s$ for any $0 < s < 1$ and we obtain, using the fact that $r^{-2\nu}[h^\delta - h^\delta(0)]$ is bounded in L^∞ norm independently on δ (because of (IV.35)),

$$\begin{aligned}\int_{B_1\setminus B_s} r^{-2\nu-2}|\nabla h^\delta|^2 &\le C + s^{-2\nu-2}\left|\int_{\partial B_s} [h^\delta - h^\delta(0)]\ \frac{\partial h^\delta}{\partial r}\right| \\ +2\nu&\left|\int_s^1 \rho^{-1}\int_{\partial B_\rho} \rho^{-2\nu-2}[h^\delta - h^\delta(0)]\ \frac{\partial h^\delta}{\partial r}\right| \quad .\end{aligned} \tag{IV.36}$$

For any $s \le \rho \le 1$ we have

$$\begin{aligned}\frac{1}{\rho^{2+2\nu}}\int_{\partial B_\rho} h^\delta - h^\delta(0)\ \frac{\partial h^\delta}{\partial r} &= \frac{1}{\rho^{1+2\nu}}\int_{\partial B_\rho} \frac{h^\delta - h^\delta_\rho}{\rho}\ \frac{\partial h^\delta}{\partial r} \\ +\frac{1}{\rho^{2+2\nu}}[h^\delta_\rho - h^\delta(0)]\int_{\partial B_\rho} \frac{\partial h^\delta}{\partial r} &\quad ,\end{aligned} \tag{IV.37}$$

where h^δ_ρ denotes the average of h^δ on ∂B_ρ. In one hand we use Poincaré inequality to deduce that

$$\left|\frac{1}{\rho^{1+2\nu}}\int_{\partial B_\rho} \frac{h^\delta - h^\delta_\rho}{\rho}\ \frac{\partial h^\delta}{\partial r}\right| \le C\rho\int_{\partial B_\rho} \frac{|\nabla h^\delta|^2}{\rho^{2+2\nu}} \tag{IV.38}$$

In the other hand we have, using (IV.31), (IV.33) and (IV.35),

$$\left|\frac{1}{\rho^{2+2\nu}}[h^\delta_\rho - h^\delta(0)]\int_{\partial B_\rho} \frac{\partial h^\delta}{\partial r}\right| \le C\rho^{\mu-2\nu} \quad , \tag{IV.39}$$

for any $\mu < 1$. Denote $Y(s) := \int_{B_1\setminus B_s} \frac{|\nabla h^\delta|^2}{\rho^{2+2\nu}}$ and combining (IV.36)...(IV.39), for ν small enough we get

$$Y(s) \le -Cs\ Y'(s) + C \quad .$$

Therefore if we denote $W(s) := Y(s) - C$, we have that $s^{\frac{1}{C}} W(s)$ is decreasing in s. Assume there exists $s_0 > 0$ such that $W(s_0) > 0$, this would imply that, for $0 < s < s_0$ $Y(s) \geq \frac{s_0^{\frac{1}{C}} W(s_0)}{s^{\frac{1}{C}}} + C$ but from (IV.32) we have that $Y(s) \leq \frac{C}{s^{2\nu}}$. This yields a contradiction for ν small enough. Therefore we have obtained the existence of $\nu > 0$ and C independent on δ such that

$$\int_{B_1} r^{-2\nu-2} |\nabla h^\delta|^2 \leq C \quad . \tag{IV.40}$$

As above, from this estimate we deduce the following pointwise estimate for ∇h^δ :

$$|\nabla h^\delta|(x) \leq C|x|^\nu \quad , \tag{IV.41}$$

where C is independent of δ. We then deduce from that estimate an L^p bound for $\frac{\nabla h^\delta}{r^2}$, for $p(2-\nu) < 2$, which is independent of δ and the equation (IV.31) passes to the limit as δ tends to 0. We have then constructed a solution of i) and ii) in the statement of lemma IV.2. The uniqueness is a direct consequence of lemma A.4. Lemma IV.2 is proved. ∎

We have the following lemma

Lemma IV.3 *Let $h_k^{\varepsilon_n}$ be the solution given by lemma IV.2, for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$ and an integer n_α such that, for $n \geq n_\alpha$, if j_k realizes the* $\min_{j \neq k} |z_k - z_j|$,

$$\frac{1}{C_\alpha} \exp\left(-(1+\alpha)|z_k - z_{j_k}|\right) \leq h_k^{\varepsilon_n}(z_{j_k}) \leq C_\alpha \exp\left(-(1-\alpha)|z_k - z_{j_k}|\right) \tag{IV.42}$$

Proof of lemma IV.3 : (In this proof again we drop the superscript ε_n). For any $-1 < \delta < +1$ we denote by G_k^δ the solution of

$$-\Delta G_k^\delta + (1+\delta) G_k^\delta = 2\pi\ \delta_{z_k} \tag{IV.43}$$

Denote by $L^\star := \min(1, \min_{l \neq j}(|z_l - z_j|))$. From lemma IV.1 we know that $L^\star$ is bounded from below independently of n. There are precise pointwise estimates for the Bessel Kernels G_k^δ in the litterature and in particular it is a well known fact that for any $\alpha > 0$ there exits C_α such that for any z verifying $|z - z_k| > L^\star/2$, we have

$$\frac{1}{C_\alpha} \exp\left(-(1+\alpha)\sqrt{1+\delta}|z_k - z|\right) \leq h_k^{\varepsilon_n}(z) \leq C_\alpha \exp\left(-(1-\alpha)\sqrt{1+\delta}|z_k - z|\right) \tag{IV.44}$$

Moreover, there exists C_1 and C_2, two positive constants such that, for $|z - z_k| < 1$, we have

$$\log \frac{1}{|z - z_k|} - C_1 \leq G_k^\delta(z) \leq \log \frac{1}{|z - z_k|} + C_2 \quad . \tag{IV.45}$$

We shall now slightly modify G_k^δ before to proceed to a comparison with h_k. Let $\eta > 0$ be a cut-off parameter smaller than $L^\star/2$ and to be fixed later. We also introduce a familly of cut-off functions $\chi_\eta \in C_0^\infty(\cup_{j\neq k} B_\eta(z_j))$ verifying

$$\chi_\eta \equiv 1 \text{ on } \cup_{j\neq k} B_{\frac{\eta}{2}}(z_j) \quad \text{ and } \quad \|\nabla^l \chi\|_\infty \le \frac{C}{\eta^l} \quad ,$$

for some fixed C. Let $\tilde{G}_\delta^k := (1-\chi_\eta) G_\delta^k + \chi_\eta G_\delta^k(z_j)$ in $B_\eta(z_j)$ for $j \neq k$ and $\tilde{G}_\delta^k := G_\delta^k$ in $\mathbb{R}^2 \setminus \cup_{j\neq k} B_\eta(z_j)$. Denote

$$E_\delta^k := -\Delta \tilde{G}_\delta^k + (1+\delta)\tilde{G}_\delta^k = \left[G_\delta^k - G_\delta^k(z_j)\right] \Delta\chi_\eta + 2\nabla\chi_\eta \cdot \nabla \tilde{G}_\delta^k + \chi_\eta (1+\delta) \tilde{G}_\delta^k(z_j)$$

$$\text{in } B_\eta(z_j) \quad j \neq k \quad .$$

A standard application of the Harnack inequality to G_k^δ solving (IV.43) in $B_{L^\star}(z_j)$ yields

$$\|\tilde{G}_\delta^k\|_{L^\infty(B_{L^\star/2}(z_j))} \le C\ \tilde{G}_\delta^k(z_j) \quad .$$

Bootstraping this information in (IV.43), we have $\|\nabla \tilde{G}_\delta^k\|_{L^\infty(B_{L^\star/4}(z_j))} \le C\, \tilde{G}_\delta^k(z_j)$. Then for $\eta < L^\star/4$ we have for any $j \neq k$

$$\|E_\delta^k\|_{L^\infty(B_\eta(z_j))} \le \frac{C}{\eta}\ G_\delta^k(z_j) \quad . \tag{IV.46}$$

Introduce $\hat{G}_\delta^k$ the solution of

$$-div\left(\frac{1}{|\Phi|^2}\nabla \hat{G}_\delta^k\right) + \hat{G}_\delta^k = E_\delta^k \qquad \text{in } \mathbb{R}^2 \quad . \tag{IV.47}$$

From (IV.46) we deduce that, for any $p < 2$, $\|E_\delta^k\|_{L^p(\mathbb{R}^2)} \le C\eta^{\frac{2}{p}-1} \sum_{j\neq k} |G_\delta^k(z_j)|$. We may now apply Lemma A.6 to $\hat{G}_\delta^k$ and deduce that for any $p > 1$ we have

$$\|\hat{G}_\delta^k\|_{L^\infty(\mathbb{R}^2)} \le C_p \eta^{\frac{2}{p}-1} \sum_{j\neq k} |G_\delta^k(z_j)| \quad . \tag{IV.48}$$

Let $\overline{G}_\delta^k := \tilde{G}_\delta^k - \hat{G}_\delta^k$. Choosing η small enough we can ensure, for j_k realizing the $\min_{j\neq k} |z_k - z_j|$, that

$$3G_\delta^k(z_{j_k}) \ge 2\overline{G}_\delta^k(z_{j_k}) \ge G_\delta^k(z_{j_k}) \quad . \tag{IV.49}$$

$\overline{G}_\delta^k$ solves

$$\begin{aligned} &-div\left(\frac{\nabla \overline{G}_\delta^k}{|\Phi|^2}\right) + \overline{G}_\delta^k \\ &= -div\left(\left[\frac{1}{|\Phi|^2} - 1\right]\nabla \tilde{G}_\delta^k\right) - \delta \tilde{G}_\delta^k \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \setminus \{z_k\}) \quad . \end{aligned} \tag{IV.50}$$

Let $L^\star/2 > \eta_0 >$ given by (IV.20) such that

$$\frac{2}{\eta_0} \ge h_k^\delta \ge \frac{1}{2\eta_0} \qquad \text{on } \partial B_{\eta_0}(z_k) \quad . \tag{IV.51}$$

Since $\nabla \tilde{G}_k^\delta$ is supported in $\mathbb{R}^2 \setminus \cup_{j\ne k} B_{\eta/2}(z_j)$, we have, for ε_n small enough

$$\left| div\left(\left[\frac{1}{|\Phi|^2}-1\right]\nabla\tilde{G}_\delta^k\right)\right|(z) \le C_\eta \varepsilon_n^2 [|\nabla \tilde{G}_\delta^k|(z) + |\Delta \tilde{G}_\delta^k|(z)] \quad .$$

By the mean of Harnack inequalities we deduce that for any $z \in \mathbb{R}^2 \setminus B_{\eta_0}(z_k)$

$$\|G_\delta^k\|_{L^\infty(B_{\eta_0/2}(z))} \le C G_\delta^k(z) \quad . \tag{IV.52}$$

Moreover, bootstraping this information in the equation solved by G_δ^k, we get for any $l \in \mathbb{N}$ and $z \in \mathbb{R}^2 \setminus B_{\eta_0}(z_k)$

$$\|\nabla^l G_\delta^k\|_{L^\infty(B_{\eta_0/2}(z))} \le C_l G_\delta^k(z) \quad . \tag{IV.53}$$

Then we deduce that, for n large enough, we have, for any $\delta \in (-1,0)\cup(0,1)$

$$\left| div\left(\left[\frac{1}{|\Phi|^2}-1\right]\nabla\tilde{G}_\delta^k\right)\right|(z) \le \frac{|\delta|}{2} G_\delta^k(z) \tag{IV.54}$$

Combining (IV.50) and (IV.54) we get,

$$\text{for } \delta \in (0,1) \quad -div\left(\frac{\nabla \overline{G}_\delta^k}{|\Phi|^2}\right) + \overline{G}_\delta^k \le 0 \qquad \text{in } \mathbb{R}^2 \setminus B_{\eta_0}(z_k) \tag{IV.55}$$

and

$$\text{for } \delta \in (-1,0) \quad -div\left(\frac{\nabla \overline{G}_\delta^k}{|\Phi|^2}\right) + \overline{G}_\delta^k \ge 0 \qquad \text{in } \mathbb{R}^2 \setminus B_{\eta_0}(z_k) \tag{IV.56}$$

From (IV.51) we deduce the existence of Λ such that for any $-1 < \delta < 0$

$$\Lambda G_\delta^k \ge h \ge h_k \qquad \text{on } \partial B_{\eta_0}(z_k)$$

Therefore we have

$$\begin{cases} -div\left(\dfrac{\nabla[\Lambda \overline{G}_\delta^k - h_k]}{|\Phi|^2}\right) + [\Lambda \overline{G}_\delta^k - h_k] \ge 0 \qquad \text{in } \mathbb{R}^2 \setminus B_{\eta_0}(z_k) \\ \Lambda \overline{G}_\delta^k - h_k \ge 0 \qquad \text{on } \partial B_{\eta_0}(z_k) \end{cases} \tag{IV.57}$$

We then deduce that

$$h_k \le \Lambda \overline{G}_\delta^k \tag{IV.58}$$

Combining this fact with (IV.49) we deduce that for any given $0 < \delta < 1$, for n large enough, we have $h_k(z_{j_k}) \le C G_k^\delta(z_{j_k})$ where we recall that j_k realizes the $\min(|z_k - z_j|)$ for $j \neq k$. Thus, the upperbound in (IV.42) is proved. In order to prove the lowerbound, we first observe from the fact that $h_l \leq \Lambda \overline{G}_\delta^l$ for any $l \neq k$ that, on $\partial B_\eta(z_k)$, h_l is uniformly bounded independently of η for $l \neq k$. Then, since $h = \sum_l h_l$, because of (IV.20), we deduce the existence of η_1 and λ, independent of n such that, for n large enough, $h_k \geq \lambda G_l^k$ on $\partial B_{\eta_1}(z_k)$. Using (IV.55) for η_0 replaced by η_1, we deduce

$$
\begin{aligned}
& -div\left(\frac{\nabla[h_k - \lambda \overline{G}_\delta^k]}{|\Phi|^2}\right) + [h_k - \lambda \overline{G}_\delta^k] \geq 0 \qquad \text{in } \mathbb{R}^2 \setminus B_{\eta_1}(z_k) \\
& h_k - \lambda \overline{G}_\delta^k \geq 0 \qquad \text{on } \partial B_{\eta_1}(z_k) \quad .
\end{aligned} \tag{IV.59}
$$

This implies that $h_k \geq \lambda \overline{G}_\delta^k$ on $\mathbb{R}^2 \setminus B_{\eta_1}(z_k)$ for any given $0 < \delta < 1$ and n large enough. Lemma IV.3 is proved. ■

We will adopt the following notations : denote by $H_{i,k} := h_k^{\varepsilon_n}(z_i)$, $H_k := h_k^{\varepsilon_n}(z_{i_k})$ where i_k realizes the $\min\{|z_i - z_k|\}$ for $i \neq k$ and $H_\star := \max_k H_k$. Denote also by $d(z) := \min_k |z - z_k|$

We now establish the following pointwise estimates.

