

Sequences of Smooth Global Isothermic Immersions.

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Abstract : *In the present work we study the behavior of sequences of smooth global isothermic immersions of a given closed surface and having a uniformly bounded total curvature. We prove that, if the conformal class of this sequence is bounded in the Moduli space of the surface, it weakly converges in $W^{2,2}$ away from finitely many points, modulo extraction of a subsequence, to a possibly branched weak isothermic immersion of this surface. Moreover, if this limit happens to be smooth away from the branched points, we give an optimal description of the possible loss of strong compactness of such a subsequence by proving that, beside possibly finitely many atomic concentrations, the defect measure associated to the L^2 norm of the second fundamental form is "transported" along exceptional directions given by some holomorphic quadratic forms associated the limiting surface. We give examples where such a loss of compactness, invariant along such exceptional directions, eventually happen.*

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I Introduction to Global Isothermic Immersions.

I.1 The origin of isothermic in the XIXth century's surface geometry in \mathbb{R}^3 and its generalization to arbitrary codimensions.

The notion of isothermic surfaces has been introduced in the second half of the XIX century and was in particular studied by E. Bour, E.B. Christoffel and G. Darboux in the context of conjugated families of surfaces. The issue was to find pairs of distinct, non homothetic, immersions into \mathbb{R}^3 , $\vec{\Phi}$ and \vec{L} of the 2 dimensional disc D^2 "dual" to each other in the following sense¹ :

$$\partial_{x_i} \vec{\Phi} \quad \text{is parallel to} \quad \partial_{x_i} \vec{L} \quad \text{for } i = 1, 2 \quad (\text{I.1})$$

and the two induced metric on D^2 are conformal to each other :

$$\vec{L}^* g_{\mathbb{R}^3} = e^{2u} \vec{\Phi}^* g_{\mathbb{R}^3} \quad (\text{I.2})$$

where $g_{\mathbb{R}^3}$ is the standard metric on \mathbb{R}^3 and u is an arbitrary function on D^2 .

E. Bour and E.B. Christoffel proved respectively in [Bou] and [Chr] that the non trivial solutions to this question are immersions which posses around every point conformal (or isothermic) coordinates such that the coordinate directions are principal (or curvature lines). In other words if $\vec{n}_{\vec{\Phi}}$ denotes the Gauss Map of such an immersion

$$\vec{n}_{\vec{\Phi}} := \frac{\partial_{x_1} \vec{\Phi} \times \partial_{x_2} \vec{\Phi}}{|\partial_{x_1} \vec{\Phi} \times \partial_{x_2} \vec{\Phi}|}$$

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¹Darboux formulated the problem this way (see [Da2]) : *Proposons nous de rechercher tous les cas dans lesquels la correspondance par plan tangents parallèles établie entre deux surfaces peut donner une représentation conforme ou un tracé géographique de l'une des surfaces sur l'autre.*

around each point there exists (x_1, x_2) coordinates such that the induced metric is conformal

$$\vec{\Phi}^* g_{\mathbb{R}^3} = e^{2\lambda} [dx_1^2 + dx_2^2] \quad (\text{I.3})$$

and

$$\langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle = \langle \partial_{x_2} \vec{n}_{\vec{\Phi}}, \partial_{x_1} \vec{\Phi} \rangle = 0 \quad (\text{I.4})$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^3 , which also means that the second fundamental form is diagonal in these conformal coordinates :

$$\vec{\mathbb{I}} = -e^{-2\lambda} \left[\langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_1} \vec{\Phi} \rangle dx_1^2 + \langle \partial_{x_2} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle dx_2^2 \right] \vec{n}_{\vec{\Phi}} \quad .$$

where $e^\lambda = |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$. If (I.3) and (I.4) hold one says that the curvature lines are *isothermic* and, following Darboux, such a surface is called *isothermic surface*. Since that time example of isothermic surfaces were known such as *axially symmetric surfaces* or *constant mean curvature surfaces* including of course *minimal surfaces*.

In order to extend the notion of isothermic surfaces to immersions into \mathbb{R}^n for an arbitrary $n > 2$ we need to reformulate the pair of constraints (I.1) and (I.2) or equivalently the pair of constraints (I.3) and (I.4) but also to relax slightly this assumption.

We recall the definition of the *Weingarten form* \vec{h}_0 of an immersion $\vec{\Phi}$ into \mathbb{R}^3 , in an arbitrary choice of complex coordinates,

$$\begin{aligned} \vec{h}_0 &:= -e^{-2\lambda} \langle \partial_z \vec{n}_{\vec{\Phi}}, \partial_z \vec{\Phi} \rangle dz \otimes dz \\ &= -\frac{e^{-2\lambda}}{4} \left[\langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_1} \vec{\Phi} \rangle - \langle \partial_{x_2} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle - 2i \langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle \right] \vec{n}_{\vec{\Phi}} dz \otimes dz \end{aligned}$$

where $z = x_1 + ix_2$ and $\partial_z := 2^{-1} [\partial_{x_1} - i\partial_{x_2}]$.

As observed in [Ri3] we have the following result.

Proposition I.1 *A conformal immersion $\vec{\Phi}$ of the disc D^2 into \mathbb{R}^3 satisfies, around each point, except possibly a discrete subset of D^2 , (I.4) in some other local conformal chart (y_1, y_2) if and only if there exists a non zero holomorphic function $f(z)$ on D^2 such that*

$$\Im \left(\overline{f(z)} \vec{H}_0 \right) = 0 \quad , \quad (\text{I.5})$$

where $\vec{H}_0 := -4^{-1} e^{-2\lambda} \left[\langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_1} \vec{\Phi} \rangle - \langle \partial_{x_2} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle - 2i \langle \partial_{x_1} \vec{n}_{\vec{\Phi}}, \partial_{x_2} \vec{\Phi} \rangle \right] \vec{n}_{\vec{\Phi}}$ is the expression of \vec{h}_0 in the given conformal parametrization $\vec{\Phi}$ on the disc D^2 . \square

Indeed, while changing conformal coordinates and taking $w(z) := y_1(z) + iy_2(z)$ the expression of \vec{h}_0 in these new coordinates becomes

$$\vec{H}'_0 \circ w = |\partial_z w|^2 (\partial_z w)^{-2} \vec{H}_0 \quad . \quad (\text{I.6})$$

Away from the zeros of f , taking $w(z) := \sqrt{f(z)}$, (I.5) becomes

$$\Im(\vec{H}'_0) = 0 \quad ,$$

which is exactly (I.4).

We introduce on the space $\wedge^{1-0}D^2 \otimes \wedge^{1-0}D^2$ of $1-0 \otimes 1-0$ form on D^2 the following hermitian product² depending on the conformal immersion $\vec{\Phi}$

$$(\psi_1 dz \otimes dz, \psi_2 dz \otimes dz)_{WP} := e^{-4\lambda} \overline{\psi_1(z)} \psi_2(z)$$

where $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. We observe that for a conformal change of coordinate $w(z)$ (i.e. w is holomorphic in z) and for ψ'_i satisfying

$$\psi'_i \circ w dw \otimes dw = \psi_i dz \otimes dz$$

one has, using the conformal immersion $\vec{\Phi} \circ w$ in the l.h.s.

$$(\psi'_1 dw \otimes dw, \psi'_2 dw \otimes dw)_{WP} = (\psi_1 dz \otimes dz, \psi_2 dz \otimes dz)_{WP}$$

Using this change of coordinate rule, (I.5) is equivalent to the following intrinsic characterization : there exists an holomorphic section³ q of the bundle $\wedge^{1-0}D^2 \otimes \wedge^{1-0}D^2$ such that

$$\mathfrak{S}(q, \vec{h}_0)_{WP} = 0 \quad . \quad (I.7)$$

In codimension larger than 1 principal directions are not defined anymore and the XIXth century definition of isothermic immersions into \mathbb{R}^3 cannot be extended in a straightforward way for immersions into \mathbb{R}^m ($m > 3$). However, considering a smooth immersions $\vec{\Phi}$ of an arbitrary 2-dimensional manifold Σ into \mathbb{R}^m one can still produce the global *Weingarten form* using local conformal charts as being the following global section of $\mathbb{R}^m \otimes \wedge^{1-0}\Sigma \otimes \wedge^{1-0}\Sigma$:

$$\begin{aligned} \vec{h}_0 &:= 2 e^{-2\lambda} \pi_{\vec{n}}(\partial_{z_2}^2 \vec{\Phi}) dz \otimes dz \\ &= \frac{e^{-2\lambda}}{2} \pi_{\vec{n}} \left(\partial_{x_1}^2 \vec{\Phi} - \partial_{x_2}^2 \vec{\Phi} - 2i \partial_{x_1 x_2}^2 \vec{\Phi} \right) dz \otimes dz \end{aligned} \quad (I.8)$$

where $\pi_{\vec{n}}$ is the orthogonal projection onto the plane orthogonal to $\vec{\Phi}_*T\Sigma$. We can now introduce the natural generalization of global smooth isothermic surfaces into arbitrary euclidian space \mathbb{R}^m .

Definition I.1 *Let $\vec{\Phi}$ be a smooth immersion of a two dimensional manifold Σ^2 into \mathbb{R}^m . One says that $\vec{\Phi}$ is **global isothermic** if there exists an holomorphic quadratic form q of the riemann surface issued from Σ^2 equipped with the pull back metric $g := \vec{\Phi}^*g_{\mathbb{R}^m}$ of the standard metric $g_{\mathbb{R}^m}$ of \mathbb{R}^m such that*

$$\mathfrak{S}(q, \vec{h}_0)_{WP} = 0 \quad . \quad (I.9)$$

where \vec{h}_0 is the *Weingarten form* of the immersion $\vec{\Phi}$ given by (I.8). □

I.2 The role of Isothermic surfaces in the calculus of variations of the Willmore Lagrangian.

In this work we are interested with analysis properties of Smooth global isothermic immersions. One of the main reasons why looking at the analysis of global isothermic immersions comes from the fact that they may arise as degenerate critical point to the conformal constrained Willmore problem, as it has been shown in [Ri3]. In his 3 volumes book on differential geometry published by Springer around 1929 Wilhelm Blaschke, (see in particular the third volume [Bla]) proposed a theory merging *minimal surface theory* and *conformal invariance*. This theory consists in studying the variations of the now so called

²This hermitian product integrated on D^2 is the *Weil Peterson product*.

³In complex coordinates $q = f(z) dz \otimes dz$ where f is holomorphic and q is called an holomorphic quadratic form.

