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Lower Semi-continuity of the Index in the Viscosity Method for Minimal Surfaces

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The goal of the present work is two-fold. First we prove the existence of a Hilbert manifold structure on the space of immersed oriented closed surfaces with three derivatives in L^2 in an arbitrary compact submanifold M^m of an Euclidian space $\mathbb{R}^Q$. Second, using this Hilbert manifold structure, we prove a lower semi-continuity property of the index for sequences of conformal immersions, critical points to the viscous approximation of the area satisfying a Struwe entropy estimate and a bubble tree strongly converging in $W^{1,2}$ to a limiting minimal surface as the viscous parameter is going to zero.

1 Introduction

Let M^m be a smooth m-dimensional submanifold of a Euclidian space $\mathbb{R}^Q$ and denote by π_{M^m} the orthogonal projection onto M^m defined in a neighborhood of M^m.

Let Σ be an arbitrary closed oriented 2D manifold. We define the *Sobolev space* of maps between Σ and M^m with three derivatives in L^2 as follows:

$$W^{3,2}(\Sigma, M^m) := \left\{ u \in W^{3,2}(\Sigma, \mathbb{R}^Q) \quad ; \quad u(x) \in M^m \quad \forall\, x \in \Sigma \right\},$$

where the Sobolev space of $W^{3,2}$ functions on Σ is defined with respect to any arbitrary reference metric (they are all equivalent due to the compactness of Σ).

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Since $3 \times 2 = 6 > 2 = \dim(\Sigma)$ the space $W^{3,2}(\Sigma, M^m)$ inherits a natural *Hilbert manifold* structure (see [11] lecture 2). Within this manifold we are considering the open subset of $W^{3,2}$-immersions

$$\mathrm{Imm}^{3,2}(\Sigma, M^m) := \left\{ \vec{\Phi} \in W^{3,2}(\Sigma, M^m) \quad ; \quad d\vec{\Phi} \wedge d\vec{\Phi} \neq 0 \quad \mathrm{in}\,\Sigma \right\}.$$

Since there is no ambiguity on the regularity we are choosing we shall simply omit the superscripts $3, 2$ and denote $\mathrm{Imm}(\Sigma, M^m)$ for $\mathrm{Imm}^{3,2}(\Sigma, M^m)$. For a given oriented closed surface Σ we denote by $b(\Sigma)$ the sum of the 1st three *Betti Numbers* of Σ

$$b(\Sigma) := b_0(\Sigma) + b_1(\Sigma) + b_2(\Sigma).$$

There are obviously finitely many classes modulo diffeomorphisms of surfaces Σ such that $b(\Sigma) \leq b$. We will work from now on with one fixed representative in each of these classes.

For any $b \in \mathbb{N}^*$ we denote by $\mathrm{Imm}_b(M^m)$ the *Hilbert manifold* obtained by taking the disjoined union of the *Hilbert manifold* of $W^{3,2}$-immersions of the finitely many surfaces such that $b(\Sigma) \leq b$.

Starting from $\mathrm{Imm}_b(M^m)$ we are constructing a *Hilbert manifold* of $W^{3,2}$-immersed surfaces in the following way. We are first marking each surface Σ by fixing respectively 3, 1, or 0 distinct points on each component of Σ of genus respectively 0, 1, and > 1. We are then considering the quotients of $\mathrm{Imm}(\Sigma, M^m)$ by $\mathrm{Diff}^*_+(\Sigma)$, the positive $W^{3,2}$-diffeomorphisms of Σ preserving the points we have fixed and isotopic to the identity. Then we denote by

$$\mathfrak{M}_b(M^m) := \bigsqcup_{b(\Sigma)\leq b} \mathrm{Imm}(\Sigma, M^m)/\mathrm{Diff}^*_+(\Sigma) \quad \text{and} \quad \mathfrak{M}(M^m) := \bigsqcup_{b(\Sigma)<+\infty} \mathrm{Imm}(\Sigma, M^m)/\mathrm{Diff}^*_+(\Sigma).$$

Our 1st main result is the following.

Theorem 1.1. For any $b \in \mathbb{N} \cup \{\infty\}$ there exists a *Hilbert manifold* Structure on $\mathfrak{M}_b(M^m)$ such that the canonical projection

$$\Pi \;\; \mathrm{Imm}_b(M^m) \longrightarrow \mathfrak{M}_b(M^m)$$

is a smooth map. □

Since $\mathrm{Diff}^*_+(\Sigma)$ misses to be a *Banach–Lie group* but is only a *topological group* (On the space of $W^{3,2}$-diffeomorphisms the right multiplication is smooth but the left multiplication is not differentiable, the inverse mapping is not C^1, there is no canonical chart in the neighborhood of the identity, the exponential map is continuous but not C^1, it is not locally surjective in a neighborhood of the identity, the Bracket operation in the Tangent space to the identity is not continuous..., etc. See a description of all these "pathological behavior" for instance in [4] or [8].) with a *Hilbert manifold* structure the existence of a differentiable Hilbert structure on the quotient $\mathrm{Imm}(\Sigma,M^m)/\mathrm{Diff}^*_+(\Sigma)$ is not the result of classical consideration and deserves to be studied with care (Progresses in this direction are given in [2] for $W^{3,2}$-embeddings but we are not going to follow this approach and the one we choose is more specific to surfaces but more precise too).

We shall in fact make the previous theorem more precise and to that aim we introduce some notations. Let Σ be a closed oriented 2D manifold, and $\vec{\Phi}\in W^{3,2}_{imm}(\Sigma,M^m)$ and let $g_{\vec{\Phi}} := \vec{\Phi}^*g_{M^m}$. Denote by $\wedge^{1,0}\Sigma$ the canonical bundle of 1-0 forms over the Riemann surface issued from $(\Sigma,g_{\vec{\Phi}})$ and denote by $P_{\vec{\Phi}}$ the L^2 projection orthogonal projection from $(\wedge^{(1,0)}\Sigma)^{\otimes^2}$ onto the space of *holomorphic quadratic forms* $\mathrm{Hol}_Q(\Sigma,g_{\vec{\Phi}})$ on $(\Sigma,g_{\vec{\Phi}})$ and by $P^\perp_{\vec{\Phi}} := \mathrm{Id}-P_{\vec{\Phi}}$. Define the linear map

$$\begin{array}{rcl} D^*_{\vec{\Phi}}\ :\ T_{\vec{\Phi}}\mathrm{Imm}(\Sigma,M^m) & \longrightarrow & W^{2,2}\left((\wedge^{(1,0)}\Sigma)^{\otimes^2}\right) \\[2mm] \vec{w} & \longrightarrow & P^\perp_{\vec{\Phi}}\left(\partial\vec{w}\,\dot{\otimes}\,\partial\vec{\Phi}\right) \end{array} ,$$

where in local complex coordinates for $\vec{\Phi}$ we denote

$$\partial\vec{w}\,\dot{\otimes}\,\partial\vec{\Phi} := \partial_z\vec{w}\cdot\partial_z\vec{\Phi}\ \ dz\otimes dz \quad .$$

We are now going to prove the following theorem

Theorem 1.2. Let $\vec{\Phi}\in\mathrm{Imm}(\Sigma,M^m)$, then there exists an open neighborhood $\mathcal{O}_{\vec{\Phi}}$ of $\vec{\Phi}$ in $\mathrm{Imm}(\Sigma,M^m)$ invariant under the action of $\mathrm{Diff}^*_+(\Sigma)$ and two smooth maps on $\mathcal{O}_{\vec{\Phi}}$, equivariant under the action of $\mathrm{Diff}^*_+(\Sigma)$,

$$\left\{\begin{array}{rcl} \vec{w}_{\vec{\Phi}}\ :\ \mathcal{O}_{\vec{\Phi}} & \longrightarrow & \mathrm{Ker}(D^*_{\vec{\Phi}})\subset T_{\vec{\Phi}}\mathrm{Imm}(\Sigma,M^m) \\[3mm] \Psi_{\vec{\Phi}}\ :\ \mathcal{O}_{\vec{\Phi}} & \longrightarrow & \mathrm{Diff}^*_+(\Sigma) \end{array}\right.$$

satisfying

$$\forall\, \vec{\Xi} \in \mathcal{O}_{\vec{\Phi}} \qquad \vec{\Xi} \circ \Psi_{\vec{\Phi}}(\vec{\Xi}) = \pi_{M^m}\left(\vec{\Phi} + \vec{w}_{\vec{\Phi}}(\vec{\Xi})\right),$$

where π_{M^m} is the orthogonal projection onto M^m and $\mathcal{T}_{\vec{\Phi}} := (\vec{w}_{\vec{\Phi}}, \Psi_{\vec{\Phi}})$ realizes a diffeomorphism from $\mathcal{O}_{\vec{\Phi}}$ onto $U_{\vec{\Phi}} \times \mathrm{Diff}^*_+(\Sigma)$, where $U_{\vec{\Phi}}$ is a neighborhood of 0 in $\mathrm{Ker}(D^*_{\vec{\Phi}})$. Moreover, the map $\mathcal{T}_{\vec{\Phi}}$ satisfies the following equivariance property: $\forall\, \vec{\Xi} \in \mathcal{O}_{\vec{\Phi}}$ and for all $\Psi_0 \in \mathrm{Diff}^*_+(\Sigma)$ one has

$$\Psi_{\vec{\Phi}}(\vec{\Xi} \circ \Psi_0) = \Psi_0^{-1} \circ \Psi_{\vec{\Phi}}(\vec{\Xi}) \quad \text{and} \quad \vec{w}_{\vec{\Phi}}(\vec{\Xi} \circ \Psi_0) = \vec{w}_{\vec{\Phi}}(\vec{\Xi}) \quad .$$

The space $\mathfrak{M}_\Sigma(M^m) := \mathrm{Imm}(\Sigma, M^m)/\mathrm{Diff}^*_+(\Sigma)$ is Hausdorff and defines a Hilbert manifold such that the projection map

$$\mathrm{Imm}(\Sigma, M^m) \longrightarrow \mathfrak{M}_\Sigma(M^m)$$

defines a $\mathrm{Diff}^*_+(\Sigma)$-bundle for which $(\mathcal{T}_{\vec{\Phi}})_{\vec{\Phi}}$ represents a local trivialization. □

Remark 1.1. The condition

$$D^*_{\vec{\Phi}} \vec{w} := P^\perp_{\vec{\Phi}}\left(\partial \vec{w} \,\dot{\otimes}\, \partial \vec{\Phi}\right) = 0$$

corresponds (Observe that $\partial \vec{w} \,\dot{\otimes}\, \partial \vec{\Phi} = 0$ is the condition which, starting from a conformal immersion $\vec{\Phi}$, preserves the conformality of $\vec{\Phi} + t\vec{w}$ at the 1st order for the same Riemman structure on Σ. Similarly, $d^*_A a = 0$ is the 1st-order condition, which, starting from a Coulomb gauge $d^*_A A = 0$, preserves the Coulomb condition for $A + ta$ for the same covariant co-differentiation d^*_A. Moreover, it is well known that the conformality of an immersion $\vec{\Phi}$ can be interpreted as a Coulomb condition (see for instance [12]) with respect to the action of the "gauge group" $\mathrm{Diff}^*_+(\Sigma)$.) to the Coulomb slice condition

$$d^*_A a = 0$$

in the gauge theory while studying the Hilbert bundle structure of the quotient of H^s connections by the Gauge group for $s > n/2$ and away from reducible connections (see [5]). □

Once this *Hilbert bundle* structure will be established we shall be considering the following application to the *viscosity method* for the area functional introduced by the author in [13].