Lemma IV.4 *Let $|\Phi^{\varepsilon_n}|$ given by lemma IV.1 and let $h_k^{\varepsilon_n}$ be given by lemma IV.2, the following identities hold :*

i) For any $k = 1 \cdots N$ and for any z such that $|z - z_k| \geq L^\star/2$, for any $\alpha \in (0,1)$ there exists $C_\alpha > 0$ and n_0 such that for $n \geq n_0$

$$
\begin{aligned}
& 0 \leq h_k^{\varepsilon_n}(z) \leq C_\alpha \ \exp\left(-(1-\alpha)|z - z_k|\right) \\
& + C \ H_\star \ \exp\left(-(1-\alpha) d(z)\right) \quad .
\end{aligned} \tag{IV.60}
$$

ii) For any $k = 1 \cdots N$ and for any z such that $d(z) \geq L^\star/2$, for any $\alpha \in (0,1)$ there exists $C_\alpha > 0$ and n_0 such that for $n \geq n_0$

$$
\begin{aligned}
& |\nabla h_k^{\varepsilon_n}|(z) \leq C_\alpha \ \exp\left(-(1-\alpha)|z - z_k|\right) \\
& + C \ H_\star \ \exp\left(-(1-\alpha) d(z)\right) \quad .
\end{aligned} \tag{IV.61}
$$

iii) For any z such that $d(z) \geq L^\star/2$ and for any $\alpha \in (0,1)$, there exists $C_\alpha > 0$ and n_0 such that for $n \geq n_0$

$$
1 - |\Phi^{\varepsilon_n}|(z) \leq C_\alpha \ \varepsilon_n^2 \ \exp\left(-\sqrt{2}(1-\alpha) \ d(z)\right) \quad . \tag{IV.62}
$$

iv) For any z such that $d(z) \geq L^\star/2$ and for any $\alpha \in (0,1)$ there exists $C_\alpha > 0$ and n_0 such that for $n \geq n_0$

$$
|\nabla |\Phi^{\varepsilon_n}||(z) \leq C_\alpha \ \varepsilon_n \ \exp\left(-\sqrt{2}(1-\alpha) \ d(z)\right) \quad . \tag{IV.63}
$$

■

Proof of lemma IV.4 : In this proof again we drop the superscripts ε_n. From (IV.58) we deduce that , for any $-1 < \delta < 0$

$$h_k(z) \le C_\alpha \ \exp(-(1-\alpha)\ |z-z_k|) + |\hat{G}^k_\delta| \quad .$$

From (IV.48) we have $\|\hat{G}^k_\delta\|_\infty \le H_\star$ and since it solves $-div\left(\frac{\nabla \hat{G}^k_\delta}{|\Phi|^2}\right) + \hat{G}^k_\delta = 0$ in $\mathbb{R}^2 \setminus \cup_j B_{L^\star/2}(z_j)$ we deduce that $|\hat{G}^k_\delta|(z) \le C\ H_\star \exp(-(1-\alpha)\ d(z))$. We have then established (IV.60).

There exists, C independent of n, such that

$$|\nabla|\Phi|^2| + ||\Phi|^2 - 1| \le C\varepsilon_n^2 \qquad \text{in } \mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j)$$

A standard bootstrapping of (IV.60) in the equation (IV.9) yields that for any $z \in \mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j)$, we have

$$\|\nabla h\|_{L^\infty(B_{L^\star/4}(z))} \le C\ \|h\|_{L^\infty(B_{L^\star/3}(z))}$$

this implies (IV.61).

Recall that $|\Phi|$ solves

$$-\Delta(1-|\Phi|) + \frac{|\Phi|\ |1+|\Phi||}{\varepsilon_n^2}\ (1-|\Phi|) = \frac{|\nabla h|^2}{|\Phi|^3} \quad . \tag{IV.64}$$

Then, from (IV.61), we deduce that in $\mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j)$

$$-\Delta(1-|\Phi|) + \frac{|\Phi|\ |1+|\Phi||}{\varepsilon_n^2}\ (1-|\Phi|) \le C\ \exp\left(-2(1-\alpha)\ d(z)\right) \quad . \tag{IV.65}$$

Let G solving

$$-\Delta G + 2G = 2\pi \sum_{j=1}^N \delta_{z_j} \qquad \text{in } \mathbb{R}^2 \quad .$$

For any positive number Λ, we have in $\mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j)$

$$\begin{aligned} &-\Delta(1-|\Phi|-\Lambda\varepsilon_n^2 G) + \frac{|\Phi|\ (1+|\Phi|)}{\varepsilon_n^2}\ (1-|\Phi|-\Lambda\varepsilon_n^2 G) \\ &\le C\ \exp\left(-2(1-\alpha)\ d(z)\right) - |\Phi|\ (1+|\Phi|)\Lambda G + 2\Lambda\varepsilon_n^2 G \quad . \end{aligned}$$

Since, for any $\alpha > 0$ there exists C_α such that $G \ge C_\alpha\ \exp(-(\sqrt{2}+\alpha)\ d(z))$, we have that for n large enough and Λ large enough $C\ \exp\left(-2(1-\alpha)\ d(z)\right) - |\Phi|\ (1+|\Phi|)\Lambda G + 2\Lambda\varepsilon_n^2 G \le 0$. There exists Λ large enough and independent of n such that $\Lambda\varepsilon_n^2 G \ge 1 - |\Phi|$ on $\cup_{j=1}^N \partial B_{L^\star/2}(z_j)$. We finally obtain that

$$\begin{aligned} &-\Delta(1-|\Phi|-\Lambda\varepsilon_n^2 G) + \frac{|\Phi|\ (1+|\Phi|)}{\varepsilon_n^2}\ (1-|\Phi|-\Lambda\varepsilon_n^2 G) \le 0 \quad \text{in } \mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j) \\ &(1-|\Phi|-\Lambda\varepsilon_n^2 G) \le 0 \qquad \text{on } \cup_{j=1}^N \partial B_{L^\star/2}(z_j) \quad . \end{aligned}$$

We then deduce by the maximum principle that

$$1-|\Phi| \leq \Lambda\varepsilon_n^2 G \qquad \text{in } \mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j) \quad .$$

This implies (IV.62).

(IV.63) is a consequence of (IV.62) the equation (IV.65) and the following Gagliardo type interpolation inequality estblished in ([2]). For any $z_0 \in \mathbb{R}^2 \setminus \cup_{j=1}^N B_{L^\star/2}(z_j)$ we dilate Φ as follows $\tilde{\Phi}(z) := \Phi(\varepsilon_n z + z_0)$ and we have

$$\|\nabla|\tilde{\Phi}|\|^2_{L^\infty(B_1(0))} \leq C\|1-|\tilde{\Phi}|\|_{L^\infty(B_1(0))}\|\mathcal{L}(1-|\tilde{\Phi}|)\|_{L^\infty(B_1(0))}+C\|1-|\tilde{\Phi}|\|^2_{L^\infty(B_1(0))}$$

where $\mathcal{L} := -\Delta\ + |\tilde{\Phi}|\ (1+|\tilde{\Phi}|)$. Lemma IV.4 is proved. ∎

Proof of theorem I.1.

We argue by contradiction and assume that there exists a sequence $\varepsilon_n \to 0$ and a sequence of minimizing configuration $(A^{\varepsilon_n}, \Phi^{\varepsilon_n})$ in the $N-$homotopy class for any given $N > 1$. We extract a subsequence, that we keep denoting ε_n such that the consequences of Lemma IV.1 are fullfield. Most of the time we forget the superscript ε_n in this proof. The fact that $h = \sum_{i=1}^N h_i$ (where the h_i are given by lemma IV.2) is a direct consequence of Lemma A.4. We proceed to the following decomposition of the minimal energy $\mathcal{A}^N_{\varepsilon_n}$:

$$\begin{aligned}
\mathcal{A}^N_{\varepsilon_n} &= \int_{\mathbb{R}^2} |\nabla_A\Phi|^2 + |h|^2 + \frac{1}{2\varepsilon_n^2}|1-|\Phi|^2|^2 \\
&= \int_{\mathbb{R}^2} \frac{|\nabla h|^2}{|\Phi|^2} + |h|^2 + |\nabla|\Phi||^2 + \frac{1}{2\varepsilon_n^2}|1-|\Phi|^2|^2 \\
&= \int_{\mathbb{R}^2} \sum_{i=1}^N \left[\frac{|\nabla h_i|^2}{|\Phi|^2} + |h_i|^2\right] + \int_{\mathbb{R}^2} \sum_{k\neq j} \frac{\nabla h_j\cdot\nabla h_k}{|\Phi|^2} + h_j\ h_k \\
&+ \int_{\mathbb{R}^2} |\nabla|\Phi||^2 + \frac{1}{2\varepsilon_n^2}|1-|\Phi|^2|^2 \\
&= \int_{\mathbb{R}^2} \sum_{i=1}^N \left[\frac{|\nabla h_i|^2}{|\Phi|^2} + |h_i|^2\right] + \sum_{j\neq k} H_{j,k} \\
&+ \int_{\mathbb{R}^2} |\nabla|\Phi||^2 + \frac{1}{2\varepsilon_n^2}|1-|\Phi|^2|^2 \quad .
\end{aligned} \tag{IV.66}$$

We define $|\Phi_i|$ in the following way

$$|\Phi_i| := |\Phi| \qquad \text{in } B_{L/2}(z_i)$$

$$|\Phi_i| := 1 \qquad \text{in } \mathbb{R}^2 \setminus B_{L/2+L^\star/4}(z_i)$$

with a smooth interpolation inbetween so that, from (IV.62) and (IV.63), we have for any $0<\alpha<1$, $|\nabla|\Phi_i|| \leq C_\alpha\ \varepsilon_n\ \exp(-(1-\alpha)\sqrt{2}\frac{L}{2})$ in $B_{L/2+L^\star/2}(z_i) \setminus$

$B_{L/2}(z_i)$ (recall that L denotes $\min_{k\neq l}|z_k - z_l|$). Define a configuration (A_i, Φ_i) as follows

$$A_i := \frac{\nabla^\perp h_i}{|\Phi|^2} + d(arg(z - z_i)) \quad ,$$

and $\Phi_i := |\Phi_i| \ \exp(i\phi_i)$ with

$$\phi_i := arg(z - z_i) \quad .$$

So that $dA_i = h_i$ and

$$|\Phi|^2 \ (d\phi_i - A_i) = \frac{|\Phi|^2}{|\Phi_i|^2} \ \langle i\Phi_i, \nabla_{A_i}\Phi_i\rangle = -\nabla^\perp h_i \quad .$$

Therefore, in particular we have $\frac{|\nabla h_i|^2}{|\Phi|^2} = \frac{|\Phi|^2}{|\Phi_i|^2}\left[|\nabla_{A_i}\Phi_i|^2 - |\nabla|\Phi_i||^2\right]$. Thus this implies

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left[\frac{|\nabla h_i|^2}{|\Phi|^2} + |h_i|^2\right] \\
&\geq \int_{\mathbb{R}^2} |\nabla_{A_i}\Phi_i|^2 + |dA_i|^2 - \int_{B_{L/2}(z_i)} |\nabla|\Phi||^2 \\
&- \int_{\mathbb{R}^2\setminus B_{L/2}(z_i)} [1 - |\Phi|^2] \ \frac{|\nabla h_i|^2}{|\Phi|^2} - \int_{B_{L/2+L^\star/4}(z_i)\setminus B_{L/2}(z_i)} |\nabla|\Phi||^2 + |1 - |\Phi||^2 \quad .
\end{aligned} \tag{IV.67}$$

Using (IV.62) we first have, for any $\alpha \in (0,1)$ and n large enough

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus\cup_k B_{\frac{L}{2}}(z_k)} [1 - |\Phi|^2]\frac{|\nabla h_i|^2}{|\Phi|^2} \\
&\leq C_\alpha \ \varepsilon_n^2 \ \exp(-\frac{1}{\sqrt{2}}(1-\alpha)L) \int_{\mathbb{R}^2\setminus\cup_k B_{\frac{L}{2}}(z_k)} \frac{|\nabla h_i|^2}{|\Phi|^2} \quad .
\end{aligned} \tag{IV.68}$$

Moreover, using (IV.61), we have

$$\int_{\mathbb{R}^2\setminus B_{\frac{L}{2}}(z_i)} \frac{|\nabla h_i|^2}{|\Phi|^2} \leq C_\alpha \int_{L/2}^{\infty} \exp(-2(1-\alpha) \ r) \ r \, dr + C \ H_\star^2 \quad . \tag{IV.69}$$

Since $H_\star$ is bounded, combining (IV.68), (IV.69) and the lower bound for $H_\star$ given by lemma IV.3 we have

$$\int_{\mathbb{R}^2\setminus\cup_k B_{\frac{L}{2}}(z_k)} [1 - |\Phi|^2]\frac{|\nabla h_i|^2}{|\Phi|^2} \leq C\varepsilon_n^2 \ H_\star \quad . \tag{IV.70}$$

For $k \neq i$ and any $\eta > 0$ we have for n large enough, using (IV.61) and (IV.62),

$$\begin{aligned}
&\int_{B_{\frac{L}{2}}(z_k)\setminus B_\eta(z_k)} [1 - |\Phi|^2]\frac{|\nabla h_i|^2}{|\Phi|^2} \\
&\leq C_\alpha \varepsilon_n^2 \int_{B_{\frac{L}{2}}(z_k)\setminus B_\eta} \exp(-\sqrt{2}(1-\alpha) \ |z - z_k| - 2(1-\alpha) \ |z - z_i|) \quad .
\end{aligned} \tag{IV.71}$$

Observe that $\min_z \sqrt{2}|z-z_k|+2|z-z_i| \geq \sqrt{2}|z_k-z_i|$. Using again the lower bound for $H_\star$ given by lemma IV.3, we deduce that

$$\int_{B_{\frac{L}{2}}(z_k)\setminus B_\eta(z_k)} [1-|\Phi|^2]\frac{|\nabla h_i|^2}{|\Phi|^2} \leq C\varepsilon_n^2\ H_\star \tag{IV.72}$$

Moreover, again for $k \neq i$, using the equation solved by h_i in $B_{L^\star/2}(z_k)$ (IV.60) and (IV.61)

$$\int_{B_\eta(z_k)} \frac{|\nabla h_i|^2}{|\Phi|^2} \leq \int_{\partial B_\eta(z_k)} |h_i| \left|\frac{\partial h_i}{\partial r}\right| \leq C\eta\ H_\star \tag{IV.73}$$

Then combining (IV.67)...(IV.73) we have

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|\nabla h_i|^2}{|\Phi|^2}+|h_i|^2 \ &\geq \int_{\mathbb{R}^2} |\nabla_{A_i}\Phi_i|^2+|dA_i|^2 - \int_{B_{L/2}(z_i)} |\nabla|\Phi||^2 \\ &-C\ (\eta+C_\eta\varepsilon_n^2)\ H_\star\end{aligned} \tag{IV.74}$$

Combining this with (IV.66) we get

$$\begin{aligned}\mathcal{A}_{\varepsilon_n}^N \geq \sum_{i=1}^N \int_{\mathbb{R}^2} |\nabla_{A_i}\Phi_i|^2+|dA_i|^2+\frac{(1-|\Phi_i|^2)^2}{2\varepsilon_n^2}+\sum_{j\neq k} H_{j,k} \\ -(\eta+C_\eta\varepsilon_n^2)\ H_\star - \int_{\mathbb{R}^2\setminus\cup_{i=1}^N B_{L/2}(z_i)} |\nabla|\Phi_i||^2+\frac{(1-|\Phi_i|^2)^2}{2\varepsilon_n^2} \quad .\end{aligned} \tag{IV.75}$$

Using (IV.62), (IV.63) and lemma IV.3, we get for any $0<\alpha<1$ and n large enough

$$\int_{\mathbb{R}^2\setminus\cup_{i=1}^N B_{L/2}(z_i)} |\nabla|\Phi_i||^2+\frac{(1-|\Phi_i|^2)^2}{2\varepsilon_n^2} \leq C_\alpha\varepsilon_n^2\ \exp\left(-\sqrt{2}(1-\alpha)\ L\right) \leq \varepsilon_n^2 H_\star\ . \tag{IV.76}$$

Combining (IV.75) and (IV.76) we finally obtain

$$\mathcal{A}_{\varepsilon_n}^N \geq \sum_{i=1}^N \mathcal{A}_{\varepsilon_n}(A_i,\Phi_i)+\sum_{j\neq k} H_{j,k} - (\eta+C_\eta\varepsilon_n^2)\ H_\star \quad .$$

Since the h_k are positive functions, $\sum_{j\neq k} H_{j,k} \geq H_\star > 0$ and we deduce that

$$\mathcal{A}_{\varepsilon_n}^N > N\mathcal{A}_{\varepsilon_n}^1 \quad . \tag{IV.77}$$

A simple glueing argument of degree 1 vortices realizing $\mathcal{A}_{\varepsilon_n}^1$ and going faraway apart gives

$$\mathcal{A}_{\varepsilon_n}^N \leq N\mathcal{A}_{\varepsilon_n}^1 \quad . \tag{IV.78}$$

This contradicts (IV.77) therefore $\mathcal{A}_{\varepsilon_n}^N$ cannot be achieved for $N\neq\pm1,0$. This proves theorem I.1. ■

Observe that bootstraping this information and the approach in part III we have also proved that for ε sufficiently small

$$\mathcal{A}_N^\varepsilon = N\mathcal{A}_1^\varepsilon \quad . \tag{IV.79}$$

Indeed it suffices to follow a minimizing sequences as in part III and to observe the vortices going to ∞. Corollary I.1 is proved. ■

V The Minimizers of the Action in the ± 1-Homotopy classes are axially symmetric

We have proved in part III the existence of a minimizer in the ± 1 homotopy classes for large λ. In lemma IV.1 we gave a general description of the behavior of the minimizers as $\lambda \to +\infty$.