Willmore Lagrangian for surfaces. This lagrangian, W , is given by the L^2 norm of the mean curvature vector \vec{H} of an arbitrary immersion $\vec{\Phi}$ into the euclidian space \mathbb{R}^m ($m \geq 3$) of a given 2-dimensional abstract manifold Σ and integrated with respect to the induced metric⁴ g

$$W(\vec{\Phi}) := \int_{\Sigma} |\vec{H}|^2 dvol_g \quad . \quad (\text{I.10})$$

Immersiones satisfying $W(\vec{\Phi}) < +\infty$ are called *immersions of finite total curvature*.

Minimal immersions, satisfying $\vec{H} \equiv 0$, are clearly critical points to W . Blaschke observed⁵ moreover the following conformal invariance of the lagrangian W : for any conformal diifeomorphism Ψ of $\mathbb{R}^m \cup \{\infty\}$ into itself which does not send any point of $\Phi(\Sigma)$ to infinity one has

$$W(\Psi \circ \vec{\Phi}) = W(\vec{\Phi}) \quad . \quad (\text{I.11})$$

Hence, as a consequence, any composition of a minimal surface with a conformal diffeomorphism is still a critical point of W without being necessarily minimal anymore. Though the space of critical points of W happens to be much broader than such compositions, Blaschke decided nevertheless to call such an immersion a *conformal minimal immersion*⁶. *Conformal minimal immersions* are nowadays known under the denomination *Willmore surfaces*. Example of such surfaces are given for instance by minimal surfaces in \mathbb{R}^m or stereographic projections into \mathbb{R}^m of minimal surfaces in S^m , constant mean curvature surfaces in \mathbb{R}^3 and the compositions of all these surfaces with conformal transformations . It has been proven in [Ri2] that an immersion $\vec{\Phi}$ is a critical point to W if and only if it satisfies

$$d^{*g} \left[d\vec{H} - 3 D\vec{H} + \star(*_g d\vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right] = 0 \quad (\text{I.12})$$

where $*_g$ is the Hodge operator on Σ associated to the induced metric $g := \vec{\Phi}^* g_{\mathbb{R}^m}$, $D\vec{H}$ is the covariant differentiation of the section \vec{H} of the normal bundle $(\vec{\Phi}_* T\Sigma)^\perp$, it is also given by

$$D\vec{H} := \pi_{\vec{n}}(d\vec{H})$$

where $\pi_{\vec{n}}$ denotes the orthogonal projection onto the fibers of $(\vec{\Phi}_* T\Sigma)^\perp$. Finally \star denotes the Hodge operator from $\wedge^p \mathbb{R}^m$ into $\wedge^{m-p} \mathbb{R}^m$ for the canonical metric of \mathbb{R}^m satisfying

$$\forall \alpha \beta \in \wedge^p \mathbb{R}^m \quad \alpha \wedge \star \beta = (\alpha, \beta) \varepsilon_1 \wedge \cdots \wedge \varepsilon_m$$

where ε_i is the canonical basis of \mathbb{R}^m and (\cdot, \cdot) denotes the canonical scalar product on $\wedge^p \mathbb{R}^m$. In conformal coordinates for the induced metric g equation (I.12) becomes.

$$div \left(\nabla \vec{H} - 3 \pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right) = 0 \quad . \quad (\text{I.13})$$

While exploring the existence and properties of critical points to the Willmore energy (I.11) , or in other words while proceeding to the calculus of variation of the Lagrangian W , it is natural to raise the question of the conformal class such an immersion defines on the abstract 2-manifold Σ . As a channel of consequences it is then natural to explore minimizers or critical points to W when the conformal class induced by $\vec{\Phi}^* g_{\mathbb{R}^m}$ is fixed. Assuming such a critical point is a non degenerate point for the conformal class constraint, it has been proved in [Ri3] that $\vec{\Phi}$ satisfies this time

$$d^{*g} \left[d\vec{H} - 3 D\vec{H} + \star(*_g d\vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right] = \mathfrak{S}(q, \vec{h}_0)_{WP} \quad (\text{I.14})$$

⁴ $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ where $g_{\mathbb{R}^m}$ is the canonical flat metric of \mathbb{R}^m .

⁵This invariance was proved by Wilhelm Blaschke for $m = 3$ and later on generalized by Bang-Yen Chen to arbitrary m

⁶Probably in order to insist on the merging of the two requirements for this theory to include minimal surfaces and conformal invariance

for some holomorphic quadratic differential q associated to the fixed conformal class. q plays here the role of a Lagrange multiplier. Equation (I.14) has been called *Constrained Willmore equation* (see [BPP] for instance) but in order to avoid any ambiguity with the other constrained problems for the Willmore lagrangian (such as the Isoperimetric ratio for instance - see [Sy]) we prefer to call equation (I.14) the *Constrained-conformal Willmore equation*.

Examples of solutions to (I.14), which are not necessarily solutions⁷ to (I.12) are given for instance by **parallel mean curvature surfaces** : surfaces that generalize to arbitrary codimensions the **constant mean curvature equation** and that are characterized by the following condition

$$D\vec{H} = \pi_{\vec{n}}(d\vec{H}) \equiv 0 \quad . \quad (\text{I.15})$$

Indeed, the Codazzi-Mainardi identity for a general conformal immersion $\vec{\Phi}$ of the disc D^2 reads (see [Ri1])

$$e^{-2\lambda} \partial_{\bar{z}} \left(e^{2\lambda} \vec{H}_0 \cdot \vec{H} \right) = \vec{H} \cdot \partial_z \vec{H} + \vec{H}_0 \cdot \partial_{\bar{z}} \vec{H} \quad , \quad (\text{I.16})$$

where $z = x_1 + ix_2$ and $\partial_z := 2^{-1}(\partial_{x_1} - i\partial_{x_2})$. Since we are assuming (I.15) we have then

$$f(z) := e^{2\lambda} \vec{H}_0 \cdot \vec{H} \quad \text{is holomorphic.} \quad (\text{I.17})$$

In [Ri1] it is proven that, for a general conformal immersion $\vec{\Phi}$ of the disc D^2 , one has

$$\text{div} \left(\nabla \vec{H} - 3 \pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right) = -8 \Re \left(\partial_{\bar{z}} \left[\pi_{\vec{n}}(\partial_z \vec{H}) + e^{2\lambda} \vec{H}_0 \cdot \vec{H} e^{-2\lambda} \partial_{\bar{z}} \vec{\Phi} \right] \right) \quad (\text{I.18})$$

Assuming (I.15), (I.18) becomes

$$\text{div} \left(\nabla \vec{H} - 3 \pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right) = -8 \Re \left(f(z) \partial_{\bar{z}} \left[e^{-2\lambda} \partial_{\bar{z}} \vec{\Phi} \right] \right) \quad . \quad (\text{I.19})$$

For a general conformal immersion $\vec{\Phi}$ of the disc D^2 , one has (see [Ri1])

$$\vec{H}_0 = 2 \partial_z \left[e^{-2\lambda} \partial_{\bar{z}} \vec{\Phi} \right] \quad . \quad (\text{I.20})$$

Hence (I.19) becomes

$$\text{div} \left(\nabla \vec{H} - 3 \pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right) = \Im \left(4 i f(z) \overline{\vec{H}_0} \right) \quad , \quad (\text{I.21})$$

which is exactly the *constrained-conformal Willmore equation* (I.14) written in complex coordinates.

If instead the critical point is a degenerate point of the conformal constraint, it is proved in [Ri3] that there exists a non trivial holomorphic quadratic differential q such that

$$\Im(q, \vec{h}_0)_{WP} \equiv 0 \quad (\text{I.22})$$

in other words, $\vec{\Phi}$ is *isothermic*.

We have proven in [Ri3] (see propositions I.2 and I.3) that, if $\vec{\Phi}$ is isothermic, away from the zeros of q , there exists locally complex coordinates $z = x_1 + ix_2$ in which the condition (I.7) reads

$$\frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_1} \right] = 0 \quad . \quad (\text{I.23})$$

⁷Surfaces of non-zero constant mean curvature in \mathbb{R}^3 which are Willmore have to be umbilic and then coincide with a plane or a round sphere.

where $e^\lambda = |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$ is the conformal factor.

Making a similar choice of conformal coordinates for the induced metric g equation (I.14) becomes.

$$\operatorname{div} \left(\nabla \vec{H} - 3 \pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H}) \right) = Q \left(\frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_1} \right] \right) . \quad (\text{I.24})$$

where⁸ $Q := |q|_{WP} \in \mathbb{R}^+$ (The WP -norm is taken with respect to the constant scalar curvature metric of volume 1 on Σ).

The **Isothermic equation** (I.23) is an **hyperbolic** equation whereas the **Constrained-conformal Willmore equation** (I.24) is an **elliptic** one. One passes from (I.24) to (I.23) in particular when the norm of the Lagrange-multiplier goes to infinity $Q = \varepsilon^{-2} \rightarrow +\infty$. Precisely in [Ri3] section IV we have proven the following result

Theorem I.1 [Ri3] *Let $\vec{\Phi}_k$ be a sequence of conformal immersion from D^2 into \mathbb{R}^m satisfying asymptotically the constrained-conformal equation :*

$$\begin{aligned} & \operatorname{div} \left(\nabla \vec{H}_k - 3 \pi_{\vec{n}_k}(\nabla \vec{H}_k) + \star(\nabla^\perp \vec{n}_{\vec{\Phi}_k} \wedge \vec{H}_k) \right) - Q_k \left(\frac{\partial}{\partial x_1} \left[e^{-2\lambda_k} \frac{\partial \vec{\Phi}_k}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda_k} \frac{\partial \vec{\Phi}_k}{\partial x_1} \right] \right) \\ & \longrightarrow 0 \quad \text{strongly in } (W^{2,2} \cap W^{1,\infty})^* \end{aligned} \quad (\text{I.25})$$

for some sequence $Q_k \in \mathbb{R}^+$. Assume

$$\|\lambda_k\|_{L^\infty(D^2)} + \|\nabla \vec{n}_{\vec{\Phi}_k}\|_{L^2(D^2)} \leq C < +\infty . \quad (\text{I.26})$$

If

$$\limsup_{k \rightarrow +\infty} Q_k < +\infty$$

then, modulo extraction of a subsequence, $\vec{\Phi}_k$ converges weakly⁹ in $W_{loc}^{2,2}$ to a C^∞ constrained conformal immersion (i.e. satisfying (I.24) for some $Q \in \mathbb{R}^+$).

Alternatively, if instead,

$$\limsup_{k \rightarrow +\infty} Q_k = +\infty$$

there exists a subsequence of $\vec{\Phi}_k$ converging weakly in $W_{loc}^{2,2}$ to a conformal lipschitz $W_{loc}^{2,2}$ isothermic immersion (i.e. satisfying (I.23)) \square

In this sense the **isothermic surface equation** should be seen as an **hyperbolic degeneracy** of the **constrained conformal equation** which represents some **viscous approximation** of the first one.