For any immersion $\vec{\Phi} \in \mathrm{Imm}(\Sigma, M^m)$ we denote

$$F(\vec{\Phi}) := \int_\Sigma \left[1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^2 \, dvol_{g_{\vec{\Phi}}},$$

where $\vec{\mathbb{I}}_{\vec{\Phi}} = \pi_{\vec{n}}(\nabla^M d\vec{\Phi})$ is the 2nd fundamental form of the immersion $\vec{\Phi}$ in M^m.

Observe that

$$\forall\, b \in \mathbb{N} \qquad \exists\, C_b > 0 \qquad F(\vec{\Phi}) < C_b \quad \Longrightarrow \quad \mathrm{b}\,(\Sigma) \le b. \tag{1.1}$$

This is a direct consequence of *Gauss–Bonnet theorem* and *Cauchy–Schwartz inequality*.

It is clear that $F(\vec{\Phi})$ only depends on the equivalence class $[\vec{\Phi}]$ of $\vec{\Phi}$ in $\mathfrak{M}_\Sigma(M^m)$. Since F is a smooth functional on $\mathrm{Imm}(\Sigma, M^m)$ (see [13]) it descends to a smooth functional on $\mathfrak{M}_\Sigma(M^m)$. We shall prove the following theorem.

Proposition 1.1. Let $[\vec{\Phi}]$ be a critical point of F in $\mathfrak{M}(M^m)$. Then the 2nd derivative of F at $[\vec{\Phi}]$ defines a Fredholm and elliptic operator. □

The *viscosity method* consists in studying the variations of the *area Lagrangian*

$$\mathrm{Area}(\vec{\Phi}) = \int_\Sigma dvol_{g_{\vec{\Phi}}}$$

by considering relaxations of the form

$$A^\sigma(\vec{\Phi}) := \mathrm{Area}(\vec{\Phi}) + \sigma^2 F(\vec{\Phi}) = \mathrm{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \left[1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2\right]^2 \, dvol_{g_{\vec{\Phi}}},$$

where $\sigma > 0$. The work [13] has been devoted to the asymptotic analysis of sequences of critical points of A^{σ_k}, with uniformly bounded A^{σ_k} energy and satisfying *Struwe's entropy condition*

$$\sigma_k^2 F(\vec{\Phi}_k) = o\left(\frac{1}{\log \sigma_k^{-1}}\right) \qquad \text{as } \sigma_k \text{ goes to zero.} \tag{1.2}$$

It is proved in these two works that, modulo extraction of a subsequence, the immersions $\vec{\Phi}_k$ *varifold converges* toward a 2D *integer rectifiable stationary varifold* $\mathbf{v}_\infty$ of M^m that is parametrized. In [14] and [10] the regularity of the *parametrized integer rectifiable stationary varifold* is established. Hence, we have the following theorem.

Theorem 1.3. [[13], [14], [10]] Let $\vec{\Phi}_k$ be a sequence of immersions of a closed surface Σ, critical points of A^{σ_k} and such that

$$\limsup_{k\to+\infty} A^{\sigma_k}(\vec{\Phi}_k) < +\infty \quad \text{and} \quad \sigma_k^2 \int_{\Sigma^g} (1 + |\vec{\mathbb{I}}_{\vec{\Phi}_k}|^2)^2 \, dvol_{g_{\vec{\Phi}_k}} = o\left(\frac{1}{\log \sigma_k^{-1}}\right).$$

Then there exists a subsequence $\vec{\Phi}_{k'}$ such that the corresponding associated varifold (The associated varifold $\mathbf{v}$ associated to an immersion $\vec{\Phi}$ of Σ^g is given by

$$\forall\, \phi \in C^0(G_2TM^m) \quad \mathbf{v}(\phi) := \int_{\Sigma^g} \phi(\vec{\Phi}_* T_x \Sigma^g) \, dvol_{\vec{\Phi}^* g_{M^m}})$$

$\mathbf{v}_k$ converges toward the varifold $\mathbf{v}_\infty$ associated to a smooth, possibly branched, conformal minimal immersion $\vec{\Psi}_\infty$ of a constant Gauss curvature nodal surface (S_∞, h) equipped with a locally constant odd multiplicity $N_\infty \in C^\infty(S_\infty, 2\,\mathbb{N}+1)$ and such that

$$\text{genus}\,(S_\infty) \le g \quad \text{and} \quad \lim_{k\to+\infty} \text{Area}(\vec{\Phi}_k) = \frac{1}{2}\int_{S_\infty} N_\infty \, |d\vec{\Psi}_\infty|_h^2 \, dvol_h.$$

□

The question of comparing the *Morse index* of the limiting surface for the area with the *Morse index* of the sequence $\vec{\Phi}_k$ for the relaxed functionals A^{σ_k} was left open in these works. The 2nd main result of the present work is the following *lower semi-continuity* of the index.

Theorem 1.5. Let $\vec{\Phi}_k$ be a sequence of immersions of a closed surface Σ, critical points of A^{σ_k} and such that

$$\limsup_{k\to+\infty} A^{\sigma_k}(\vec{\Phi}_k) < +\infty \quad \text{and} \quad \sigma_k^2 \int_{\Sigma^g} (1 + |\vec{\mathbb{I}}_{\vec{\Phi}_k}|^2)^2 \, dvol_{g_{\vec{\Phi}_k}} = o\left(\frac{1}{\log \sigma_k^{-1}}\right).$$

Then there exists a subsequence $\vec{\Phi}_{k'}$ such that the corresponding immersed surface converges in varifolds toward a parametrized integer rectifiable stationary varifold $\mathbf{v}_\infty := (S_\infty, \vec{\Psi}_\infty, N_\infty)$. If $N_\infty \equiv 1$ then we have

$$\text{Ind}(\vec{\Psi}_\infty) \le \liminf_{k\to\infty} \text{Ind}^{\sigma_{k'}}(\vec{\Phi}_{k'}), \tag{1.3}$$

where $\mathrm{Ind}(\vec{\Psi}_\infty)$ is the maximal dimension of a subspace of $T_{[\vec{\Psi}_\infty]}\mathfrak{M}$ on which $D^2\mathrm{Area}(\vec{\Psi}_\infty)$ is strictly negative and $\mathrm{Ind}^{\sigma_k}(\vec{\Phi}_k)$ is the maximal dimension of a subspace of $T_{[\vec{\Phi}_k]}\mathfrak{M}$ on which $D^2A^{\sigma_k}(\vec{\Phi}_k)$ is strictly negative. □

Remark 1.2. After the present work has been completed, the author in collaboration with Alessandro Pigati proved that the condition $N_\infty \equiv 1$ always holds for the varifold limit of sequences of critical points of F_{σ_k} satisfying the entropy condition (2) (see [9]). Hence, the lower semi-continuity of the index always holds. Combining this result with the main theorem of [7] the authors in [9] establish that the Morse index of any minimal surface realizing the minmax of a k-dimensional homological (or cohomological) family obtained by the viscosity method is bounded by k. □

The paper is organized as follows. In the next section we are proving Theorem 1.2 from which we deduce Theorem 1.1. In a short intermediate section we establish Proposition 1.3. Then, in Section 4, we are proving the lower semi-continuity of the index in the *viscosity method* (i.e., Theorem 1.5).

2 A Proof of Theorem 1.2.

Let Σ^g be a closed connected oriented surface of genus g. Let $\mathrm{Diff}_+(\Sigma^g)$ be the topological group of positive $W^{3,2}$-diffeomorphisms of Σ, isotopic to the identity. This can be seen as an open subspace of $W^{3,2}(\Sigma,\Sigma)$ that itself defines a Hilbert manifold (see [11]). For $g = 0$ we are marking three distinct points, which we denote a_1, a_2, a_3, for $g = 1$ we are marking one point that we denote a, and for $g > 1$ no point is marked. We denote by $\mathrm{Diff}^*_+(\Sigma^g)$ the subgroup of $\mathrm{Diff}_+(\Sigma^g)$, which is fixing the marked points. In particular for $g > 1$ we have $\mathrm{Diff}^*_+(\Sigma^g) = \mathrm{Diff}_+(\Sigma^g)$. We have the following lemma.

Lemma 2.1. The action of $\mathrm{Diff}^*_+(\Sigma^g)$ on $\mathrm{Imm}(\Sigma^g, M^m)$ is free. □

Proof of Lemma 2.1. We first claim that every element in $\mathrm{Diff}^*_+(\Sigma^g)$ possess at least one fixed point. This is included in the definition for $g = 0, 1$. For $g > 1$ we have that for any diffeomorphism Ψ isotopic to the identity the *Lefschetz number* $L(\Psi)$ is given by definition by

$$L(\Psi) = \mathrm{Tr}(\Psi|H_0(\Sigma^g)) - \mathrm{Tr}(\Psi|H_1(\Sigma^g)) + \mathrm{Tr}(\Psi|H_2(\Sigma^g)) = 2 - 2g.$$

Hence, for $g > 1$ we have $L(\Psi) \neq 0$ and then Ψ must have at least one fixed point. Due to Lemma 1.3 in [3] we deduce that for any $g \in \mathbb{N}$ the action of $\mathrm{Diff}_+^*(\Sigma^g)$ on $\mathrm{Imm}(\Sigma^g, M^m)$ is *free*. $\square$

Proof of Theorem 1.2. Let $\vec{\Phi} \in \mathrm{Imm}(\Sigma, M^m)$. A basis of neighborhoods of $\vec{\Phi}$ is given by

$$\mathcal{V}^{\varepsilon}_{\vec{\Phi}} := \left\{ \vec{\Xi} = \pi_{M^m}\left(\vec{\Phi} + \vec{v}\right) \quad ; \quad \vec{v} \in \Gamma^{3,2}(\vec{\Phi}^* TM^m) \ \text{ and } \|\vec{v}\|_{W^{3,2}} < \varepsilon \right\},$$

for $\varepsilon > 0$ small enough and where $\Gamma^{3,2}(\vec{\Phi}^* TM^m)$ denotes the $W^{3,2}$-sections of the pullback bundle $\vec{\Phi}^* TM^m$, which is the subvector space of $\vec{v} \in W^{3,2}(\Sigma, \mathbb{R}^Q)$ such that $\vec{v}(x) \in T_{\vec{\Phi}(x)}\Sigma^g$ for any $x \in \Sigma^g$.