Let $(A^\varepsilon, \Phi^\varepsilon)$ be a minimizer of $\mathcal{A}_{YMH}$ in the $+1$ homotopy class for $\lambda = \frac{2}{\varepsilon^2}$. (again we will most of the time omitt the superscript ε). We may translate the unique zero of $|\Phi|$ so that $|\Phi|^{-1}(0) = \{0\}$. We then have

$$-\mathrm{div}\left(\frac{\nabla h}{|\Phi|^2}\right) + h = 2\pi\delta_0 \qquad \text{on } \mathbb{R}^2 \quad . \tag{V.1}$$

Bootstrapping these informations and using the upper bound on the energy of (A,Φ), we may deduce like in Lemma 12.6 of [10] the important pointwise estimates for any $x \in \mathbb{C}$

$$\forall l \geq 1 \quad |\nabla^l h|(x) \leq C_l \max(|x|,\varepsilon)^{-l} \quad \text{ and } \quad |\nabla^l |\Phi|| \leq C_l \varepsilon^2 \max(|x|,\varepsilon)^{-l-2} \quad . \tag{V.2}$$

We will use the following gauge

$$A(x) = \frac{1}{2\pi}\left(\begin{array}{c} -\dfrac{\partial}{\partial x_2}\displaystyle\int_{\mathbb{R}^2} \log|x-y| \ h(y)\, dy + C_1 \\ \\ \dfrac{\partial}{\partial x_1}\displaystyle\int_{\mathbb{R}^2} \log|x-y| \ h(y)\, dy + C_2 \end{array}\right) \quad , \tag{V.3}$$

where C_1 and C_2 are chosen such that $A(0) = 0$. Because of the exponential decrease at infinity of h it is clear that $A(x)$ converges to a limiting vector $A(\infty)$ at infinity. Standard results on singular integrals yield for any $1 < p < 2$

$$\left(\int_{\mathbb{R}^2} |\nabla A|^{p_*}\right)^{\frac{1}{p_*}} \leq C \left(\int_{\mathbb{R}^2} |\nabla h|^p\right)^{\frac{1}{p}} \quad ,$$

where $\frac{1}{p_*} = \frac{1}{p} - \frac{1}{2}$ and Morrey estimates yields

$$|A(x) - A(y)| \leq C\ |x-y|^\alpha \left(\int_{\mathbb{R}^2} |\nabla h|^p\right)^{\frac{1}{p}} \quad ,$$

where $\alpha = 1 - \frac{2}{p_*} = 2 - \frac{2}{p}$. Thus for any $0 < \alpha < 1$ we have

$$|A(z)| \leq C_\alpha \ |z|^\alpha \qquad \text{in } \mathbb{R}^2 \tag{V.4}$$

where C_α is independent of ε. Using the fact that $-div(\frac{\nabla h}{|\Phi|^2}) + h = 0$ in $B_{2r} \setminus B_r$ for any $r > 0$ and that $\|h\|_{W^{1,p}}$ is bounded independently of ε we get that $|\nabla h| \leq \frac{C_\beta}{r^{1+\beta}}$ for any $\beta > 0$. Bootstrping this information in the PDE $\Delta A = \nabla^\perp h$ in any annulus of the form $B_{2r} \setminus B_r$ we establish by the mean also of (V.4)

$$|\nabla A(x)| \leq C_\alpha \ |z|^{\alpha - 1} \qquad \text{in } \mathbb{R}^2 \tag{V.5}$$

for any $\alpha < 1$. These estimates (V.4) and (V.5) will be used systematically in the sequel.

Since div$A$=0, A and Φ satisfy the equation on $\mathbb{R}^2$

$$\Delta\Phi + \Phi\frac{(1 - |\Phi|^2)}{\varepsilon^2} - |A|^2\Phi - 2iA.\nabla\Phi = 0 \quad . \tag{V.6}$$

Since $|\nabla_A\Phi| \leq \frac{C}{\varepsilon}$ on B_1 and (V.4) holds this implies

$$|\nabla\Phi| \leq \frac{C}{\varepsilon} \qquad \text{in } B_1 \tag{V.7}$$

Denote by $B = S_\varepsilon(r)\, d\theta$ and $\Psi = \rho_\varepsilon(r) \exp(i\theta)$ the axially symmetric solutions considered in [1], we have

$$\Delta\rho + \rho\left(\frac{1 - \rho^2}{\varepsilon^2} - |\nabla\theta - B|^2\right) = 0 \qquad \text{on } \mathbb{R}^2 \setminus \{0\} \tag{V.8}$$

Argueing as in [8] we consider $w = \frac{\Phi}{\rho}$, and verify the following equation on $\mathbb{R}^2 \setminus \{0\}$:

$$\begin{aligned} &\Delta w + 2\frac{\rho'}{\rho}\frac{\partial w}{\partial r} + \frac{w}{r^2} - w\rho^2\frac{(|w|^2 - 1)}{\varepsilon^2} - w(|A|^2 - |B|^2) \\ &\qquad -2w\nabla\theta \cdot B - 2iwA_r\frac{\rho'}{\rho} - 2iA \cdot \nabla w = 0 \end{aligned} \tag{V.9}$$

We apply the Pohozaev argument (see also [10] 12.3.3) to this equation on $T = B_R \setminus B_s$ and we find after integration by parts and making s tends to zero we obtain

$$
\begin{aligned}
&\frac{1}{4}\int_{B_R}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\rho^2\right)\frac{(1-|w|^2)^2}{\varepsilon^2}+2\int_{B_R} r\frac{\rho'}{\rho}\left|\frac{\partial w}{\partial r}\right|^2\\
&=\frac{1}{2}\int_{B_R} r\frac{\partial|w|^2}{\partial r}[|A|^2-|B|^2]+2\int_{B_R} r|w|^2\frac{\partial\phi}{\partial r}A_r\frac{\rho'}{\rho}\\
&+\int_{B_R} r\frac{\partial|w|^2}{\partial r}[B\cdot\nabla\theta-A\cdot\nabla\phi]+\frac{R}{4}\int_{\partial B_R}\rho^2\frac{(1-|w|^2)^2}{\varepsilon^2}\\
&+\int_{B_R} r\frac{\partial\phi}{\partial r}A\cdot\nabla|w|^2-\frac{1}{2R}\int_{\partial B_R}|w|^2+\frac{R}{2}\int_{\partial B_R}\left|\frac{\partial w}{\partial\tau}\right|^2-\left|\frac{\partial w}{\partial r}\right|^2 \quad ,
\end{aligned}
\tag{V.10}
$$

where ϕ is the multivalued function on $\mathbb{R}^2\setminus\{0\}$ with $\nabla\phi=\frac{\Phi_1\nabla\Phi_2-\Phi_2\nabla\Phi_1}{|\Phi|^2}$. To control the boundary term we will need the following lemma :

Lemma V.1 *For $|x|\geq 1$ the following bound holds*

$$
\left|A(x)-A(\infty)-d\theta+\frac{1}{2\pi}\nabla^{\perp}\left[\frac{x}{|x|^2}\cdot\int_{\mathbb{R}^2} yh(y)\right]\right|\leq\frac{C}{|x|^3}
\tag{V.11}
$$

Proof of Lemma V.1

Let $\xi=\frac{1}{2\pi}(\log r)*h$, since $\deg_\infty\frac{\Phi}{|\Phi|}=+1$, we have $\int_{\mathbb{R}^2}h=2\pi$ (h still denotes $*dA$). Thus

$$
\begin{aligned}
&\xi(x)-\log|x|=\frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{|x-y|}{|x|}\ h(y)dy\\
&=\frac{1}{2\pi}\int_{\frac{|x-y|}{|x|}\leq\frac{1}{2}}\log\frac{|x-y|}{|x|}\ h(y)dy+\frac{1}{4\pi}\int_{\frac{|x-y|}{|x|}\geq\frac{1}{2}}\log\frac{|x-y|^2}{|x|^2}\ h(y)dy \quad .
\end{aligned}
$$

On the one hand we have

$$
\left|\frac{1}{2\pi}\int_{\frac{|x-y|}{|x|}\leq\frac{1}{2}}\log\frac{|x-y|}{|x|}\ h(y)dy\right|\leq C_\alpha\ |x|^2\ \exp\left(-\frac{(1-\alpha)}{2}|x|\right)\quad ,
\tag{V.12}
$$

on the other hand we have $|\log(1-s)+s|\leq C\,s^2$ for $|s|\leq\frac{3}{4}$

$$
\left|\log\frac{|x-y|^2}{|x|^2}+2\frac{x.y}{|x|^2}-\frac{|y|^2}{|x|^2}\right|\leq C\ \left(\left|\frac{x.y}{|x|^2}\right|^2+\frac{|y|^4}{|x|^4}\right)\quad ,
\tag{V.13}
$$

and since for $\left|\frac{y}{x}\right|\geq\frac{1}{2}$ $|h(y)|\leq C\exp(-\frac{1}{4}|x|)\exp(-\frac{1}{4}|y|)$, we obtain, combining (V.12) and (V.13)

$$
\left|\xi(x) - \frac{1}{2\pi}\log|x| - \frac{1}{2\pi}\frac{x}{|x|^2}.\int_{\mathbb{R}^2} yh(y)dy\right| \le \frac{C}{|x|^2} \tag{V.14}
$$

Since $\frac{x}{|x|^2}$ is harmonic on $\mathbb{R}^2\setminus\{0\}$ we have

$$
\begin{aligned}
\|\nabla^\perp\zeta\|^2_{L^\infty(B_{|x|/4})} &\le \|\zeta\|_{L^\infty(B_{|x|/2})}\|\Delta\zeta\|_{L^\infty(B_{|x|/2})} + \frac{1}{|x|^2}\|\zeta\|^2_{L^\infty(B_{|x|/2})}\\
&\le C_\alpha\,|x|^{-2}\exp(-(1-\alpha)|x|) + \frac{C}{|x|^6} \quad ,
\end{aligned} \tag{V.15}
$$

where

$$
\zeta(x) = \xi(x) - \frac{1}{2\pi}\log|x| - \frac{1}{2\pi}\frac{x}{|x|^2}.\int_{\mathbb{R}^2} yh(y)\,dy
$$

and (V.15) together with $A - A(\infty) = \nabla^\perp\xi$ implies Lemma V.1. ∎

Because of the exponential decays of $1-|w|^2$, $|\nabla|w||$ and $(iw;\nabla_A w)$ (see Proposition 12.2 and section 12.3.3of [10]) we have, as $R\to+\infty$

$$
\begin{aligned}
&-\frac{1}{2R}\int_{\partial B_R}|w|^2 + \frac{R}{2}\int_{\partial B_R}\left|\frac{\partial w}{\partial\tau}\right|^2 - \left|\frac{\partial w}{\partial r}\right|^2\\
&\simeq -\frac{|\partial B_R|}{2R} + \frac{R}{2}\int_{\partial B_R}|A.\tau|^2 - |A.\nu|^2 \quad .
\end{aligned} \tag{V.16}
$$

Denoting $a = \int_{\mathbb{R}^2} yh(y)$, we have from Lemma V.1

$$
\begin{aligned}
A(r\,e^{i\theta})\cdot\tau &= -A_1(\infty)\sin\theta + A_2(\infty)\cos\theta + \frac{1}{r}\\
&\quad -\frac{a_1}{2\pi r^2}\cos\theta - \frac{a_2}{2\pi r^2}\sin\theta + O(\frac{1}{r^3})
\end{aligned}
$$

and

$$
\begin{aligned}
A(r\,e^{i\theta})\cdot\nu &= A_1(\infty)\cos\theta + A_2(\infty)\sin\theta\\
&\quad +\frac{a_1}{2\pi r^2}\sin\theta - \frac{a_2}{2\pi r^2}\cos\theta + O(\frac{1}{r^3}) \quad .
\end{aligned}
$$

Combining these two expressions for $A\cdot\tau$ and $A\cdot\nu$ and (V.16) we obtain finally

$$
\begin{aligned}
\lim_{R\to+\infty} \quad &-\frac{1}{2R}\int_{\partial B_R}|w|^2 + \frac{R}{2}\int_{\partial B_R}\left|\frac{\partial w}{\partial\tau}\right|^2 - \left|\frac{\partial w}{\partial r}\right|^2\\
&= \frac{1}{2\pi}\sum_{i=1}^2\int_{\mathbb{R}^2}\log|x|\frac{\partial h}{\partial x_i}(x)\cdot\int_{\mathbb{R}^2} x_i h(x) = -iA(\infty)\cdot a \quad ,
\end{aligned} \tag{V.17}
$$

And (V.10) becomes

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\rho^2\right)\frac{(1-|w|^2)^2}{\varepsilon^2}+2\int_{\mathbb{R}^2}r\frac{\rho'}{\rho}\left|\frac{\partial w}{\partial r}\right|^2\\
&=\frac{1}{2}\int_{\mathbb{R}^2}r\frac{\partial|w|^2}{\partial r}[|A|^2-|B|^2]+2\int_{\mathbb{R}^2}r|w|^2\frac{\partial\phi}{\partial r}A_r\frac{\rho'}{\rho}\\
&+\int_{\mathbb{R}^2}r\frac{\partial|w|^2}{\partial r}[B\cdot\nabla\theta-A\cdot\nabla\phi]+\int_{\mathbb{R}^2}r\frac{\partial\phi}{\partial r}A.\nabla|w|^2\\
&-\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^2}\log|x|\frac{\partial h}{\partial i}(x)\ \int_{\mathbb{R}^2}x_i h(x)
\end{aligned}\tag{V.18}$$

Let $k=*dB$ be the magnetic field of the axially symmetric solution. $\Xi=h-k$ satisfies on $\mathbb{C}$

$$-\Delta\Xi+2\frac{\nabla\rho}{\rho}.\nabla\Xi+\rho^2\Xi=\rho^2\left[\mathrm{div}\left(\left(\frac{1}{|\Phi|^2}-\frac{1}{\rho^2}\right)\nabla h\right)\right]\tag{V.19}$$

