A Quantization Property for Moving Line Vortices

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Abstract : We prove that if the energy of a time evolving three dimensional Ginzburg-Landau configuration is small enough on a given parabolic ball then no line vortex is passing through the ball of half radius.

I Introduction

In this paper we study the asymptotic of the Ginzburg-Landau evolution equation in 3 dimensions

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{u_\varepsilon}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ u_\varepsilon(0,x) = u_\varepsilon^0(x) \qquad \text{in } \mathbb{R}^3. \end{cases} \tag{I.1}$$

It is the gradient flow for the Ginzburg-Landau Functional

$$E_\varepsilon(u) = \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \tag{I.2}$$

This asymptotic for the corresponding static problem has been extensively studied since the work of F.Bethuel, H.Brezis and F.Hélein, [2] in two dimension (see [12], [13], [9], [10], [15]...in higher dimensions). The asymptotic for

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the equation (I.1) itself in 2 dimension was initiated by the first author in [7] and developped in [8], [6]...

The main difficulty for understanding the limiting behavior of sequences of solutions u_ε of (I.1) as $\varepsilon \to 0$ comes from the possible existence of dimension 1 vanishing set for u_ε around which $u_\varepsilon/|u_\varepsilon|$ has a non zero degree. Such set is known as a Ginzburg-Landau vortex. Zero sets for u_ε are problematic in general. Indeed, as long as $|u_\varepsilon| > 0$ in some ball B_1, if we write $u_\varepsilon = |u_\varepsilon| \exp(i\phi)$, the phase ϕ of u_ε solves the nice parabolic equation

$$\frac{\partial \phi}{\partial t} - \frac{\mathrm{div}(|u|^2 \nabla \phi)}{|u|^2} = 0 \tag{I.3}$$

This equation clearly degenerates as u get close to zero (as we approach the vortices).

The least initial energy required to get line vortices is of the order $\log \frac{1}{\varepsilon}$ (see [13]), thus we will assume that

$$E_\varepsilon(u_\varepsilon^0) = O(\log \frac{1}{\varepsilon})$$

which implies, because of the decrease of the energy under the gradient flow,

$$E_\varepsilon(u_\varepsilon(x,t)) \le E_\varepsilon(u_\varepsilon^0) = O(\log \frac{1}{\varepsilon}) \tag{I.4}$$

In order to identify the location of the zero set of $|u|$, which is responsible for the degeneracy of the equation (I.3), we introduce, as in the static case, the bad set

$$\mathcal{B}_\varepsilon = \left\{ (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \qquad |u_\varepsilon(x,t)| \le 1/2 \right\}$$

(1/2 plays the role of a barrier number and could be replaced by any number strictly between 0 and 1).

Our first main result in the present paper is to establish a relation between the bad set and the energy concentration set. This result is known as the *eta-compactness property* and was proved in the static case in [12], [13], [9] and [10]. Precisely we have

Theorem I.1 *($\eta-$compactness property) Let u_ε be a solution of (I.1) verifying $|u_\varepsilon| \le 1$ and $|\nabla u_\varepsilon| \le \frac{C_0}{\varepsilon}$, then there exist ε_0 and η depending only on C_0 such that for all $x^0 \in \mathbb{R}^3$, for all $t^0 > \rho^2 > 0$ and $\varepsilon < \varepsilon_0$ if*

$$\frac{1}{\rho^3} \int_{t^0-\rho^2}^{t^0} \int_{\mathbb{R}^3} \left[|\nabla u_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \right] \; e^{\frac{|x-x^0|^2}{4(\tau - t^0)}} d\tau \, dx \le \eta \log \frac{\rho}{\varepsilon} \tag{I.5}$$

Then

$$|u_\varepsilon(x^0,t^0)| \geq \frac{1}{2}$$

As it is explained in [12], [13], [10]...for the static case, the eta-compactness is the key step for proving the compactness of the sequence u_ε in $W^{1,p}$. The first step in order to establish this compactness consists of controling the size of the bad set $\mathcal{B}_\varepsilon$. From the a-priori bound (I.4) we deduce the bound for the Higgs part of the energy

$$\int_{\mathbb{R}^3\times\{t\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C\log\frac{1}{\varepsilon} \tag{I.6}$$

which implies

$$\left|\mathcal{B}_\varepsilon \cap [0,1]\times\mathbb{R}^3\right| \leq C\varepsilon^2\ \log\frac{1}{\varepsilon} \tag{I.7}$$

In fact, although the bound in $\log 1/\varepsilon$ is optimal for the Dirichlet part of the energy, it was established for the static problem in 2 dimensions (see [2]), by the mean of the Pohozaev identity, that is is not optimal at all for the Higgs part of the energy and that one should expect instead

$$\int_{\mathbb{R}^3\times\{t\}} \frac{(1-|u|^2)^2}{\varepsilon^2} = O(1)$$

and that

$$|\mathcal{B}_\varepsilon| \simeq \varepsilon^2 \tag{I.8}$$

It happens that, in higher dimension, for the static case, such a splitting in the orders of the various terms of the Ginzburg-Landau Energy is still valid and that (I.8) gives rapidely the desired compactness result (see [10]).

In this paper we prove that the same phenomenon holds for the Ginzburg-Landau evolution equation. As a consequence of the eta-compactness property above we have

Theorem I.2 *Let u_ε be a solution of (I.1), assume that $E_\varepsilon(u_\varepsilon^0) = O(\log 1/\varepsilon)$ and $|\nabla u| \leq C/\varepsilon^2$, then we have*

$$\left|\mathcal{B}_\varepsilon \cap [0,1]\times\mathbb{R}^3\right| \leq C\varepsilon^2 \tag{I.9}$$

In a forthcoming paper we will explain how to deduce from (I.9), under the same hypothesis, $W^{1,p}$ estimates for u_ε independent on ε. Such a compactness combined with the $\eta-$compactness property permits to prove that the limiting defect measure

$$\lim_{\varepsilon\to 0}\frac{\mathrm{e}_\varepsilon(u_\varepsilon)}{\log 1/\varepsilon}=\mu_t$$

verifies the following density lower-bound

$$\lim sup_{\rho\to 0}\frac{\mu_t(B_\rho(x))}{\rho}\ge\eta>0 \qquad \text{for } \mu_t-\text{a.e. } x\ , \quad \mathcal{L}^1-\text{a.e. } t\ge 0 \quad (\text{I.10})$$

This density lower bound was the missing ingredient in [1] to prove that the limiting defect measure μ_t evolves along a Brakke flow.

The proof of theorem I.1 as the same structure as the one for the eta-compactness of static Ginzburg-Landau vortices in [10] but involves new technical difficulties due to the parabolic nature of the equation. The previous existing proofs of the eta-compactness in [13] and [9] were only valid for minimizers of the Ginzburg-Landau Functional and they were extensively using comparison arguments. The main achivement of [10] was to present arguments that avoid comparison ones and that were valid for any solution of the Ginzburg-Landau equations. The crucial idea was to introduce the use of spaces which were not so common in calculus of variations such as Lorentz spaces $L^{2,\infty}-L^{2,1}$. Although there were already introduced 10 years ago in this context by F.Hélein in the proof of the regularity of harmonic maps from a surface into a two dimensional manifold, not much has been done by the mean of this spaces since Hélein's proof in non linear elliptic or parabolic PDE. Here, like in [10], this $L^{2,\infty}-L^{2,1}$ spaces are used in dimension larger than the conformal one ($n>2$), this is made possible by a combination of arguments based on the use of monotonicity formulas and the mean value formula in order to extract a “good slice” of dimension 2 (the conformal one) where these spaces start to be efficient. In a forthcoming paper we will explain how dimension reduction arguments of this kind repeated several times permit to extend the results of [10] or theorem I.1 above in dimension larger than 3. Short after developping this use of Lorentz Spaces in non conformal dimension for Ginzburg-Landau Equations we realised that it could also be very efficient in another context to prove quantization results for defect measures of sequences of stationary harmonic maps (see [11]). We are convinced that the techniques developped below could be very fruitfuhl for other problems in non-linear elliptic or parabolic PDE.

II Proof of the eta-compactness property.

We assume in this section that all the hypothesis of theorem I.1 are fulfilled.

II.1 The monotonicity formula

One of the key property satisfied by any solution of (I.1) is the following increasing property of the conformal invariant density of energy on parabolic balls.