For any $\vec{v} \in \Gamma^{3,2}(\vec{\Phi}^* TM^m)$ we consider the tensor in $\Gamma^{2,2}((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)})$ given by

$$\overline{D}^*_{\vec{\Phi}}\vec{v} \llcorner g^{-1}_{\vec{\Phi}} \quad \text{where} \quad g^{-1}_{\vec{\Phi}} = e^{-2\lambda}\,[\partial_z \otimes \partial_{\overline{z}} + \partial_{\overline{z}} \otimes \partial_z],$$

where

$$\overline{D}^*_{\vec{\Phi}}\vec{v} = \overline{P}^{\perp}_{\vec{\Phi}}\left(\overline{\partial}\vec{v} \,\dot{\otimes}\, \overline{\partial}\vec{\Phi}\right).$$

$\overline{P}_{\vec{\Phi}}$ is the L^2 projection orthogonal projection from $(\wedge^{(0,1)}\Sigma)^{\otimes^2}$ onto the space of *anti-holomorphic quadratic forms* $\mathrm{AHol}_Q(\Sigma, g_{\vec{\Phi}})$ on $(\Sigma, g_{\vec{\Phi}})$ and by $\overline{P}^{\perp}_{\vec{\Phi}} := \mathrm{Id} - \overline{P}_{\vec{\Phi}}$, and where $\llcorner$ is the contraction operator between covariant and contravariant tensors. In particular we have in local complex coordinates

$$(b\ d\overline{z} \otimes d\overline{z}) \llcorner g^{-1}_{\vec{\Phi}} = e^{-2\lambda}\ b\ d\overline{z} \otimes \partial_z.$$

We denote

$$\mathcal{I} := \left\{ \overline{D}^*_{\vec{\Phi}}\vec{v} \llcorner g^{-1}_{\vec{\Phi}} \ ; \quad \vec{v} \in \Gamma^{3,2}(\vec{\Phi}^* TM^m) \right\}. \tag{2.4}$$

Recall the definition of the $\overline{\partial}$ operator on $\wedge^{(1,0)}\Sigma^g$ given in local coordinates by

$$\overline{\partial}\left(a\,\partial_z\right) = \partial_{\overline{z}} a\ d\overline{z} \otimes \partial_z.$$

Denote $\mathrm{Hol}_1(\Sigma^g)$ the finite-dimensional subspace of $\Gamma^{3,2}(\wedge^{(1,0)}\Sigma^g)$ made of holomorphic sections (Due to the Riemann–Hurwitz theorem, the holomorphic tangent bundle $T^{(1,0)}\Sigma^g$, which is the inverse of the canonical bundle of the Riemann surface defined by $(\Sigma, g_{\vec{\Phi}})$, has a degree given by

$$\deg\left(T^{(1,0)}\Sigma^g\right) = \int_{\Sigma^g} c_1\left(T^{(1,0)}\Sigma^g\right) = 2 - 2g;$$

therefore, $\mathrm{Hol}_1(\Sigma^g) \neq 0 \Rightarrow g < 2$.) of $T^{(1,0)}\Sigma^g$. We shall now prove the following lemma.

Lemma 2.2. Under the previous notations we have that

$$\forall\, \vec{v} \in W^{3,2}(\Sigma, \mathbb{R}^Q) \quad \exists!\, f \in (\mathrm{Hol}_1(\Sigma^g))^{\perp} \cap \Gamma^{3,2}(\wedge^{(1,0)}\Sigma^g) \qquad \text{s. t.} \quad \overline{\partial} f = \overline{D}^*_{\vec{\Phi}}\vec{v} \llcorner g_{\vec{\Phi}}^{-1}. \tag{2.5}$$

Moreover,

$$\|f\|_{W^{3,2}} \leq C_{\vec{\Phi}}\ \|\vec{v}\|_{W^{3,2}}. \tag{2.6}$$

□

Proof of Lemma 2.2. We have for any $\alpha = a\,\partial_z \in \Gamma^{3,2}((T\Sigma^g)^{1,0})$ and $\beta = b\, d\overline{z}\otimes\partial_z \in \Gamma^{2,2}((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)})$

$$\begin{aligned}
\int_{\Sigma^g} \left\langle \overline{\partial}\left(a\,\partial_z\right), b\ \mathrm{d}\overline{z} \otimes \partial_z\right\rangle_{g_{\vec{\Phi}}}\ \mathrm{dvol}_{g_{\vec{\Phi}}} &= \Re\left[\frac{i}{2}\int_{\Sigma^g} \partial\overline{a}\ b\ e^{2\lambda}\ \mathrm{d}z \wedge \mathrm{d}\overline{z}\right]\\
&= \Re\left[\frac{i}{2}\int_{\Sigma^g} \mathrm{d}[\overline{a}\ b\ e^{2\lambda}] \wedge \mathrm{d}\overline{z}\right] - \Re\left[\frac{i}{2}\int_{\Sigma^g} \overline{a}\ \partial_z[b\ e^{2\lambda}]\ \mathrm{d}z \wedge \mathrm{d}\overline{z}\right]\\
&= \int_{\Sigma^g} \left\langle \alpha, \left(\partial\left(\beta \llcorner g_{\vec{\Phi}}\right)\llcorner_2\ g_{\vec{\Phi}}^{-1}\right)\llcorner g_{\vec{\Phi}}^{-1}\right\rangle_{g_{\vec{\Phi}}}\ \mathrm{dvol}_{g_{\vec{\Phi}}},
\end{aligned}$$

where

$$g_{\vec{\Phi}}^{-1} = e^{-2\lambda}\ [\partial_z \otimes \partial_{\overline{z}} + \partial_{\overline{z}} \otimes \partial_z] \quad , \quad (b\ \mathrm{d}z \otimes \mathrm{d}\overline{z} \otimes \mathrm{d}\overline{z})\llcorner_2\, g_{\vec{\Phi}}^{-1} = e^{-2\lambda}\ b\ \mathrm{d}\overline{z}, \quad \text{and}$$
$$(e^{-2\lambda}\ b\ \mathrm{d}\overline{z})\llcorner g_{\vec{\Phi}}^{-1} = e^{-4\lambda}\ b\ \partial_z,$$

where $\llcorner$ and $\llcorner_2$ are, respectively, again the simple and double contractions between covariant and contravariant tensors. Hence, we have proved that the adjoint of $\overline{\partial}$ on $\Gamma((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)})$ is given by

$$\overline{\partial}^*\ :\ \beta \in \Gamma((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)}) \longrightarrow \overline{\partial}^*\beta = \left(\partial\left(\beta\llcorner g_{\vec{\Phi}}\right)\llcorner_2\ g_{\vec{\Phi}}^{-1}\right)\llcorner g_{\vec{\Phi}}^{-1} \in \Gamma(\wedge^{(1,0)}\Sigma^g).$$

We have $\mathrm{Im}\,\overline{\partial} = (\mathrm{Ker}\,\overline{\partial}^*)^{\perp}$. We have that

$$\mathrm{Ker}\,\overline{\partial}^* = \left\{\beta \in \Gamma((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)}) \quad ; \quad \beta\llcorner g_{\vec{\Phi}} \in \mathrm{AHol}_Q(\Sigma^g, g_{\vec{\Phi}}))\right\}. \tag{2.7}$$

Observe that

$$\forall\,\gamma \in \Gamma\left(\left((T^*\Sigma^g)^{(0,1)}\right)^{\otimes^2}\right)\ \forall\,\beta \in \Gamma((T^*\Sigma^g)^{(0,1)} \otimes (T\Sigma^g)^{(1,0)}) \quad \left\langle\gamma\llcorner g_{\vec{\Phi}}^{-1}, \beta\right\rangle_{g_{\vec{\Phi}}} = \langle\gamma, \beta\llcorner g_{\vec{\Phi}}\rangle_{g_{\vec{\Phi}}}.$$

This implies

$$\gamma \llcorner g_{\vec{\Phi}}^{-1} \in \operatorname{Im}\overline{\partial} \quad \Longleftrightarrow \quad \gamma \in \left(\mathrm{AHol}_Q(\Sigma^g, g_{\vec{\Phi}})\right)^{\perp}. \tag{2.8}$$

We deduce (2.5) from (2.8) and (2.6) follows by classical estimates for elliptic complexes in Sobolev spaces. □

Continuation of the proof of Theorem 1.2. To $f \in (\mathrm{Hol}_1(\Sigma^g))^{\perp} \cap \Gamma^{3,2}(\wedge^{(1,0)}\Sigma^g)$ solving (2.5) we assign

$$X := 2\,\Re(f) = 2\,\Re((f_1 + i f_2)\,\partial_z) = (f_1\,\partial_{x_1} + f_2\,\partial_{x_2}) = X_1\,\partial_{x_1} + X_2\,\partial_{x_2}.$$

Observe that, if we denote $\vec{X} := \mathrm{d}\vec{\Phi}\cdot X$, we have

$$\overline{\partial}\left(\vec{X}\cdot\overline{\partial}\vec{\Phi}\llcorner g_{\vec{\Phi}}^{-1}\right) = \overline{\partial}\left(e^{2\lambda}(X_1 + iX_2)\,d\overline{z}\llcorner g_{\vec{\Phi}}^{-1}\right) = \overline{\partial} f.$$

Observe also that, since $\vec{X}$ is tangent to the immersion $\vec{X}\cdot\overline{\partial}\left(\overline{\partial}\vec{\Phi}\llcorner g_{\vec{\Phi}}^{-1}\right) = 0$. Indeed in local conformal coordinates we have

$$\overline{\partial}\left(\overline{\partial}\vec{\Phi}\llcorner g_{\vec{\Phi}}^{-1}\right) = \partial_{\overline{z}}(e^{-2\lambda}\,\partial_{\overline{z}}\vec{\Phi})\;\mathrm{d}\overline{z}\otimes\partial_z,$$

and $\vec{h}^0 := \partial_{\overline{z}}(e^{-2\lambda}\,\partial_{\overline{z}}\vec{\Phi})\;\mathrm{d}\overline{z}\otimes\mathrm{d}\overline{z} = e^{-2\lambda}\,\pi_{\perp}(\partial^2_{\overline{z}^2}\vec{\Phi})\;\mathrm{d}\overline{z}\otimes\mathrm{d}\overline{z}$ is nothing but the trace-free part of the 2nd fundamental form (We denote by $\pi_\perp$ the orthogonal projection onto $(\vec{\Phi}_* T\Sigma^g)^{\perp}$ in $T\mathbb{R}^Q$.) of the immersion viewed as an immersion into $\mathbb{R}^Q$ and by which is then orthogonal to the immersion. Hence,

$$\overline{\partial} f = (\overline{\partial}\vec{X}\cdot\overline{\partial}\vec{\Phi})\llcorner g_{\vec{\Phi}}^{-1}.$$

Using $\operatorname{Im}\overline{\partial} = (\operatorname{Ker}\overline{\partial}^*)^{\perp}$ and the characterization of $\operatorname{Ker}\overline{\partial}^*$ given by (4), we have $\overline{\partial}\vec{X}\cdot\overline{\partial}\vec{\Phi} = P^{\perp}_{\vec{\Phi}}(\overline{\partial}\vec{X}\cdot\overline{\partial}\vec{\Phi})$ and hence

$$\overline{\partial} f = \overline{D}^*_{\vec{\Phi}}\vec{X}\llcorner g_{\vec{\Phi}}^{-1}. \tag{2.9}$$

We denote

$$\begin{cases} \mathcal{X}^{3,2}(S^2) := \left\{X \in \Gamma^{3,2}(TS^2)\ ;\ X(a_i) = 0 \quad i = 1,2,3\right\}, \\[2mm] \mathcal{X}^{3,2}(T^2) := \left\{X \in \Gamma^{3,2}(TT^2)\ ;\ X(a) = 0\right\}, \\[2mm] \mathcal{X}^{3,2}(\Sigma^g) = \Gamma^{3,2}(\Sigma^g) \quad \text{for} g > 1. \end{cases}$$

The space of the holomorphic vector field on $T^{(1,0)}S^2$ is a 3D complex vector space given in $\mathbb{C}$, after the stereographic projection, by

$$h(z) = (\alpha + \beta\,z + \gamma\,z^2)\;\partial_z \quad \text{where}(\alpha,\beta,\gamma) \in \mathbb{C}^3.$$

Whereas the space of the holomorphic vector field on $T^{(1,0)}T^2$ is a one-dimensional complex vector space given in $\mathbb{C}$ by

$$h(z) = \alpha\ \partial_z \quad \text{where } \alpha \in \mathbb{C},$$

while for $g > 1$ we have $\mathrm{Hol}_1(\Sigma^g) = \{0\}$. Hence, for any $g \in \mathbb{N}$ and any

$$f \in (\mathrm{Hol}_1(\Sigma^g))^{\perp} \cap \Gamma^{3,2}(\wedge^{(1,0)}\Sigma^g) \quad \exists\,!\ h_f \in \mathrm{Hol}_1(\Sigma^g) \quad \text{s. t.} \quad \Re(f + h_f) \in \mathcal{X}^{3,2}(\Sigma^g); \tag{2.10}$$

moreover, the map $f \to h_f$ from $(\mathrm{Hol}_1(\Sigma^g))^{\perp} \cap \Gamma^{3,2}(\wedge^{(1,0)}\Sigma^g)$ into $\mathrm{Hol}_1(\Sigma^g)$ is linear and smooth. Hence, we can summarize what we have proved so far in the following lemma.