Let χ be a cut-off function such that $|\nabla\chi|\le\frac{C}{\varepsilon}$, $\chi\equiv 1$ on $B_{\varepsilon/2}(0)$, $\chi\equiv 0$ on $\mathbb{R}^2\setminus B_\varepsilon$. Let us decompose the r.h.s. of (V.19) in the following way :

$$\begin{aligned}
\mathrm{div}\left(\left(\frac{1}{|\Phi|^2}-\frac{1}{\rho^2}\right)\nabla h\right)\ &=\mathrm{div}\left(\left(\frac{1}{|\Phi|^2}-\frac{1}{\rho^2}\right)\chi\nabla h\right)\\
&+\mathrm{div}\left(\left(\frac{1}{|\Phi|^2}-\frac{1}{\rho^2}\right)(1-\chi)\nabla h\right)\quad .
\end{aligned}\tag{V.20}$$

We compute

$$\begin{aligned}
\rho^2\mathrm{div}\left(\left(\frac{1}{|\Phi|^2}-\frac{1}{\rho^2}\right)\chi\nabla h\right)&=\frac{1-|w|^2}{|w|^2}\nabla\chi.\nabla h\\
+(\rho^2-|\Phi|^2)\chi h-\frac{2}{|w|}\frac{\partial|w|}{\partial r}\frac{\partial h}{\partial r}\chi&-\frac{2}{|w|}\frac{\partial|w|}{\partial\tau}\frac{\partial h-k}{\partial\tau}\chi\quad ,
\end{aligned}\tag{V.21}$$

where $\tau=r\,\nabla\theta$. We have used in (V.21) $\frac{\partial k}{\partial\tau}=0$. Observe that

$$\frac{\partial|w|}{\partial\tau}=\frac{1}{\rho}\frac{\partial|\Phi|}{\partial\tau}=\frac{1}{\rho}\left[\frac{\partial|\Phi|}{\partial\tau}-\frac{\partial\rho}{\partial\tau}+\left(1-\frac{|\Phi|}{\rho}\right)\frac{\partial\rho}{\partial\tau}\right]\quad .$$

We claim that, as $\varepsilon\to 0$

$$\varepsilon^2\|\nabla(|\Phi|-\rho)\|_{L^\infty(B_\varepsilon)}+\left\|\frac{|\Phi|-\rho}{\rho}\right\|_{L^\infty(B_\varepsilon)}=o(1)\quad .\tag{V.22}$$

Dilating $1 \to \frac{1}{\varepsilon}$ and passing from ρ and Φ to $\tilde{\rho}$ and $\tilde{\Phi}$, we saw that $\tilde{\rho}$ and $\tilde{\Phi}$ converge to the same limiting map, modulo a rotation, in $C^1_{\text{loc}}(\mathbb{R}^2)$. Let us modify $\tilde{\Phi}$ into $\exp(i\alpha_\varepsilon)\tilde{\Phi}$ in order to ensure that $u = \exp(i\alpha_\varepsilon)\tilde{\Phi}$ and $v = \tilde{\rho}\exp(i\theta)$ converge to the same map in $C^1_{\text{loc}}(\mathbb{R}^2)$, thus $\|\nabla(u-v)\|_{L^\infty(B_1)} \to 0$ and since $u(0) = v(0) = 0$ we have $\|\frac{u-v}{r}\|_{L^\infty(B_1)} \to 0$. We have also

$$|\nabla(|u| - |v|)| \leq |\nabla(u-v)| + \frac{|u-v|}{|u|+|v|}|\nabla v|$$

Going back to the original scale and using the fact that u and v have maximal rank 2 at 0, we deduce (V.22).

Using Lemma A.4 for a ν between 0 and 1 we deduce from (V.19), (V.20) and (V.21) after some computations decomposing $\mathbb{R}^2$ into $B_\varepsilon(0)$ and $\mathbb{R}^2 \setminus B_\varepsilon(0)$ (see [10], Lemma 12.7).

$$\begin{aligned} \int_{\mathbb{R}^2} r^{2-2\nu}\frac{|\nabla h - k|^2}{\rho^2} \quad &\leq C \int_{\mathbb{R}^2} \max(r,\varepsilon)^{-2\nu}(|\Phi| - \rho)^2 \\ &+ C \int_{B_\varepsilon(0)} \varepsilon^{2-2\nu}\left|\frac{\partial w}{\partial r}\right|^2 \quad . \end{aligned} \tag{V.23}$$

By means of this inequality, we will bound in particular the l.h.s. of (V.18) by a $o(1)$ times the r.h.s of the same inequality (V.18). This will imply that $\Phi = \rho\exp(i(\theta + \alpha_\varepsilon))$ for sufficiently small ε and some $\alpha_\varepsilon \in \mathbb{R}$ which is the conclusion of Theorem I.2.

Let us write the first term of the r.h.s. of (V.18) in the following way

$$\begin{aligned} \int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r} \quad \left[|A|^2 - |B|^2\right] &= \int_{B_\varepsilon(0)} r\frac{\partial|w|^2}{\partial r}\left[|A|^2 - |B|^2\right] \\ &+ \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} r\frac{\partial|w|^2}{\partial r}\left[|A|^2 - |B|^2\right] \quad . \end{aligned} \tag{V.24}$$

We have

$$\begin{aligned} &\int_{B_\varepsilon(0)} r\frac{\partial|w|^2}{\partial r}\left[|A|^2 - |B|^2\right] \\ &\leq 2\left(\int_{B_\varepsilon(0)}\left|\frac{\partial w}{\partial r}\right|^2\right)^{1/2}\left(\int_{B_\varepsilon(0)} r^2\,\big||A| + |B|\big|^2\,|A-B|^2\right)^{1/2} \quad . \end{aligned} \tag{V.25}$$

Since $(A-B)(0) = 0$ and $A - B = \frac{1}{2\pi}\log r * \nabla^\perp(h-k)$+constant , using Lemma A.3 and (V.23) we have, for $0 < \nu < 1$,

$$
\int_{\mathbb{R}^2}\sum_{k=0}^{2} r^{-2\nu+2(k-1)}\left|\nabla^k(A-B)\right|^2
$$

$$
\leq C_\nu\int_{\mathbb{R}^2}\max(r,\varepsilon)^{-2\nu}(|\Phi|-\rho)^2+C_\nu\int_{B_\varepsilon(0)}\varepsilon^{2-2\nu}\left|\frac{\partial w}{\partial r}\right|^2 \quad . \tag{V.26}
$$

Thus

$$
\int_{B_\varepsilon(0)} r\frac{\partial|w|^2}{\partial r}\left[|A|^2-|B|^2\right]\leq o(1)\times(\text{ l.h.s. of (V.18) }) \quad . \tag{V.27}
$$

At this stage we have used $r\rho'(r)/\rho(r)\leq c>0$ where c is independent of ε. On the other hand, outside B_ε integrating by parts we have

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} r\frac{\partial|w|^2}{\partial r}\left[|A|^2-|B|^2\right] &= -\int_{\partial B_{\mu\varepsilon}}\mu\varepsilon(|w|^2-1)\left[|A|^2-|B|^2\right]\\
&-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}\frac{|w|^2-1}{r}\frac{\partial}{\partial r}\left[r^2(|A|^2-|B|^2)\right] \quad ,
\end{aligned} \tag{V.28}
$$

where $\mu\in(1/2,1)$ is chosen using the mean value formula such that

$$
\begin{aligned}
\frac{\varepsilon}{2}\int_{\partial B_{\mu\varepsilon}}||w|^2-1|\,\left||A|^2-|B|^2\right| &\leq \int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}||w|^2-1|\,\left||A|^2-|B|^2\right|\\
&\leq C\varepsilon^\alpha\left(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}||w|^2-1|^2\right)^{1/2}\left(\int_{B_\varepsilon}|A-B|^2\right)^{1/2} \quad .
\end{aligned} \tag{V.29}
$$

For the second term in the r.h.s of (V.28) we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}\frac{|w|^2-1}{r}\frac{\partial}{\partial r}\left[r^2(|A|^2-|B|^2)\right]\right|\\
&\leq C\left(\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} r^{2+2\nu}||w|^2-1|^2\right)^{1/2}\\
&\times\left(\int_{\mathbb{R}^2} r^{-2-2\nu}|A-B|^2+r^{-2\nu}|\nabla(A-B)|^2\right)^{1/2}
\end{aligned} \tag{V.30}
$$

In order to bound the quantity above we will need the following lemma.

Lemma V.2 *For any $p \geq 0$ the following bound holds*

$$\int_{\mathbb{R}^2} r^p \left(||\Phi| - \rho|^2 + |\nabla(h-k)|^2\right)$$
$$\leq C_p \int_{\mathbb{R}^2} ||\Phi| - \rho|^2 + |\nabla(h-k)|^2$$

Combining this lemma (V.28), (V.29) and (V.30) we get

$$\int_{\mathbb{R}^2 \setminus B_{\mu\varepsilon}} r \frac{\partial |w|^2}{\partial r} \left[|A|^2 - |B|^2\right] \leq o(1) \times (\text{ l.h.s of (V.18) }) \quad . \tag{V.31}$$

Before to bound the other terms in the r.h.s. of (V.18) we need to prove lemma V.1.

Proof of lemma V.2 :

Outside B_1 we have on the one hand

$$\begin{cases} -\Delta h + 2\dfrac{\nabla|\Phi|}{|\Phi|}.\nabla h + |\Phi|^2 h = 0 \\ \\ -\Delta k + 2\dfrac{\nabla \rho}{\rho}.\nabla k + \rho^2 k = 0 \quad , \end{cases}$$

on the other hand

$$\begin{cases} \Delta\rho + \rho\left(\dfrac{1-\rho^2}{\varepsilon^2} - \dfrac{|\nabla k|^2}{\rho^4}\right) = 0 \\ \\ \Delta|\Phi| + |\Phi|\left(\dfrac{1-|\Phi|^2}{\varepsilon^2} - \dfrac{|\nabla h|^2}{|\Phi|^4}\right) = 0 \quad . \end{cases}$$

Let $\zeta = h - k$ and $\tau = |\Phi| - \rho$

$$\begin{cases} -\Delta\zeta + 2\dfrac{\nabla\rho}{\rho}.\nabla\zeta + \rho^2\zeta = -2\nabla\tau \cdot \dfrac{\nabla h}{|\Phi|} + 2\tau \\ frac\nabla\rho\rho \cdot \frac{\nabla h}{|\Phi|} - \tau(|\Phi| + \rho)h \quad , \\ \\ -\Delta\tau + \dfrac{c_\varepsilon}{\varepsilon^2}\tau = -\nabla\zeta.\dfrac{\nabla(h+k)}{|\Phi|^3} \quad , \end{cases} \tag{V.32}$$

where $c_\varepsilon = |\Phi|^2 + \rho^2 + \rho|\Phi| - 1 - \varepsilon^2 H(|\Phi|, \rho, \nabla h, \nabla k)$ whith $H(|P|, \rho, \nabla h, \nabla k)$ remains uniformly bounded in $\mathbb{R} \setminus B_1$, thus $1 \leq c_\varepsilon \leq 3$ for ε sufficiently small. Multiplying the two equations of (V.32) respectively by $r^p\zeta$ and $r^p\tau$ and integrating on $\mathbb{R}^2 \setminus B_R$ for R sufficiently large (independent of ε) we get

$$\int_{\mathbb{R}^2\setminus B_R} r^p \left[|\nabla\zeta|^2+|\zeta|^2+|\nabla\tau|^2+\frac{|\tau|^2}{\varepsilon^2}\right]$$

$$\leq C_{R,p}\int_{\partial B_R}\left[|\nabla\zeta|^2+|\zeta|^2+|\nabla\tau|^2+\frac{|\tau|^2}{\varepsilon^2}\right] \quad .$$

Using the mean value formula on R and bootstrapping again in (V.32) we obtain the result and Lemma V.1 is proved. ∎

We bound now the second term in the r.h.s. of (V.18). Like for the first term we decompose the integral over $\mathbb{R}^2$ into the integral over B_ε and the integral over $\mathbb{R}^2\setminus B_\varepsilon$. We have first, using $B_r=0$, (V.26) and $C^{-1}\leq r\rho'(r)/\rho(r)\leq C$ for $0<r<\varepsilon$

$$\begin{aligned}
&\int_{B_\varepsilon} r|w|^2\frac{\partial\phi}{\partial r}A_r\frac{\rho'}{\rho}\leq C\int_{B_\varepsilon}\frac{r\rho'}{\rho}\left|\frac{\partial w}{\partial r}\right||A_r-B_r| \\
&\leq C\left(\int_{B_\varepsilon}\left|\frac{\partial w}{\partial r}\right|^2\right)^{\frac12}\left(\int_{B_\varepsilon}|A-B|^2\right)^{\frac12}\leq o(1)\times(\text{ l.h.s. of (V.18) })
\end{aligned} \tag{V.33}$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus B_\varepsilon} r|w|^2\frac{\partial\phi}{\partial r}A_r\frac{\rho'}{\rho}\leq C\int_{\mathbb{R}^2\setminus B_\varepsilon}\left[\left|\frac{\partial h}{\partial\tau}-\frac{\partial k}{\partial\tau}\right||A-B|+|A-B|^2\right]\frac{r\rho'}{\rho} \\
&\leq C\left(\int_{\mathbb{R}^2\setminus B_\varepsilon} r^{2-2\nu}|\nabla(h-k)|^2\right)^{\frac12}\left(\int_{\mathbb{R}^2\setminus B_\varepsilon}\left(\frac{r\rho'}{\rho}\right)^2 r^{2\nu-2}|A-B|^2\right)^{\frac12} \\
&+C\int_{\mathbb{R}^2\setminus B_\varepsilon}|A-B|^2\frac{r\rho'}{\rho}\leq o(1)\,(\text{ l.h.s. of (V.18) }) \quad ,
\end{aligned} \tag{V.34}$$

where we have used the exponential decay of ρ'. Thus combining (V.33) and (V.34) we finally bound the second term of the r.h.s. of (V.18) by the desired quantity :

$$\left|\int_{\mathbb{R}^2\setminus B_\varepsilon} r|w|^2\frac{\partial\phi}{\partial r}A_r\frac{\rho'}{\rho}\right|\leq o(1)\times(\text{ l.h.s of (V.18) }) \quad . \tag{V.35}$$

The third term of the r.h.s. of (V.18) is decomposed in the following way

$$\int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}\left[B.\nabla\theta - A.\nabla\phi\right] = \int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}(B-A).\nabla\theta$$

$$+\int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}A\cdot\frac{\nabla^\perp k - h}{\rho^2} + \int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}A.(B-A) \tag{V.36}$$

$$+\int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}\ A\cdot\frac{\nabla^\perp h}{|\Phi|^2}(1-|w|^2)$$

The two last term in the r.h.s. of (V.36) can be bounded by the desired quantity as we did before, so we only have to deal with the two first. Once again we decompose each of the integrals over $\mathbb{R}^2$ as the sum of the integral over B_ε and the integral over $\mathbb{R}^2\setminus B_\varepsilon$. We have first, as in (V.26),

$$\begin{aligned}&\int_{B_\varepsilon} r\frac{\partial|w|^2}{\partial r}(B-A).\nabla\theta\\ &\le\left(\int_{B_\varepsilon}\left|\frac{\partial w}{\partial r}\right|^2\right)^{\frac12}\left(\int_{B_\varepsilon}|A-B|^2\right)^{\frac12}\le o(1)\ (\text{ l.h.s of (V.18) })\quad .\end{aligned} \tag{V.37}$$