Lemma II.1 *(Monotonicity formula) Let u be a solution of (I.1) and* (t^0, x^0) *be any pair in* $\mathbb{R}_+^* \times \mathbb{R}^3$*, the following identity holds*

$$\begin{aligned}
&\frac{d}{d\rho}\left[\frac{1}{\rho^3}\int_{-\rho^2+t^0}^{t^0}\int_{\mathbb{R}^3}\left[|\nabla u_\varepsilon|^2+\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\right]\ e^{\frac{|x-x^0|^2}{4(t-t^0)}}\right]\\
&=\frac{1}{\rho^4}\int_{-\rho^2+t^0}^{t^0}\int_{\mathbb{R}^3}\left[\frac{1}{t^0-t}\left|\sum_{i=1}^3(x_i-x_i^0)\frac{\partial u}{\partial x_i}+2(t-t^0)\frac{\partial u}{\partial t}\right|^2\right.\\
&\quad\left.+\frac{(1-|u|^2)^2}{\varepsilon^2}\right]\ e^{\frac{|x-x^0|^2}{4(t-t^0)}}
\end{aligned} \tag{II.1}$$

Proof of lemma II.1 : To simplify the presentation we take $x^0 = 0$ and $t^0 = 0$, assuming that (I.1) is solved for some negative t. So we will have $t < 0$ below. Following the approach of M.Struwe [16], we multiply (I.1) by $\left(r\frac{\partial u}{\partial r} + 2t\frac{\partial u}{\partial t}\right) e^{\frac{r^2}{4t}}$. In one hand integration by parts on $\mathbb{R}^3$ yields

$$\begin{aligned}
\int_{\mathbb{R}^3} r\frac{\partial u}{\partial r}\,\Delta u\ e^{\frac{r^2}{4t}} &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\ e^{\frac{r^2}{4t}}+\frac{1}{4}\int_{\mathbb{R}^3}\frac{r^2}{t}|\nabla u|^2\ e^{\frac{r^2}{4t}}\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\frac{r^2}{t}\left|\frac{\partial u}{\partial r}\right|^2\ e^{\frac{r^2}{4t}}
\end{aligned} \tag{II.2}$$

In the other hand, integrating by parts again we have

$$\begin{aligned}\int_{\mathbb{R}^3} 2t\frac{\partial u}{\partial t}\,\Delta u\; e^{\frac{r^2}{4t}} &= -\int_{\mathbb{R}^3} t\frac{\partial}{\partial t}|\nabla u|^2\; e^{\frac{r^2}{4t}} \\ &\quad -\int_{\mathbb{R}^3} r\frac{\partial u}{\partial r}\,\frac{\partial u}{\partial t}\; e^{\frac{r^2}{4t}}\end{aligned} \tag{II.3}$$

Thus integrating in time (II.3) yields

$$\begin{aligned}&\int_{-\rho^2}^0\int_{\mathbb{R}^3} 2t\,\frac{\partial u}{\partial t}\Delta u\; e^{\frac{r^2}{4t}} = \int_{-\rho^2}^0\int_{\mathbb{R}^3}(1+\frac{r^2}{4\rho^2})|\nabla u|^2\; e^{\frac{r^2}{4t}} \\ &-\rho^2\int_{\mathbb{R}^3\times\{-\rho^2\}}|\nabla u|^2\; e^{\frac{r^2}{4t}} - \int_{-\rho^2}^0\int_{\mathbb{R}^3} r\frac{\partial u}{\partial r}\,\frac{\partial u}{\partial t}\; e^{\frac{r^2}{4t}}\end{aligned} \tag{II.4}$$

Then combining (II.1) and (II.4) we obtain

$$\begin{aligned}&\int_{-\rho^2}^0\int_{\mathbb{R}^3}\Delta u\;\left(r\frac{\partial u}{\partial r}+2t\,\frac{\partial u}{\partial t}\right) = +\frac{3}{2}\int_{-\rho^2}^0\int_{\mathbb{R}^3}|\nabla u|^2\; e^{\frac{r^2}{4t}} \\ &-\frac{1}{2}\int_{-\rho^2}^0\int_{\mathbb{R}^3}\frac{r^2}{t}\left|\frac{\partial u}{\partial t}\right|^2\; e^{\frac{r^2}{4t}} - \int_{-\rho^2}^0\int_{\mathbb{R}^3} r\frac{\partial u}{\partial r}\frac{\partial u}{\partial t}\; e^{\frac{r^2}{4t}} - \rho^2\int_{\mathbb{R}^3\times\{-\rho^2\}}|\nabla u|^2\; e^{\frac{r^2}{4t}}\end{aligned} \tag{II.5}$$

Using the fact that

$$\begin{aligned}&-\int_{-\rho^2}^0\int_{\mathbb{R}^3}\frac{\partial u}{\partial t}\left(r\frac{\partial u}{\partial r}+2t\frac{\partial u}{\partial t}\right)\; e^{\frac{r^2}{4t}} = -2\int_{-\rho^2}^0\int_{\mathbb{R}^3} t\left|\frac{\partial u}{\partial t}\right|^2\; e^{\frac{r^2}{4t}} \\ &-\int_{-\rho^2}^0\int_{\mathbb{R}^3} r\frac{\partial u}{\partial r}\frac{\partial u}{\partial t}\; e^{\frac{r^2}{4t}}\end{aligned} \tag{II.6}$$

summing (II.5) and (II.6) we get

$$\begin{aligned}&\int_{-\rho^2}^0\int_{\mathbb{R}^3}\left(-\frac{\partial u}{\partial t}+\Delta u\right)\;\left(r\frac{\partial u}{\partial r}+2t\frac{\partial u}{\partial t}\right)\; e^{\frac{r^2}{4t}} \\ &= \int_{-\rho^2}^0\int_{\mathbb{R}^3}\frac{1}{2|t|}\left|r\frac{\partial u}{\partial r}+2t\frac{\partial u}{\partial t}\right|^2\; e^{\frac{r^2}{4t}} + \frac{3}{2}\int_{-\rho^2}^0\int_{\mathbb{R}^3}|\nabla u|^2\; e^{\frac{r^2}{4t}} \\ &-\rho^2\int_{\mathbb{R}^3\times\{-\rho^2\}}|\nabla u|^2\; e^{\frac{r^2}{4t}}\end{aligned} \tag{II.7}$$

Some more integration by parts yield also

$$\begin{aligned}&\int_{-\rho^2}^0\int_{\mathbb{R}^3} u\frac{(1-|u|^2)}{\varepsilon^2}\left(r\frac{\partial u}{\partial r}+2t\frac{\partial u}{\partial t}\right)\,e^{\frac{r^2}{4t}}\\&=\frac{5}{4}\int_{-\rho^2}^0\int_{\mathbb{R}^3}\frac{(1-|u|^2)^2}{\varepsilon^2}\,e^{\frac{r^2}{4t}}-\frac{\rho^2}{2}\int_{\mathbb{R}^3\times\{-\rho^2\}}\frac{(1-|u|^2)^2}{\varepsilon^2}\,e^{\frac{r^2}{4t}}\end{aligned}\tag{II.8}$$

Using the fact that, for any function f, one has

$$\frac{d}{d\rho}\left[\frac{1}{\rho^3}\int_{-\rho^2}^0\int_{\mathbb{R}^3} f\right]=-\frac{3}{\rho^4}\int_{-\rho^2}^0\int_{\mathbb{R}^3} f+\frac{2}{\rho^2}\int_{\mathbb{R}^3\times\{-\rho^2\}} f\tag{II.9}$$

Summing (II.7) and (II.8) and using (II.9) gives (II.1).

II.2 Extracting a "good time".

We are proving now theorem I.1. Again we will take $x^0=0$ and $t^0=0$ to simplify notations. Let t be the time given by the statement of theorem I.1. Integrating (II.1) between ε and ρ and using (I.5) we obtain

$$\int_\varepsilon^\rho\frac{1}{s^4}\int_{-s^2}^0\int_{\mathbb{R}^3}\mathrm{j}_\varepsilon(u_\varepsilon)\leq\eta\log\frac{\rho}{\varepsilon}\tag{II.10}$$

where j_ε denotes

$$\mathrm{j}_\varepsilon(u_\varepsilon)=\frac{1}{|t|}\left|r\frac{\partial u}{\partial r}+2t\frac{\partial u}{\partial t}\right|^2+\frac{(1-|u|^2)^2}{\varepsilon^2}\tag{II.11}$$

In the same spirit as [13] or [10], by the mean of the mean value formula we extract a $\rho_0\in[\varepsilon;\rho]$ such that

$$\frac{1}{\rho_0^3}\int_{-\rho_0^2}^0\int_{\mathbb{R}^3}\mathrm{j}_\varepsilon(u_\varepsilon)\,e^{\frac{r^2}{4t}}\leq\eta\tag{II.12}$$

Using again the mean value formula we can in fact simultaneously ensure that (II.12) holds (replacing eventually η by $C\eta$ - C being independant of ε, η, t and u) with the additional assumption that

$$\frac{1}{\rho_0}\int_{\mathbb{R}^3\times\{-\rho_0^2\}}\mathrm{j}_\varepsilon(u_\varepsilon)\,e^{\frac{r^2}{4t}}\leq C\eta\tag{II.13}$$

Let E be the following energy

$$E = \frac{1}{\rho_0^3} \int_{-\rho_0^2} \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \tag{II.14}$$

where

$$\mathrm{e}_\varepsilon(u_\varepsilon) = |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

The goal of our proof is to establish that for η chosen small enough E is of the order $O(\eta^{1/2})$ (independantly of ε) that yields directly to the proof of theorem I.1 as it is explained in II.5.