Remark I.1 *An interesting issue is to understand if the solution to (I.23) that are obtained as weak limits of the viscous approximation (I.24) enjoy some additional regularity properties which are not shared with the arbitrary $W^{2,2}$ conformal solutions to (I.23) .* \square

⁸By dilating these conformal coordinates one can always make $Q = 1$ in (I.24) - except when $q = 0$ of course - but we prefer to normalize the conformal coordinates for them not to degenerate as the Weil-Petersson norm of the Lagrange multiplier $|q|_{WP}$ would go either to 0 or $+\infty$

⁹in this case the weak $W_{loc}^{2,2}$ convergence should even be strong.

I.3 Weak Global Isothermic Immersions.

The previous result, theorem I.1, shows the importance of enlarging the class of *smooth global isothermic immersions* to a wider class of *weak global isothermic immersions*. For analysis reasons it is also needed to enlarge the class of C^1 immersions while studying critical points to the Willmore functional (I.10). In [Ri3] the author introduced the framework of *weak immersion with finite total curvature* (or simply *weak immersions*).

Let g_0 be a reference smooth metric on Σ . One defines the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^m)$ of measurable maps from Σ into \mathbb{R}^m in the following way

$$W^{k,p}(\Sigma, \mathbb{R}^m) = \left\{ f \text{ meas. } \Sigma \rightarrow \mathbb{R}^m \text{ s.t. } \sum_{l=0}^k \int_{\Sigma} |\nabla^l f|_{g_0}^p \, dvol_{g_0} < +\infty \right\}$$

Since Σ is assumed to be compact it is not difficult to see that this space is independent of the choice we have made of g_0 .

First we need to have a weak first fundamental form that is we need $\vec{\Phi}^* g_{\mathbb{R}^m}$ to define an L^∞ metric with a bounded inverse. The last requirement is satisfied if we assume that $\vec{\Phi}$ is in $W^{1,\infty}(\Sigma)$ and if $d\vec{\Phi}$ has maximal rank 2 at every point with some uniform quantitative control of "how far" $d\vec{\Phi}$ is from being degenerate : there exists $c_0 > 0$ s.t.

$$|d\vec{\Phi} \wedge d\vec{\Phi}|_{g_0} \geq c_0 > 0 \quad . \quad (I.27)$$

where $d\vec{\Phi} \wedge d\vec{\Phi}$ is a 2-form on Σ taking values into 2-vectors from \mathbb{R}^m and given in local coordinates by $2 \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \, dx \wedge dy$. The condition (I.1) is again independent of the choice of the metric g_0 . For a Lipschitz immersion satisfying (I.1) we can define the Gauss map as being the following measurable map in $L^\infty(\Sigma)$ taking values in the Grassmanian of oriented $m - 2$ -planes in \mathbb{R}^m .

$$\vec{n}_{\vec{\Phi}} := \star \frac{\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}}{|\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}|} \quad .$$

We then introduce the space \mathcal{E}_Σ of *weak immersions of Σ with total finite curvature* as being the following space :

$$\mathcal{E}_\Sigma := \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^m) \quad \text{s.t. } \vec{\Phi} \text{ satisfies (I.27) for some } c_0 \\ \text{and} \quad \int_{\Sigma} |d\vec{n}|_g^2 \, dvol_g < +\infty \end{array} \right\} \quad .$$

Where $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ is the pull back by $\vec{\Phi}$ of the flat canonical metric $g_{\mathbb{R}^m}$ of \mathbb{R}^m and $dvol_g$ is the volume form associated to g .

The analysis of \mathcal{E}_Σ shows that for completeness purposes (see [Ri3]) one has to relax the fact that $\vec{\Phi}$ is globally an immersion by requiring only that $\vec{\Phi}$ is an immersion away from finitely many points. We then define the space of *branched weak immersions with finite total curvature* in the following way

$$\mathcal{F}_\Sigma := \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^m) \quad \text{s.t. } \exists a_1 \cdots a_N \in \Sigma \text{ s.t.} \\ \forall K \text{ compact in } \Sigma \setminus \{a_1 \cdots a_N\} \quad \vec{\Phi} \text{ satisfies (I.27) on } K \text{ for some } c_0(K) > 0 \\ \text{and} \quad \int_{\Sigma} |d\vec{n}|_g^2 \, dvol_g < +\infty \end{array} \right\} \quad .$$

It is proved in [Ri3] (see also [Ri1]) that any *weak immersion* $\vec{\Phi}$ in \mathcal{E}_Σ defines a smooth conformal structure on Σ : more precisely, following Toro, Müller-Sverak, Hélein's works on immersions with finite total curvature one proves (see [Ri1]) that for any $\vec{\Phi} \in \mathcal{E}_\Sigma$ and for any $p \in \Sigma$ there exists a neighborhood U containing p and a bilipshitz homeomorphism Ψ from D^2 into U such that $\vec{\Phi} \circ \Psi$ satisfies the weak conformal condition

$$\begin{cases} \partial_{x_1}(\vec{\Phi} \circ \Psi) \cdot \partial_{x_2}(\vec{\Phi} \circ \Psi) = 0 & \text{a. e. in } D^2 \\ |\partial_{x_1}(\vec{\Phi} \circ \Psi)| = |\partial_{x_2}(\vec{\Phi} \circ \Psi)| & \text{a. e. in } D^2 \end{cases}$$

moreover $\vec{\Phi} \circ \Psi$ is $W^{2,2}$ on D^2 . Hence Σ is equipped with a system of charts such that the transition functions satisfy the Cauchy-Riemann conditions almost everywhere and thus are holomorphic. This defines the conformal structure induced by $\vec{\Phi}$. The same can be done for any element of \mathcal{F}_Σ using also Huber theorem about the conformal structure of a metric of finite total curvature on a closed surface minus finitely many points.

We can now give the definition of a weak global isothermic immersion as the natural extension of definition I.1.

Definition I.2 *Let Σ^2 be a closed two dimensional manifold. One says that a weak immersion $\vec{\Phi}$ in \mathcal{E}_Σ (resp. a weak branched immersion in \mathcal{F}_Σ) is **weakly global isothermic** if there exists an holomorphic quadratic form q of the riemann surface issued from Σ^2 equipped with the conformal structure defined by $\vec{\Phi}$ such that*

$$\mathfrak{S}(q, \vec{h}_0)_{WP} = 0 \quad . \quad (\text{I.28})$$

where \vec{h}_0 is the Weingarten form of the immersion $\vec{\Phi}$ given by (I.8). □

Remark I.2 *Observe that for any $\vec{\Phi}$ in \mathcal{E}_Σ the Weingarten $1 - 0 \otimes 1 - 0$ form h_0 is a well defined L^2 section of $\wedge^{(1,0)}\Sigma \otimes \wedge^{(1,0)}\Sigma$ and therefore the function $\mathfrak{S}(q, \vec{h}_0)_{WP}$ is a well efined L^2 function on Σ for any holomorphic quadratic form q .*

The following characterization of weak global isothermic immersion has been given in [Ri3] (proposition I.3).

Proposition I.2 *A weak immersion $\vec{\Phi}$ is global isothermic if and only if around every point there exists a L^2 \mathbb{R}^m valued map \vec{L} such that the following two conditions are satisfied*

$$\begin{cases} d\vec{\Phi} \cdot d\vec{L} := [\partial_{x_1}\vec{\Phi} \cdot \partial_{x_2}\vec{L} - \partial_{x_2}\vec{\Phi} \cdot \partial_{x_1}\vec{L}] dx_1 \wedge dx_2 = 0 \\ d\vec{\Phi} \wedge d\vec{L} := [\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{L} - \partial_{x_2}\vec{\Phi} \wedge \partial_{x_1}\vec{L}] dx_1 \wedge dx_2 = 0 \end{cases} \quad (\text{I.29})$$

\vec{L} is called a **Darboux transform** of $\vec{\Phi}$. □

An elementary observation shows that property (I.29) is invariant under the action of transformations that preserves angles infinitesimally in \mathbb{R}^m . From this observation we deduce the following fundamental property.

Proposition I.3 *Let $\vec{\Phi}$ be a weak isothermic immersion of \mathcal{E}_Σ (resp. weak branched isothermic immersion of \mathcal{F}_Σ). Let Ξ be a conformal transformation of $\mathbb{R}^m \cup \{\infty\}$. Then $\Xi \circ \vec{\Phi}$ is still a weak isothermic immersion of \mathcal{E}_Σ (resp. weak branched isothermic immersion of \mathcal{F}_Σ). □*

In [BR] the following proposition is proved

Proposition I.4 *A weak immersion $\vec{\Phi} \in \mathcal{E}_\Sigma$ is constrained-conformal Willmore if and only if, around every point, there exists an L^2 \mathbb{R}^m -valued map \vec{L} such that the following two conditions are satisfied*

$$\begin{cases} d\vec{\Phi} \cdot d\vec{L} := [\partial_{x_1} \vec{\Phi} \cdot \partial_{x_2} \vec{L} - \partial_{x_2} \vec{\Phi} \cdot \partial_{x_1} \vec{L}] dx_1 \wedge dx_2 = 0 \\ d\vec{\Phi} \wedge d\vec{L} := [\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{L} - \partial_{x_2} \vec{\Phi} \wedge \partial_{x_1} \vec{L}] dx_1 \wedge dx_2 = 2 (-1)^m d(\star(\vec{n} \llcorner \vec{H})) \wedge \llcorner d\vec{\Phi} \end{cases} \quad (\text{I.30})$$

where \llcorner is the standard contraction operator between a p -vectors and a q -vectors ($p \geq q$) given by

$$\forall \vec{a} \in \wedge^p \mathbb{R}^m, \quad \forall \vec{b} \in \wedge^q \mathbb{R}^m, \quad \forall \vec{c} \in \wedge^{p-q} \mathbb{R}^m \quad \langle \vec{a} \llcorner \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{b} \wedge \vec{c} \rangle \quad .$$

and

$$d(\star(\vec{n} \llcorner \vec{H})) \wedge \llcorner d\vec{\Phi} := [\partial_{x_1} (\star(\vec{n} \llcorner \vec{H})) \llcorner \partial_{x_2} \vec{\Phi} - \partial_{x_2} (\star(\vec{n} \llcorner \vec{H})) \llcorner \partial_{x_1} \vec{\Phi}] dx_1 \wedge dx_2 \quad .$$

□

In [Ri2] it is proven that weak immersion $\vec{\Phi} \in \mathcal{E}_\Sigma$ which are *constrained-conformal Willmore* are in fact C^∞ .