Integrating by parts now the integral of the same quantity, but over $\mathbb{R}^2\setminus B_{\mu\varepsilon}$, yields

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} r\frac{\partial|w|^2}{\partial r}(B-A).\nabla\theta &= -\int_{\partial B_{\mu\varepsilon}}\mu\varepsilon(|w|^2-1)(B-A).\nabla\theta\\ &-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)\frac1r\frac{\partial}{\partial r}\left(r^2(B-A).\nabla\theta\right)\quad .\end{aligned} \tag{V.38}$$

Here μ is chosen between $1/2$ and 1, using the mean value formula, such that

$$\begin{aligned}&\frac{\varepsilon}{2}\int_{\partial B_{\mu\varepsilon}}(|w|^2-1)(B-A).\nabla\theta\le\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}(|w|^2-1)(B-A).\nabla\theta\\ &\le\left(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}\left||w|^2-1\right|^2\right)^{\frac12}\left(\int_{B_\varepsilon}\frac{|A-B|^2}{r^2}\right)^{\frac12}\le o(1)\times(\text{ l.h.s of (V.18) })\quad ,\end{aligned} \tag{V.39}$$

and we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} (|w|^2-1)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2(B-A).\nabla\theta\right)\\
&\le C\left(\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} \left||w|^2-1\right|^2 r^{2\nu}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}\sum_{k=0}^{1} r^{-2\nu+2(k-1)}|\nabla^k(A-B)|^2\right)^{\frac{1}{2}}\\
&\le o(1)\times(\text{ l.h.s of (V.18) })\quad ,
\end{aligned}
\tag{V.40}
$$

where we have used Lemma V.2, (V.23) and (V.26) . Combining (V.37), (V.38), (V.39) and (V.40) we obtain

$$
\left|\int_{\mathbb{R}^2} r\frac{\partial|w|^2}{\partial r}(B-A).\nabla\theta\right|\le o(1)\times(\text{ l.h.s of (V.18) })
\tag{V.41}
$$

Thus we have bounded the first term in the r.h.s. of (V.36) by the right quantity. The second integral in the r.h.s. of (V.36) is also decomposed into the sum of the integral of the same quantity over B_ε and the integral over $\mathbb{R}^2\setminus B_\varepsilon$. First we have

$$
\left|\int_{B_\varepsilon} r\frac{\partial|w|^2}{\partial r}A.\frac{\nabla^\perp k-h}{\rho^2}\right|\le C\left(\int_{B_\varepsilon}\left|\frac{\partial w}{\partial r}\right|^2\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon}\frac{r^2|A|^2}{\rho^2}\frac{|\nabla h-k|^2}{\rho^2}\right)^{\frac{1}{2}}\quad .
\tag{V.42}
$$

Since $A\le C_\alpha r^\alpha$ for any $0<\alpha<1$ we have $\frac{r^2A^2}{\rho^2}=O(r^{2-2\nu})$. Thus, by (V.23)

$$
\left|\int_{B_\varepsilon} r\frac{\partial|w|^2}{\partial r}A.\frac{\nabla^\perp k-h}{\rho^2}\right|\le o(1)\,(\text{ l.h.s of (V.18) })\quad .
\tag{V.43}
$$

In the same spirit of the estimates established above we consider the same integral, but on $\mathbb{R}^2\setminus B_{\mu\varepsilon}$ for some $\mu\in(1/2,1)$, and we integrate by parts

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} r\frac{\partial|w|^2}{\partial r}A.\frac{\nabla^\perp(k-h)}{\rho^2}&=\mu\varepsilon\int_{\partial B_{\mu\varepsilon}}(|w|^2-1)A\cdot\frac{\nabla^\perp(k-h)}{\rho^2}\\
&-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2A\cdot\frac{\nabla^\perp(k-h)}{\rho^2}\right)\quad .
\end{aligned}
\tag{V.44}
$$

Choose μ in $(1/2,1)$, once again using the mean value formula, such that the following holds, recalling (V.23) again :

$$
\begin{aligned}
&\frac{\varepsilon}{2}\int_{\partial B_{\mu\varepsilon}}(|w|^2-1)A\cdot\frac{\nabla^\perp(k-h)}{\rho^2}\\
&\leq\left(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}\left||w|^2-1\right|^2\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon}|A|^2|\nabla(h-k)|^2\right)^{\frac{1}{2}}\\
&\leq o(1)\times(\text{ l.h.s. of (V.18) })\quad .
\end{aligned}
\tag{V.45}
$$

The second integral in the r.h.s. of (V.44) is decomposed in the following way

$$
\begin{aligned}
&-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2A.\frac{\nabla^\perp(k-h)}{\rho^2}\right)\\
&=-2\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)A.\frac{\nabla^\perp(k-h)}{\rho^2}-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)r\frac{\partial A}{\partial r}.\frac{\nabla^\perp k-h}{\rho^2}\\
&+2\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)\frac{\rho'}{\rho^3}rA.\nabla^\perp(k-h)\\
&-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}}(|w|^2-1)\frac{rA}{\rho^2}\cdot\frac{\partial}{\partial r}\nabla^\perp(k-h)
\end{aligned}
\tag{V.46}
$$

The most delicate term to deal with appears to be the last term. We have

$$
\int_{B_{4s}\setminus B_{2s}}|\nabla^2(h-k)|^2\leq C\int_{B_{8s}\setminus B_s}|\Delta(h-k)|^2+C\int_{B_{8s}\setminus B_s}\frac{|h-k|^2}{r^4}\quad .
$$

Thus

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus B_\varepsilon}|\nabla^2(h-k)|^2r^{4-2\nu}\quad&\leq C\int_{\mathbb{R}^2\setminus B_{\varepsilon/2}}|\nabla(|\Phi|-\rho)|^2r^{4-2\nu}|\nabla h|^2\\
&+o(1)\times(\text{ l.h.s of (V.18) })
\end{aligned}
\tag{V.47}
$$

(V.2) implies that $|r|\nabla h||\leq\frac{Cr}{\max(r,\varepsilon)}$. $|\Phi|-\rho$ verifies on $\mathbb{R}^2\setminus B_{\varepsilon/4}$

$$
-\Delta(|\Phi|-\rho)+\frac{c_\varepsilon}{\varepsilon^2}(|\Phi|-\rho)=-\frac{|\nabla h|^2}{|\Phi|^3}+\frac{|\nabla k|}{|\Phi|^3}\quad ,
$$

where $3\geq c_\varepsilon\geq 1$ (in fact this holds perhaps only on $\mathbb{R}^2\setminus B_{R\varepsilon}$ for some sufficiently large R independent of ε but lets say $R=1/4$ in order to simplify the presentation). Thus we deduce

$$\int_{B_{4s}\setminus B_{2s}} |\nabla(|\Phi|-\rho)|^2 \leq C\int_{B_{8s}\setminus B_s} \frac{||\Phi|-\rho|^2}{r^2}$$
$$+C\int_{B_{8s}\setminus B_s} r^2(|\nabla k|^2+|\nabla h|^2)|\nabla(h-k)|^2 \quad , \tag{V.48}$$

for $s \geq \frac{\varepsilon}{4}$. Combining (V.47) and (V.48) we get

$$\int_{\mathbb{R}^2\setminus B_\varepsilon} |\nabla^2(h-k)|^2 r^{4-2\nu} \leq o(1)\times (\text{ l.h.s. of (V.18) }) \tag{V.49}$$

and we can bound

$$\int_{\mathbb{R}^2\setminus B_\varepsilon} (|w|^2-1)\frac{rA}{\rho^2}\cdot\frac{\partial}{\partial r}\nabla^\perp k - h$$
$$\leq \left(\int_{\mathbb{R}^2\setminus B_\varepsilon} ||w|^2-1|^2 r^{-2+2\nu}|A|^2\right)^{\frac{1}{2}}\left(\int_{B_{2r}\setminus B_r} |\nabla^2 h - k|^2 r^{4-2\nu}\right)^{\frac{1}{2}} \tag{V.50}$$
$$\leq o(1)\times (\text{ l.h.s of (V.18) }) \quad .$$

The other terms in the r.h.s. of (V.46) can be bounded like before. So finally combining (V.43)...(V.50) we have proved

$$\left|\int_{\mathbb{R}^2} r\frac{\partial |w|^2}{\partial r}A\cdot\frac{\nabla^\perp(k-h)}{\rho^2}\right| \leq o(1)\times(\text{ l.h.s of (V.18) }) \quad . \tag{V.51}$$

Lets bound now the fourth term in the r.h.s. of (V.18). Once again we write the integral over $\mathbb{R}^2$ as the sum of the integral over B_ε and the integral over $\mathbb{R}^2\setminus B_\varepsilon$. Clearly

$$\int_{B_\varepsilon} r\frac{\partial\phi}{\partial r}A\cdot\nabla|w|^2 \leq C\left(\int_{B_\varepsilon}\left|\frac{\partial w}{\partial r}\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(\int_{B_\varepsilon} r^2|A|^2\left|\frac{\partial|w|}{\partial\tau}\right|^2+\int_{B_\varepsilon}|A-B|^2r^2\left|\frac{\partial|w|}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad , \tag{V.52}$$

where we have used $\frac{\partial\phi}{\partial r}=\Im\left(\frac{\rho}{|\Phi|}\frac{\partial w}{\partial r}\right)$ and $B_r=0$. We know that $\left|\frac{\partial|w|}{\partial r}\right|=\frac{o(1)}{r^2}$ for $0<r\leq\varepsilon$, see (V.21). So, with (V.26) we conclude

$$\left|\int_{B_\varepsilon}|A-B|^2r^2\left|\frac{\partial|w|}{\partial r}\right|^2\right| \leq o(1)\times(\text{ l.h.s. of (V.18) }) \tag{V.53}$$

Moreover, $\left|\frac{\partial |w|}{\partial \tau}\right| \leq \frac{|\nabla(|\Phi|-\rho)|^2}{\rho^2}$. Since $|A| \leq C_\alpha r^\alpha$ (for any $0<\alpha<1$), we have to bound $\int_{B_\varepsilon} \varepsilon^2 r^{2\alpha} |\nabla|\Phi|-\rho|^2$. We write

$$
\begin{aligned}
-\Delta(|\Phi|-\rho) \; &= |\Phi|\frac{1-|\Phi|^2}{\varepsilon^2} - \rho\frac{1-\rho^2}{\varepsilon^2} \\
&+ \left[\frac{|\nabla k|^2}{\rho^3} - \frac{|\nabla h|^2}{|\Phi|^3}\right] \quad .
\end{aligned}
$$

Since $\nabla^\perp = (i\Phi;\nabla_A\Phi)$, we have $|\nabla h| \leq |\Phi||\nabla_A\Phi|$, then, in $B_\mu\varepsilon$, where $\mu \geq 1$ will be determined later independent of ε, we have using (IV.2) $|\nabla h| \leq C\frac{|\Phi|}{\varepsilon} \leq C\frac{r}{\varepsilon^2}$ in $B_{\mu\varepsilon}$. Similarly we have of course $|\nabla k| \leq C\frac{r}{\varepsilon^2}$ in $B_{\mu\varepsilon}$. Multiplying this equation by $r^{2\alpha}(|\Phi|-\rho)$ and integrating by parts on $B_{\mu\varepsilon}$, gives

$$
\begin{aligned}
&\int_{B_{\mu\varepsilon}} r^{2\alpha}|\nabla(|\Phi|-\rho)|^2 \leq \frac{C}{\varepsilon^2}\int_{B_{\mu\varepsilon}} r^{2\alpha}||\Phi|-\rho|^2 \\
&+\int_{B_{\mu\varepsilon}} r^{2\alpha}||\Phi|-\rho||\nabla(h-k)|\left[\frac{|\nabla h|}{|\Phi|^3}+\frac{|\nabla k|}{\rho^3}\right] + C\mu\varepsilon^{1+2\alpha}\int_{\partial B_{\mu\varepsilon}}(|\Phi|-\rho)\frac{\partial}{\partial r}(|\Phi|-\rho) \quad .
\end{aligned}
$$

First we have, choosing $\alpha > \frac{1}{2}$ and using (V.23)

$$
\begin{aligned}
&\int_{B_{\mu\varepsilon}} r^{2\alpha}||\Phi|-\rho||\nabla h-k|\left[\frac{|\nabla h|}{|\Phi|^3}+\frac{|\nabla k|}{\rho^3}\right] \\
&\leq C\left(\int_{B_{\mu\varepsilon}}||\Phi|-\rho|^2\right)^{\frac{1}{2}}\left(\int_{B_{\mu\varepsilon}}\frac{r^{4\alpha}}{\varepsilon^2\rho^2}\frac{|\nabla(h-k)|^2}{\rho^2}\right)^{\frac{1}{2}} \\
&\leq o(1)\times(\text{ l.h.s. of (V.18) })
\end{aligned}
$$

where we have used $\frac{r^{4\alpha}}{\varepsilon^2\rho^2} \leq Cr^{4\alpha-2} \leq Cr^{2-2\nu}$ since α can be taken as close to 1 as we want. For R sufficiently large c_ε in (V.32) verifies $1 \leq c_\varepsilon \leq 3$ in $\mathbb{R}^2\setminus B_{R\varepsilon}$, and we easily get, testing (V.32)

$$
\begin{aligned}
&\textstyle\int_{B_{7R\varepsilon/4}\setminus B_{5R\varepsilon/4}}|\nabla(|\Phi|-\rho)|^2 \leq C\int_{B_{2R\varepsilon}\setminus B_{R\varepsilon}}\frac{||\Phi|-\rho|^2}{\varepsilon^2} \\
&+C\int_{B_{2R\varepsilon}\setminus B_{R\varepsilon}}[|\nabla h|^2+|\nabla k|^2]|\nabla(h-k)|^2
\end{aligned}
$$

We use once again the mean value formula to choose $\mu \in (\frac{5R}{4},\frac{7R}{4})$ such that

$$
\frac{R\varepsilon}{2}\int_{\partial B_{\mu\varepsilon}}|\nabla(|\Phi|-\rho)|^2+\frac{||\Phi|-\rho|^2}{\varepsilon^2} \leq \int_{B_{2R\varepsilon}\setminus B_{R\varepsilon}}|\nabla(|\Phi|-\rho)|^2+\frac{||\Phi|-\rho|^2}{\varepsilon^2} \quad .
$$

Finally this yields

$$\int_{B_{\mu\varepsilon}} r^{2\alpha}|\nabla|\Phi|-\rho|^2 \le o(1)\times (\text{ l.h.s. of (V.18) }) \quad . \tag{V.54}$$

Combining (V.52), (V.53) and (V.54) we obtain

$$\int_{B_\varepsilon} r\frac{\partial\phi}{\partial r}A.\nabla|w|^2 \le o(1)\times (\text{ l.h.s. of (V.18) }) \quad . \tag{V.55}$$

The integral of the same quantity outside $B_{\mu\varepsilon}$ for some $\mu\in(1/2,1)$, that will be chosen later using the mean value formula as above, can be decomposed in the following way, once again using integration by parts and $\nabla\phi = A - \frac{\nabla^\perp h}{|\Phi|^2}$ as well as $\frac{\partial k}{\partial\tau}=0$ and $div\, A = 0$,

$$\begin{aligned}-\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} r\frac{\partial\phi}{\partial r}A\cdot\nabla|w|^2 &= \int_{\partial B_{\mu\varepsilon}}\frac{r}{\rho^2}\,A\cdot\nu\,(|w|^2-1)\frac{\partial(h-k)}{\partial\tau}\\ &+\int_{\mathbb{R}^2\setminus B_{\mu\varepsilon}} A.\nabla\left(\frac{r}{\rho^2}\frac{\partial(h-k)}{\partial\tau}\right)(|w|^2-1)\end{aligned} \tag{V.56}$$

As we said, μ is chosen between 1/2 and 1 using the mean value formula such that $\frac{\varepsilon}{2}\int_{\partial B_{\mu\varepsilon}}\ldots \le \int_{B_\varepsilon\setminus B_{\varepsilon/2}}\ldots$. Observe now that both integrals in the r.h.s. of (V.56) were already bounded $o(1)\times$ (l.h.s of (V.18)).