Using the mean value formula again we can choose $\rho \in [\rho_0/2; \rho_0]$ such that simultaneously we have

$$\frac{1}{\rho} \int_{\mathbb{R}^3 \times \{\rho\}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq 2E \qquad \text{and} \qquad \frac{1}{\rho} \int_{\mathbb{R}^3 \times \{\rho\}} \mathrm{j}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\eta \tag{II.15}$$

Using the monotonicity formula (II.1), the bound (II.12) we deduce that for this $\rho \in [\rho_0/2; \rho_0]$

$$E = \frac{1}{\rho_0^3} \int_{-\rho_0^2} \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq \frac{1}{\rho^3} \int_{-\rho^2} \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} + C\eta$$

we consider the case where $E >> C\eta$, otherwise we would go directly to section II.5, then we deduce that

$$E \leq 2\frac{1}{\rho^3} \int_{-\rho^2} \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \tag{II.16}$$

We can take our ρ_0 to be this new ρ, in other words it means that the ρ_0 choosen such that (II.12) and (II.13) hold can be chosen such that we also have

$$\frac{1}{\rho_0} \int_{\mathbb{R}^3 \times \{-\rho_0^2\}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq CE = C\frac{1}{\rho_0^3} \int_{-\rho_0^2}^0 \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}}. \tag{II.17}$$

We also need to have a bound of $\frac{1}{\rho_0} \int_{-\rho_0^2}^0 \int_{\mathbb{R}^3} \frac{r^2}{4|t|} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}}$. To that purpose we prove the following lemma

Lemma II.2 *Let u be a solution of (I.1) then the following inequality holds for any $0 < t^0$ and $\rho \leq \sqrt{t^0}$*

$$\begin{aligned}
&\int_{t^0-\rho^2}^{t^0}\int_{\mathbb{R}^3}\left(1+\frac{r^2}{4(t^0-t)}\right)\ \left[|\nabla u|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right]\ e^{\frac{r^2}{t-t^0}}\\
&+\int_{t^0-\rho^2}^{t^0}\int_{\mathbb{R}^3}\frac{1}{2(t^0-t)}\left|r\frac{\partial u}{\partial r}+2(t-t^0)\frac{\partial u}{\partial t}\right|^2\ e^{\frac{r^2}{t-t^0}}\\
&=\rho^2\int_{\mathbb{R}^3}e^{-\frac{r^2}{4\rho^2}}\left[|\nabla u|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right]\\
&+\int_{t^0-\rho^2}^{t^0}\int_{\mathbb{R}^3}\frac{r}{2(t^0-t)}\frac{\partial u}{\partial r}\left[r\frac{\partial u}{\partial r}+2(t-t^0)\frac{\partial u}{\partial t}\right]
\end{aligned}\tag{II.18}$$

Proof of lemma II.2: Again we shift the time by t^0 to simplify the notations. Multiplying (I.1) by $2t\frac{\partial u}{\partial t}e^{\frac{r^2}{4t}}$ and integrating on $\mathbb{R}^3\times[-\rho^2;0]$, we get after integration by part in space

$$\begin{aligned}
0=&\int_{-\rho^2}^{0}\int_{\mathbb{R}^3}2t\left|\frac{\partial u}{\partial t}\right|^2\ e^{\frac{r^2}{4t}}+\int_{-\rho^2}^{0}\int_{\mathbb{R}^3}t\frac{\partial}{\partial t}|\nabla u|^2\ e^{\frac{r^2}{4t}}\\
&+\int_{-\rho^2}^{0}\int_{\mathbb{R}^3}r\frac{\partial u}{\partial r}\frac{\partial u}{\partial t}\ e^{\frac{r^2}{4t}}+\int_{-\rho^2}^{0}\int_{\mathbb{R}^3}t\frac{\partial}{\partial t}\frac{(1-|u|^2)^2}{2\varepsilon^2}
\end{aligned}\tag{II.19}$$

Integrating by part in t this time, one gets (II.18) and lemma II.2 is proved.

Combining (II.17) and (II.18) we get that

$$\frac{1}{\rho_0^3}\int_{-\rho_0^2}^{0}\int_{\mathbb{R}^3}\frac{r^2}{|t|}\mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}}\leq CE+\eta^{\frac{1}{2}}\left[\frac{1}{\rho_0^3}\int_{-\rho_0^2}^{0}\int_{\mathbb{R}^3}\frac{r^2}{|t|}\left|\frac{\partial u}{\partial r}\right|^2\right]^{\frac{1}{2}}\tag{II.20}$$

and since we only consider the case where $E >> \eta$ (otherwise we go dierctly to subsection II.5) we deduce that

$$\frac{1}{\rho_0^3}\int_{-\rho_0^2}^{0}\int_{\mathbb{R}^3}\frac{r^2}{|t|}\mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}}\leq CE\tag{II.21}$$

Exactly like above when we established (II.17), modulo a slight change of ρ_0, by the mean of the mean value argument we may assume that

$$\frac{1}{\rho_0}\int_{\mathbb{R}^3\times\{-\rho_0^2\}}\frac{r^2}{|t|}\mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}}\leq C\,E\tag{II.22}$$

Observe that the monotonicity identity implies that

$$\frac{2}{\rho_0}\int_{\mathbb{R}^3\times\{-\rho_0^2\}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \geq \frac{3}{\rho_0^3}\int_{-\rho_0^2}^0\int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} = 2E \tag{II.23}$$

Thus, in order to bound E by something small (depending on η) we will just need to bound the corresponding quantity (in the left-hand-side of (II.23)) for the time $t=-\rho_0^2$. A time $-\rho_0^2$ verifying (II.12), (II.13), (II.17) and (II.22) is called a "good choice of time".

II.3 The Hodge decomposition of $u\wedge du$ at the "good time"

From now on the time is fixed to be $t=-\rho_0^2$ for which we have (II.12), (II.13), (II.17) and (II.22).

Let $g_{ij}=\delta_{ij}\, e^{-\frac{r^2}{2\rho_0^2}}$ be a new metric on $B_{\lambda\rho_0}(0)$, where λ will be fixed later on independantly on ε, u, ρ_0. If α is a 1-form on $B_{\lambda\rho_0}(0)$ and β a 2-form on $B_{\lambda\rho_0}(0)$ we have

$$*_g\alpha = e^{-\frac{r^2}{4\rho_0^2}}\ *\alpha \qquad \text{and} \qquad *_g\,\beta = e^{\frac{r^2}{4\rho_0^2}}\ *\beta \tag{II.24}$$

where $*$ is the Hodge operator for the canonical metric on $\mathbb{R}^3$. We denote by d_g^* the adjoint of d for the metric g.

Following [10] we introduce on $\mathbb{R}^3\times\{-\rho_0^2\}$

$$\begin{cases} a(x)=\dfrac{1}{|u|^2} & \text{for } |u|\geq 1/2\\[2mm] a(x)=4 & \text{for } |u|\leq 1/2\end{cases}$$

We proceed to the following Hodge decomposition of $a(x)\,u\wedge du$ on $B_{\lambda\rho_0}(0)$: Let H be the function given by

$$\begin{cases} \Delta_g H = d_g^*[a(x)\,u\wedge du] & \text{in } B_{\lambda\rho_0}(0)\\[2mm] \dfrac{\partial H}{\partial\nu} = u\wedge\dfrac{\partial u}{\partial\nu} & \text{on } \partial B_{\lambda\rho_0}(0)\end{cases} \tag{II.25}$$

we have

$$d_g^*[dH - a(x)u \wedge du] = 0 \qquad \text{on } B_{\lambda\rho_0}$$

Then there exists a 2-form ψ such that

$$d_g^*\psi = -dH + a(x)\, u \wedge du$$

II.3.1 An L^p estimate for $d_g^*\psi$ on $B_{\lambda\rho_0}$.

From (II.25) we deduce that $d_g^*\psi$ solves

$$\begin{cases} dd_g^*\psi = d[a(x)\, u \wedge du] & \text{in } B_{\lambda\rho_0}(0) \\ \\ d *_g \psi = 0 \qquad \text{on } \partial B_{\lambda\rho_0}(0) \end{cases}$$

From Lemma IV.1 of [10] we deduce that there exists $C_{\lambda,p}$ depending only on λ and p such that for any $p < 3/2$

$$\begin{aligned} \left[\frac{1}{\rho_0^{3-p}} \int_{B_{\lambda\rho_0}} |d_g^*\psi|^p\right]^{\frac{1}{p}} & \leq \frac{C_{\lambda,p}}{\rho_0} \int_{B_{\lambda\rho_0}} |d(a(x)\, u \wedge du)| \\ & \leq \frac{C_{\lambda,p}}{\rho_0} \int_{B_{\lambda\rho_0} \cap \{x\ ;\ |u(x)| \leq \frac{1}{2}\}} |\nabla u|^2 \\ & \leq \frac{C_{\lambda,p}}{\rho_0} \frac{1}{\varepsilon^2} \left| B_{\lambda\rho_0} \cap \{x\ ;\ |u(x)| \leq \frac{1}{2}\}\right| \end{aligned} \tag{II.26}$$

From (II.13) we have $\frac{1}{\rho_0} \int_{B_{\lambda\rho_0}}(0) \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C_\lambda \eta$, then

$$\frac{1}{\rho_0} \int_{B_{\lambda\rho_0} \cap \{x\ ;\ |u| \leq 1/2\}} \frac{1}{\varepsilon^2} \leq C_\lambda \eta \tag{II.27}$$

Then we deduce

Lemma II.3 *for any $p < 3/2$ we have*

$$\left[\frac{1}{\rho_0^{3-p}} \int_{B_{\lambda\rho_0}} |d_g^*\psi|^p\right]^{\frac{1}{p}} \leq C_{\lambda,p}\ \eta \tag{II.28}$$

II.3.2 An L^2-Weighted estimate for dH on $B_{\lambda\rho_0}(0)$.