Minimal surfaces in \mathbb{R}^m -satisfying $\vec{H} = 0$ - clearly solve (I.13). This means that they are Willmore and, a fortiori, they are special cases of *constrained-conformal Willmore*. Therefore, from proposition I.4, they satisfy (I.30). But since $\vec{H} = 0$ the right hand side of (I.30) is zero. Thus minimal surfaces are also satisfying (I.29) and are then **isothermic**.

More generally **parallel mean curvature surfaces**, surfaces satisfying (I.15), are also **constrained-conformal Willmore** and not necessarily Willmore, as we proved in the previous subsection, and they are also **isothermic**. Indeed, it is proven in [BR] (equation (II.6)) that, in conformal coordinates,

$$\partial_{x_1} (\star(\vec{n} \llcorner \vec{H})) \llcorner \partial_{x_2} \vec{\Phi} - \partial_{x_2} (\star(\vec{n} \llcorner \vec{H})) \llcorner \partial_{x_1} \vec{\Phi} = (-1)^{m-1} \nabla \vec{\Phi} \wedge \nabla \vec{H} \quad (\text{I.31})$$

For *parallel mean curvature surfaces*, which satisfy (I.15), we have

$$\nabla \vec{\Phi} \wedge \nabla \vec{H} = \nabla \vec{\Phi} \wedge \pi_T(\nabla \vec{H}) = \nabla^\perp \vec{H} \cdot \nabla \vec{\Phi} \vec{e}_1 \wedge \vec{e}_2 = -2 \operatorname{div}(\vec{H} \cdot \nabla^\perp \vec{\Phi}) \vec{e}_1 \wedge \vec{e}_2 = 0 \quad . \quad (\text{I.32})$$

Other examples of weak isothermic immersions which are not smooth and then not necessarily *constrained-conformal Willmore* are easy to produce : take a non necessarily smooth simple closed lipshitz curve $\gamma : S^1 \rightarrow \mathbb{R}^2$ such that

$$\int_{S^1} \kappa^2 dl < +\infty$$

where κ is the curvature distribution of that curve and dl the length 1-form on S^1 induced by the immersion γ . Identify the plane \mathbb{R}^2 with the vertical plane in \mathbb{R}^3 given by $\{x_2 = 0\}$ and rotate that curve around the x_3 vertical axis. One proves that this generates a *weak global isothermic immersion* : **axially symmetric surfaces** are **isothermic**. We saw in proposition I.3 that being isothermic is a conformally invariant property and therefore any composition of the obtained axially surface with a diffeomorphism of \mathbb{R}^3 generates another isothermic surface.

It is proven in [Ri3] (see the proof of lemma III.1) that the space of weak immersion \mathcal{E}_Σ of controlled conformal class has a nice weak closure property modulo renormalization and branched points. Precisely one has the following weak closure lemma.

Lemma I.1 [Ri3] *Let Σ be a closed two-dimensional manifold. Let $\vec{\Phi}_k$ be a sequence of elements in \mathcal{E}_Σ such that $W(\vec{\Phi}_k)$ is uniformly bounded. Assume that the conformal class of the conformal structure c_k (i.e. complex structure of Σ) defined by $\vec{\Phi}_k$ remains in a compact subspace of the Moduli space of Σ . Then,*

modulo extraction of a subsequence, the sequence c_k converges to a smooth limiting complex structure c_∞ ; and there exist a sequence of Lipschitz diffeomorphisms f_k of Σ such that $\vec{\Phi}_k \circ f_k$ is conformal from (Σ, c_k) into \mathbb{R}^m . Moreover, there exists a sequence Ξ_k of conformal diffeomorphisms of $\mathbb{R}^m \cup \{\infty\}$ and at most finitely many points $\{a_1, \dots, a_N\}$ such that

$$\limsup_{k \rightarrow +\infty} \mathcal{H}(\Xi_k \circ \vec{\Phi}_k \circ f_k(\Sigma)) < +\infty \quad , \quad \Xi_k \circ \vec{\Phi}_k \circ f_k(\Sigma) \subset B_R(0) \quad (\text{I.33})$$

for some $R > 0$ independent of k , and

$$\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } (W_{loc}^{2,2} \cap W_{loc}^{1,\infty})^*(\Sigma \setminus \{a_1, \dots, a_N\}) \quad . \quad (\text{I.34})$$

The convergences are understood with respect to h_k , which is the constant scalar curvature metric of unit volume attached to the conformal structure c_k .

Furthermore, there holds

$$\forall K \text{ compact subset of } \Sigma \setminus \{a_1, \dots, a_N\} \quad \limsup_{k \rightarrow +\infty} \| \log |d\vec{\xi}_k|_{h_k} \|_{L^\infty(K)} < +\infty \quad . \quad (\text{I.35})$$

Finally, $\vec{\xi}_\infty$ is an element of \mathcal{F}_Σ , a weak immersion of $\Sigma \setminus \{a_1, \dots, a_N\}$, and conformal from (Σ, c_∞) into \mathbb{R}^m . \square

Following the arguments of [Ri3] proof of lemma IV.1 one establishes the following weak closure result for weak isothermic immersions.

Theorem I.2 [Ri3] *Let Σ be a closed two-dimensional manifold. Let $\vec{\Phi}_k$ be a sequence of **weak global isothermic immersions** such that $W(\Phi_k)$ is uniformly bounded. Assume that the conformal classes c_k defined by $\vec{\Phi}_k$ converge to a limiting structure c_∞ in the Moduli space of Σ . Then, modulo extraction of a subsequence, there exists a sequence of Lipschitz diffeomorphisms f_k of Σ and a sequence Ξ_k of conformal diffeomorphisms of $\mathbb{R}^m \cup \{\infty\}$ such that $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k$ is a weak conformal isothermic immersion converging weakly in $W_{loc}^{2,2}$ on Σ minus finitely many points to $\vec{\xi}_\infty$ a , possibly branched at these points, conformal **weak global isothermic immersion** for the limiting conformal structure c_∞ on Σ . \square*

I.4 Weakly converging smooth global isothermic immersions - Main result.

The goal of the present paper is to present a result regarding the lack of strong compactness and the geometric structure of the defect measure for sequences of smooth global isothermic immersions weakly converging to another smooth global isothermic immersion. Our main result is the following

Theorem I.3 *Let Σ be a closed two-dimensional manifold. Let $\vec{\Phi}_k$ be a sequence of **smooth global isothermic immersions** such that $W(\Phi_k)$ is uniformly bounded. Assume that the conformal classes c_k defined by $\vec{\Phi}_k$ converge to a limiting structure c_∞ in the Moduli space of Σ . Then, modulo extraction of a subsequence, there exists a sequence of Lipschitz diffeomorphisms f_k of Σ and a sequence Ξ_k of conformal diffeomorphisms of $\mathbb{R}^m \cup \{\infty\}$ and finitely many points $\{a^1 \cdots a^n\}$ such that $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k$ is a conformal global isothermic immersion satisfying*

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W_{loc}^{2,2}(\Sigma \setminus \{a^1 \cdots a^n\}) \quad (\text{I.36})$$

where $\vec{\xi}_\infty$ a weak, possibly branched at the a^i , conformal **weak global isothermic immersion** for the limiting conformal structure c_∞ on Σ . If moreover $\vec{\xi}_\infty$ is **smooth** away from the points a^i then the following convergence holds

$$|d\vec{\eta}_{\vec{\xi}_k}|_{g_k}^2 dvol_{g_k} \rightharpoonup |d\vec{\eta}_{\vec{\xi}_\infty}|_{g_\infty}^2 dvol_{g_\infty} + \nu + \sum_{i=1}^n \alpha^i \delta_{a^i} dvol_{g_\infty} \quad \text{weakly in } \mathcal{M}(\Sigma) \quad (\text{I.37})$$

where $\mathcal{M}(\Sigma)$ is the space of Radon measures on Σ and ν , the non atomic part of the defect measure, satisfies the following condition : around every point different from the a^i there exists a conformal coordinate chart $z = x_1 + ix_2$ such that, simultaneously the following holds

$$0 = \Im(\vec{H}_0) = -\frac{1}{2}\pi_{\vec{n}} \left(\frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\xi}_\infty}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\xi}_\infty}{\partial x_1} \right] \right) \quad (\text{I.38})$$

where \vec{H}_0 is the expression in the z coordinates of the Weingarten form $\vec{h}_0 = \vec{H}_0 dz \otimes dz$, and

$$\nu = \nu_1(x_1) \wedge dx_2 + dx_1 \wedge \nu_2(x_2) \quad (\text{I.39})$$

where $\nu_i(x_i)$ are Radon measures on the x_i axis and $\nu_i(x_i) \wedge dx_{i+1}$ is the product of this Radon measure with the Lebesgue measure on the x_{i+1} axis. \square

Remark I.3 In codimension 1 the coordinates directions in which (I.38) happens are **principal directions**. The theorem says that the defect measure associated to the lack of strong compactness of the sequence of isothermic immersions **"propagates" uniformly along principal directions, modulo possible concentration points.** \square

Remark I.4 The result is optimal in the sense that it is not difficult to produce examples where (I.39) indeed happens. Consider a family of simple closed curves in the plane of fixed length, such that, the normal parametrization, $\gamma_k(s)$, weakly converges in $W^{2,2}(S^1)$ with a non zero defect measure $\mu(s)$

$$|\dot{\gamma}_k|^2(s) ds \rightharpoonup |\dot{\gamma}_\infty|^2(s) ds + \mu(s)$$

By identifying the 2-plane with the vertical plane in \mathbb{R}^3 given by $\{x_2 = 0\}$ and by rotating the sequence of curves around the x_3 axis we obtain a weakly converging family of isothermic surfaces with a non zero defect measure satisfying (I.39). \square

II Entropies for Isothermic Surfaces.

One of the main tool for proving theorem I.3 is the computation of entropies for isothermic surfaces. Precisely the goal of the present section is to establish the following proposition.