So it just remains to bound the last term in the r.h.s. of (V.18). We write

$$\int_{\mathbb{R}^2}\log|x|\frac{\partial h}{\partial x_1} = \int_{\mathbb{R}^2}\log|x|\frac{\partial h-k}{\partial 1} = \int_{B_1}\ldots+\int_{\mathbb{R}^2\setminus B_1}\ldots$$

We bound the two integrals in the r.h.s of the above identity in the following way :

$$\left|\int_{B_1}\log|x|\frac{\partial(h-k)}{\partial x_1}\right| \le \left(\int_{B_1} r^{2\nu-2}\log^2 r\right)^{\frac12}\left(\int_{B_1} r^{2-2\nu}|\nabla(h-k)|^2\right)^{\frac12}$$

and

$$\left|\int_{\mathbb{R}^2\setminus B_1}\log|x|\frac{\partial(h-k)}{\partial x_1}\right| \le \left(\int_{\mathbb{R}^2\setminus B_1}\frac{1}{r^3}\log^2 r\right)^{\frac12}\left(\int_{\mathbb{R}^2\setminus B_1} r^3|\nabla(h-k)|^2\right)^{\frac12} \quad .$$

So we easily see, using lemma V.2 again, in this way that the last term in the r.h.s. of (V.18) is also bounded by $o(1)\times$ (l.h.s of (V.18)) like the four first other terms. Inspecting the l.h.s. of (V.18) this yields that $|\Phi|\equiv\rho$, w is independant on r and $h=k$ for ε small enough. This is the desired result and Theorem I.2 is proved. ∎

A Appendix

Lemma A.1 *Let Ω be a regular domain of $\mathbb{R}^2$ and u a regular map from Ω into $\mathbb{C}$. Assume there exists a positive number μ such that $|u| \geq 1$ for any $x \in \Omega$ such that $dist(x, \partial\Omega) < \mu$ and denote by d the degree of $u/|u|$ from $\partial\Omega$ into S^1. Then for any $\varepsilon > 0$ we have*

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \pi|d|\log\frac{\mu}{\varepsilon} - C|d|\log(|d|+1) \quad ,$$

where C is some universal constant.

Proof of Lemma A.1. We start by following the proof of Theorem 2 of [14]. Let $\Omega_t = \{|u| \geq t\}$, $\omega_t = \Omega \setminus \Omega_t$ and denote $\gamma_t = \partial\omega_t$. Following the notations of [14] we have

$$\Theta(t) = \int_{\Omega_t} \left|\nabla \frac{u}{|u|}\right|^2 \quad .$$

By the coarea formula

$$E_\varepsilon(u) = \int_0^{+\infty} \left[\int_{\gamma_t} |\nabla|u||(x) + \frac{(1-t^2)^2}{2\varepsilon^2|\nabla|u||(x)}\ d\mathcal{H}^1\right] - t^2\Theta'(t)\ dt \quad .$$

Denoting by $|\omega_t|$ the inner radius of ω_t we have $\lambda(\gamma_t) = \lambda(\partial\omega_t) \geq |\omega_t|$ where $\lambda(\gamma)$ is the length of γ. We have

$$-\int_0^{+\infty} \frac{t^2}{2}\Theta'(t) = \int_0^{\infty} t\ \Theta(t) \geq \int_0^1 t\ \Theta(t) \quad ,$$

where we have performed an integration by parts and where we have used that u is bounded in L^∞. We use now the heart of Sandiers paper which is the proposition in the first part of [14] with $F_t(A) = \frac{1}{2}\int_{A\setminus\omega_t} |\nabla\frac{u}{|u|}|^2$ and $f_t(r,s,d) = \pi|d|\log(s/r)$ and we get

$$-\int_0^{+\infty} \frac{t^2}{2}\Theta'(t) \geq \int_0^1 t\ \Theta(t) \geq \int_0^1 2t\ \pi|d|\log\frac{\mu}{|\omega_t|} \quad .$$

Argueing like in [14] we have, granting the previous lower bound,

$$E_\varepsilon(u) \geq \int_0^1 |\omega_t|\frac{(1-t^2)}{\sqrt{2}\varepsilon} - 2\pi t|d|\log|\omega_t| + 2\pi t|d|\log\mu \quad ,$$

which is exactly (4) of [14] where $C(\Omega, K)$ is replaced by $2\pi|d|\log\mu$. We then end the proof of Lemma A.1 like in [14]. ∎

Lemma A.2 *Let (A,Φ) be a sequence of configurations as $\varepsilon\to 0$ such that*

$$\int_{\mathbb{C}}|\nabla_A\Phi|^2+\frac{(1-|\Phi|^2)^2}{2\varepsilon^2}+|dA|^2\le C\log\frac{1}{\varepsilon}\quad ,$$

where C is independent on ε, then

$$deg_\infty(A,\Phi)$$

is bounded independently of ε.

Proof of lemma A.2. The proof is very similar to the proof of lemma III.1. We first choose a lattice $\mathcal{L}^\eta_{x_0}$ such that

$$\int_{\mathcal{L}^\eta_{x_0}} a_\varepsilon(A,\Phi)\ d\mathcal{H}^1\le\frac{C}{\eta}\log\frac{1}{\varepsilon}$$

for ε small enough. Like in the proof of Lemma III.1 this implies that $|\Phi|\ge 1/2$ on $\mathcal{L}^\eta_{x_0}$ for ε small enough. We consider here also a connected component Λ_j of $\mathbb{R}^2\setminus\mathcal{L}^\eta_{x_0}$ such thatd_j =deg$(\Phi/|\Phi|;\partial\Lambda_j)\neq 0$ and we deduce exactly like in the proof of Lemma III.1 that for any $0<\delta$

$$\int_{\Lambda_j} a_\varepsilon(A,\Phi)\ge 2\pi|d_j|(1-\delta)\left(1-C\delta-C\frac{\eta^2}{\delta}\right)\log\frac{1}{\varepsilon}+o(\log\frac{1}{\varepsilon})\quad .$$

where $o_\varepsilon(\log\frac{1}{\varepsilon})$ above only depends on δ, η and ε but not on j, $|d_j|$...etc. summing over j we get the desired result. ■

Lemma A.3 *Let $f\in L^2(\mathbb{R}^2)$ be a smooth function and $u=\frac{1}{2\pi}\log r*f+c$ such that $u(0)=0$. Then $\forall 0<\nu<1$ we have*

$$\int_{\mathbb{R}^2} r^{-2\nu-2}|u|^2+r^{-2\nu}|\nabla u|^2+r^{-2\nu+2}|\nabla^2u|^2\le C\int_{\mathbb{R}^2} r^{-2\nu+2}|f|^2$$

Proof of Lemma A.3 Write $f=\sum_{n\in\mathbb{Z}}f_n(r)\exp(in\theta)$ and assume first $f_0\equiv f_1\equiv 0$. Since $u=\frac{1}{2\pi}\log r*f+c$, we have $u_0\equiv u_1\equiv 0$. Let $0<\nu<1$. In one hand we have

$$\begin{aligned}\int_{B_R\setminus B_s} r^{-2\nu}u\,f\ &=\int_{B_R\setminus B_s} r^{-2\nu}u\,\Delta u=-\int_{B_R\setminus B_s} r^{-2\nu}|\nabla u|^2\\ &-\frac{1}{2}\int_{B_R\setminus B_s}\nabla r^{-2\nu}\cdot\nabla|u|^2+\int_{\partial(B_R\setminus B_s)} r^{-2\nu}u\,\partial_\nu u\quad .\end{aligned}\tag{A.1}$$

Since $u_0\equiv 0$

$$\int_{\partial B_s} s^{-2\nu-1}|u|^2\le\int_{\partial B_s} s^{-2\nu+1}\left|\frac{\partial u}{\partial\tau}\right|^2\longrightarrow 0\quad\text{ as }\quad s\to 0\quad ,$$

and we have

$$-\frac{1}{2}\int_{B_R}\nabla r^{-2\nu}.\nabla|u|^2 = 2\nu^2\int_{B_R} r^{-2-2\nu}|u|^2 + \nu\int_{\partial B_R} r^{-2\nu-1}|u|^2 \tag{A.2}$$

(A.1) becomes

$$\begin{aligned}\int_{B_R} r^{-2\nu} u\, f \quad &= -\int_{B_R} r^{-2\nu}|\nabla u|^2 + 2\nu^2\int_{B_R} r^{-2-2\nu}|u|^2 \\ &+\nu R\int_{\partial B_R} r^{-2\nu-2}|u|^2 + \int_{\partial B_R} r^{-2\nu} u\,\frac{\partial u}{\partial r} \quad .\end{aligned} \tag{A.3}$$

On the other hand we have

$$\begin{aligned}\int_{B_R} f\frac{\partial u}{\partial r} r^{-2\nu+1} \quad &= \int_{B_R}\Delta u\frac{\partial u}{\partial r} r^{-2\nu+1} \\ &= \nu\left[\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2 - \int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2\right] \\ &+\frac{R}{2}\left[\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2 - \int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2\right] \quad .\end{aligned} \tag{A.4}$$

Now, (A.3)+$\frac{\nu}{2}$(A.4) yields

$$\begin{aligned}&\left(1-\frac{\nu^2}{2}\right)\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2 + \left(1+\frac{\nu^2}{2}\right)\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2 - 2\nu^2\int_{B_R} r^{-2-2\nu}|u|^2 \\ &= \nu R\int_{\partial B_R} r^{-2\nu-2}|u|^2 - \frac{\nu R}{4}\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{\nu R}{4}\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2 \\ &+\int_{\partial B_R} r^{-2\nu} u\,\frac{\partial u}{\partial r} - \frac{\nu}{2}\int_{B_R} f\frac{\partial u}{\partial r} r^{-2\nu+1} - \int_{B_R} r^{-2\nu} u\, f \quad .\end{aligned} \tag{A.5}$$

Since $u = \sum_{|n|\geq 2} u_n(r)\exp(in\theta)$, we have

$$\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2 \geq 4\int_{\partial B_R} r^{-2\nu-2}|u|^2 \quad . \tag{A.6}$$

Thus, for every $M > 0$,

$$
\begin{aligned}
&\left(1-\frac{\nu^2}{2}\right)\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2+\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2\\
&\leq R\left(\frac{\nu}{4}+\frac{M}{4}\right)\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2+\frac{R}{4M}\int_{\partial B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2 \qquad (A.7)\\
&-\frac{\nu}{2}\int_{B_R} f\frac{\partial u}{\partial r} r^{-2\nu+1}-\int_{B_R} r^{-2\nu} u\, f \quad .
\end{aligned}
$$

Since $0<\nu<1$ we have $1-\frac{\nu^2}{2}>\frac{\nu+1}{4}$. Thus there exists $M>1$ such that $1-\frac{\nu^2}{2}>\frac{\nu+M}{4}$. So there exists $\eta<1$ such that

$$
Y(R)\leq R\,\eta\, Y'(R)+D^{1/2}Y^{1/2}(R) \tag{A.8}
$$

where

$$
Y(R)=\left(1-\frac{\nu^2}{2}\right)\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2+\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial \tau}\right|^2
$$

and $D=C\int_{\mathbb{R}^2} r^{-2\nu+2}|f|^2$. This implies that there exist $\eta_0<1$ and $C>1$ such that, for $R>0$,

$$
(Y(R)-CD)\leq R\eta_0\left(Y(R)-CD\right)' \quad .
$$

Thus

$$
\left((Y(R)-CD)R^{-\frac{1}{\eta_0}}\right)'\geq 0 \quad .
$$

Assume there exists R_0 such that $Y(R_0)>CD$, then

$$
\underline{\lim}\frac{Y(R)}{R^{\frac{1}{\eta_0}}}>0 \quad . \tag{A.9}
$$

Take $f\in C_0^\infty(\mathbb{R}^2)$. Then we have $\int_{\mathbb{R}^2}|\nabla u|^p<+\infty$ for any $p>2$ and we obtain that $\int_{B_R\setminus B_1} r^{-2\nu}|\nabla u|^2\leq CR^\alpha$ for some $\alpha<1$. This contradicts (A.9). And we get the result for any f by density. Thus the lemma is established for $|n|\geq 2$.

For $n=0$, i.e. f and u depending only on r from (A.4) we have

$$
\nu\int_{B_R} r^{-2\nu}\left|\frac{\partial u}{\partial r}\right|^2\leq -\frac{R^{2-2\nu}}{2}|\dot{u}_0(R)|^2+\int_{B_R} r^{-2\nu+1}\left|\frac{\partial u}{\partial r}\right||f| \quad . \tag{A.10}
$$

Then we get the result for $n=0$.

For $n=1$ an explicit computation yields, provided the integral exist, e.g. has compact support in $[0,+\infty)$,

$$
u(r)=\left[-\frac{r}{2}\int_r^{+\infty} f_1(s)\,ds-\frac{1}{2r}\int_0^r s^2 f_1(s)\,ds\right]\exp(i\theta) \quad .
$$

Thus

$$|\dot{u}_1(R)|^2 \leq C\left[R^{2\nu-2}\int_0^{+\infty} s^{-2\nu+3} f_1^2(s)\,ds\right] \tag{A.11}$$

and

$$|u(R)|^2 \leq CR^{2\nu}\int_0^{+\infty} s^{-2\nu+3} f_1^2(s)\,ds \quad . \tag{A.12}$$

(A.3) and (A.4) imply, taking into account $\left|\frac{\partial u}{\partial \tau}\right| = \frac{1}{r}|u|$ for $n = 1$,

$$\begin{aligned}\int_{B_R} r^{-2\nu}|\nabla u|^2 \quad &\leq C\int_{B_R} r^{-2\nu+1}|f||\nabla u| \\ &+CR^{-2\nu}|u_1(R)|^2 + CR^{-2\nu+2}|\dot{u}_1(R)|^2\end{aligned} \tag{A.13}$$

Combining (A.11), (A.12) and (A.13) we get the result as $R \to +\infty$. ∎

Lemma A.4 *Let ξ be a smooth solution in $W^{1,2}(\mathbb{R}^2)$ of*

$$-\Delta\xi + 2\frac{\nabla|\Phi^{\varepsilon_n}|}{|\Phi^{\varepsilon_n}|}.\nabla\xi + |\Phi^{\varepsilon_n}|^2\xi = f + |\Phi^{\varepsilon_n}|^2 div g \qquad on\ \mathbb{R}^2 \quad , \tag{A.14}$$

where f and g are bounded functions, $|\Phi^{\varepsilon_n}|$ is the modulus of a minimizer of $\mathcal{A}_{YMH}$ in a given homotopy class given by lemma IV.1 or the modulus of a degree ± 1 solution of YMH equation. Then there exists $1 > \nu_0 > 0$ independant of ε, f and g such that $\forall \nu_0 < \nu < 1$ there exists η_0 such that for any $\eta \in [\eta_0/2, \eta_0]$ we have, for n large enough, independent of ξ, f and g

$$\begin{aligned}\int_{\mathbb{R}^2} r^{2-2\nu}\ \frac{|\nabla\xi|^2}{\rho^2} \leq C\Bigg[&\int_{\mathbb{R}^2} r^{4-2\nu}\frac{|f|^2}{\rho^2} + \int_{\mathbb{R}^2} r^{2-2\nu}\rho^2|g|^2 \\ &+ \left(\int_{\partial B_\eta}\frac{1}{\rho^2}\frac{\partial\xi}{\partial\nu} - \int_{B_\eta}\xi\right)^2 + \left(\int_{\partial B_\eta} g.\nu\right)^2\Bigg] \quad .\end{aligned} \tag{A.15}$$

Proof of Lemma A.4. We make the proof of Lemma A.4 simply for $|\Phi^{\varepsilon_n}|(x) = \rho^{\varepsilon_n}(|x|)$ the modulus of the axially symetric solution $(B, \Psi = \rho(|x|)\,e^{i\theta})$ of the YMH equation. General Φ^{ε_n} given by lemma IV.1 vanishe at isolated points on a model based on ρ^{ε_n}. Therefore one extends the proof to the general case by introducing glueing methods involving standard Dirichlet-Neuman problems, as explained in [10]. (we will often ommit the superscipt ε_n in the proof).