Lemma 2.3. Let $\vec{\Phi}$ be a $W^{3,2}$-immersion. Then the following holds:

$$\forall\, \vec{v} \in \Gamma^{3,2}(\vec{\Phi}^*TM^m) \quad \exists\,!\ X \in \mathcal{X}^{3,2}(\Sigma^g) \quad \text{s. t.}$$
$$\overline{\partial}\left(X - i\,X^{\perp}\right) = \overline{D}^*_{\vec{\Phi}}\vec{X} \llcorner g^{-1}_{\vec{\Phi}} = \overline{D}^*_{\vec{\Phi}}\vec{v} \llcorner g^{-1}_{\vec{\Phi}},$$

where $\vec{X} = d\vec{\Phi}\cdot X$ and such that

$$\|X\|_{W^{3,2}} \leq C_{\vec{\Phi}}\ \|\vec{v}\|_{W^{3,2}}.$$

□

End of the proof of Theorem 1.2. Let g_0 be a smooth reference metric on Σ^g and denote by $\exp^{g_0}$ the smooth exponential map from $T\Sigma$ into Σ associated to g_0. Let $\varepsilon > 0$ be small and denote

$$\mathcal{X}^{3,2}_{\varepsilon}(\Sigma^g) := \left\{X \in \mathcal{X}^{3,2}(\Sigma^g)\ ;\ \|X\|_{W^{3,2}} < \varepsilon\right\}$$

and

$$\mathcal{D}_{\varepsilon} := \left\{\Psi \in \mathrm{Diff}^*_+(\Sigma)\ ; \quad \exists X \in \mathcal{X}^{3,2}_{\varepsilon}(\Sigma^g) \quad \text{s.t} \quad \Psi(x) = \exp^{g_0}_x(X(x))\right\}.$$

We define

$$\begin{aligned} \Lambda_{\vec{\Phi}}\ :\ \mathcal{V}^{\varepsilon}_{\vec{\Phi}} \times \mathcal{D}_{\varepsilon} \quad &\longrightarrow \quad \Gamma^{2,2}((T^*\Sigma)^{(0,1)} \otimes (T^*\Sigma)^{(0,1)}) \\ (\vec{\Xi}, \Psi) \quad &\longrightarrow \quad \overline{D}^*_{\vec{\Phi}}\left(\vec{\Xi}\circ\Psi\right) \llcorner g^{-1}_{\vec{\Phi}}. \end{aligned}$$

The map is clearly C^1 and Lemma 2.3 gives that

$$\partial_\Psi \Lambda_{\vec{\Phi}}\Big|_{(\vec{\Phi},0)} \cdot X = \overline{D}^*_{\vec{\Phi}}\left(d\vec{\Phi}\cdot X\right) \llcorner g^{-1}_{\vec{\Phi}}$$

realizes an isomorphism between $\mathcal{X}^{3,2}$ and $\mathcal{I}$ (defined in (1)). The implicit function theorem gives then the existence of a C^1 map $\Psi_{\vec{\Phi}}(\vec{\Xi})$ defined in a neighborhood of $\vec{\Phi}$ such that

$$\overline{D}^*_{\vec{\Phi}}\left(\vec{\Xi}\circ\Psi_{\vec{\Phi}}(\vec{\Xi})\right)\llcorner g^{-1}_{\vec{\Phi}} = 0,$$

and we denote $\vec{w}_{\vec{\Phi}}(\vec{\Xi}) := \vec{\Xi}\circ\Psi_{\vec{\Phi}}(\vec{\Xi}) - \vec{\Phi}$.

For any element $\Psi_0 \in \mathrm{Diff}^*_+(\Sigma)$ close to the identity and $\vec{\Xi}$ close enough to $\vec{\Phi}$ one has trivially

$$\overline{D}^*_{\vec{\Phi}}\left(\vec{\Xi}\circ\Psi_0\circ\Psi_0^{-1}\circ\Psi_{\vec{\Phi}}(\vec{\Xi})\right)\llcorner g^{-1}_{\vec{\Phi}} = 0.$$

Because of the local uniqueness of $\Psi_{\vec{\Phi}}(\vec{\Xi})$ given by the implicit function theorem, we deduce the equivariance property

$$\Psi_{\vec{\Phi}}(\vec{\Xi}\circ\Psi_0) = \Psi_0^{-1}\circ\Psi_{\vec{\Phi}}(\vec{\Xi}) \quad \text{and} \quad \vec{w}_{\vec{\Phi}}(\vec{\Xi}\circ\Psi_0) := \vec{\Xi}\circ\Psi_{\vec{\Phi}}(\vec{\Xi}) - \vec{\Phi} = \vec{w}_{\vec{\Phi}}(\vec{\Xi}).$$

We then naturally extend, by equivariance, the map $\mathcal{T}_{\vec{\Phi}} := (\vec{w}_{\vec{\Phi}}, \Psi_{\vec{\Phi}})$ on a neighborhood $\mathcal{O}_{\vec{\Phi}}$ of $\vec{\Phi}$ invariant under the action of $\mathrm{Diff}^*_+(\Sigma)$.

We are now proving the Hausdorff property for $\mathfrak{M}_g(\Sigma^g, M^m) := \mathrm{Imm}(\Sigma^g, M^m)/\mathrm{Diff}^*_+(\Sigma^g)$. Following classical considerations (see the arguments in [15, proof of Lemma 2.9.9]) it suffices to prove that

$$\Gamma := \left\{(\vec{\Phi}, \vec{\Phi}\circ\Psi) \quad ; \quad \vec{\Phi}\in \mathrm{Imm}(\Sigma^g, M^m) \text{ and } \Psi\in\mathrm{Diff}^*_+(\Sigma^g)\right\}$$

is closed in $(\mathrm{Imm}(\Sigma^g, M^m))^2$. This follows from the 1st part of the proof of the theorem. Let $(\vec{\Phi}_k, \vec{\Xi}_k := \vec{\Phi}_k\circ\Psi_k) \to (\vec{\Phi}_\infty, \vec{\Xi}_\infty)$ in $W^{3,2}$. For k large enough both $\vec{\Phi}_k$ and $\vec{\Xi}_k$ are included in $\mathcal{O}_{\vec{\Phi}_\infty}$. Because of the continuity of the map $\vec{w}_{\vec{\Phi}_\infty}$ we have, respectively,

$$\vec{w}_{\vec{\Phi}_\infty}(\vec{\Phi}_k) \to \vec{w}_{\vec{\Phi}_\infty}(\vec{\Phi}_\infty) = 0 \quad \text{and} \quad \vec{w}_{\vec{\Phi}_\infty}(\vec{\Xi}_k) \to \vec{w}_{\vec{\Phi}_\infty}(\vec{\Xi}_\infty).$$

The equivariance of $\vec{w}_{\vec{\Phi}_\infty}$ gives $\vec{w}_{\vec{\Phi}_\infty}(\vec{\Xi}_k) = \vec{w}_{\vec{\Phi}_\infty}(\vec{\Phi}_k)$, hence $\vec{w}_{\vec{\Phi}_\infty}(\vec{\Xi}_\infty) = 0$. Thus, $\vec{\Xi}_\infty\circ\Psi_{\vec{\Phi}_\infty} = \vec{\Phi}_\infty$ and this shows that Γ is closed and then $\mathfrak{M}(\Sigma^g, M^m)$ defines a Hausdorff *Hilbert manifold* and Theorem 1.2 is proved. □

3 A Proof of Proposition 1.1

From [6] (see also an alternative approach in [1]) we know that under the assumptions that $\vec{\Phi}$ is a critical point of F, it defines a smooth immersion in conformal coordinates. We shall be working in the chart in the neighborhood of $[\vec{\Phi}]$ in $\mathfrak{M}(\Sigma^g, M^m)$ given by $\vec{w}_{\vec{\Phi}}$ from Theorem 1.2. In other words we identify

$$T_{[\vec{\Phi}]}\mathfrak{M} \simeq \left\{\vec{w} \in \Gamma^{3,2}(\vec{\Phi}^* TM^m) \quad ; \quad P^{\perp}_{\vec{\Phi}}\left(\overline{\partial}\vec{w} \dot{\otimes} \overline{\partial}\vec{\Phi}\right) = 0\right\}. \tag{3.11}$$

For such a $\vec{w}$ we denote by $q_{\vec{w}}$ the holomorphic quadratic form given by

$$\overline{\partial}\vec{w} \dot{\otimes} \overline{\partial}\vec{\Phi} = q_{\vec{w}}.$$

After contracting with the tensor $g^{-1}_{\vec{\Phi}}$, this equation becomes

$$\overline{\partial}\left(\vec{w} \cdot \overline{\partial}\vec{\Phi} \llcorner g^{-1}_{\vec{\Phi}}\right) = -\pi_{\perp}(\vec{w}) \cdot \vec{h}^0 + q_{\vec{w}} \llcorner g^{-1}_{\vec{\Phi}},$$

where we recall that $\pi_{\perp}$ is the orthogonal projection onto the normal space to $\vec{\Phi}_* T\Sigma$ in $T_{\Phi}\mathbb{R}^Q$ and $\vec{h}^0$ is the trace-free part of the 2nd fundamental form of the immersion in $\mathbb{R}^Q$, which is orthogonal to the tangent plane of the immersion and given in local coordinates by

$$\vec{h}^0_{\vec{\Phi}} = \partial_{\overline{z}}\left(e^{-2\lambda}\partial_{\overline{z}}\vec{\Phi}\right) \, \mathrm{d}\overline{z} \otimes \partial_z.$$

Observe that since $\vec{w}$ is tangent to $T_{\vec{\Phi}}M^m$ we have $\pi_{\perp}(\vec{w}) = \pi_{\vec{n}}(\vec{w})$ where $\pi_{\vec{n}}$ is the orthogonal projection onto the normal space to $\vec{\Phi}_* T\Sigma$ in $T_{\vec{\Phi}}M^m$. Using the characterization of $\mathrm{Im}\,\overline{\partial} = (\mathrm{Ker}\,\overline{\partial}^*)^{\perp}$ given by (5) we deduce

$$\overline{\partial}\left(\vec{w} \cdot \overline{\partial}\vec{\Phi} \llcorner g^{-1}_{\vec{\Phi}}\right) = -\mathfrak{P}_{\vec{\Phi}}\left(\pi_{\vec{n}}(\vec{w}) \cdot \vec{h}^0_{\vec{\Phi}}\right),$$

where $\mathfrak{P}_{\vec{\Phi}}$ is the orthogonal projection onto $(\mathrm{Hol}_Q(\Sigma^g, g_{\vec{\Phi}}) \llcorner g^{-1}_{\vec{\Phi}})^{\perp}$. Denote $\vec{X}_{\vec{w}}$ the projection of $\vec{w}$ onto the tangent plane (i.e., $\vec{X}_{\vec{w}} = \vec{w} - \pi_{\vec{n}}(\vec{w})$) and let $X_{\vec{w}}$ be the vector field on Σ such that $\mathrm{d}\vec{\Phi} \cdot X_{\vec{w}} = \vec{X}_{\vec{w}}$. Following the computations from the previous subsection we deduce

$$\overline{\partial}\left(X_{\vec{w}} - i X^{\perp}_{\vec{w}}\right) = -\mathfrak{P}_{\vec{\Phi}}\left(\pi_{\vec{n}}(\vec{w}) \cdot \vec{h}^0_{\vec{\Phi}}\right). \tag{3.12}$$