(II.25) reads

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}}\dfrac{\partial H}{\partial x_i}\right]=\dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}}\ a(x)\ u\wedge\dfrac{\partial u}{\partial x_i}\right] & \text{on } B_{\lambda\rho_0}(0)\\ \dfrac{\partial H}{\partial \nu}=u\wedge\dfrac{\partial u}{\partial \nu} \qquad \text{on } \partial B_{\lambda\rho_0}(0) & \end{array}\right. \tag{II.29}$$

Let K be the solution of

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}}\dfrac{\partial K}{\partial x_i}\right]=\dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}}\ \ u\wedge\dfrac{\partial u}{\partial x_i}\right] & \text{on } B_{\lambda\rho_0}(0)\\ \dfrac{\partial K}{\partial \nu}=u\wedge\dfrac{\partial u}{\partial \nu} \qquad \text{on } \partial B_{\lambda\rho_0}(0) & \end{array}\right. \tag{II.30}$$

Multiplying the equation solved by $H-K$ and integrating by parts yields

$$\begin{aligned}\int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}|\nabla(H-K)|^2 &= \int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ (a(x)-1)\ u\wedge\frac{\partial u}{\partial x_i}.\frac{\partial H-K}{\partial x_i}\\ &\quad -\int_{\partial B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ (H-K)\,(a(x)-1)\ u\wedge\frac{\partial u}{\partial r}\end{aligned}$$

$H-K$ is chosen such that $\int_{\partial B_{\lambda\rho_0}} H-K=0$. In one hand we have

$$\begin{aligned}&\left|\int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ (a(x)-1)\ u\wedge\frac{\partial u}{\partial x_i}\ \frac{\partial H-K}{\partial x_i}\right|\le\\ &\left(\int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{\frac{1}{2}}\ \left(\int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ |\nabla H-K|^2\right)^{\frac{1}{2}}\end{aligned}$$

In the other hand

$$\begin{aligned}\frac{1}{(\lambda\rho_0)^2}\int_{\partial B_{\lambda\rho_0}}|H-K|^2 &\le \frac{1}{\lambda\rho_0}\|H-K\|^2_{H^{\frac{1}{2}}(\partial B_{\lambda\rho_0})}\\ &\le \frac{1}{\lambda\rho_0}\int_{B_{\lambda\rho_0}\setminus B_{\lambda\rho_0/2}}|\nabla(H-K)|^2\end{aligned}$$

where we have used the fact that $\int_{\partial B_{\lambda\rho_0}} H - K = 0$ which permits to prove that $\int_{B_{\lambda\rho_0}\setminus B_{\lambda\rho_0/2}} |H-K|^2 \le C(\lambda\rho_0)^2 \int_{B_{\lambda\rho_0}} |\nabla(H-K)|^2$ (see (II.40) and (II.41) below). So, using (II.14) we deduce

$$\frac{1}{\rho_0}\int_{B_{\lambda\rho_0}} e^{-\frac{r^2}{4\rho_0^2}}\ |\nabla H - K|^2 \le C\eta + C_\lambda \int_{\partial B_{\lambda\rho_0}} \frac{(1-|u|^2)^2}{\varepsilon^2}\ e^{-\frac{r^2}{4\rho_0^2}} \tag{II.31}$$

Again C is a constant that does not depend on ε, η, u, ρ_0, whereas C_λ may depend on λ.

We need the following estimate for K.

Lemma II.4 *Let K satisfying (II.30), the following inequality holds*

$$\begin{aligned}\int_{B_{\lambda\rho_0}} |\nabla K|^2\ e^{-\frac{r^2}{4\rho_0^2}} &\le C\lambda\rho_0 \int_{\partial B_{\lambda\rho_0}} \left|\frac{\partial u}{\partial r}\right|^2\ e^{-\frac{r^2}{4\rho_0^2}} \\ &\quad + C\lambda^3 e^{\lambda}\rho_0\,\eta\end{aligned} \tag{II.32}$$

Proof of Lemma II.4. In order to get rid of the parameter ρ_0 during the proof of the lemma, we proceed to the following scaling

$$\tilde K(x) = K(\rho_0 x) \qquad \tilde u(x) = u(\rho_0 x)$$

$\tilde K$ solves

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\dfrac{\partial \tilde K}{\tilde x_i}\right] = \dfrac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\tilde u \wedge \dfrac{\partial \tilde u}{\partial x_i}\right] & \text{in} \quad B_\lambda(0) \\ \dfrac{\partial \tilde K}{\partial r} = \tilde u \wedge \dfrac{\partial \tilde u}{\partial r} \qquad \text{on} \quad \partial B_\lambda(0) & \end{array}\right. \tag{II.33}$$

We can choose $\tilde K$ such that $\int_{\partial B_\lambda} \tilde K = 0$. Then standard Poincaré inequality yields

$$\int_{\partial B_\lambda} |\tilde K|^2\ e^{-\frac{\lambda^2}{4}} \le C\lambda^2 \int_{\partial B_\lambda} \left|\frac{\partial \tilde K}{\partial \tau}\right|^2\ e^{-\frac{\lambda^2}{4}} \tag{II.34}$$

Multiplying (II.33) by $x_k \frac{\partial \tilde K}{\partial x_k}$ and integrating over $B_\lambda(0)$ we obtain, after integration by parts

$$\begin{aligned}
&\int_{B_\lambda} x_k \frac{\partial K}{\partial x_k}\ \left[\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial x_i}\right]\right]\\
&=\frac{1}{2}\int_{B_\lambda} e^{-r^2}|\nabla\tilde{K}|^2-\int_{B_\lambda} r^2 e^{-\frac{r^2}{4}}\ |\nabla\tilde{K}|^2\\
&+\lambda\int_{\partial B_\lambda}\left|\frac{\partial\tilde{K}}{\partial r}\right|^2\ e^{-\frac{r^2}{4}}-\int_{\partial B_\lambda}\frac{r}{2}e^{-\frac{r^2}{4}}\ |\nabla\tilde{K}|^2
\end{aligned}\tag{II.35}$$

Thus

$$\begin{aligned}
\lambda\int_{\partial B_\lambda}|\nabla\tilde{K}|^2\ e^{-\frac{r^2}{4}}\ &\le\int_{B_\lambda}|\nabla\tilde{K}|^2+\lambda\int_{\partial B_\lambda}\left|\frac{\partial\tilde{K}}{\partial r}\right|^2\ e^{-\frac{r^2}{4}}\\
&+\lambda^2\int_{B_\lambda} e^{r^2}\ \left|\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\ \tilde{u}\wedge\frac{\partial\tilde{u}}{\partial x_i}\right]\right|^2
\end{aligned}\tag{II.36}$$

Multiplying now (II.33) by $\tilde{K}$, after integration by parts, one gets

$$\begin{aligned}
&\int_{B_\lambda}|\nabla\tilde{K}|^2\ e^{-\frac{r^2}{4}}\le\left(\frac{1}{\lambda}\int_{\partial B_\lambda}|\tilde{H}|^2\ e^{-\frac{r^2}{4}}\right)^{\frac{1}{2}}\ \left(\lambda\int_{\partial B_\lambda}\left|\frac{\partial\tilde{K}}{\partial r}\right|^2\right)^{\frac{1}{2}}\\
&+\left(\int_{B_\lambda}|\tilde{K}|^2\ e^{-\frac{r^2}{4}}\right)^{\frac{1}{2}}\ \left(\int_{B_\lambda} e^{r^2}\left|\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial x_i}\right]\right|^2\right)^{\frac{1}{2}}
\end{aligned}\tag{II.37}$$

Using (II.34) and (II.36) we get the existence of C independent of λ such that

$$\begin{aligned}
&\int_{B_\lambda}|\nabla\tilde{K}|^2\ e^{-\frac{r^2}{4}}\le C\lambda\int_{\partial B_\lambda}\left|\frac{\partial\tilde{K}}{\partial r}\right|^2+C\lambda^2\int_{B_\lambda} e^{r^2}\left|\frac{\partial}{\partial x_i}\ \left[e^{-\frac{r^2}{4}}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial x_i}\right]\right|^2\\
&+\left(\int_{B_\lambda}|\tilde{K}|^2\ e^{-\frac{r^2}{4}}\right)^{\frac{1}{2}}\ \left(\int_{B_\lambda} e^{r^2}\left|\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial x_i}\right]\right|^2\right)^{\frac{1}{2}}
\end{aligned}\tag{II.38}$$

It remains to bound $\int_{B_\lambda}|\tilde{K}|^2\ e^{-\frac{r^2}{4}}$. For any $0\le r\le\lambda$ we introduce the notation

$$\tilde{K}_r=\frac{1}{|\partial B_r|}\int_{\partial B_r}\tilde{K}=\frac{1}{4\pi}\int_{S^2}\tilde{K}(r,\sigma)\,d\sigma$$