Proposition II.1 Let $\vec{\Phi}$ be a smooth conformal immersion of D^2 into \mathbb{R}^m satisfying

$$\frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_1} \right] = 0 \quad . \quad (\text{II.1})$$

where $e^\lambda = |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$ is the conformal factor. Then the following conservation laws hold

$$\begin{cases} \frac{\partial}{\partial x_1} \left[\left(\frac{\partial \vec{n}_\Phi}{\partial x_2} \llcorner \vec{e}_2 \right)^2 + \left| \frac{\partial \lambda}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda}{\partial x_2} \right|^2 \right] + \frac{\partial}{\partial x_2} \left[2 \frac{\partial \lambda}{\partial x_1} \frac{\partial \lambda}{\partial x_2} \right] = 0 \\ \frac{\partial}{\partial x_2} \left[\left(\frac{\partial \vec{n}_\Phi}{\partial x_1} \llcorner \vec{e}_1 \right)^2 + \left| \frac{\partial \lambda}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda}{\partial x_1} \right|^2 \right] + \frac{\partial}{\partial x_1} \left[2 \frac{\partial \lambda}{\partial x_1} \frac{\partial \lambda}{\partial x_2} \right] = 0 \end{cases} \quad (\text{II.2})$$

where \vec{e}_i is the unit orthonormal Coulomb frame of $\vec{\Phi}_*TD^2$ given by $\vec{e}_i := e^{-\lambda} \partial_{x_i} \vec{\Phi}$ and \llcorner is the following standard contraction operator between a p -vector and a q -vector ($p \geq q$) giving a $p - q$ -vector

$$\forall \vec{a} \in \wedge^p \mathbb{R}^m, \quad \forall \vec{b} \in \wedge^q \mathbb{R}^m, \quad \forall \vec{c} \in \wedge^{p-q} \mathbb{R}^m \quad \langle \vec{a} \llcorner \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{b} \wedge \vec{c} \rangle \quad .$$

\square

Remark II.1 *The proof of proposition II.1 we give below is using the smoothness of the isothermic immersion and, a-priori (II.2) does not necessarily hold for general isothermic weak immersion in \mathcal{E}_Σ . \square*

Proof of proposition II.1.

A classical computation (see [BR]) gives

$$\vec{H}_0 = 2 \partial_z [e^{-\lambda} \vec{e}_z] \quad (\text{II.3})$$

where $\partial_z := 2^{-1}(\partial_{x_1} - i\partial_{x_2})$ and $\vec{e}_z := 2^{-1}(\vec{e}_1 - i\vec{e}_2)$. Observe that this identity implies

$$\Im(\vec{H}_0) = 2^{-1} \frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_2} \right] + 2^{-1} \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_1} \right] \quad (\text{II.4})$$

Our assumption is then equivalent to

$$\Im(\vec{H}_0) = 0 \quad (\text{II.5})$$

Since $\pi_{\vec{n}}(\vec{H}_0) = \vec{H}_0$, where $\pi_{\vec{n}}$ denotes the orthogonal projection onto the $m-2$ plane perpendicular to \vec{e}_1 and \vec{e}_2 , we deduce from (II.4) and (II.5) that

$$0 = \pi_{\vec{n}} \left(\frac{\partial}{\partial x_1} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda} \frac{\partial \vec{\Phi}}{\partial x_1} \right] \right) \quad (\text{II.6})$$

which itself implies

$$\pi_{\vec{n}} \left(\frac{\partial^2 \vec{\Phi}}{\partial x_1 \partial x_2} \right) = 0 \quad (\text{II.7})$$

Observe that we have

$$\partial_{x_2} \vec{n} \llcorner \partial_{x_1} \vec{\Phi} = \partial_{x_2} (\vec{n} \llcorner \partial_{x_1} \vec{\Phi}) - \vec{n} \llcorner \partial_{x_1 x_2}^2 \vec{\Phi} = -\vec{n} \llcorner \partial_{x_1 x_2}^2 \vec{\Phi} = \partial_{x_1} (\vec{n} \llcorner \partial_{x_2} \vec{\Phi}) - \vec{n} \llcorner \partial_{x_1 x_2}^2 \vec{\Phi} = \partial_{x_1} \vec{n} \llcorner \partial_{x_2} \vec{\Phi} \quad (\text{II.8})$$

where we have used that $\vec{n} \llcorner \partial_{x_1} \vec{\Phi} = 0$ and $\vec{n} \llcorner \partial_{x_2} \vec{\Phi}$. Inserting (II.7) in this identity gives

$$\frac{\partial \vec{n}}{\partial x_2} \llcorner \frac{\partial \vec{\Phi}}{\partial x_1} = \frac{\partial \vec{n}}{\partial x_1} \llcorner \frac{\partial \vec{\Phi}}{\partial x_2} = 0 \quad (\text{II.8})$$

We have

$$\begin{aligned} \partial_{x_1} (\partial_{x_2} \vec{n} \llcorner \vec{e}_2) &= \partial_{x_1} (e^{-\lambda} \partial_{x_2} \vec{n} \llcorner \partial_{x_2} \vec{\Phi}) \\ &= -\partial_{x_1} \lambda \partial_{x_2} \vec{n} \llcorner \vec{e}_2 + e^{-\lambda} \partial_{x_1 x_2}^2 \vec{n} \llcorner \partial_{x_2} \vec{\Phi} + e^{-\lambda} \partial_{x_2} \vec{n} \llcorner \partial_{x_1 x_2}^2 \vec{\Phi} \\ &= -\partial_{x_1} \lambda \partial_{x_2} \vec{n} \llcorner \vec{e}_2 - e^{-\lambda} \partial_{x_1} \vec{n} \llcorner \partial_{x_2}^2 \vec{\Phi} + e^{-\lambda} \partial_{x_2} \vec{n} \llcorner \pi_T (\partial_{x_1 x_2}^2 \vec{\Phi}) \end{aligned} \quad (\text{II.9})$$

where we have used (II.7) and (II.8). In one hand we have

$$\begin{aligned} \pi_T (\partial_{x_1 x_2}^2 \vec{\Phi}) &= 2^{-1} e^{-\lambda} \left[\partial_{x_2} (|\partial_{x_1} \vec{\Phi}|^2) \vec{e}_1 + \partial_{x_1} |\partial_{x_2} \vec{\Phi}|^2 \vec{e}_2 \right] \\ &= \partial_{x_2} \lambda \partial_{x_1} \vec{\Phi} + \partial_{x_1} \lambda \partial_{x_2} \vec{\Phi} \end{aligned} \quad (\text{II.10})$$

Thus using (II.8) we have

$$e^{-\lambda} \partial_{x_2} \vec{n} \llcorner \pi_T (\partial_{x_1 x_2}^2 \vec{\Phi}) = \partial_{x_1} \lambda e^{-\lambda} \partial_{x_2} \vec{n} \llcorner \partial_{x_2} \vec{\Phi} = \partial_{x_1} \lambda \partial_{x_2} \vec{n} \llcorner \vec{e}_2 \quad (\text{II.11})$$

and (II.9) becomes

$$\partial_{x_1}(\partial_{x_2}\vec{n}\llcorner\vec{e}_2) = -e^{-\lambda} \partial_{x_1}\vec{n}\llcorner\partial_{x_2}^2\vec{\Phi} \quad (\text{II.12})$$

In the other hand

$$\langle \partial_{x_1}\vec{n}\llcorner\pi_{\vec{n}}(\partial_{x_2}^2\vec{\Phi}), \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle = \langle \partial_{x_1}\vec{n}, \pi_{\vec{n}}(\partial_{x_2}^2\vec{\Phi}) \wedge (\partial_{x_2}\vec{n}\llcorner\vec{e}_2) \rangle = 0 \quad (\text{II.13})$$

Indeed, if $m = 3$ $\partial_{x_1}\vec{n}$ is perpendicular to the vector \vec{n} to which $\pi_{\vec{n}}(\partial_{x_2}^2\vec{\Phi})$ is parallel and, in the case when $m > 3$, one easily verifies that

$$(\partial_{x_2}\vec{n}\llcorner\vec{e}_2)\llcorner\vec{e}_i = 0 \quad \text{for } i = 1, 2 \quad ,$$

thus $\pi_{\vec{n}}(\partial_{x_2}^2\vec{\Phi}) \wedge (\partial_{x_2}\vec{n}\llcorner\vec{e}_2)$ is parallel to \vec{n} which proves (II.13).

Combining now (II.12) and (II.13) we obtain

$$\langle \partial_{x_1}(\partial_{x_2}\vec{n}\llcorner\vec{e}_2), \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle = -e^{-\lambda} \langle \partial_{x_1}\vec{n}\llcorner\pi_T(\partial_{x_2}^2\vec{\Phi}), \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle \quad .$$

Using two more times (II.8) this gives

$$\begin{aligned} \langle \partial_{x_1}(\partial_{x_2}\vec{n}\llcorner\vec{e}_2), \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle &= -e^{-\lambda} \langle \partial_{x_1}\vec{n}\llcorner\vec{e}_1, \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle \langle \vec{e}_1, \partial_{x_2}^2\vec{\Phi} \rangle \\ &= \left[\langle \partial_{x_1}\vec{n}\llcorner\vec{e}_1, \partial_{x_2}\vec{n}\llcorner\vec{e}_2 \rangle - |\partial_{x_1}\vec{n}\llcorner\vec{e}_2|^2 \right] \partial_{x_1}\lambda \\ &= e^{2\lambda} K \partial_{x_1}\lambda = -\Delta\lambda \partial_{x_1}\lambda = -\partial_{x_1}(|\partial_{x_1}\lambda|^2/2) - \partial_{x_2}(\partial_{x_1}\lambda \partial_{x_2}\lambda) + \partial_{x_1}(|\partial_{x_2}\lambda|^2/2) \end{aligned} \quad (\text{II.14})$$

where K is the Gauss curvature and where we have used the Liouville equation. (II.14) gives the first equation of (II.2). The second equation is established in a similar way. The proof of proposition II.1 is complete. \square

III A lemma in Compensation Compactness Theory

In order to prove the main theorem I.3 we shall need a compactness result related to some quantities present in the expressions (II.2) of the entropies. This result is based on a compensation phenomenon observed first in [De] (see also [Ge] and [EM]) in the framework of the analysis of 2-dimensional perfect incompressible fluids.