Denote

$$\pi_T \;:\; \Gamma^{3,2}(\vec{\Phi}^*TM^m) \longrightarrow \Gamma^{3,2}((T\Sigma)^{(1,0)})$$
$$\vec{w} \longrightarrow X_{\vec{w}} - i\, X_{\vec{w}}^{\perp}.$$

In view of the expression of the 2nd derivative D^2F given by (20) we have that, modulo compact operators (remembering that $\vec{\Phi}$ is smooth), we are reduced (The sum of a *Fredholm operator* with a *compact operator* is *Fredholm*.) to study the Fredholm nature of the operator generated by the following quadratic form:

$$Q_{\vec{\Phi}}(\vec{w}) = \int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})\, \left|\pi_{\vec{n}}\left(D^{g_{\vec{\Phi}}} d\vec{w}\right)\right|^2_{g_{\vec{\Phi}}}\, dvol_{g_{\vec{\Phi}}} + 2\int_\Sigma \left|\left\langle \vec{\mathbb{I}}_{\vec{\Phi}}, D^{g_{\vec{\Phi}}} d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\right|^2 dvol_{g_{\vec{\Phi}}}$$

combined with (2). Hence, the symbols of the generated operator, in local conformal coordinates, is given by

$$\left\{\begin{array}{l} 2e^{-2\lambda}\pi_{\vec{n}} \circ \left[(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})|\xi|^4 + 2e^{-4\lambda}\sum_{i,j,k,l} \vec{\mathbb{I}}_{kl} \otimes \vec{\mathbb{I}}_{kl}\, \xi_i\, \xi_j\, \xi_k\, \xi_l\right] \circ \pi_{\vec{n}} \\ \\ (\xi_1 + i\,\xi_2) \circ \pi_T. \end{array}\right.$$

This is clearly the symbol defining an *elliptic operator* on $\Gamma^{3,2}(\vec{\Phi}^*TM^m)$ and D^2F is *Fredholm* on $T_{[\vec{\Phi}]}\mathfrak{M}$. This concludes the proof of Proposition 1.3.

4 A Proof of Theorem 1.4, the Lower Semi-continuity of the Index

We shall assume that Σ^g is connected. We shall present the computations for $M^m = S^m$. The general constraint generates lower-order terms whose abundance could mask the true reason why the theorem is true whereas the same terms in the $M^m = S^m$ case are easier to present. The 1st part of the theorem is the main results of [13]. It remains to prove the inequality (3) under the assumption of Theorem 1.5. The 1st derivative of the area of an immersion (possibly branched) of a closed surface Σ^g into $\mathbb{R}^Q$ is given by (see [13])

$$D\mathrm{Area}(\vec{\Phi}) \cdot \vec{w} = \int_{\Sigma^g} \left\langle d\vec{\Phi}\,;\, d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\, \mathrm{dvol}_{g_{\vec{\Phi}}}$$

and the 2nd derivative (A reader familiar to the rich literature in geometry on minimal surface theory in three dimensions might not immediately recognize the most commonly used expression of the 2nd derivative of the area by the mean of the *Jacobi field*. This classical presentation of D^2Area has the advantage to "reduce" this operator to an operator on function by introducing the decomposition $\vec{w} = w\,\vec{n}$. This decomposition however is not "analytically" favorable since $\vec{n}$ has *a priori* one degree of regularity less than $\vec{w}$. This observation is at the base of the analysis of the *Willmore functional* as it has been developed by the author in the recent years.)

$$D^2\text{Area}(\vec{\Phi})\cdot(\vec{w},\vec{w}) = \int_\Sigma \left[\left\langle d\vec{w}\,;\,d\vec{w}\right\rangle_{g_{\vec{\Phi}}} + \left|\left\langle d\vec{\Phi}\,;\,d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\right|^2 - 2^{-1}\left|d\vec{\Phi}\dot{\otimes}d\vec{w} + d\vec{w}\dot{\otimes}\,d\vec{\Phi}\right|^2\right]\,d\text{vol}_{g_{\vec{\Phi}}},$$

where we recall that in coordinates $d\vec{\Phi}\,\dot{\otimes}\,d\vec{w} := \sum_{i,j}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}\;dx_i\otimes dx_j$.

Since we are assuming $N_\infty \equiv 1$ a.e. on S_∞ and, following the proof of the main theorem of [13], we can extract a subsequence that we keep denoting $\vec{\Phi}_k$ such that we have a *bubble tree* strong $W^{1,2}$ convergence of $\vec{\Phi}_k$ toward a minimal (possibly branched) immersion $\vec{\Psi}_\infty$ of a surface S_∞, which is the union of nodal surfaces and spheres. More precisely, if one denotes $\{S^j_\infty\}_{j\in J}$ to be the connected components of S_∞, for every $j\in J$ there exists N^j points $\{a^{j,l}\}_{l=1\cdots N^j}$ (containing in particular the possible branched points of $\vec{\Psi}_\infty$ and a converging sequence of constant scalar curvature metrics h^j_k of volume one and for any $\delta>0$ a sequence of conformal embeddings ϕ^j_k from $(S^j_\infty\setminus\cup_{l=1}^{N^j}B_\delta(a^{j,l}),h^j_k)$ into $(\Sigma^g,g_{\vec{\Phi}_k})$ such that

$$\vec{\Psi}^j_k := \vec{\Phi}_k\circ\phi^j_k \longrightarrow \vec{\Psi}_\infty \qquad \text{strongly in} W^{1,2}_{loc}(S^j_\infty\setminus\cup_{l=1}^{N^j}B_\delta(a^{j,l})). \tag{4.13}$$

For δ small enough and k' large enough the subdomains $\Omega^j_k(\delta) := \phi^j_k(S^j_\infty\setminus\cup_{l=1}^{N^j}B_\delta(a^{j,l}))$ are disjoint and

$$\lim_{\delta\to 0}\lim_{k\to+\infty}\text{Area}\left(\vec{\Phi}_k\left(\Sigma^g\setminus\bigcup_{j\in J}\Omega^j_k(\delta)\right)\right) = 0.$$

Let $\vec{w}_1\cdots\vec{w}_N$ a family of N independent smooth vectors in $W^{3,2}(\vec{\Psi}^*_\infty TM^m)$ representing N independent directions in $T_{[\vec{\Psi}_\infty]}\mathfrak{M}$ on the span of which D^2Area is strictly negative. We can assume without loss of generality that the $\vec{w}_i$ are C^∞. One modifies each of these vectors in the following way. For each $i\in\{1\cdots Q\}$ for each $j\in J$ and each $l\in\{1\cdots N^j\}$ we introduce (after identifying for each j and l the tangent planes to M^m around $\vec{\Phi}_\infty(a^{j,l})$ with the one at exactly $\vec{\Phi}_\infty(a^{j,l})$)

$$\vec{w}^\delta_i(x) = \begin{cases} \vec{w}_i(x) & \text{for} |a^{j,l}-x|\geq\sqrt{\delta} \\ \vec{w}_i(x)\;\chi^\delta(|x-a^{j,l}|) & \text{for } \delta\leq|a^{j,l}-x|\leq\sqrt{\delta} \\ 0 & \text{for } |a^{j,l}-x|\leq\delta, \end{cases}$$

where we take $\chi^{\delta}(s)$ to be a slight smoothing of $\log(s/\delta)/\log(1/\sqrt{\delta})$. A short computation gives that

$$\vec{w}_i^{\delta} \longrightarrow \vec{w}_i \quad \text{strongly in} \quad W^{1,2}(S_\infty, \mathbb{R}^Q).$$

Therefore, in view of the explicit expression of $D^2\text{Area}(\vec{\Psi}_\infty) \cdot (\vec{w}, \vec{w})$, there exists δ small enough such that $\vec{w}_1^{\delta} \cdots \vec{w}_N^{\delta}$ realizes a family of N independent smooth vectors in $W^{3,2}(\vec{\Psi}_\infty^* TM^m)$ on the span of which D^2Area is strictly negative. We fix such a δ.

Let $\rho > 0$ small enough such that for any $z \in M^m$ the map $\vec{\Psi}_\infty$ is injective on each components of $\vec{\Psi}_\infty^{-1}(\overline{B_\rho^Q(p)}) \subset S_\infty^j$. Let $\{\chi_s(p)\}_{s\in\{1\cdots N\}}$ be a finite smooth partition of unity of $M^m \subset \mathbb{R}^Q$ such that the support of every χ_s is included in an m-ball of radius ρ. We denote the connected components of $\vec{\Psi}_\infty^{-1}(\text{Supp}(\chi_s))$ in S_∞ by Ω_s^t for $t = 1 \cdots n_s$ and ω_s^t are the corresponding characteristic functions. We have that $\chi_s(\vec{\Psi}_\infty(x))\, \omega_s^t(x)$ is smooth for any $s \in \{1 \cdots N\}$ and any $t \in \{1 \cdots n_s\}$ and moreover

$$\mathrm{d}(\chi_s(\vec{\Psi}_\infty(x))\, \omega_s^t(x)) = \mathrm{d}(\chi_s(\vec{\Psi}_\infty(x)))\, \omega_s^t(x).$$

We can then write each $\vec{w}_i^{\delta}$ in the form

$$\vec{w}_i^{\delta}(x) = \sum_{s=1}^{N} \chi_s(\vec{\Psi}_\infty(x)) \sum_{t=1}^{n_s} \vec{v}_{i,s}^{\,t}(\vec{\Psi}_\infty(x))\, \omega_s^t,$$

where $\vec{v}_{i,s}^{\,t}$ are smooth functions (This is due to the fact that $\vec{\Psi}_\infty$ is smooth embedding on each open set Ω_s^t). For any $s =\in \{1 \cdots N\}$ since the components $\overline{\Omega_s^t}$ are disjoint to each other for $t \in \{1 \cdots n_s\}$ we can include them in strictly larger disjoint open sets $\overline{\Omega_s^t} \subset \tilde{\Omega}_s^t$; moreover, because of the strong $W^{1,2}$-convergence of $\vec{\Phi}_k$ toward $\vec{\Phi}_\infty$ in $S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})$

$$\|\vec{\Psi}_k - \vec{\Psi}_\infty\|_{L^\infty(\partial\tilde{\Omega}_s^t)} \longrightarrow 0. \tag{2}$$

We denote $\tilde{\omega}_s^t$ the characteristic functions of $\tilde{\Omega}_s^t$: $\tilde{\omega}_s^t := \mathbf{1}_{\tilde{\Omega}_s^t}$. We still have of course

$$\vec{w}_i^{\delta}(x) = \sum_{s=1}^{N} \chi_s(\vec{\Psi}_\infty(x)) \sum_{t=1}^{n_s} \vec{v}_{i,s}^{\,t}(\vec{\Psi}_\infty(x))\, \tilde{\omega}_s^t.$$