We have

$$
\begin{aligned}
|\tilde{K}_r| = |\tilde{K}_r - \tilde{K}_\lambda| \quad &= \frac{1}{4\pi}\left|\int_{S^2} \tilde{K}(r,\sigma) - \tilde{K}(\lambda,\sigma)\ d\sigma\right| \\
&\leq \frac{\lambda^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\ \left(\int_{S^2}\int_r^\lambda \left|\frac{\partial \tilde{K}}{\partial r}\right|^2 (s,\sigma)\ ds\, d\sigma\right)^{\frac{1}{2}}
\end{aligned}
\tag{II.39}
$$

Integrating (II.39) to the square multiplied by $e^{-\frac{r^2}{4}}\ r^2$ we get

$$
\begin{aligned}
\int_0^\lambda |\tilde{K}_r|^2\ e^{-\frac{r^2}{4}} r^2\, dr \quad & \int_0^\lambda \lambda^2\, r^2\ \int_r^\lambda \int_{S^2} \left|\frac{\partial \tilde{K}}{\partial r}\right|^2 (s,\sigma)\ d\sigma\, ds\, dr \\
&\leq \lambda^3 e^{\lambda^2} \int_{S^2}\int_0^\lambda \left|\frac{\partial \tilde{K}}{\partial r}\right|^2 (s,\sigma)\ e^{-s^2} s^2\, ds\, d\sigma \\
&\leq \lambda^3 e^{\lambda^2} \int_{B_\lambda} |\nabla \tilde{K}|^2\ e^{-\frac{r^2}{4}}
\end{aligned}
\tag{II.40}
$$

Writting $|\tilde{K}|^2 \leq 2|\tilde{K}-\tilde{K}_r|^2 + 2|\tilde{K}_r|^2$, using Poincaré inequality (II.34) and (II.40) we finally get

$$
\int_{B_\lambda} |\tilde{K}|^2\ e^{-\frac{r^2}{4}} \leq C_\lambda \int_{B_\lambda} |\nabla \tilde{K}|^2\ e^{-\frac{r^2}{4}}
\tag{II.41}
$$

Combining now (II.38) and (II.41) we obtain the existence of C independent on λ such that

$$
\begin{aligned}
\int_{B_\lambda} |\nabla \tilde{K}|^2\ e^{-\frac{r^2}{4}} \quad &\leq C\lambda \int_{\partial B_\lambda} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 e^{-\frac{r^2}{4}} \\
&+ C\lambda^3\, e^\lambda \int_{B_\lambda}\quad e^{r^2} \left|\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4}}\, \tilde{u} \wedge \frac{\partial \tilde{u}}{\partial x_i}\right]\right|^2
\end{aligned}
\tag{II.42}
$$

Going back to the original scale (II.42) reads

$$
\begin{aligned}
\int_{B_\lambda} |\nabla K|^2\ e^{-\frac{r^2}{4\rho_0^2}} \quad &\leq C\lambda \int_{\partial B_{\lambda\rho_0}} \left|\frac{\partial u}{\partial r}\right|^2 e^{-\frac{r^2}{4\rho_0^2}} \\
&+ C\lambda\, e^\lambda (\lambda\rho_0)^2 \int_{B_{\lambda\rho_0}} e^{\frac{r^2}{4\rho_0^2}} \left|\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}} u \wedge \frac{\partial u}{\partial x_i}\right]\right|^2
\end{aligned}
\tag{II.43}
$$

One of the main observation is the following identity

$$\frac{\partial}{\partial x_i}\left[e^{-\frac{r^2}{4\rho_0^2}} u\wedge \frac{\partial u}{\partial x_i}\right] = -u\wedge\left[r\frac{\partial u}{\partial r} - 2\rho_0^2\frac{\partial u}{\partial t}\right]\frac{e^{-\frac{r^2}{4\rho_0^2}}}{2\rho_0^2} \tag{II.44}$$

Using the bound (II.13) on the weighted L^2-norm of the conformal derivative $r\frac{\partial u}{\partial r} + 2t\frac{\partial u}{\partial t}$, (II.45) implies

$$\int_{B_\lambda} |\nabla K|^2\ e^{-\frac{r^2}{4\rho_0^2}} \le C\lambda\rho_0 \int_{\partial B_{2\lambda\rho_0}} \left|\frac{\partial u}{\partial r}\right|^2\ e^{-\frac{r^2}{4\rho_0^2}} + C\lambda^3 e^\lambda \rho_0\eta \tag{II.45}$$

This proves Lemma II.4.

Using now the mean value formula (modulo eventually a slight change of λ) we may always assume that

$$\lambda\rho_0 \int_{\partial B_{\lambda\rho_0}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{-\frac{r^2}{4\rho_0^2}} \le C\int_{B_{2\lambda\rho_0}\setminus B_{\lambda\rho_0/2}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{-\frac{r^2}{4\rho_0^2}} \le \frac{C}{\lambda^2}\ \rho_0 E$$

where we have used (II.22). Using this fact, (II.31) and lemma II.4 we finally obtain

$$\int_{B_{\lambda\rho_0}} |\nabla H|^2\ e^{-\frac{r^2}{4\rho_0^2}} \le \frac{C}{\lambda^2}\ \rho_0 E + C\lambda^3 e^\lambda\ \rho_0\eta \tag{II.46}$$

II.4 Estimating the energy on the "main ball".

Combining (II.22) and (II.23) we deduce the existence of $\Lambda_0 > 0$ independant on ε, η, u, ρ_0 such that

$$\frac{1}{\rho_0}\int_{B_{\Lambda_0\rho_0}(0)} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{-\frac{r^2}{4\rho_0^2}} \ge \frac{E}{6} \tag{II.47}$$

We choose such a radius and we call $B_{\Lambda_0\rho_0}(0)$ the "main ball". In order to bound E it suffices to bound then

$$\frac{1}{\rho_0}\int_{B_{\Lambda_0\rho_0}(0)} \mathrm{e}_\varepsilon(u_\varepsilon)$$

II.4.1 Covering the "main ball"

We claim that there exists a covering $B_{\delta_0\rho_0}(x^i)$ $(i \in I)$ of $B_{\Lambda_0\rho_0}$ such that

i) δ_0 is independant on ρ_0, ε, η, u, E.

ii) $\forall x \in B_{\Lambda_0\rho_0}$ $\text{Card}\{i\ ;\ x \in B_{2\delta_0\rho_0}(x^i)\} \leq C_0$, where C_0 is a universal constant.

iii) $\forall \delta \in [\delta_0, 2\delta_0]$ the following bound holds

$$\frac{1}{\rho_0}\int_{B_{\Lambda_0\rho_0}(0)} \mathrm{e}_\varepsilon(u_\varepsilon) \leq C \sum_{i\in I} \int_{\partial B_{\delta\rho_0}(x^i)} \mathrm{e}_\varepsilon(u_\varepsilon) + C\eta \tag{II.48}$$

where C is independant of η, ε, ρ_0, u, E.

The existence of such a covering is a consequence of the following computation.

Call r^i the distance to x^i, we have

$$\begin{aligned} 0 &= \int_{B_\rho(x^i)} r^i \frac{\partial u}{\partial r^i}\left(-\frac{\partial u}{\partial t} + \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right) \\ &= -\int_{B_\rho(x^i)} r^i \frac{\partial u}{\partial r^i}\frac{\partial u}{\partial t} + \frac{1}{2}\int_{B_\rho(x^i)} |\nabla u|^2 \\ &\quad -\int_{\partial B_\rho(x^i)} \frac{\rho}{2}|\nabla u|^2 + \rho\int_{\partial B_\rho(x^i)} \left|\frac{\partial u}{\partial r}\right|^2 \end{aligned}$$

Thus

$$\rho\int_{\partial B_\rho(x^i)} |\nabla u|^2 + 2\int_{B_\rho(x^i)} r^i\frac{\partial u}{\partial r^i}\frac{\partial u}{\partial t} \geq \int_{B_\rho(x^i)} |\nabla u|^2$$

We have

$$\int_{B_\rho(x^i)} r^i\frac{\partial u}{\partial r^i}\frac{\partial u}{\partial t} \leq \frac{1}{4}\int_{B_\rho(x^i)} |\nabla u|^2 + \int_{B_\rho(x^i)} (r^i)^2\left|\frac{\partial u}{\partial t}\right|^2$$

Thus combining the two previous identities we have

$$\int_{B_\rho(x^i)} |\nabla u|^2 \leq 4\int_{B_\rho(x^i)} (r^i)^2\left|\frac{\partial u}{\partial t}\right|^2 + \rho\int_{\partial B_\rho(x^i)} |\nabla u|^2 \tag{II.49}$$