Lemma III.1 *Let α_k and β_k be two sequences of functions in $W^{1,2}(D^2, \mathbb{R})$*

$$\limsup_{k \rightarrow +\infty} \|\nabla\alpha_k\|_{L^2(D^2)} + \|\nabla\beta_k\|_{L^2(D^2)} < +\infty \quad (\text{III.1})$$

Let φ_k be the sequence of solutions in $W^{1,2}(D^2, \mathbb{R})$ of

$$\begin{cases} \Delta\varphi_k = \partial_{x_1}\alpha_k \partial_{x_2}\beta_k - \partial_{x_2}\alpha_k \partial_{x_1}\beta_k & \text{in } D^2 \\ \varphi_k = 0 & \text{on } \partial D^2 \end{cases} \quad (\text{III.2})$$

Then there exists a subsequence $\varphi_{k'}$ and two Radon measures μ and ν such that

$$\begin{cases} |\partial_{x_1}\varphi_{k'}|^2 - |\partial_{x_2}\varphi_{k'}|^2 \rightharpoonup |\partial_{x_1}\varphi_\infty|^2 - |\partial_{x_2}\varphi_\infty|^2 + \mu & \text{in } \mathcal{D}'(D^2) \\ \partial_{x_1}\varphi_{k'} \partial_{x_2}\varphi_{k'} \rightharpoonup \partial_{x_1}\varphi_\infty \partial_{x_2}\varphi_\infty + \nu & \text{in } \mathcal{D}'(D^2) \end{cases} \quad (\text{III.3})$$

where

$$\begin{cases} \Delta\varphi_\infty = \partial_{x_1}\alpha_\infty \partial_{x_2}\beta_\infty - \partial_{x_2}\alpha_\infty \partial_{x_1}\beta_\infty & \text{in } D^2 \\ \varphi_\infty = 0 & \text{on } \partial D^2 \end{cases} \quad (\text{III.4})$$

and α_∞ (resp. β_∞) is the weak limit in $W^{1,2}$ of $\alpha_{k'}$ (resp. $\beta_{k'}$). Moreover both μ and ν are atomic inside D^2 : there exists $p_i \in D^2$ for $i \in \mathbb{N}$, and $q_j \in D^2$ for $j \in \mathbb{N}$ such that

$$\mu = \sum_{i \in \mathbb{N}} c_i \delta_{p_i} \quad \text{and} \quad \nu = \sum_{j \in \mathbb{N}} d_j \delta_{q_j} \quad \text{in } \mathcal{D}'(D^2) \quad , \quad (\text{III.5})$$

where

$$\sum_{i \in \mathbb{N}} |c_i| = |\mu|(D^2) < +\infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} |d_j| = |\nu|(D^2) < +\infty \quad . \quad (\text{III.6})$$

□

Proof of lemma III.1.

Let $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ be two Whitney type extension on the whole plane \mathbb{R}^2 of respectively α_k and β_k satisfying

$$\|\nabla \tilde{\alpha}_k\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla \alpha_k\|_{L^2(D^2)} \quad \text{and} \quad \|\nabla \tilde{\beta}_k\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla \beta_k\|_{L^2(D^2)} \quad (\text{III.7})$$

where C is independent of the two sequences α_k and β_k (take for instance in $\mathbb{R}^2 \setminus D^2$ respectively $\tilde{\alpha}_k(x) := \alpha(x/|x|^2)$ and $\tilde{\beta}_k(x) := \beta_k(x/|x|^2)$). Introduce

$$\tilde{\varphi}_k := \frac{1}{2\pi} \log |x| * \left[\partial_{x_1} \tilde{\alpha}_k \partial_{x_2} \tilde{\beta}_k - \partial_{x_2} \tilde{\alpha}_k \partial_{x_1} \tilde{\beta}_k \right] \quad . \quad (\text{III.8})$$

From Wente theorem (see [We] and the exposition in [He]) we know that both φ_k and $\tilde{\varphi}_k$ are uniformly bounded in $W^{1,2} \cap L^\infty$ and we have in particular

$$\|\varphi_k\|_{L^\infty(\mathbb{R}^2)} + \|\tilde{\varphi}_k\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \varphi_k\|_{L^2(\mathbb{R}^2)} + \|\nabla \tilde{\varphi}_k\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla \alpha_k\|_{L^2(D^2)} \|\nabla \beta_k\|_{L^2(D^2)} \quad (\text{III.9})$$

Hence the difference $v_k = \tilde{\varphi}_k - \varphi_k$, which is harmonic in D^2 , is strongly precompact in every $C_{loc}^l(D^2)$ for $l \in \mathbb{N}$ and since we don't care about concentration of the measures at the boundary ∂D^2 , it suffices to prove the results of the lemma (identities III.3...III.6) for $\tilde{\varphi}_k$ in D^2 , this will imply the corresponding identities for φ_k in D^2

We present the proof of the lemma for the quantity $\partial_{x_1} \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k$ (the proof for the other quantity $|\partial_{x_1} \tilde{\varphi}_k|^2 - |\partial_{x_2} \tilde{\varphi}_k|^2$ being identical).

To shorten a bit the notation we write

$$\omega_k := \partial_{x_1} \tilde{\alpha}_k \partial_{x_2} \tilde{\beta}_k - \partial_{x_2} \tilde{\alpha}_k \partial_{x_1} \tilde{\beta}_k = \Delta \tilde{\varphi}_k \quad .$$

Because of the uniform bounds given by (III.9) combined with the assumption (III.1), we can extract a subsequence still denoted $\tilde{\varphi}_k$ such that

$$\tilde{\varphi}_k \rightharpoonup \tilde{\varphi}_\infty \quad \text{weakly in } W^{1,2}(\mathbb{R}^2)$$

and, due to the jacobian structure, we can pass to the limit in (III.8) :

$$\tilde{\varphi}_\infty := \frac{1}{2\pi} \log |x| * \left[\partial_{x_1} \tilde{\alpha}_\infty \partial_{x_2} \tilde{\beta}_\infty - \partial_{x_2} \tilde{\alpha}_\infty \partial_{x_1} \tilde{\beta}_\infty \right]$$

where $\tilde{\alpha}_\infty$ and $\tilde{\beta}_\infty$ are weak $W^{1,2}$ -limits of respectively $\tilde{\alpha}_k$ and $\tilde{\beta}_k$. Moreover we can also ensure that

$$\partial_{x_1} \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k \rightharpoonup \gamma \quad \text{weakly in } \mathcal{M}(D^2)$$

where $\mathcal{M}(D^2)$ denotes the space of Radon measures. It remains now to identify the Radon measure γ .

Let ψ be an arbitrary function in $C_0^\infty(D^2)$, denoting by Δ^{-1} the convolution with $(2\pi)^{-1} \log|x|$ we have

$$\begin{aligned} \int_{D^2} \psi(x) \partial_{x_1} \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k dx &= \int_{\mathbb{R}^2} \psi(x) \partial_{x_1} \Delta^{-1} \omega_k \partial_{x_2} \Delta^{-1} \omega_k dx \\ &= - \int_{\mathbb{R}^2} \partial_{x_1} \psi(x) \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k dx - \int_{\mathbb{R}^2} \psi(x) \Delta^{-1} \omega_k \partial_{x_1 x_2}^2 \Delta^{-1} \omega_k dx \\ &= - \int_{\mathbb{R}^2} \partial_{x_1} \psi(x) \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k dx + \int_{\mathbb{R}^2} [\Delta^{-1}(\psi \omega_k) - \psi(x) \Delta^{-1} \omega_k] \partial_{x_1 x_2}^2 \tilde{\varphi}_k dx \\ &\quad + \int_{\mathbb{R}^2} \psi(x) \omega_k(x) \partial_{x_1 x_2}^2 \Delta^{-2} \omega_k(x) dx \end{aligned} \tag{III.10}$$

We shall now pass to the limit in the three terms in the r.h.s. of (III.10).

The first term of the r.h.s. of (III.10). Since $\tilde{\varphi}_k \rightharpoonup \tilde{\varphi}_\infty$ weakly in $W^{1,2}(D^2)$, from Rellich Kondrachoff theorem $\tilde{\varphi}_k$ converges strongly to $\tilde{\varphi}_\infty$ in $L^2(D^2)$ therefore

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \partial_{x_1} \psi(x) \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k dx = \int_{\mathbb{R}^2} \partial_{x_1} \psi(x) \tilde{\varphi}_\infty \partial_{x_2} \tilde{\varphi}_\infty dx \quad . \tag{III.11}$$

The second term of the r.h.s. of (III.10). Observe first that

$$\Delta [\Delta^{-1}(\psi \omega_k) - \psi(x) \Delta^{-1} \omega_k] = -\Delta \psi \tilde{\varphi}_k - 2\nabla \psi \nabla \tilde{\varphi}_k \quad . \tag{III.12}$$

Since $\tilde{\varphi}_k \rightharpoonup \tilde{\varphi}_\infty$ weakly in $W^{1,2}(\mathbb{R}^2)$, we have that

$$\Delta [\Delta^{-1}(\psi \omega_k) - \psi(x) \Delta^{-1} \omega_k] \rightharpoonup \Delta [\Delta^{-1}(\psi \omega_\infty) - \psi(x) \Delta^{-1} \omega_\infty] \quad \text{weakly in } L^2(\mathbb{R}^2) \tag{III.13}$$

Hence, using again Rellich-Kondrachoff we deduce that

$$[\Delta^{-1}(\psi \omega_k) - \psi(x) \Delta^{-1} \omega_k] \longrightarrow [\Delta^{-1}(\psi \omega_\infty) - \psi(x) \Delta^{-1} \omega_\infty] \quad \text{strongly in } W^{1,2}(\mathbb{R}^2) \tag{III.14}$$

Since

$$\partial_{x_1 x_2}^2 \tilde{\varphi}_k \rightharpoonup \partial_{x_1 x_2}^2 \tilde{\varphi}_\infty \quad \text{weakly in } H^{-1}(\mathbb{R}^2) \tag{III.15}$$

Combining (III.14) and (III.15) gives

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} [\Delta^{-1}(\psi \omega_k) - \psi(x) \Delta^{-1} \omega_k] \partial_{x_1 x_2}^2 \tilde{\varphi}_k dx = \int_{\mathbb{R}^2} [\Delta^{-1}(\psi \omega_\infty) - \psi(x) \Delta^{-1} \omega_\infty] \partial_{x_1 x_2}^2 \tilde{\varphi}_\infty dx \quad . \tag{III.16}$$

The third term of the r.h.s. of (III.10). This is of course the most delicate term in which the specificity of the bilinearity $\partial_{x_1} \tilde{\varphi}_k \partial_{x_2} \tilde{\varphi}_k$ we are considering plays a role.

From [Ste] we have that the Kernel associated to the operator $\partial_{x_1 x_2}^2 \Delta^{-2}$ **is bounded in L^∞** . Indeed one has that the Fourier multiplier associated to the operator $\partial_{x_1 x_2}^2 \Delta^{-2}$ is given by

$$\widehat{\partial_{x_1 x_2}^2 \Delta^{-2}} = -\frac{\xi_1 \xi_2}{|\xi|^4} \tag{III.17}$$

which as to be understood either as in a singular integral sense or in distributional sense as being the following tempered distribution in $\mathcal{S}'(\mathbb{R}^2)$

$$\begin{aligned} \forall \phi(\xi) \in \mathcal{S}(\mathbb{R}^2) \quad \left\langle pv \left(-\frac{\xi_1 \xi_2}{|\xi|^4} \right); \phi(\xi) \right\rangle &= -\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(0)} \frac{\xi_1 \xi_2}{|\xi|^4} \phi(\xi) d\xi \\ &= \int_{\mathbb{R}^2} \frac{\xi_1 \xi_2}{|\xi|^4} (\phi(0) - \phi(\xi)) d\xi \end{aligned}$$

Since the homogeneous polynomial $\xi_1 \xi_2$ is harmonic we can apply theorem 5 in 3.3 of [Ste] and deduce the existence of a universal constant c_0 such that the inverse of the Fourier transform of $\xi_1 \xi_2 / |\xi|^4$ is given by

$$-\widehat{\frac{\xi_1 \xi_2}{|\xi|^4}}^{-1} = c_0 \frac{x_1 x_2}{|x|^2} .$$

Hence

$$\int_{\mathbb{R}^2} \psi(x) \omega_k(x) \partial_{x_1 x_2}^2 \Delta^{-2} \omega_k(x) dx = c_0 \int_{\mathbb{R}^2} \psi(x) \omega_k(x) \omega_k(y) \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \quad (\text{III.18})$$

If the kernel $(x_1 - y_1)(x_2 - y_2)/|x - y|^2$ would have been continuous up to the diagonal $x = y$ (or even VMO on \mathbb{R}^4) we could have easily pass to the limit in this integral, since $\omega_k(x) \omega_k(y)$ is uniformly bounded in the local Hardy space $\mathcal{H}_{loc}^1(\mathbb{R}^4)$, it converges weakly in particular in Radon measure to $\omega_\infty(x) \omega_\infty(y)$. We shall however make use of the fact that $(x_1 - y_1)(x_2 - y_2)/|x - y|^2$ is bounded in L^∞ in order to pass to the limit in (III.18) modulo possible concentration points.