Because of (2), we have that

$$\text{dist}\left(\vec{\Psi}_k(\partial\tilde{\Omega}_s^t), \vec{\Psi}_\infty(\partial\tilde{\Omega}_s^t\right) \longrightarrow 0.$$

Since χ_s is zero in an open neighborhood of each $\vec{\Psi}_\infty(\partial\tilde{\Omega}_s^t)$ and since $\vec{\Psi}_k$ is smooth, we have for every s and t and for k large enough

$$\chi_s \circ \vec{\Psi}_k \equiv 0 \quad \text{in an open neighborhood} U_{k,s}^t \text{ of } \partial\tilde{\Omega}_s^t. \tag{4.15}$$

Hence, in particular we have for every s and t and k large enough

$$\mathrm{d}(\chi_s(\vec{\Psi}_k(x))\ \tilde{\omega}_s^t(x)) = \mathrm{d}(\chi_s(\vec{\Psi}_k(x)))\ \tilde{\omega}_s^t(x).$$

It is then clear that

$$\vec{w}_{i,k}^\delta(x) := \sum_{s=1}^N \chi_s(\vec{\Psi}_k(x)) \sum_{t=1}^{n_s} \vec{v}_{t,s}(\vec{\Psi}_k(x))\ \tilde{\omega}_s^t \ \longrightarrow\ \vec{w}_i^\delta(x) \text{ strongly in} W_{loc}^{1,2}(S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})). \tag{4}$$

Using the compositions with the maps $(\phi^{j,k})^{-1}$ we extend the $\vec{w}_{i,k}^\delta$, which we still denote $\vec{w}_{i,k}^\delta$ to the whole of Σ^g by taking $\vec{w}_{i,k}^\delta = 0$ on $\Sigma^g \setminus \bigcup_{j\in J} \Omega_{k'}^j(\delta)$. We see $\vec{w}_{i,k}^\delta$ as vectors in $\mathbb{R}^Q$ and we denote by $\pi_{k'}^j$ the map from $S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})$ into the space of projection matrices that to $x \in S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})$ assigns the orthogonal projection from $T_{\vec{\Psi}_{k'}^j(x)}\mathbb{R}^Q$ into $T_{\vec{\Psi}_{k'}^j(x)}M^m$. In other words, let P_z be the C^1 map from M^m into the space of $Q \times Q$ matrices that assigns the orthogonal projection onto T_zM^m, we have $\pi_{k'}^j(x) := P_{\vec{\Psi}_{k'}^j(x)}$ and we have

$$\pi_{k'}^j \longrightarrow P_{\vec{\Psi}_\infty} \qquad \text{strongly in } \ W_{loc}^{1,2}(S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})). \tag{5}$$

On $S_\infty^j \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})$ we denote $\vec{u}_{i,k'}^\delta(x) := \pi_k^j(x)(\vec{w}_{i,k}^\delta)$. Because of (5) we have

$$\vec{u}_{i,k}^\delta \longrightarrow \vec{w}_i^\delta \qquad \text{strongly in } \ W^{1,2}(S_\infty^j). \tag{6}$$

Consider now the symmetric matrix

$$\begin{aligned}
&D^2\mathrm{Area}(\vec{\Phi}_k)(\vec{u}_{i,k'}^\delta, \vec{u}_{i',k}^\delta) = \\
&\quad \sum_{j=1}^{\mathrm{card}(J)} \int_{S_\infty^j} \left[\left\langle \mathrm{d}\vec{u}_{i,k}^\delta\,;\, \mathrm{d}\vec{u}_{i',k}^\delta \right\rangle_{g_{\vec{\Psi}_k^j}} + \left\langle \mathrm{d}\vec{\Psi}_k^j\,;\, \mathrm{d}\vec{u}_{i,k}^\delta \right\rangle_{g_{\vec{\Psi}_k^j}} \left\langle \mathrm{d}\vec{\Psi}_k^j\,;\, \mathrm{d}\vec{u}_{i',k}^\delta \right\rangle_{g_{\vec{\Psi}_k^j}} \right] \mathrm{dvol}_{g_{\vec{\Psi}_k^j}} \\
&- 2^{-1} \sum_{j=1}^{\mathrm{card}(J)} \int_{S_\infty^j} \left\langle \mathrm{d}\vec{\Psi}_k^j \dot{\otimes}\, \mathrm{d}\vec{u}_{i,k}^\delta + \mathrm{d}\vec{u}_{i,k}^\delta \dot{\otimes}\, \mathrm{d}\vec{\Psi}_{k'}^j, \mathrm{d}\vec{\Psi}_k^j \dot{\otimes}\, \mathrm{d}\vec{u}_{i',k}^\delta + \mathrm{d}\vec{u}_{i',k}^\delta \dot{\otimes}\, \mathrm{d}\vec{\Psi}_k^j \right\rangle \mathrm{dvol}_{g_{\vec{\Psi}_k^j}}.
\end{aligned}$$

Let f and g be two smooth functions supported on $M^m \setminus \cup_{l=1}^{N^j} \Psi_\infty(B_\delta(a^{j,l}))$ then one has

$$\int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}_k)), \mathrm{d}(g(\vec{\Psi}_k)) >_{g_{\vec{\Psi}_k}} \ \mathrm{dvol}_{g_{\vec{\Psi}_k}} = \int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}_k)), \mathrm{d}(g(\vec{\Psi}_k)) >_{h^j_k} \ \mathrm{dvol}_{h^j_k},$$

and since h^j_k converges in any norms toward h^j_∞, because of the strong $W^{1,2}$ convergence of $\vec{\Psi}_k$ on $S^j_\infty \setminus \cup_{l=1}^{N^j} B_\delta(a^{j,l})$ one has

$$\int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}_k)), \mathrm{d}(g(\vec{\Psi}_k)) >_{g_{\vec{\Psi}_k}} \ \mathrm{dvol}_{g_{\vec{\Psi}_k}} \longrightarrow \int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}_\infty)), \mathrm{d}(g(\vec{\Psi}_\infty)) >_{g_{\vec{\Psi}_\infty}} \ \mathrm{dvol}_{g_{\vec{\Psi}_\infty}}. \quad (7)$$

In a conformal chart for h^j_k we denote $e^{\lambda^j_{k'}} := |\partial_{x_1} \vec{\Psi}^j_{k'}| = |\partial_{x_2} \vec{\Psi}^j_{k'}|$. Because of the strong $W^{1,2}$ convergence (1) we have

$$e^{\lambda^j_{k'}} \longrightarrow e^{\lambda^j_\infty} = |\partial_{x_1} \vec{\Psi}_\infty| = |\partial_{x_2} \vec{\Psi}_\infty| \quad \text{a. e. in} \quad S^j_\infty.$$

Since $e^{\lambda^j_\infty} > 0$ almost everywhere on S^j_∞ we have $e^{-\lambda^j_{k'}} \longrightarrow e^{-\lambda^j_\infty}$ almost everywhere and then for $i = 1, 2$

$$\partial_{x_i} \vec{\Psi}^j_k / e^{\lambda^j_k} \longrightarrow \partial_{x_i} \vec{\Psi}^j_\infty / e^{\lambda^j_\infty} \quad \text{almost everywhere.}$$

Let f,g, ϕ, and ψ be four arbitrary smooth functions on M^m. Assume that both f and g are supported on $M^m \setminus \cup_{l=1}^{N^j} \vec{\Psi}_\infty(B_\delta(a^{j,l}))$. One has in local conformal coordinates

$$\begin{aligned} &< \mathrm{d}(f(\vec{\Psi}^j_k)) \otimes \mathrm{d}(\phi(\vec{\Psi}^j_k)), \mathrm{d}(g(\vec{\Psi}^j_k)) \otimes \mathrm{d}(\psi(\vec{\Psi}^j_k)) >_{g_{\vec{\Psi}^j_k}} \ \mathrm{dvol}_{g_{\vec{\Psi}^j_k}} = \\ &\sum_{\mu,\nu=1,2} e^{-2\lambda^j_k} \partial_{x_\mu} f(\vec{\Psi}^j_k)\ \partial_{x_\nu} \phi(\vec{\Psi}^j_k)\ \partial_{x_\mu} g(\vec{\Psi}^j_k)\ \partial_{x_\nu} \psi(\vec{\Psi}^j_k)\ \mathrm{d}x_1 \wedge \mathrm{d}x_2. \end{aligned}$$

Because of the above

$$e^{-2\lambda^j_k} \partial_{x_\mu} f(\vec{\Psi}^j_k)\ \partial_{x_\nu} \phi(\vec{\Psi}^j_k)\ \partial_{x_\mu} g(\vec{\Psi}^j_k)\ \partial_{x_\nu} \psi(\vec{\Psi}^j_k) \longrightarrow e^{-2\lambda^j_\infty} \partial_{x_\mu} f(\vec{\Psi}^j_\infty)\ \partial_{x_\nu} \phi(\vec{\Psi}^j_\infty)\ \partial_{x_\mu} g(\vec{\Psi}^j_\infty)\ \partial_{x_\nu} \psi(\vec{\Psi}^j_\infty)$$

almost everywhere and we have moreover

$$|e^{-2\lambda^j_k} \partial_{x_\mu} f(\vec{\Psi}^j_k)\ \partial_{x_\nu} \phi(\vec{\Psi}^j_k)\ \partial_{x_\mu} g(\vec{\Psi}^j_k)\ \partial_{x_\nu} \psi(\vec{\Psi}^j_k)| \leq C\, |\nabla \vec{\Psi}^j_k|^2 \to |\nabla \vec{\Psi}^j_\infty|^2 \quad \text{strongly in } L^1.$$

Hence, the generalized dominated convergence theorem implies

$$\begin{aligned}
&\int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}^j_k)) \otimes \mathrm{d}(\phi(\vec{\Psi}^j_k)), \mathrm{d}(g(\vec{\Psi}^j_k)) \otimes \mathrm{d}(\psi(\vec{\Psi}^j_k)) >_{g_{\vec{\Psi}^j_k}} \ \mathrm{dvol}_{g_{\vec{\Psi}^j_k}} \\
&\longrightarrow \quad \int_{S^j_\infty} < \mathrm{d}(f(\vec{\Psi}^j_\infty)) \otimes \mathrm{d}(\phi(\vec{\Psi}^j_\infty)), \mathrm{d}(g(\vec{\Psi}^j_\infty)) \otimes \mathrm{d}(\psi(\vec{\Psi}^j_\infty)) >_{g_{\vec{\Psi}^j_\infty}} \ \mathrm{dvol}_{g_{\vec{\Psi}^j_\infty}}.
\end{aligned}$$

Similarly we also have

$$\begin{aligned}
&\int_{S^j_\infty} \left\langle \mathrm{d}(f(\vec{\Psi}^j_k)), \mathrm{d}(g(\vec{\Psi}^j_k))\right\rangle_{g_{\vec{\Psi}_k}} \left\langle \mathrm{d}(\phi(\vec{\Psi}^j_k)), \mathrm{d}(\psi(\vec{\Psi}^j_k))\right\rangle_{g_{\vec{\Psi}_k}} \ \mathrm{dvol}_{g_{\vec{\Psi}^j_k}} \\
&\longrightarrow \quad \int_{S^j_\infty} \left\langle \mathrm{d}(f(\vec{\Psi}^j_\infty)), \mathrm{d}(g(\vec{\Psi}^j_\infty))\right\rangle_{g_{\vec{\Psi}_\infty}} \left\langle \mathrm{d}(\phi(\vec{\Psi}^j_\infty)), \mathrm{d}(\psi(\vec{\Psi}^j_\infty))\right\rangle_{g_{\vec{\Psi}_\infty}} \ \mathrm{dvol}_{g_{\vec{\Psi}^j_\infty}}.
\end{aligned}$$