Take a Vitali covering of $B_{\Lambda_0\rho_0}(0)$ by balls $B_{2\delta_0\rho_0}$, such that the balls of half radius $B_{\delta_0\rho_0}(x^i)$ also cover $B_{\Lambda_0\rho_0}(0)$. Any point of $B_{\Lambda_0\rho_0}(0)$ is then covered by a number of balls $B_{2\delta_0\rho_0}$ which is bounded by a universal constant and the number of balls needed depends only on Λ_0 and δ_0. We apply then (II.49) to each ball of this covering for any radius δ between δ_0 and $2\delta_0$ and summing over i we get

$$\int_{B_{\Lambda_0\rho_0}(0)} |\nabla u|^2 \leq C\int_{B_{\lambda_0\rho_0}(0)} \delta_0^2\rho_0^2\left|\frac{\partial u}{\partial t}\right|^2 + \sum_{i\in I}\delta\rho_0\int_{\partial B_{\delta\rho_0}(x^i)}|\nabla u|^2$$

From (II.13) we have

$$\int_{B_{\Lambda_0\rho_0}(0)} \rho_0\left|\frac{\partial u}{\partial t}\right|^2 \leq C\eta + \frac{C}{\rho_0}\int_{B_{\Lambda_0\rho_0}(0)}\left|\frac{\partial u}{\partial r}\right|^2$$

Thus, combining the two previous inequalities, we obtain

$$\int_{B_{\Lambda_0\rho_0}(0)} |\nabla u|^2 \leq C\eta + C\delta_0^2\int_{B_{\Lambda_0\rho_0}(0)}|\nabla u|^2 + \sum_{i\in I}\delta\rho_0\int_{\partial B_{\delta\rho_0}(x^i)}|\nabla u|^2$$

For δ_0 sufficiently small, independant of ρ_0, u, η, ε, we establish (II.48).

Finally in order to bound E it remains to bound the energy of u on the spheres $\partial B_\delta(x^i)$ for some δ well chosen between δ_0 and $2\delta_0$.

II.4.2 Bounding the $L^{2,\infty}$ norm of $d_g^*\psi$ on $\partial B_{\delta\rho_0}(x^i)$.

Because of (II.14) we can find $\Lambda_1\in[\Lambda_0;2\Lambda_0]$ such that

$$\int_{\partial B_{\Lambda_1\rho_0}(0)}\frac{(1-|u|^2)^2}{\varepsilon^2}\leq Ce^{2\Lambda_0^2}\,\eta$$

Thus for η small enough, since $|\nabla|u||\leq C/\varepsilon$, we can ensure that

$$|u|\geq 1/2\qquad\text{on }\partial B_{\Lambda_1\rho_0}(0)$$

Take χ_{Λ_1} to be a characteristic function of $B_{\Lambda_1\rho_0}(0)$, then we have

$$d\left[\chi_{\Lambda_1}\ a(x)\,u\wedge du\right]=\chi_{\Lambda_1}d\left[a(x)\ u\wedge du\right]\tag{II.50}$$

In order to fix ψ itself (once $d_g^*\psi$ is given by $a(x)u\wedge du-dH$) we can impose the restriction of ψ to $\partial B_{\Lambda_1\rho_0}(0)$ to be zero with $d\psi=0$ on $B_{\Lambda_1\rho_0}(0)$. Then we may control the L^2 norm of ψ by the mean of the $L^{6/5}$ norm of $d_g^*\psi$, precisely we have

$$\frac{1}{\rho_0^3}\int_{B_{\Lambda_1\rho_0}(0)}|\psi|^2\le C\left[\frac{1}{\rho_1^{3-6/5}}\int_{B_{\Lambda_1\rho_0}(0)}|d_g^*\psi|^{\frac{6}{5}}\right]^{\frac{5}{6}}\le C_\lambda\eta \tag{II.51}$$

On $B_{\Lambda_1\rho_0}$ ψ verifies

$$d\left[e^{\frac{r^2}{4\rho_0^2}}d^*\left[e^{\frac{r^2}{4\rho_0^2}}\psi\right]\right]=d\left[\chi_{\Lambda_1}\ a(x)\,u\wedge du\right]$$

Let $\hat{\psi}$ be the solution of

$$\begin{aligned}\hat{\psi}\ &=-\frac{1}{4\pi|x|}\star e^{-\frac{r^2}{2\rho_0^2}}d\left[\chi_{\Lambda_1}\ a(x)\,u\wedge du\right]\\ &=-\frac{1}{4\pi|x|}\star e^{-\frac{r^2}{2\rho_0^2}}\chi_{\Lambda_1}d\left[a(x)\,u\wedge du\right]\end{aligned}\tag{II.52}$$

where we have used (II.50). Since

$$\chi_{\Lambda_1}d[a(x)\ u\wedge du]=4du\wedge du\ \chi_{\{x\in B_{\Lambda_1\rho_0}(0)\ ;\ |u(x)|\le 1/2\}}$$

where $\chi_{\{x\in B_{\Lambda_1\rho_0}(0)\ ;\ |u(x)|\le 1/2\}}$ denotes the set characteristic function of $\{x\in B_{\Lambda_1\rho_0}(0)\ ;\ |u(x)|\le 1/2\}$, using (II.13), we have

$$\begin{aligned}\|\chi_{\Lambda_1}d[a(x)\ u\wedge du]\|_{L^1}\ &\le C\|\nabla u\|_{L^\infty}^2\,|\{x\in B_{\Lambda_1\rho_0}\ ;\ |u(x)|\le 1/2\}|\\ &\le\frac{C}{\varepsilon^2}\int_{B_{\Lambda_1\rho_0}}(1-|u|^2)^2\le C\rho_0\eta\end{aligned}$$

By standard results on singular operators we get that for any $p<3/2$, we have

$$\frac{1}{\rho_0^{3-p}}\int_{B_{\Lambda_1\rho_0}(0)}|\nabla\hat{\psi}|^p\le C\eta \tag{II.53}$$

More crucialy, using lemma IV.2 of [10], we deduce the existence of a constant C independant on ε, η, ρ_0, E and u such that for every $i\in I$

$$\left|\left\{r\ ;\ \delta_0\rho_0\le r\le 2\delta_0\rho_0\quad \frac{1}{\rho_0^{1/2}}\|\nabla\hat{\psi}\|_{L^{2,\infty}(\partial B_r(x^i))}\le C\eta^{\frac{1}{2}}\right\}\right|\ge\frac{19}{20}\delta_0\rho_0 \quad \text{(II.54)}$$

Using the explicit expression for Δ_g it is not difficult to see that $\Delta(\psi-\hat{\psi})$ can be expressed in terms of first order derivatives at most of ψ and $\hat{\psi}$ so using (II.28), (II.51) and (II.53) we see that we can replace $\nabla\hat{\psi}$ by $d_g^*\psi$ in (II.54). Then we get for any $i\in I$

$$\left|\left\{r\ ;\ \delta_0\rho_0\le r\le 2\delta_0\rho_0\quad \frac{1}{\rho_0^{1/2}}\|d_g^*\psi\|_{L^{2,\infty}(\partial B_r(x^i))}\le C\eta^{\frac{1}{2}}\right\}\right|\ge\frac{19}{20}\delta_0\rho_0 \quad \text{(II.55)}$$

II.4.3 Bounding the $L^{2,1}-$norm of $d_g^*\psi$ on spheres $\partial B_{\delta\rho_0}(x^i)$.

(I.1) implies

$$u\wedge\frac{\partial u}{\partial t}=u\wedge\Delta u=d^*(u\wedge du).$$

Thus, the computation of $-\Delta(u\wedge du)$ yields

$$-\Delta(u\wedge du)=d\left(u\wedge\frac{\partial u}{\partial t}\right)+d^*\left[du\wedge du\right] \quad \text{(II.56)}$$

Using standard extension theorems we may find $\hat{u}\in W^{1,2}(\mathbb{R}^3,\mathbb{C})$ such that $\hat{u}=u$ on $B_{\Lambda_1\rho_0}(0)$ verifying

$$\|\nabla\hat{u}\|_{L^2(\mathbb{R}^3)}\le C\|\nabla u\|_{L^2(B_{\Lambda_1\rho_0})}$$

Let ξ be the 2-form in $\mathbb{R}^3$ given by

$$\xi=-\frac{1}{4\pi|x|}\star\left[2d\hat{u}_1\wedge d\hat{u}_2\right]$$

As it was observed in [10], using the result of R.Coifman, P.-L.Lions, Y.Meyer and S.Semmes, we have

$$\|d\hat{u}_1\wedge d\hat{u}_2\|_{\mathcal{H}^1(\mathbb{R}^3)}\le C\|du_1\|_{L^2(B_{\Lambda_1\rho_0})}\ \|du_2\|_{L^2(B_{\Lambda_1\rho_0})}\le C\rho_0\ E$$

where $\mathcal{H}^1(\mathbb{R}^3)$ denotes the Hardy norm in $\mathbb{R}^3$. Standard results on singular integrals and Hardy spaces imply (see [4]) that

$$\|\nabla^2 \xi\|_{L^1(\mathbb{R}^3)} \leq \|\Delta \xi\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq C \ \rho_0 \ E \tag{II.57}$$

From the definition of ξ and the identity (II.56) above we deduce that

$$-\Delta \left(u \wedge du - d^* \xi\right) = d \left(u \wedge \frac{\partial u}{\partial t} \right)$$