Let χ be a cut-off function in $C_0^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that χ is equal to 1 on $[0, 1]$ and equal to zero on $[2, +\infty)$ and $0 \leq \chi \leq 1$. For $\varepsilon > 0$ we denote $\chi_\varepsilon(t) := \chi(t/\varepsilon)$.

We write

$$\begin{aligned} &\int_{\mathbb{R}^4} \psi(x) \omega_k(x) \omega_k(y) \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \\ &= \int_{\mathbb{R}^4} \psi(x) \omega_k(x) \omega_k(y) \chi_\varepsilon(|x - y|) \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \\ &+ \int_{\mathbb{R}^4} \psi(x) \omega_k(x) \omega_k(y) [1 - \chi_\varepsilon(|x - y|)] \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy . \end{aligned} \quad (\text{III.19})$$

Since $[1 - \chi_\varepsilon(|x - y|)] (x_1 - y_1)(x_2 - y_2)/|x - y|^2$ is continuous on \mathbb{R}^4 we have

$$\begin{aligned} &\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^4} \psi(x) \omega_k(x) \omega_k(y) [1 - \chi_\varepsilon(|x - y|)] \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \\ &= \int_{\mathbb{R}^4} \psi(x) \omega_\infty(x) \omega_\infty(y) [1 - \chi_\varepsilon(|x - y|)] \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \end{aligned} \quad (\text{III.20})$$

And then

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^4} \psi(x) \omega_k(x) \omega_k(y) [1 - \chi_\varepsilon(|x - y|)] \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \\ &= \int_{\mathbb{R}^4} \psi(x) \omega_\infty(x) \omega_\infty(y) \frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} dx dy \end{aligned} \quad (\text{III.21})$$

Combining (III.11), (III.16) and (III.21) we obtain that

$$|\langle \gamma - \partial_{x_1} \tilde{\varphi}_\infty \partial_{x_2} \tilde{\varphi}_\infty ; \psi \rangle| \leq \liminf_{\varepsilon \rightarrow 0} \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^4} |\psi(x)| |\omega_k(x)| |\omega_k(y)| \chi_\varepsilon(|x-y|) dx dy \quad (\text{III.22})$$

Modulo extraction of a subsequence we can assume that the sequence of measures $|\omega_k(x)| dx$ converges weakly to a non negative Radon measure $\zeta(x)$ and we have

$$|\langle \gamma - \partial_{x_1} \tilde{\varphi}_\infty \partial_{x_2} \tilde{\varphi}_\infty ; \psi \rangle| \leq \liminf_{\varepsilon \rightarrow 0} \langle |\psi(x)| \chi_\varepsilon(|x-y|); \zeta(x) \zeta(y) \rangle \quad (\text{III.23})$$

Denote by $A_\zeta := \sum_{j \in \mathbb{N}} \zeta_j \delta_{q_j}$ the atomic part of ζ :

$$\langle \zeta(y) - A_\zeta(y) ; \chi_\varepsilon(|x-y|) \rangle \rightarrow 0 \quad \zeta \text{ a. e. } x \quad .$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \langle |\psi(x)| \chi_\varepsilon(|x-y|); \zeta(x) \zeta(y) \rangle = \lim_{\varepsilon \rightarrow 0} \sum_{j \in \mathbb{N}} \zeta_j \langle |\psi(x)| \chi_\varepsilon(|x-q_j|); \zeta(x) \rangle = \sum_{j \in \mathbb{N}} \zeta_j^2 |\psi(q_j)|$$

Hence (III.23) implies

$$|\langle \gamma - \partial_{x_1} \tilde{\varphi}_\infty \partial_{x_2} \tilde{\varphi}_\infty ; \psi \rangle| \leq \sum_{j \in \mathbb{N}} \zeta_j^2 |\psi(q_j)| \quad (\text{III.24})$$

which shows that $\gamma - \partial_{x_1} \tilde{\varphi}_\infty \partial_{x_2} \tilde{\varphi}_\infty$ is atomic. This implies the lemma for the bilinearity $\partial_{x_1} \varphi_\infty \partial_{x_2} \varphi_\infty$. The same applies to the bilinearity $|\partial_{x_1} \varphi_\infty|^2 - |\partial_{x_2} \varphi_\infty|^2$ since in the estimation of the third term in the r.h.s of the identity corresponding to (III.10) one uses that $\xi_1^2 - \xi_2^2$ is harmonic and thus the kernel associated to $(\partial_{x_1}^2 - \partial_{x_2}^2) \Delta^{-2}$ is also bounded in L^∞ due to theorem 5 in section 3.3 of [Ste]. \square

IV Proof of the main theorem I.3.

Let $\vec{\Phi}_k$ be a sequence of global weak isothermic immersions of an abstract closed surface Σ into \mathbb{R}^m such that the conformal class to which the induced metric $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$ does not degenerate. This means that there exists a sequence of constant scalar curvature metric h_k of volume 1, precompact for any C^l norm of Σ (equipped with some fixed arbitrary reference metric g_0) and a diffeomorphism f_k of Σ such that

$$\vec{\Phi}_k \circ f_k \quad : \quad (\Sigma, h_k) \longrightarrow \mathbb{R}^m \quad \text{is conformal} \quad . \quad (\text{IV.1})$$

Modulo extraction of a subsequence we can assume that

$$h_k \rightarrow h_\infty \quad \text{in } C^l(\Sigma) \quad \forall l \in \mathbb{N} \quad , \quad (\text{IV.2})$$

where h_∞ is a constant scalar curvature of volume 1 on Σ .

We assume moreover that

$$\limsup_{k \rightarrow +\infty} W(\vec{\Phi}_k) = \limsup_{k \rightarrow +\infty} \int_{\Sigma} |\vec{H}_{\vec{\Phi}_k}|^2 dvol_{g_k} < +\infty$$

Following the normalization lemma A.4 and lemma III.1 of [Ri3], we deduce the existence of a sequence of Möbius transformation Ξ_k of $\mathbb{R}^m \cup \{\infty\}$ (i.e. Ξ_k are conformal diffeomorphism of $\mathbb{R}^m \cup \{\infty\}$) such that $\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ f_k$ satisfies the following conditions (up to subsequence)

i)

$$\exists R > 0 \quad , \quad \forall k \quad \vec{\xi}_k(\Sigma) \subset B_R(0) \quad .$$

ii)

$$\exists a_1 \cdots a_N \quad \text{s.t.} \quad \vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{in } W_{loc}^{2,2}(\Sigma \setminus \{a_1 \cdots a_N\})$$

iii)

$$\forall K \text{ compact of } \Sigma \setminus \{a_1 \cdots a_N\} \quad \limsup_{k \rightarrow +\infty} \|\log |d\vec{\xi}_k|_{h_k}\|_{L^\infty(K)} < +\infty$$

These 3 conditions ensure that the weak limiting map $\vec{\xi}_\infty$ is a weak possibly branched conformal immersion in the space \mathcal{F}_Σ .

Assuming now that $\vec{\Phi}_k$ are *weak global isothermic immersions* in \mathcal{E}_Σ then, due to the conformal invariance proved in proposition I.3, $\vec{\xi}_k$ are also *weak global isothermic immersions*. Thus there exists a sequence of non zero holomorphic quadratic differentials q_k for the sequence of riemann surfaces (Σ, h_k) satisfying

$$\mathfrak{S}(q_k, \vec{h}_{0,k})_{WP} = 0 \quad (\text{IV.3})$$

where the Weil-Peterson norm is taken with respect to h_k . Because of the linearity of equation (IV.3) with respect to q_k we can normalize q_k in such a way that

$$\forall k \in \mathbb{N} \quad \int_{\Sigma} (q_k, q_k)_{WP} \, d\text{vol}_{h_k} = 1 \quad (\text{IV.4})$$

The space P_k of holomorphic quadratic forms of (Σ, h_k) is a finite dimensional space of fixed dimension (depending on Σ only) of the space $\Gamma(T^*\Sigma \otimes T^*\Sigma)$ of smooth sections of $T^*\Sigma \otimes T^*\Sigma$. Since h_k converges to h_∞ we can extract a subsequence such that P_k converges to P_∞ and we can extract a subsequence such that q_k converges in any C^l norm towards q_∞ for any $l \in \mathbb{N}$.

The holomorphic quadratic form q_∞ satisfy also (IV.4), moreover, due to the weak convergence of $\vec{h}_{0,k}$ towards $\vec{h}_{0,\infty}$ in $L_{loc}^2(\Sigma \setminus \{a_1 \cdots a_N\})$,

$$\mathfrak{S}(q_\infty, \vec{h}_{0,\infty})_{WP} = 0 \quad \text{in } \Sigma \setminus \{a_1 \cdots a_N\} \quad . \quad (\text{IV.5})$$

This implies that $\vec{\xi}_\infty$ is a weak, possibly branched, conformal isothermic immersion of (Σ, h_∞) into \mathbb{R}^m .

In an arbitrary strongly converging conformal chart $\phi_k : D^2 \setminus (\Sigma, h_k)$ the equation satisfied by $\vec{\xi}_k \circ \phi_k$ reads (omitting to write explicitly the composition with ϕ_k)

$$\mathfrak{S}(f_k(z), \overline{\vec{H}_{0,k}}) = 2 \mathfrak{S} \left(f_k(z), \partial_{\bar{z}} \left[e^{-2\lambda_k} \partial_{\bar{z}} \vec{\xi}_k \right] \right)$$

where f_k is the expression of q_k in this chart $f_k(z) \, dz \otimes dz = q_k$.