We have

$$\vec{u}_{i,k} := \sum_{s=1}^N \chi_s(\vec{\Psi}_k(x)) \sum_{t=1}^{n_s} P_{\vec{\Psi}_k}\left(\vec{v}^{\,t}_{i,s}(\vec{\Psi}_k(x))\right) \ \omega^t_s.$$

Because of (3) we have obviously from (20) that for any choice of s, t, s', t'

$$\begin{aligned}
&D^2F(\vec{\Phi})\left(\chi_s(\vec{\Psi}_k(x))P_{\vec{\Psi}_k}\left(\vec{v}^{\,t}_{i,s}(\vec{\Psi}_k(x))\right) \ \tilde{\omega}^t_s, \chi_{s'}(\vec{\Psi}_k(x))P_{\vec{\Psi}_k}\left(\vec{v}^{\,t'}_{i,s'}(\vec{\Psi}_k(x))\right) \ \tilde{\omega}^{t'}_{s'}\right) \\
&\quad \le \int_{\tilde{\Omega}^t_s \cap \tilde{\Omega}^{t'}_{s'}} (1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})^2 \ d vol_{g_{\vec{\Phi}}}.
\end{aligned}$$

Combining all the above gives

$$D^2\mathrm{Area}(\vec{\Phi}_k)(\vec{u}^{\,\delta}_{i,k}, \vec{u}^{\,\delta}_{i',k}) \quad \longrightarrow \quad D^2\mathrm{Area}(\vec{\Phi}_\infty)(\vec{w}^{\,\delta}_i, \vec{w}^{\,\delta}_{i'}). \tag{4.20}$$

Hence, for k large enough $(D^2A(\vec{\Phi}_k)(\vec{u}^{\,\delta}_{i,k}, \vec{u}^{\,\delta}_{i',k}))_{i,i'=1\cdots N}$ defines a strictly negative quadratic form.

Using now Lemma A.1 below we deduce that for any $i, i' \in \{1 \cdots N\}$

$$\sigma_k^2 \left| D^2F(\vec{\Phi}_k)(\vec{u}^{\,\delta}_{i,k}, \vec{u}^{\,\delta}_{i',k})\right| \le C\sigma_k^2 \left[F(\vec{\Phi}_k) + \mathrm{Area}(\vec{\Phi}_k)^{1/4} \ F(\vec{\Phi}_k)^{3/4}\right] = o(1). \tag{4.21}$$

Combining (8) and (9) we obtain that for k large enough $(D^2A^{\sigma_k}(\vec{\Phi}_k)(\vec{u}^{\,\delta}_{i,k}, \vec{u}^{\,\delta}_{i',k}))_{i,i'=1\cdots N}$ defines a strictly negative quadratic form. This implies inequality (1.3) and Theorem 1.4 is proved. □

Lemma A.1. Let M^m be a closed submanifold of the euclidian space $\mathbb{R}^Q$. For any $W^{2,4}$-immersion $\vec{\Phi}$ of an oriented closed surface Σ we denote

$$F(\vec{\Phi}) := \int_\Sigma \left(1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right)^2 \, dvol_{g_{\vec{\Phi}}},$$

where $\vec{\mathbb{I}}_{\vec{\Phi}}$ is the 2nd fundamental form of the immersion into M^m. The Lagrangian F is C^2 and there exists a constant C depending only on M^m such that for any perturbation $\vec{w}$ of the form $\vec{v} \circ \vec{\Phi}$ one has

$$|DF(\vec{\Phi})(\vec{v}(\vec{\Phi}))| \le C \int_\Sigma \left(1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right) \left[(1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \, |\partial\vec{v}|(\vec{\Phi}) + |\vec{\mathbb{I}}_{\vec{\Phi}}|_{g_{\vec{\Phi}}} \, |\partial^2\vec{v}|(\vec{\Phi})\right] \, dvol_{g_{\vec{\Phi}}}, \tag{A.1}$$

and

$$|D^2F(\vec{\Phi})(\vec{v}(\vec{\Phi}), \vec{v}(\vec{\Phi}))| \le C \int_\Sigma \left(1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right) \left[(1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \, |\partial\vec{v}|^2(\vec{\Phi}) + \, |\partial^2\vec{v}|^2(\vec{\Phi})\right] \, dvol_{g_{\vec{\Phi}}}. \tag{A.2}$$

□

Proof of Lemma A.1. We give the proof of inequalities (1) and (2) in the case of immersions into $\mathbb{R}^Q$. The terms coming from the fact we restrict to immersions into $M^m \subset \mathbb{R}^Q$ are of lower order and do not contribute in clarifying the argument and the successive estimates.

In local coordinates we denote the 2nd fundamental form

$$\vec{\mathbb{I}}_{\vec{\Phi}} = \pi_{\vec{n}}\left(d^2\vec{\Phi}\right) = \pi_{\vec{n}}\left(\partial^2_{x_i x_j}\vec{\Phi}\right) \, dx_i \otimes dx_j$$

we have

$$|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}} := \left|\pi_{\vec{n}}\left(d^2\vec{\Phi}\right)\right|^2_{g_{\vec{\Phi}}} = \sum_{i,j,k,l} g^{ik} g^{jl} \pi_{\vec{n}} \partial^2_{x_i x_j}\vec{\Phi} \cdot \pi_{\vec{n}} \partial^2_{x_k x_l}\vec{\Phi}. \tag{A.3}$$

Denote π_T the projection onto the tangent plane of the immersion. We have in local coordinates

$$\pi_T(\vec{X}) = \sum_{i,j=1}^2 g^{ij} \, \partial_{x_i}\vec{\Phi} \cdot \vec{X} \, \partial_{x_j}\vec{\Phi}. \tag{A.4}$$

Hence,

$$\pi_{\vec{n}} \frac{d\pi_{\vec{n}}}{dt}\bigg|_{t=0}(\vec{X}) = -\sum_{i,j=1}^2 g^{ij} \, \partial_{x_i}\vec{\Phi} \cdot \vec{X} \, \pi_{\vec{n}}\left(\partial_{x_j}\vec{w}\right). \tag{A.5}$$

We have clearly

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w} + \partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi}.$$

Hence,

$$\frac{\mathrm{d}g^{ij}}{\mathrm{d}t} = -\,g^{ik}g^{jl}\left[\partial_{x_k}\vec{\Phi}\cdot\partial_{x_l}\vec{w} + \partial_{x_k}\vec{w}\cdot\partial_{x_l}\vec{\Phi}\right] := -2\,(\mathrm{d}\vec{\Phi}\dot{\otimes}_S\mathrm{d}\vec{w})^{ij}. \tag{A.6}$$

We have then

$$\left.\frac{\mathrm{d}|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}}{\mathrm{d}t}\right|_{t=0} = 2\left\langle\pi_{\vec{n}}\left(\mathrm{d}^2\vec{\Phi}\right), \pi_{\vec{n}}\left(D^{g_{\vec{\Phi}}}\mathrm{d}\vec{w}\right)\right\rangle_{g_{\vec{\Phi}}} - 4\left(g\otimes(\mathrm{d}\vec{\Phi}\dot{\otimes}_S\mathrm{d}\vec{w})\llcorner\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}}\right), \tag{A.7}$$

where $\llcorner$ is the contraction operator between 4-contravariant and 4-covariant tensors and

$$D^{g_{\vec{\Phi}}}\mathrm{d}\vec{w} := \left[\partial^2_{x_ix_j}\vec{w} - \sum_{rs=1}^{2} g^{rs}\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_s}\vec{w}\right]\mathrm{d}x_i\otimes\mathrm{d}x_j. \tag{A.8}$$

This gives in particular that

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})^2\;\mathrm{d}vol_{g_{\vec{\Phi}}}\right|_{t=0} = DF(\vec{\Phi})(\vec{w})\\
&= 4\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})\left[\left\langle\vec{\mathbb{I}}_{\vec{\Phi}}, D^{g_{\vec{\Phi}}}\mathrm{d}\vec{w}\right\rangle_{g_{\vec{\Phi}}} - 2\left(g\otimes(\mathrm{d}\vec{\Phi}\dot{\otimes}_S\mathrm{d}\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}}\right)\right]\mathrm{d}vol_{g_{\vec{\Phi}}}\\
&+\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})^2\left\langle\mathrm{d}\vec{\Phi};\mathrm{d}\vec{w}\right\rangle_{g_{\vec{\Phi}}}\;\mathrm{d}vol_{g_{\vec{\Phi}}}.
\end{aligned} \tag{A.9}$$

For $\vec{w} := \vec{v}\left(\vec{\Phi}\right)$ we have

$$\begin{aligned}
&\partial^2_{x_ix_j}\vec{w} - \sum_{rs=1}^{2} g^{rs}\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_s}\vec{w} = \sum_{\alpha,\beta=1}^{Q}\partial^2_{z_\alpha z_\beta}\vec{v}(\vec{\Phi})\;\partial_{x_i}\vec{\Phi}^\alpha\;\partial_{x_j}\vec{\Phi}^\beta\\
&+\sum_{\alpha=1}^{Q}\partial_{z_\alpha}\vec{v}(\vec{\Phi})\left[\partial^2_{x_ix_j}\vec{\Phi}^\alpha - \sum_{rs=1}^{2} g^{rs}\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_s}\vec{\Phi}^\alpha\right].
\end{aligned}$$

We have

$$\pi_T\left(\partial^2_{x_ix_j}\vec{\Phi}\right) = \sum_{rs=1}^{2} g^{rs}\,\partial^2_{x_ix_j}\vec{\Phi}\cdot\partial_{x_r}\vec{\Phi}\;\partial_{x_s}\vec{\Phi}. \tag{A.10}$$

Hence,

$$\partial^2_{x_ix_j}\vec{\Phi} - \sum_{rs=1}^{2} g^{rs}\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_s}\vec{\Phi} = \pi_{\vec{n}}(\partial^2_{x_ix_j}\vec{\Phi}) = \vec{\mathbb{I}}_{ij}.$$

This implies that

$$D^{g_{\vec\Phi}}\,\mathrm{d}\vec w=\sum_{\alpha,\beta=1}^{Q}\partial^2_{z_\alpha z_\beta}\vec v(\vec\Phi)\,\mathrm{d}\vec\Phi^{\,\alpha}\otimes\mathrm{d}\vec\Phi^{\,\beta}+\sum_{\alpha=1}^{Q}\partial_{z_\alpha}\vec v(\vec\Phi)\,\vec{\mathbb{I}}^{\alpha}_{ij}. \tag{A.11}$$

We deduce

$$|DF(\vec\Phi)(\vec v(\vec\Phi))|\le C\int_\Sigma\left(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right)\,\left[(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}})\;|\partial\vec v|(\vec\Phi)+|\vec{\mathbb{I}}_{\vec\Phi}|_{g_{\vec\Phi}}\;|\partial^2\vec v|(\vec\Phi)\right]\,dvol_{g_{\vec\Phi}}. \tag{A.12}$$