Following (II.13) we have

$$\rho_0 \int_{B_{\Lambda \rho_0}(0)} \left| \frac{\partial u}{\partial t} \right|^2 \leq C \ E \tag{II.58}$$

Thus we deduce, since $\frac{1}{\rho_0} \int_{B_{\Lambda_1 \rho_0}} |u \wedge du|^2 \leq C \, E$,

$$\frac{1}{\rho_0} \int_{B_{\Lambda_0 \rho_0}(0)} |\nabla (u \wedge du - d^* \xi)| \leq C \ E^{\frac{1}{2}} \tag{II.59}$$

So combining (II.57) and (II.59) we deduce

$$\frac{1}{\rho_0} \int_{B_{\Lambda_0 \rho_0}(0)} |\nabla (u \wedge du)| \leq C \left[E + E^{\frac{1}{2}} \right] \tag{II.60}$$

Since H solves

$$\frac{\partial}{\partial x_i} \left[e^{-\frac{r^2}{4\rho_0^2}} \frac{\partial H}{\partial x_i} \right] = \left[-\frac{x_i}{2\rho_0^2} u \wedge \frac{\partial u}{\partial x_i} + u \wedge \frac{\partial u}{\partial t} \right] \ e^{-\frac{r^2}{4\rho_0^2}} \tag{II.61}$$

using (II.46) and (II.13), we have

$$\frac{1}{\rho_0} \int_{B_{\Lambda_0 \rho_0}(0)} |\nabla^2 H| \leq CE + C_\lambda \eta \tag{II.62}$$

Combining (II.62) and (II.60), we get

$$\frac{1}{\rho_0} \int_{B_{\Lambda_0 \rho_0}(0)} |\nabla (d_g^* \psi)| \leq C \left[E + E^{\frac{1}{2}} \right] + C_\lambda \eta^{\frac{1}{2}} \tag{II.63}$$

Using the previous estimate, Fubini and the injection of $W^{1,1}(\partial B_{\delta\rho_0})$ into the Lorentz space $L^{2,1}(\partial B_{\delta\rho_0})$ (see [17]), in the spirit of the approach in [10], we get for any $i \in I$,

$$
\begin{aligned}
&|\{r\ ;\ \delta_0\rho_0 \le r \le 2\delta_0\rho_0 \\
&\rho_0^{-\frac{1}{2}}\|d_g^*\psi\|_{L^{2,1}(\partial B_r(x^i))} \le C\left[E + E^{\frac{1}{2}} + C_\lambda \eta^{\frac{1}{2}}\right]\Big\}\Big| \ge \frac{19}{20}\delta_0\rho_0
\end{aligned}
\tag{II.64}
$$

II.4.4 An L^2 bound of $u \wedge du$ on the spheres $\partial B_{\delta\rho_0}(x^i)$.

Combining (II.46), (II.55), (II.64) and using the interpolation between $L^{2,\infty}$ and $L^{2,1}$ into L^2 we obtain

$$
\begin{aligned}
&|\{r\ ;\ \delta_0\rho_0 \le r \le 2\delta_0\rho_0 \\
&\rho_0^{-\frac{1}{2}}\|u \wedge du\|_{L^2(\partial B_r(x^i))} \le C(\eta^{\frac{1}{4}} + \lambda^{-1})\ \left[E^{\frac{1}{2}} + E^{\frac{1}{4}} + C_\lambda \eta^{\frac{1}{4}}\right]\Big\}\Big| \ge \frac{18}{20}\delta_0\rho_0
\end{aligned}
\tag{II.65}
$$

II.4.5 An L^2 bound of $d|u|$ on the spheres $\partial B_{\delta\rho_0}(x^i)$.

Let $\Lambda \in [\Lambda_0, \Lambda_1]$ to be fixed later on. For $0 < \mu < 1$ denote $\Omega_\mu = \{x \in B_{\Lambda\rho_0}(0)\ ;\ |u|^2 \ge 1 - \mu\}$. (I.1) implies on the set where $|u| \ge 1 - \mu$

$$
-\frac{\partial |u|}{\partial t} + \Delta|u| + |u|\left(\frac{1 - |u|^2}{\varepsilon^2} - \frac{|u \wedge du|^2}{|u|^4}\right) = 0
$$

Let χ be the following cutoff function $\chi(s) \equiv 0$ on $[0, 1-2\mu]$. Multiplying the above equation by $\chi(|u|)(1 - |u|)$ integrating by parts and using also (II.58) and the fact that $\rho_0 > \varepsilon$ we obtain

$$
\begin{aligned}
\frac{1}{\rho_0}\int_{\Omega_\mu} |\nabla|u||^2 \ &\le \mu E + \frac{C}{\mu\rho_0}\int_{B_{\Lambda\rho_0}(0)} \frac{(1 - |u|^2)^2}{\varepsilon^2} \\
&+ \left(\int_{\partial B_{\Lambda\rho_0}(0)} (1 - |u|^2)^2\right)^{\frac{1}{2}}\ \left(\int_{\partial B_{\Lambda\rho_0}(0)} |\nabla|u||^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Using the mean value formula we can extract $\Lambda \in [\Lambda_0, \Lambda_1]$ such that

$$
\int_{\partial B_{\Lambda\rho_0}(0)} (1 - |u|^2)^2 \le \frac{C}{\rho_0}\int_{B_{\Lambda_1\rho_0}} (1 - |u|^2)^2
$$

and

$$\int_{\partial B_{\Lambda\rho_0}} |\nabla|u||^2 \le \frac{C}{\rho_0}\int_{B_{\Lambda_1\rho_0}} |\nabla|u||^2$$

Using now (II.13) and (II.17) we get that

$$\frac{1}{\rho_0}\int_{\Omega_\mu} |\nabla|u||^2 \le \mu\ E + \frac{C}{\mu}\eta \tag{II.66}$$

Since, by definition, $1-|u|^2 \ge \mu$ on $B_{\Lambda_0\rho_0}\setminus\Omega_\mu$ we have

$$|B_{\Lambda_0\rho_0}\setminus\Omega_\mu| \le \frac{1}{\mu^2}\int_{B_{\Lambda_0\rho_0}} (1-|u|^2)^2$$

Thus, since $\|\nabla|u|\|_\infty \le C/\varepsilon$ we deduce that

$$\frac{1}{\rho_0}\int_{B_{\Lambda_0\rho_0}\setminus\Omega_\mu} |\nabla|u||^2 \le \frac{C}{\mu^2\rho_0}\int_{B_{\Lambda_0\rho_0}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le \frac{C}{\mu^2}\eta \tag{II.67}$$

Combining (II.66) and (II.67) and choosing $\mu=\eta^{\frac{1}{4}}$ we get that

$$\frac{1}{\rho_0}\int_{B_{\Lambda_0\rho_0}} |\nabla|u||^2 \le \eta^{\frac{1}{4}}E + C\eta^{\frac{1}{2}} \tag{II.68}$$

Thus we obtain the existence of C independant on η, ρ_0, E, u, ε such that

$$\begin{aligned}&|\{r\ ;\ \delta_0\rho_0\le r\le 2\delta_0\rho_0\\ &\rho_0^{-\frac{1}{2}}\|d|u|\|_{L^2(\partial B_r(x^i))} \le C\eta^{\frac{1}{8}}\ \left[E^{\frac{1}{2}}+C\eta^{\frac{1}{8}}\right]\}\Big| \ge \tfrac{19}{20}\delta_0\rho_0\end{aligned} \tag{II.69}$$

II.5 Conclusion

Since

$$\int_{B_{2\Lambda_0\rho_0}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le C\rho_0\eta$$

we may find a constant C independent on η, ρ_0, E, u, ε such that $\forall i\in I$

$$\left|\left\{r\ ;\ \delta_0\rho_0\le r\le 2\delta_0\rho_0\quad \int_{\partial B_r(x^i)} \frac{(1-|u|^2)^2}{\varepsilon^2} \le C\eta\right\}\right| \ge \frac{19}{20}\delta_0\rho_0 \tag{II.70}$$

If η is choosen small enough, since $|\nabla|u|| \le C/\varepsilon^2$ we deduce that

$$\left|\left\{r\ ;\ \delta_0\rho_0 \le r \le 2\delta_0\rho_0 \quad |u| \ge 1/2 \quad \text{on } \partial B_{\delta\rho_0}(x^i)\right\}\right| \ge \frac{19}{20}\delta_0\rho_0 \qquad \text{(II.71)}$$

Combining now (II.65), (II.69) and (II.71) we deduce for all $i \in I$ the existence of $\delta_0 \le \delta^i \le 2\delta^i$ such that

$$\int_{\partial B_{\delta^i\rho_0}(x^i)} |\nabla u|^2 \le C(\eta^{\frac{1}{4}} + \lambda^{-2})\ (E + E^{\frac{1}{2}}) + C_\lambda \eta^{\frac{1}{2}} \qquad \text{(II.72)}$$

where C is independent on η, λ ρ_0, E, u and ε but C_λ may depend on λ. Combining now (II.47) and (II.48) we obtain

$$E \le C(\eta^{\frac{1}{4}} + \lambda^{-2})\ (E + E^{\frac{1}{2}}) + C_\lambda \eta^{\frac{1}{2}} \qquad \text{(II.73)}$$

Given any positive number $\delta > 0$ we may find λ large enough and η small enough such that (II.73) implies

$$E \le \delta \qquad \text{(II.74)}$$

Using now the monotonicity formula we have then

$$\frac{1}{\varepsilon^3}\int_{-\varepsilon^2}^0 \int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \le \delta \qquad \text{(II.75)}$$

In particular this gives

$$\frac{1}{\varepsilon^3}\int_{-\varepsilon^2}^0 \int_{B_\varepsilon(0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \le C\eta^{\frac{1}{2}} \qquad \text{(II.76)}$$

Since, again, $\|\nabla|u|\|_\infty \le C/\varepsilon$, the usual argument implies $|u|(0,0) \ge 1/2$. and theorem I.1 is proved.