Assume first $f = \sum_{|n|\geq 1} f_n \exp(in\theta)$ and $\mathrm{div}\, g = \sum_{|n|\geq 1} g_n \exp(in\theta)$. This implies $\xi = \sum_{|n|\geq 1} \xi_n \exp(in\theta)$. Multiplying the l.h.s. of equation (A.14) by $\frac{1}{\rho^2} r^{2-2\nu}\xi$ and integrating on $B_R(0) \setminus B_s(0)$ we get

$$\int_{B_R\setminus B_s} \frac{r^{2-2\nu}}{\rho^2}\xi\left(-\Delta\xi+2\frac{\nabla\rho}{\rho}.\nabla\xi+\rho^2\xi\right)$$

$$= \int_{B_R\setminus B_s} r^{2-2\nu}\frac{|\nabla\xi|^2}{\rho^2}+r^{2-2\nu}|\xi|^2-\int_{\partial(B_R\setminus B_s)}\frac{r^{2-2\nu}}{\rho^2}\xi\frac{\partial\xi}{\partial\nu} \tag{A.16}$$

$$+\int_{B_R\setminus B_s}\frac{1}{2\rho^2}\nabla r^{2-2\nu}.\nabla|\xi|^2 \quad .$$

We have

$$\int_{B_R\setminus B_s}\frac{1}{2\rho^2}\nabla r^{2-2\nu}.\nabla|\xi|^2=(1-\nu)\int_{\partial(B_R\setminus B_s)} r^{1-2\nu}\frac{|\xi|^2}{\rho^2}$$

$$-2(1-\nu)^2\int_{B_R\setminus B_s}\frac{r^{-2\nu}}{\rho^2}|\xi|^2+(2-2\nu)\int_{B_R\setminus B_s}\frac{\rho' r^{1-2\nu}}{\rho}\frac{|\xi|^2}{\rho^2} \quad . \tag{A.17}$$

Since $|n|\geq 1$

$$\forall r>0 \qquad \int_{\partial B_r}\left|\frac{\partial\xi}{\partial\tau}\right|^2\geq\int_{\partial B_r}\frac{|\xi|^2}{r^2} \quad . \tag{A.18}$$

Thus we can replace s by 0 in (A.16) and (A.17). Let

$$Y(R)=\int_{B_R} r^{2-2\nu}\frac{|\nabla\xi|^2}{\rho^2}$$

(A.16), (A.17), (A.18) and the equation for ξ yield for $1-\nu$ sufficiently small

$$Y(R)\leq\delta\, R\, Y'(R)+\int_{B_R}\frac{r^{2-2\nu}}{\rho^2}\xi(f+\rho^2\mathrm{div}g) \quad . \tag{A.19}$$

for some $0<\delta<1$ and for all $R>0$.
After integration by parts we get, for some $0<\tilde{\delta}<1$,

$$Y(R)\leq \quad \tilde{\delta}\, R\, Y'(R)+C_0\, R\int_{\partial B_R} r^{2-2\nu}|g|^2\rho^2$$

$$+C\int_{\mathbb{R}^2}\frac{r^{4-2\nu}}{\rho^2}f^2+C\int_{\mathbb{R}^2} r^{2-2\nu}|g|^2\rho^2 \quad .$$

Let

$$Z(R)=Y(R)+\frac{C_0}{\tilde{\delta}}\int_{B_R} r^{2-2\nu}|g|^2\rho^2-C\int_{\mathbb{R}^2}\frac{r^{4-2\nu}}{\rho^2}f^2+r^{2-2\nu}|g|^2\rho^2 \quad .$$

We have

$$Z(R) - \tilde{\delta}\, R\, Z'(R) \le 0$$

Assume $\exists\ R_0$ such that $Z(R_0) > 0$ then, for $R \ge R_0$ $Z(R) \ge C^{-1} R^{1/\tilde{\delta}}$ on $(0, \infty)$. Take f and g compactly supported, we necessaryly have $\int_{\mathbb{R}^2 \setminus B_1} |\nabla \xi|^2 < +\infty$ which contradicts the growth of Z. By density we prove (A.15) for any f and g such that $f_0 = 0$ and $(div\, g)_0 = 0$.

Consider now the case $f_n = 0$ and $(div\, g)_n = 0$ for $|n| \ge 1$. We have

$$-\ddot{\xi}_0 - \frac{\dot{\xi}_0}{r} + 2\frac{\dot{\rho}}{\rho}\dot{\xi}_0 + \rho^2 \xi_0 = h + \rho^2 \dot{h}_0 \tag{A.20}$$

where $h = f_0 + (div\, g)_0 = f_0 + \rho^2(\frac{h_0}{r} + \dot{h}_0)$ and where $h_0(r) = \frac{1}{2\pi r} \int_{\partial B_r} g \cdot \nu$. Multiplying (A.20) by $\frac{r^{4-2\nu}}{\rho^2} \dot{\xi}_0$ and integrating between 0 and a positive number η, which will be fixed later, we get

$$\begin{aligned}
\int_0^\eta \frac{r^{4-2\nu}}{\rho^2} \dot{\xi}_0 \quad (h + \rho^2 \dot{h}_0) &= (1-\nu) \int_0^\eta \frac{r^{3-2\nu}}{\rho^2} \left|\dot{\xi}_0\right|^2 \\
&+ \int_0^\eta \frac{\dot{\rho} r}{\rho} \frac{r^{3-2\nu}}{\rho^2} \left|\dot{\xi}_0\right|^2 - (2-\nu) \int_0^\eta r^{3-2\nu} |\xi_0|^2 \\
&- \frac{\eta^{4-2\nu}}{2} \frac{|\dot{\xi}_0|^2}{\rho^2}(\eta) + \frac{\eta^{4-2\nu}}{2} \xi_0^2(\eta) \quad .
\end{aligned} \tag{A.21}$$

Moreover since $\xi_0(r) = -\int_r^{+\infty} \frac{\partial \xi_0}{\partial r}$

$$|\xi_0|^2(r) \le \left(\int_r^{+\infty} |\dot{\xi}_0| \right)^2 \le \frac{r^{2\nu-2}}{2(1-\nu)} \int_r^{+\infty} s^{3-2\nu} |\dot{\xi}_0|^2(s)\, ds \quad . \tag{A.22}$$

In particular this implies

$$\int_0^\eta r^{3-2\nu} |\xi_0|^2 \le \frac{\eta^2}{4(1-\nu)} \int_0^{+\infty} r^{3-2\nu} |\dot{\xi}_0|^2 \quad . \tag{A.23}$$

We next multiply (A.20) by $\xi_0 r^{3-2\nu}$ and integrate by parts between η and $+\infty$ to obtain

$$\begin{aligned}
&\eta^{3-2\nu} \xi_0(\eta) \dot{\xi}_0(\eta) + \int_\eta^{+\infty} |\dot{\xi}_0|^2 r^{3-2\nu} \\
&+2 \int_\eta^{+\infty} \left[(1-\nu) + \frac{r\dot{\rho}}{\rho} \right] r^{2-2\nu} \xi_0 \dot{\xi}_0 + \int_\eta^{+\infty} r^{3-2\nu} \rho^2 \xi_0^2 \\
&= \int_\eta^{+\infty} r^{3-2\nu} \xi_0 (h + \rho^2 \dot{h}_0) \quad ,
\end{aligned} \tag{A.24}$$

which gives

$$
\begin{aligned}
&\int_\eta^{+\infty}\left(1-\frac{1}{\eta}\left(1-\nu+\frac{r\dot\rho}{\rho}\right)\right)|\dot\xi_0|^2 r^{3-2\nu}\\
&+\int_\eta^{+\infty}\left(\rho^2-\frac{1}{\eta}\left(1-\nu+\frac{r\dot\rho}{\rho}\right)\right)|\xi_0|^2 r^{3-2\nu}\\
&\le |\eta^{3-2\nu}\xi_0(\eta)\dot\xi_0(\eta)|+\int_\eta^{+\infty} r^{3-2\nu}\xi_0(h+\rho^2\dot h_0) \quad .
\end{aligned}
\tag{A.25}
$$

We have, using the fact that $\rho \le 1$,

$$
\begin{aligned}
|\dot\xi_0(\eta)|^2 &\le \frac{1}{4\pi^2\eta^2}\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi}{\partial r}\right)^2\\
&\le \frac{1}{2\pi^2\eta^2}\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2+\frac{1}{2\pi}\int_{B_\eta}|\xi_0|^2\\
&\le \frac{1}{2\pi^2\eta^2}\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2+\frac{\eta^{2\nu}}{4\nu(1-\nu)}\int_0^{+\infty} r^{3-2\nu}|\dot\xi_0|^2
\end{aligned}
\tag{A.26}
$$

where we have used (A.22) for the last inequality. From this inequality we deduce in particular

$$
\begin{aligned}
|\eta^{3-2\nu}\xi_0(\eta)\dot\xi_0(\eta)| &\le \frac{\eta^{4-2\nu}}{2}\xi_0^2(\eta)+\frac{\eta^{2-2\nu}}{2}\dot\xi_0^2(\eta)\\
&\le \frac{\eta^{4-2\nu}}{2}\xi_0^2(\eta)+\frac{1}{4\pi^2\eta^{2\nu}}\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi_0}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2\\
&+\frac{\eta^2}{8\nu(1-\nu)}\int_0^{+\infty} r^{3-2\nu}|\dot\xi_0|^2 \quad .
\end{aligned}
\tag{A.27}
$$

Combining (A.21), (A.25), (A.26) and (A.27) we obtain

$$
\begin{aligned}
&(1-\nu)\int_0^\eta \frac{r^{3-2\nu}}{\rho^2}|\dot{\xi}_0|^2-(2-\nu)\int_0^\eta r^{3-2\nu}|\xi_0|^2\\
&+\int_\eta^{+\infty}\left(1-\frac{1}{\eta}\left(1-\nu+\frac{r\dot{\rho}}{\rho}\right)\right)|\dot{\xi}_0|^2 r^{3-2\nu}\\
&+\int_\eta^{+\infty}\left(\rho^2-\frac{1}{\eta}\left(1-\nu+\frac{r\dot{\rho}}{\rho}\right)\right)|\xi_0|^2 r^{3-2\nu}\\
&\leq \frac{\eta^{4-2\nu}}{2\rho^2(\eta)}|\dot{\xi}_0|^2(\eta)+\frac{\eta^2}{8\nu(1-\nu)}\int_0^{+\infty} r^{3-2\nu}|\dot{\xi}_0|^2\\
&+\int_\eta^{+\infty} r^{3-2\nu}\xi_0(h+\rho^2\dot{h}_0)+\int_0^\eta \frac{r^{4-2\nu}}{\rho^2}\dot{\xi}_0(h+\rho^2\dot{h}_0)\\
&+\frac{1}{4\pi^2\eta^{2\nu}}\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi_0}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2 \quad .
\end{aligned}
\tag{A.28}
$$

The second term in the l.h.s. of (A.28) is bounded using (A.23) and the first term in the r.h.s of (A.28) is bounded using (A.26), this yields for $\nu\eta \geq 1-\nu$

$$
\begin{aligned}
&(1-\nu)\int_0^\eta \frac{r^{3-2\nu}}{\rho^2}|\dot{\xi}_0|^2-(2-\nu)\frac{1-\nu}{4\nu^2}\int_0^{+\infty} \frac{r^{3-2\nu}}{\rho^2}|\dot{\xi}_0|^2\\
&+\int_\eta^{+\infty}(1-\nu-\frac{\nu}{1-\nu}\frac{r\dot{\rho}}{\rho})|\dot{\xi}_0|^2 r^{3-2\nu}+\int_\eta^{+\infty}(\rho^2-\nu-\frac{\nu}{1-\nu}\frac{r\dot{\rho}}{\rho})|\xi_0|^2 r^{3-2\nu}\\
&\leq \left[\frac{(1-\nu)^3}{8\rho^2(\eta)\nu^5}+\frac{(1-\nu)}{8\nu^3}\right]\int_0^{+\infty} r^{3-2\nu}|\dot{\xi}_0|^2\\
&+\int_\eta^{+\infty} r^{3-2\nu}\xi_0(h+\rho^2\dot{g}_0)+\int_0^\eta \frac{r^{4-2\nu}}{\rho^2}\dot{\xi}_0(h+\rho^2\dot{g}_0)\\
&+C_\nu\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi_0}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2 \quad .
\end{aligned}
\tag{A.29}
$$

Clearly there exists a $\nu_0<1$ such that for any $\nu_0<\nu<1$ since $1-\rho^2(r)\lesssim\frac{\varepsilon^2}{r^2}$ and $\dot{\rho}\lesssim\frac{\varepsilon^2}{r^3}$ we have, for ε_n small enough

$$
\begin{aligned}
&\int_0^{+\infty} \frac{r^{3-2\nu}}{\rho^2}|\dot{\xi}_0|^2+\int_\eta^{+\infty}|\xi_0|^2 r^{3-2\nu}\\
&\leq C_\nu\left|\int_\eta^{+\infty} r^{3-2\nu}\xi_0(h+\rho^2\dot{g}_0)+\int_0^\eta \frac{r^{4-2\nu}}{\rho^2}\dot{\xi}_0(h+\rho^2\dot{g}_0)\right|\\
&+C_\nu\left(\int_{\partial B_\eta(0)}\frac{1}{\rho^2}\frac{\partial\xi_0}{\partial r}-\int_{B_\eta(0)}\xi_0\right)^2 \quad .
\end{aligned}
\tag{A.30}
$$

After integrations by parts in the integrals in the r.h.s. of (A.30), inserting the information that $\ddot{\xi}_0 = -\frac{\dot{\xi}_0}{r} + 2\frac{\dot{\rho}}{\rho}\dot{\xi}_0 + \rho^2\xi_0 - h - \rho^2\dot{h}_0$, using again (A.23) we get the desired result. ■

Lemma A.5 *Let $|\Phi^{\varepsilon_n}|$ given by lemma IV.1. For any function $f \in C_0^\infty(\mathbb{R}^2)$ there exists a unique solution $h \in W^{1,2}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ solving*

i)

$$\frac{\nabla h}{|\Phi^{\varepsilon_n}|^2} \in L^p(\mathbb{R}^2) \text{ for some } p > 1 \quad .$$

ii)

$$-div\left(\frac{\nabla h}{|\Phi^{\varepsilon_n}|^2}\right) + h = f \qquad in\ \mathcal{D}'(\mathbb{R}^2) \quad .$$