We now compute the 2nd derivative

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}DF(\vec\Phi_t)(\vec w)\Big|_{t=0}\\
&=4\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}})\left[\left\langle\vec{\mathbb{I}}_{\vec\Phi},D^{g_{\vec\Phi}}\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi}}-2\left(g\otimes(\mathrm{d}\vec\Phi\dot\otimes_S\mathrm{d}\vec w)\right)\llcorner\left(\vec{\mathbb{I}}_{\vec\Phi}\dot\otimes\vec{\mathbb{I}}_{\vec\Phi}\right)\right]\left\langle\mathrm{d}\vec\Phi;\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi}}\;dvol_{g_{\vec\Phi}}\\
&+\int_\Sigma|1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}}|^2\left|\left\langle\mathrm{d}\vec\Phi;\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi}}\right|^2+8\left|\left\langle\vec{\mathbb{I}}_{\vec\Phi},D^{g_{\vec\Phi}}\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi}}-2\left(g\otimes(\mathrm{d}\vec\Phi\dot\otimes_S\mathrm{d}\vec w)\right)\llcorner\left(\vec{\mathbb{I}}_{\vec\Phi}\dot\otimes\vec{\mathbb{I}}_{\vec\Phi}\right)\right|^2\,dvol_{g_{\vec\Phi}}\\
&+4\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}})\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\langle\vec{\mathbb{I}}_{\vec\Phi_t},D^{g_{\vec\Phi_t}}\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi_t}}-2\left(g_{\vec\Phi_t}\otimes(\mathrm{d}\vec\Phi_t\dot\otimes_S\mathrm{d}\vec w)\right)\llcorner\left(\vec{\mathbb{I}}_{\vec\Phi_t}\dot\otimes\vec{\mathbb{I}}_{\vec\Phi_t}\right)\right]\;dvol_{g_{\vec\Phi}}\\
&+\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}})^2\left\langle\mathrm{d}\vec w;\mathrm{d}\vec w\right\rangle_{g_{\vec\Phi}}\;dvol_{g_{\vec\Phi}}-2\int_\Sigma(1+|\vec{\mathbb{I}}_{\vec\Phi}|^2_{g_{\vec\Phi}})^2\;\left(\mathrm{d}\vec\Phi\dot\otimes_S\mathrm{d}\vec w\right)\llcorner\left(\mathrm{d}\vec\Phi\otimes\mathrm{d}\vec w\right)\;dvol_{g_{\vec\Phi}}.\end{aligned} \tag{A.13}$$

We have on one hand

$$\pi_{\vec n}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbb{I}}_{\vec\Phi_t}=\pi_{\vec n}\left(D^{g_{\vec\Phi}}\mathrm{d}\vec w\right), \tag{A.14}$$

on the other hand

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{r=1}^{2}g^{rs}_{\vec\Phi_t}\;\partial_{x_r}\vec\Phi_t\cdot\partial^2_{x_ix_j}\vec\Phi_t\right)&=\sum_{r=1}^{2}g^{rs}\;\partial_{x_r}\vec\Phi\cdot\partial^2_{x_ix_j}\vec w+g^{rs}\;\partial_{x_r}\vec w\cdot\partial^2_{x_ix_j}\vec\Phi+\frac{\mathrm{d}g^{rs}}{\mathrm{d}t}\partial_{x_r}\vec\Phi\cdot\partial^2_{x_ix_j}\vec\Phi\\
&=\sum_{r=1}^{2}g^{rs}\;\partial_{x_r}\vec\Phi\cdot\partial^2_{x_ix_j}\vec w+g^{rs}\;\partial_{x_r}\vec w\cdot\pi_{\vec n}(\partial^2_{x_ix_j}\vec\Phi)+\frac{\mathrm{d}g^{rs}}{\mathrm{d}t}\partial_{x_r}\vec\Phi\cdot\partial^2_{x_ix_j}\vec\Phi\\
&+\sum_{r,k,l=1}^{2}g^{rs}\;g^{kl}\;\partial_{x_r}\vec w\cdot\partial_{x_k}\vec\Phi\;\partial_{x_l}\vec\Phi\cdot\partial^2_{x_ix_j}\vec\Phi;\end{aligned} \tag{A.15}$$

we have

$$\begin{aligned}\sum_{r=1}^{2}\frac{\mathrm{d}g^{rs}}{\mathrm{d}t}\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}&=-\sum_{r,k,l=1}^{2}g^{rk}\,g^{sl}\,\partial_{x_r}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\left[\partial_{x_k}\vec{w}\cdot\partial_{x_l}\vec{\Phi}+\partial_{x_l}\vec{w}\cdot\partial_{x_k}\vec{\Phi}\right]\\&=-\sum_{r,k,l=1}^{2}g^{lk}\,g^{sr}\,\partial_{x_l}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_k}\vec{w}\cdot\partial_{x_r}\vec{\Phi}+g^{lk}\,g^{sr}\,\partial_{x_l}\vec{\Phi}\cdot\partial^2_{x_ix_j}\vec{\Phi}\;\partial_{x_r}\vec{w}\cdot\partial_{x_k}\vec{\Phi}.\end{aligned}\tag{A.16}$$

Combining (A.15) and (A.16) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{r=1}^{2}g^{rs}_{\vec{\Phi}_t}\;\partial_{x_r}\vec{\Phi}_t\cdot\partial^2_{x_ix_j}\vec{\Phi}_t\right)=\sum_{r=1}^{2}g^{rs}\;\partial_{x_r}\vec{\Phi}\cdot(D^{g_{\vec{\Phi}}}\mathrm{d}\vec{w})_{ij}+\sum_{r=1}^{2}g^{rs}\,\partial_{x_r}\vec{w}\cdot\vec{\mathbb{I}}_{ij}.\tag{A.17}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(D^{g_{\vec{\Phi}_t}}d\vec{w}\right)=-\sum_{i,j=1}^{2}\left[\sum_{r=1}^{2}g^{rs}\;\partial_{x_r}\vec{\Phi}\cdot(D^{g_{\vec{\Phi}}}\mathrm{d}\vec{w})_{ij}\;\partial_{x_s}\vec{w}+g^{rs}\,\partial_{x_r}\vec{w}\cdot\vec{\mathbb{I}}_{ij}\;\partial_{x_s}\vec{w}\right]\mathrm{d}x_i\otimes\mathrm{d}x_j.\tag{A.18}$$

Combining (A.6), (A.14), and (A.18) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\langle\vec{\mathbb{I}}_{\vec{\Phi}_t},D^{g_{\vec{\Phi}_t}}\mathrm{d}\vec{w}\right\rangle_{g_{\vec{\Phi}_t}}-2\left(g_{\vec{\Phi}_t}\otimes(\mathrm{d}\vec{\Phi}_t\dot{\otimes}_S d\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}_{\vec{\Phi}_t}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}_t}\right)\right]\\&=\left|\pi_{\vec{n}}\left(D^g\mathrm{d}\vec{w}\right)\right|^2_{g_{\vec{\Phi}}}-\left\langle\vec{\mathbb{I}}\,;\sum_{i,j=1}^{2}g^{ij}\partial_{x_i}\vec{\Phi}\cdot D^g\mathrm{d}\vec{w}\;\partial_{x_j}\vec{w}\right\rangle_{g_{\vec{\Phi}}}+4\Big[(\mathrm{d}\vec{\Phi}\otimes_S\mathrm{d}\vec{w})\otimes(\mathrm{d}\vec{\Phi}\otimes_S\mathrm{d}\vec{w})\Big]\llcorner(\vec{\mathbb{I}}\dot{\otimes}\vec{\mathbb{I}})\\&-4\left(g\otimes(\mathrm{d}\vec{\Phi}\otimes_S\mathrm{d}\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}\otimes\pi_{\vec{n}}(D^g\mathrm{d}\vec{w})+\pi_{\vec{n}}(D^g\mathrm{d}\vec{w})\otimes\vec{\mathbb{I}}\right)-2\;\left(g\otimes(\mathrm{d}\vec{w}\otimes_S\mathrm{d}\vec{w})\right)\llcorner(\vec{\mathbb{I}}\dot{\otimes}\vec{\mathbb{I}}).\end{aligned}\tag{A.19}$$

Combining (A.13) and (A.19) gives

$$
\begin{aligned}
&D^2F(\vec{\Phi})(\vec{w},\vec{w}) = \\
&4\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \left[\left\langle \vec{\mathbb{I}}_{\vec{\Phi}}, D^{g_{\vec{\Phi}}} d\vec{w}\right\rangle_{g_{\vec{\Phi}}} - 2\left(g_{\vec{\Phi}}\otimes(d\vec{\Phi}\dot{\otimes}_S d\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}}\right)\right] \left\langle d\vec{\Phi}; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \\
&+\int_\Sigma |1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}|^2 \left|\left\langle d\vec{\Phi}; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\right|^2 + 8\left|\left\langle \vec{\mathbb{I}}_{\vec{\Phi}}, D^{g_{\vec{\Phi}}} d\vec{w}\right\rangle_{g_{\vec{\Phi}}} - 2\left(g\otimes(d\vec{\Phi}\dot{\otimes}_S d\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}}\right)\right|^2 dvol_{g_{\vec{\Phi}}} \\
&+4\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \left[\left|\pi_{\vec{n}}\left(D^{g_{\vec{\Phi}}} d\vec{w}\right)\right|^2_{g_{\vec{\Phi}}} - \left\langle \vec{\mathbb{I}}_{\vec{\Phi}}\,;\, \sum_{i,j=1}^2 g^{ij}\partial_{x_i}\vec{\Phi}\cdot D^g_{g_{\vec{\Phi}}} d\vec{w}\ \partial_{x_j}\vec{w}\right\rangle_{g_{\vec{\Phi}}}\right] dvol_{g_{\vec{\Phi}}} \\
&+16\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \left[(d\vec{\Phi}\otimes_S d\vec{w})\otimes(d\vec{\Phi}\otimes_S d\vec{w})\right]\llcorner(\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}})\ dvol_{g_{\vec{\Phi}}} \\
&-16\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \left(g_{\vec{\Phi}}\otimes(d\vec{\Phi}\otimes_S d\vec{w})\right)\llcorner\left(\vec{\mathbb{I}}_{\vec{\Phi}}\otimes\pi_{\vec{n}}(D^{g_{\vec{\Phi}}} d\vec{w}) + \pi_{\vec{n}}(D^{g_{\vec{\Phi}}} d\vec{w})\otimes\vec{\mathbb{I}}_{\vec{\Phi}}\right)\ dvol_{g_{\vec{\Phi}}} \\
&-8\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}) \left(g_{\vec{\Phi}}\otimes(d\vec{w}\otimes_S d\vec{w})\right)\llcorner(\vec{\mathbb{I}}_{\vec{\Phi}}\dot{\otimes}\vec{\mathbb{I}}_{\vec{\Phi}})\ dvol_{g_{\vec{\Phi}}} \\
&+\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})^2 \left\langle d\vec{w}; d\vec{w}\right\rangle_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} - 2\int_\Sigma (1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})^2\ \left(d\vec{\Phi}\dot{\otimes}_S d\vec{w}\right)\llcorner\left(d\vec{\Phi}\otimes d\vec{w}\right)\ dvol_{g_{\vec{\Phi}}}.
\end{aligned}
\tag{A.20}
$$

For $\vec{w} := \vec{v}(\vec{\Phi})$, using (11), we deduce

$$
|D^2F(\vec{\Phi})(\vec{v}(\vec{\Phi}),\vec{v}(\vec{\Phi}))| \le C\int_\Sigma \left(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right)\ \left[(1+|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}})\ |\partial\vec{v}|^2(\vec{\Phi}) +\ |\partial^2\vec{v}|^2(\vec{\Phi})\right]\ dvol_{g_{\vec{\Phi}}}.
\tag{A.21}
$$

This concludes the proof of Lemma .1. □

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