III Proof of theorem I.2.

First of all we prove the following version of the $\eta-$ compactness

Theorem III.1 *Let $0 < \alpha < 1$, there exists $\eta > 0$ and $\Lambda > 0$ such that, for any ε sufficiently small, if u_ε is a solution of (I.1) verifying $|\nabla u_\varepsilon|_\infty \leq C/\varepsilon$, for any $x^0 \in \mathbb{R}^3$, $t^0 > \rho^2$ and $\rho > \varepsilon^\alpha$ if*

$$\frac{1}{\rho^3}\int_{t^0-\rho^2}^{t^0}\int_{B_{\Lambda\rho}(x^0)} e_\varepsilon(u_\varepsilon) \leq \eta \log\frac{1}{\varepsilon}$$

then

$$|u| \geq 1/2 \qquad in \quad [-\rho^2/2 + t^0, t^0] \times B_\rho(x^0)$$

In order to prove theorem III.1, we will need first the following lemma :

Lemma III.1 *Let u_ε be a solution of (I.1) verifying $E_\varepsilon(u^0) = O(\log\frac{1}{\varepsilon})$, for any $\rho > 0$, any $t^0 > \rho^2$ and any $x^0 \in \mathbb{R}^3$, we have*

$$\frac{1}{\rho^3}\int_{t^0-\rho^2}^{t^0}\int_{\mathbb{R}^3} \frac{|x-x^0|^2}{t^0-t} e_\varepsilon(u_\varepsilon)\ e^{\frac{|x-x^0|^2}{t-t^0}} \leq C\ \log\frac{1}{\varepsilon} \tag{III.1}$$

Proof of lemma III.1. Again we shift time by t^0 and space by x^0. From the monotonicity formula and (I.4) we deduce that

$$\frac{1}{8\rho^3}\int_{-4\rho^2}^{0}\int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\log\frac{1}{\varepsilon} \tag{III.2}$$

Using again the monotonicity formula and (III.2) we have

$$\int_\rho^{2\rho}\frac{1}{s^4}\int_{-s^2}^{0}\int_{\mathbb{R}^3} \mathrm{j}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\log\frac{1}{\varepsilon} \tag{III.3}$$

Using now the mean value formula, (III.2) and (III.3) we may extract a slice $\rho_1 \in [\rho; 2\rho]$ such that simultaneously we have

$$\frac{1}{\rho_1^3}\int_{-\rho_1^2}^{0}\int_{\mathbb{R}^3} \mathrm{j}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\log\frac{1}{\varepsilon} \quad \text{and} \quad \frac{1}{\rho_1}\int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\log\frac{1}{\varepsilon} \tag{III.4}$$

Using now lemma II.2 we deduce that

$$\frac{1}{\rho_1}\int_{-\rho_1^2}\int_{\mathbb{R}^3} \frac{|x|^2}{|t|}\mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{r^2}{4t}} \leq C\log\frac{1}{\varepsilon}$$

and lemma III.1 is proved.

Proof of theorem III.1. We assume that the hypothesis of the theorem hold for $x^0 = 0$ and $t^0 = 0$. take now x^1 such that $|x^1| \leq \rho$. Using the hypothesis of theorem III.1 (assuming $2 < \Lambda$) we have

$$\frac{1}{\rho^3}\int_{-\rho^2}^{0}\int_{B_{\Lambda\rho/2}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{|x-x^1|^2} t \leq \eta \log\frac{1}{\varepsilon} \tag{III.5}$$

Now by the mean of Lemma III.1 we have

$$\begin{aligned}&\frac{1}{\rho^3}\int_{-\rho^2}^{0}\int_{\mathbb{R}^3\setminus B_{\Lambda\rho/2}} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{|x-x^1|^2}{t}}\\ &\leq \frac{1}{\rho^3}\int_{-\rho^2}^{0}\int_{\mathbb{R}^3\setminus B_{\Lambda\rho/2}} \frac{4}{\Lambda^2}\frac{|x-x^1|^2}{|t|}\mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{|x-x^1|^2}{t}} \leq \frac{4}{\Lambda^2}\log\frac{1}{\varepsilon}\end{aligned} \tag{III.6}$$

Combining (III.5) and (III.6), since $\rho > \varepsilon^\alpha$ we have $\log\frac{\rho}{\varepsilon} > (1-\alpha)\log\frac{1}{\varepsilon}$ and then

$$\frac{1}{\rho^3}\int_{-\rho^2}^{0}\int_{\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon)\ e^{\frac{|x-x^1|^2}{t}} \leq \frac{\eta+\frac{4}{\lambda^2}}{(1-\alpha)}\log\frac{\rho}{\varepsilon} \tag{III.7}$$

We can then choose η and Λ such that $\frac{\eta+\frac{4}{\lambda^2}}{(1-\alpha)}$ is less than the η of theorem I.1 and we deduce that $|u|(x^1,0) \geq 1/2$ for any x^1 verifying $|x^1| \leq \rho$. Clearly $\frac{1}{\rho^3}\int_{-\rho^2}^{0}\int_{B_{\Lambda\rho}} \mathrm{e}_\varepsilon(u_\varepsilon) \leq \eta$ implies for any $\delta < 1/2$

$$\frac{1}{(1-\delta^2)^{\frac{3}{2}}\rho^3}\int_{-\rho^2}^{-\delta^2\rho^2}\int_{B_{\Lambda\sqrt{1-\delta^2}\rho}} \mathrm{e}_\varepsilon(u_\varepsilon) \leq \frac{4^3}{9}\eta\log\rho\varepsilon$$

Then choosing η a bit smaller than the previous one, we get the desired result and theorem III.1 is proven.

Proof of theorem I.2. Choose any $0 < \alpha < 1$. For any $(s,y) \in [0,1]\times\mathbb{R}^3 \cap \mathcal{B}_\varepsilon$, using the same argument as the one used to establish (II.13) we may assign $\rho_{(s,y)} > \varepsilon^\alpha$ such that

$$\frac{1}{\rho^3_{(s,y)}}\int_{-\rho^2_{(s,y)}+s}^{s}\int_{\mathbb{R}^3} \mathrm{j}_\varepsilon(u_\varepsilon)\ e^{\frac{|x-y|^2}{t-s}} \leq C \tag{III.8}$$

Consider now the collection of parabolic balls $[s-\rho^2_{(s,y)}, s]\times B_{\Lambda\rho_{(s,y)}}(y)$ (where Λ is given by theorem III.1) and extract from it a Besicovitch covering of $\mathcal{B}_\varepsilon$: there exists a universal number N such that i)

$$\mathcal{B}_\varepsilon \subset \cup_{i\in I}[s_i - \rho_i^2, s_i] \times B_{\Lambda\rho_i}(y_i)$$

ii) and

$$\forall z \in \mathcal{B}_\varepsilon \ \mathrm{Card}\left\{i \in I \ ; \ z \in [s_i - \rho_i^2, s_i] \times B_{\Lambda\rho_i}(y_i)\right\} \leq N$$

In one hand we have using (III.8)

$$\begin{aligned} |\mathcal{B}_\varepsilon| &\leq \frac{16}{9}\varepsilon^2 \int_{\mathcal{B}_\varepsilon} \frac{(1-|u|^2)^2}{\varepsilon^2} \\ &\leq \varepsilon^2 \sum_{i\in I} \int_{[s_i-\rho_i^2,s_i]\times B_{\Lambda\rho_i}(y_i)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C\varepsilon^2 \sum_{i\in I} \rho_i^3 \end{aligned} \tag{III.9}$$

In the other hand since $|u|(y_i) < 1/2$ and since $\rho_i \geq \varepsilon^\alpha$, we have using theorem III.1

$$\begin{aligned} \sum_{i\in I} \rho_i^3 \eta (1-\alpha) \log\frac{1}{\varepsilon} &\leq \sum_{i\in I} \int [s_i - \rho_i^2, s_i] \times B_{\Lambda\rho_i}(y_i) \mathrm{e}_\varepsilon(u_\varepsilon) \\ &\leq C \int_{[0,1]\times\mathbb{R}^3} \mathrm{e}_\varepsilon(u_\varepsilon) \leq C \log\frac{1}{\varepsilon} \end{aligned} \tag{III.10}$$

So we deduce that $\sum_{i\in I} \rho_i^3$ is bounded independantly on ε and theorem I.2 follows from this fact combined with (III.9).

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