Moreover if $f \geq 0$ on $\mathbb{R}^2$ and $f \not\equiv 0$ then $h > 0$ on $\mathbb{R}^2$. ■

Proof of lemma A.5. Since the result is only a quantitative one, (no estimates) (ε_n plays no particular rôle), in order to simplify the presentation of the proof, we prove the lemma where Φ_n^ε is replaced by $\overline{r} = \min(|z|, 1)$. Remember that the behavior of Φ_n^ε at each of it's zero z_k has the form $c|z|/\varepsilon_n \leq |\Phi^{\varepsilon_n}|(z) \leq C|z|/\varepsilon_n$. Let h^δ be the solution of the perturbed problem

$$-div\left(\frac{\nabla h^\delta}{\overline{r}^2 + \delta}\right) + h^\delta = f \qquad \text{in } \mathbb{R}^2 \quad . \tag{A.31}$$

First of all, multiplying the equation by h^δ and integrating by parts, it is straightforward to establish that

$$\int_{\mathbb{R}^2} \frac{|\nabla h^\delta|^2}{|\overline{r}^2 + \delta|} + |h^\delta|^2 \leq C \quad \text{indep. of } \delta \text{ as } \delta \to 0 \quad . \tag{A.32}$$

We now claim that there exists $\nu > 0$ such that

$$\int_{\mathbb{R}^2} \frac{|\nabla h^\delta|^2}{|\overline{r}^2 + \delta|^{1+\nu}} + |h^\delta|^2 \leq C \quad \text{indep. of } \delta \text{ as } \delta \to 0 \quad . \tag{A.33}$$

Multiply (A.31) by $h^\delta|\overline{r}^2 + \delta|^{-\nu}$ and integrate by parts on $B_1(0) \setminus B_s(0)$ for any $0 < s < 1$, this yields

$$\begin{aligned} &\int_{B_1\setminus B_s} \frac{|\nabla h^\delta|^2}{|r^2 + \delta|^{1+\nu}} + \frac{|h^\delta|^2}{|r^2 + \delta|^\nu} \\ &\leq C_f + \nu \int_{B_1\setminus B_s} \frac{2r\, h^\delta}{(r^2 + \delta)^{\nu+2}} \frac{\partial h^\delta}{\partial \nu} + \left| \int_{\partial B_s} \frac{h^\delta}{(r^2 + \delta)^{1+\nu}} \frac{\partial h^\delta}{\partial r} \right| \quad , \end{aligned} \tag{A.34}$$

where we have used (A.32) and C_f is a positive constant depending on f but not on δ. From (A.31) we have for any $\rho < 1$

$$\left|\int_{\partial B_\rho} \frac{\partial h^\delta}{\partial r}\right| = \left|\int_{B_\rho} h^\delta - f\right| \leq C_f\ \rho \tag{A.35}$$

where we have used Cauchy-Schwartz inequality and (A.32). Denote h^δ_ρ the averages of h^δ on the circles ∂B_ρ for any $\rho > 0$. We write

$$\int_{\partial B_\rho} h^\delta\ \frac{\partial h^\delta}{\partial r} = \int_{\partial B_\rho} [h^\delta - h^\delta_\rho]\ \frac{\partial h^\delta}{\partial r} + h^\delta_\rho \int_{\partial B_\rho} \frac{\partial h^\delta}{\partial r} \quad . \tag{A.36}$$

On one hand we have, by the mean of Poincaré inequality,

$$\left|\int_{\partial B_\rho} [h^\delta - h^\delta_\rho]\ \frac{\partial h^\delta}{\partial r}\right| \leq C\rho \int_{\partial B_\rho} |\nabla h^\delta|^2 \quad . \tag{A.37}$$

On the other hand, using (A.35), we have, for any $p > 1$,

$$\left|h^\delta_\rho \int_{\partial B_\rho} \frac{\partial h^\delta}{\partial r}\right| \leq C(r^2+\delta)^{\nu+1} r^{1-\frac{2}{p}-2\nu} \left(s \int_{\partial B_s} |h^\delta|^p\right)^{\frac{1}{p}} \quad . \tag{A.38}$$

Combining (A.34)...(A.38) we get

$$\begin{aligned}
&(1-\nu)\int_{B_1\setminus B_s} \frac{|\nabla h^\delta|^2}{(r^2+\delta)^{1+\nu}} + \int_{B_1\setminus B_s} \frac{|h^\delta|^2}{(r^2+\delta)^{\nu}}\\
&\leq C\ s \int_{\partial B_s} \frac{|\nabla h^\delta|^2}{(r^2+\delta)^{1+\delta}} + C_f + s^{1-\frac{2}{p}-2\nu}\left(s\int_{\partial B_s} |h^\delta|^p\right)^{\frac{1}{p}}\\
&+\nu \int_s^1 \rho^{-\frac{1}{p}-2\nu} \left|\int_{\partial B_\rho} |h^\delta|^p\right|^{\frac{1}{p}}
\end{aligned} \tag{A.39}$$

Because of (A.32), we know that for any $p < +\infty$, $\int_{B_1} |h^\delta|^p \leq C_f$. Denote $Z_p(s) = \int_{B_1\setminus B_s} |h^\delta|^p$ and $Y(s) = \int_{B_1\setminus B_s} \frac{|\nabla h^\delta|^2}{(r^2+\delta)^{1+\nu}} + \int_{B_1\setminus B_s} \frac{|h^\delta|^2}{(r^2+\delta)^{\nu}}$. For p large enough, (A.39) becomes

$$Y(s) \leq -Cs\ Y'(s) + C_f - Cs\ Z'(s) \tag{A.40}$$

Since $Z(s)$ is uniformly bounded, independently on δ, as $s \to 0$, we have $W(s) \leq -Cs\ W'(s)$ where $W(s) := Y(s) + C\,Z(s) - C$, where C is independent of δ. The differential inequality implies that $Cs^{\frac{1}{C}} W(s)$ is a decreasing function. Assume there exists $s_0 > 0$ such that $W(s_0) > 0$, then, for any $0 < s < s_0$ we have $W(s) > \frac{s_0^{\frac{1}{C}} W(s_0)}{s^{\frac{1}{C}}}$. Because of (A.32), we have $W(s) \leq \frac{C}{s^{2\nu}}$, therefore, for ν sufficiently small we get a contradiction. This establishes (A.33).

We claim now that there exist $C > 0$ such that

$$\forall x \in B_1(0) \setminus B_{\sqrt{\delta}}(0) \qquad |\nabla h^\delta|(x) \le C\ |x|^\nu \quad . \tag{A.41}$$

Let $\rho > \sqrt{\delta}$ and denote $\tilde{h}^\delta(x) := \frac{1}{\rho} h^\delta(\rho x)$ for $x \in B_2 \setminus B_1$. Observe that $\nabla \tilde{h}^\delta(x) = \nabla h^\delta(\rho x)$ and therefore from (A.33) we deduce

$$\int_{B_2 \setminus B_1} |\nabla \tilde{h}^\delta|^2 \le C \rho^{2\nu} \quad . \tag{A.42}$$

In this new scaling the equation (A.31) becomes

$$-div \left(\frac{1}{|y|^2 + \frac{\delta}{\rho^2}} \nabla \tilde{h}^\delta \right) + \rho^2 \tilde{h}^\delta = \rho f(\rho x) \qquad \text{in } \mathcal{D}'(B_2 \setminus B_1) \quad . \tag{A.43}$$

A standard bootstraping of (A.42) in the elliptic equation (A.43) yields $|\nabla \tilde{h}^\delta|_{L^\infty(B_{3/2} \setminus B_{5/4})} \le C\, \rho^\nu$ and (A.41) is established. From (A.41) we deduce a unifrom L^p bound for $\frac{\nabla h^\delta}{r^2+\delta^2}$ independent of δ for any p solving $(2-\nu)p < 2$. It is now possible to pass to the limit in the equation (A.31) and we obtain a solution of i) and ii) in the statement of the lemma. The uniqueness is a direct consequence of Lemma A.4. ∎

Lemma A.6 *Let $|\Phi^{\varepsilon_n}|$ be the function on $\mathbb{R}^2$ given by lemma IV.1. Let $f \in C^\infty_0(\mathbb{R}^2)$ and let u be the solution of*

$$-div \left(\frac{\nabla u}{|\Phi^{\varepsilon_n}|^2} \right) + u = f \qquad in\ \mathcal{D}'(\mathbb{R}^2) \quad , \tag{A.44}$$

given by lemma A.5. Then, for any $p > 1$, there exists $C_p > 0$ independent of n and f, such that

$$\|u\|_\infty \le C_p \|f\|_p \quad . \tag{A.45}$$

Proof of lemma A.6 : Since $u \in C^1$ we may multiply (A.44) by u and integrate by parts. This yields, for any $p > 1$,

$$\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{|\Phi|^2} + |u|^2 \le \|f\|_p\ \|u\|_{p'} \tag{A.46}$$

Using Sobolev embeddig we then deduce, for any $q > 1$ and any $p > 1$

$$\|u\|_{L^q(\mathbb{R}^2)} \le C\ \|u\|_{W^{1,2}(\mathbb{R}^2)} \le C \|f\|_{L^p(\mathbb{R}^2)} \tag{A.47}$$

To simplify the presentation we assume that we are dealing with the modulus $|\Phi^{\varepsilon_n}|$ corresponding to a minimizer of $\mathcal{A}_1^{\varepsilon_n}$ having exactly one zero at the origin for n large enough as stated in Lemma IV.1. We prove now that there exists $\nu > 0$ such that

$$\int_{\mathbb{R}^2} r^{-\nu} \frac{|\nabla u|^2}{|\Phi|^2} \le C\ \|f\|^2_{L^p(\mathbb{R}^2)} \quad . \tag{A.48}$$

For any $0 < s < 1$ we multiply (A.44) by $r^{-\nu}u$ and integrate by parts on $B_1 \setminus B_s$. We thus obtain

$$\begin{aligned}&\int_{B_1\setminus B_s} r^{-\nu}\frac{|\nabla u|^2}{|\Phi|^2}+r^{-\nu}|u|^2\\&\leq C\|f\|_p^2+\nu\left|\int_{B_1\setminus B_s} r^{-\nu-1}u\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}\right|+\left|\int_{\partial B_s} r^{-\nu}u\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}\right|\end{aligned}\tag{A.49}$$

where the cut-off at 1 is eventually replaced by an arbitrary cut-off between 1/2 and 1 obtained by applying the mean value formula in order to control the boundary term $\int_{\partial B_1} r^{-\nu-1}u\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}$ by the mean of estimates (A.47). Moreover we have also used the following bound

$$\left|\int_{B_1\setminus B_s} r^{-\nu}u\ f\right|\leq\left|\int_{B_1}|u|^{p_1}\right|^{\frac{1}{p_1}}\ \left|\int_{B_1}|f|^p\right|^{\frac{1}{p}}\ \left|\int_{B_1} r^{-\nu p_2}\right|^{\frac{1}{p_2}}\leq C\|f\|_p^2$$

(for any $p > 1$ we may find $p_1, p_2 < +\infty$ such that $\frac{1}{p_1}+\frac{1}{p}+\frac{1}{p_2}=1$ and choose $\nu > 0$ so that $\nu p_2 < 2$). For any $\rho \in [s, 1]$ we denote by u_ρ the average of u on ∂B_ρ. We write

$$\int_{\partial B_\rho} r^{-\nu}u\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}=\rho^{-\nu}\int_{\partial B_\rho}[u-u_\rho]\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}+\rho^{-\nu}\ u_\rho\ \int_{\partial B_\rho}\frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}\tag{A.50}$$

On one hand, by the mean of Poincaré inequality we have

$$\left|\rho^{-\nu}\int_{\partial B_\rho}[u-u_\rho]\ \frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}\right|\leq C\rho\int_{\partial B_\rho}\rho^{-\nu}\frac{|\nabla u|^2}{|\Phi|^2}\tag{A.51}$$

On the other hand

$$\begin{aligned}&\left|\rho^{-\nu}\ u_\rho\ \int_{\partial B_\rho}\frac{1}{|\Phi|^2}\frac{\partial u}{\partial r}\right|=\left|\rho^{-\nu}\ u_\rho\ \int_{B_\rho} u-f\right|\\&\leq C\rho^{-\nu+2-\frac{2}{p}}\ u_\rho\ \|f\|_p\leq C\rho^{-\nu+2-\frac{2}{p}-\frac{2}{q}}\|f\|_p\left|\rho\int_{\partial B_\rho}|u|^q\right|^{\frac{1}{q}}\quad,\end{aligned}\tag{A.52}$$

for any $q < \infty$. We then choose q large enough so that $2-\frac{2}{p}-\frac{2}{q} > 0$ and ν small enough so that $0 < \nu < 2-\frac{2}{p}-\frac{2}{q}$. Combining then (A.49)...(A.52) we get

$$\begin{aligned}&(1-C\nu)\int_{B_1\setminus B_s} r^{-\nu}\frac{|\nabla u|^2}{|\Phi|^2}+r^{-\nu}|u|^2\\&\leq C\|f\|_p^2+Cs\int_{\partial B_s} s^{-\nu}\frac{|\nabla u|^2}{|\Phi|^2}+C\|f\|_p\left|\rho\int_{\partial B_\rho}|u|^q\right|^{\frac{1}{q}}\quad.\end{aligned}\tag{A.53}$$

Denoting $Y(s) = \int_{B_1 \setminus B_s} r^{-\nu} \frac{|\nabla u|^2}{|\Phi|^2} + |u|^q$, we deduce from (A.53) and for ν small enough

$$Y(s) \leq C_f - CsY'(s) \tag{A.54}$$

which implies that $(s^{\frac{1}{C}}(Y(s) - C_f))$ is an increasing function. Assume there exists s_0 such that $Y(s_0) > C_f$. Then this would imply that for $s < s_0$ $Y(s) \geq \frac{s_0^{\frac{1}{C}}(Y(s_0)-C_f)}{s^{\frac{1}{C}}}$. Meanwhile, from the bound (A.46) we know that $Y(s) \leq Cs^{-\nu}$ then for ν small enough we get a contradiction. Therefore $Y(s)$ is uniformly bounded as s tends to 0. Choosing now $s_i \to 0$ such that $s_i^{1-\nu} \int_{\partial B_{s_i}} \frac{|\nabla u|^2}{|\Phi|^2} + s_i \int_{\partial B_{s_i}} |u|^q \to 0$ as $s_i \to 0$ and pluging this information in (A.53) we get (A.48).

Since $\frac{1}{|\Phi|^2} \geq 1$, (A.48) implies $\int_{B_{2r} \setminus B_r} r^{-\nu} |\nabla u|^2$, we dilate the annulus $B_{2r} \setminus B_r$ by a factor $\frac{1}{r}$ and a standard bootstraping implies that $\|\tilde{u}\|_{C^{0,\alpha}(B_{\frac{3}{2}} \setminus B_{\frac{5}{4}})} \leq r^{\frac{\nu}{2}} \|f\|_p$ for $\alpha \leq 1 - \frac{1}{p}$. Then $\|u\|_{C^{0,\alpha}(B_{\frac{3r}{2}} \setminus B_{\frac{5r}{4}})} \leq r^{\frac{\nu}{2} - \alpha} \|f\|_p$. Taking any $\alpha < \frac{\nu}{2}$ we get the desired result $\|u\|_\infty \leq C\|f\|_p$ and lemma A.6 is proved. ∎

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