Denote by $b_1 \cdots b_Q$ the isolated zeros of q_∞ in Σ . Let U be a disc included in $\Sigma \setminus \{a_1 \cdots a_N, b_1 \cdots b_Q\}$. Considering a converging sequence of conformal charts ϕ_k realizing a diffeomorphism from D^2 into U , since f_∞ the expression of $\vec{h}_{0,\infty}$ in this chart does not vanish on D^2 and since f_k converge strongly on D^2 towards f_∞ , we can introduce the new **converging** chart $w := \sqrt{f_k \circ \phi_k^{-1}}$. In this new chart the isothermic equation reads

$$\mathfrak{S}(\vec{H}_{0,k}) = \frac{\partial}{\partial x_1} \left[e^{-2\lambda_k} \frac{\partial \vec{\xi}_k}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[e^{-2\lambda_k} \frac{\partial \vec{\xi}_k}{\partial x_1} \right] = 0 \quad . \quad (\text{IV.6})$$

where $w = x_1 + ix_2$ and $e^{\lambda_k} = |\partial_{x_1} \vec{\xi}_k| = |\partial_{x_2} \vec{\xi}_k|$. since the chart is strongly converging the expression of $\vec{\xi}_k$ in this chart satisfy

$$\vec{\xi}_k(w) \rightarrow \vec{\xi}_\infty(w) \quad \text{in} \quad W^{2,2}(D^2) \quad \text{and} \quad \limsup_{k \rightarrow +\infty} \|\lambda_k(w)\|_{L^\infty(D^2)} < +\infty \quad . \quad (\text{IV.7})$$

We also choose U small enough and the subsequence in such a way that

$$\forall k \in \mathbb{N} \quad \int_{D^2} |\nabla \vec{n}_{\vec{\xi}_k}|^2 dx_1 dx_2 < \frac{8\pi}{3} \quad . \quad (\text{IV.8})$$

We can then use a result by F. Hélein (see [He] chapter 5) that gives the existence of $(\vec{e}_{1,k}, \vec{e}_{2,k}) \in (W^{1,2}(D^2, S^{m-1}))^2$ such that

$$\vec{e}_{1,k} \wedge \vec{e}_{2,k} = \star \vec{n}_{\vec{\xi}_k} \quad \int_{D^2} \sum_{i=1}^2 |\nabla \vec{e}_{i,k}|^2 < C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}_k}|^2 \quad (\text{IV.9})$$

where C is independent of k . We can use this moving frame to express the laplacian of λ_k (see [Ri1]) and we have precisely

$$-\Delta \lambda_k = \partial_{x_1} \vec{e}_{1,k} \cdot \partial_{x_2} \vec{e}_{2,k} - \partial_{x_2} \vec{e}_{1,k} \cdot \partial_{x_1} \vec{e}_{2,k} \quad \text{in} \quad D^2 \quad . \quad (\text{IV.10})$$

Let s_k be the solution of

$$\begin{cases} -\Delta s_k = \partial_{x_1} \vec{e}_{1,k} \cdot \partial_{x_2} \vec{e}_{2,k} - \partial_{x_2} \vec{e}_{1,k} \cdot \partial_{x_1} \vec{e}_{2,k} & \text{in} \quad D^2 \\ s_k = 0 & \text{on} \quad \partial D^2 \end{cases} \quad (\text{IV.11})$$

From Wente theorem (see [We] and [He]) we have

$$\|s_k\|_{L^\infty(D^2)} \leq C \|\nabla \vec{e}_{1,k}\|_{L^2(D^2)} \|\nabla \vec{e}_{2,k}\|_{L^2(D^2)} \leq C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}_k}|^2 \quad . \quad (\text{IV.12})$$

Using (IV.8) we deduce that s_k is uniformly bounded in $L^\infty(D^2)$. Combining this fact with (IV.7) we obtain that the harmonic function $v_k := \lambda_k - s_k$ is uniformly bounded in $L^\infty(D^2)$. Thus we have that

$$v_k \rightarrow v_\infty \quad \text{in} \quad C_{loc}^1(D^2) \quad \forall l \in \mathbb{N} \quad . \quad (\text{IV.13})$$

Lemma III.1 implies that there exists a subsequence and two atomic measures μ and ν such that there exists $p_i \in D^2$ for $i \in \mathbb{N}$, and $q_j \in D^2$ for $j \in \mathbb{N}$ satisfying

$$\mu = \sum_{i \in \mathbb{N}} c_i \delta_{p_i} \quad \text{and} \quad \nu = \sum_{j \in \mathbb{N}} d_j \delta_{q_j} \quad \text{in} \quad \mathcal{D}'(D^2) \quad , \quad (\text{IV.14})$$

where

$$\sum_{i \in \mathbb{N}} |c_i| = |\mu|(D^2) < +\infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} |d_j| = |\nu|(D^2) < +\infty \quad . \quad (\text{IV.15})$$

and

$$\begin{cases} |\partial_{x_1} s_{k'}|^2 - |\partial_{x_2} s_{k'}|^2 \rightarrow |\partial_{x_1} s_\infty|^2 - |\partial_{x_2} s_\infty|^2 + \mu & \text{in} \quad \mathcal{D}'(D^2) \\ \partial_{x_1} s_{k'} \partial_{x_2} s_{k'} \rightarrow \partial_{x_1} s_\infty \partial_{x_2} s_\infty + \nu & \text{in} \quad \mathcal{D}'(D^2) \end{cases} \quad . \quad (\text{IV.16})$$

Using (IV.13) we deduce

$$\begin{cases} |\partial_{x_1} \lambda_{k'}|^2 - |\partial_{x_2} \lambda_{k'}|^2 \rightarrow |\partial_{x_1} \lambda_\infty|^2 - |\partial_{x_2} \lambda_\infty|^2 + \mu & \text{in} \quad \mathcal{D}'(D^2) \\ \partial_{x_1} \lambda_{k'} \partial_{x_2} \lambda_{k'} \rightarrow \partial_{x_1} \lambda_\infty \partial_{x_2} \lambda_\infty + \nu & \text{in} \quad \mathcal{D}'(D^2) \end{cases} \quad . \quad (\text{IV.17})$$

Assuming the $\vec{\xi}_{k'}$ and $\vec{\xi}_\infty$ are smooth, since these immersions are smooth, we can apply proposition II.1 and deduce that in one hand

$$\begin{cases} \frac{\partial}{\partial x_1} \left[\left(\frac{\partial \vec{n}_{\vec{\xi}_{k'}}}{\partial x_2} \lrcorner \vec{e}_{2,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 \right] + \frac{\partial}{\partial x_2} \left[2 \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \right] = 0 \\ \frac{\partial}{\partial x_2} \left[\left(\frac{\partial \vec{n}_{\vec{\xi}_{k'}}}{\partial x_1} \lrcorner \vec{e}_{1,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 \right] + \frac{\partial}{\partial x_1} \left[2 \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \right] = 0 \end{cases} \quad (\text{IV.18})$$

and in the other hand

$$\begin{cases} \frac{\partial}{\partial x_1} \left[\left(\frac{\partial \vec{n}_{\vec{\xi}_\infty}}{\partial x_2} \lrcorner \vec{e}_{2,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 \right] + \frac{\partial}{\partial x_2} \left[2 \frac{\partial \lambda_\infty}{\partial x_1} \frac{\partial \lambda_\infty}{\partial x_2} \right] = 0 \\ \frac{\partial}{\partial x_2} \left[\left(\frac{\partial \vec{n}_{\vec{\xi}_\infty}}{\partial x_1} \lrcorner \vec{e}_{1,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 \right] + \frac{\partial}{\partial x_1} \left[2 \frac{\partial \lambda_\infty}{\partial x_1} \frac{\partial \lambda_\infty}{\partial x_2} \right] = 0 \end{cases} \quad (\text{IV.19})$$

Applying Poincaré Lemma, we deduce the existence of $A_{k'}$ and $B_{k'}$ in $W^{1,1}$ such that

$$\begin{cases} \partial_{x_2} A_{k'} = \left(\frac{\partial \vec{n}_{\vec{\xi}_{k'}}}{\partial x_2} \lrcorner \vec{e}_{2,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 \\ \partial_{x_1} A_{k'} = -2 \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \end{cases}$$

such that

$$\begin{cases} \partial_{x_1} B_{k'} = \left(\frac{\partial \vec{n}_{\vec{\xi}_{k'}}}{\partial x_1} \lrcorner \vec{e}_{1,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 \\ \partial_{x_2} B_{k'} = -2 \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \end{cases}$$

Moreover for the same reason there exist A_∞ and B_∞ in $W^{1,1}$ such that

$$\begin{cases} \partial_{x_2} A_\infty = \left(\frac{\partial \vec{n}_{\vec{\xi}_\infty}}{\partial x_2} \lrcorner \vec{e}_{2,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 \\ \partial_{x_1} A_\infty = -2 \frac{\partial \lambda_\infty}{\partial x_1} \frac{\partial \lambda_\infty}{\partial x_2} \end{cases}$$

such that

$$\begin{cases} \partial_{x_1} B_\infty = \left(\frac{\partial \vec{n}_{\vec{\xi}_\infty}}{\partial x_1} \lrcorner \vec{e}_{1,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 \\ \partial_{x_2} B_\infty = -2 \frac{\partial \lambda_\infty}{\partial x_1} \frac{\partial \lambda_\infty}{\partial x_2} \end{cases}$$

We observe that we have

$$\partial_{x_1} A_{k'} = \partial_{x_2} B_{k'} \quad \text{and} \quad \partial_{x_1} A_\infty = \partial_{x_2} B_\infty$$

Applying again Poincaré Lemma, we have the existence of $\alpha_{k'}$ and α_∞ in $W^{2,1}$ such that

$$\nabla \alpha_{k'} = (B_{k'}, A_{k'}) \quad \text{and} \quad \nabla \alpha_\infty = (B_\infty, A_\infty)$$

Thus we have

$$\begin{cases} \frac{\partial^2 \alpha_{k'}}{\partial x_2 \partial x_2} = \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_2} \llcorner \vec{e}_{2,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 \\ \frac{\partial^2 \alpha_{k'}}{\partial x_1 \partial x_1} = \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_1} \llcorner \vec{e}_{1,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 \\ \frac{\partial^2 \alpha_{k'}}{\partial x_1 \partial x_2} = -2 \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \end{cases} \quad (\text{IV.20})$$

and

$$\begin{cases} \frac{\partial^2 \alpha_\infty}{\partial x_2 \partial x_2} = \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_2} \llcorner \vec{e}_{2,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 \\ \frac{\partial^2 \alpha_\infty}{\partial x_1 \partial x_1} = \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_1} \llcorner \vec{e}_{1,\infty} \right)^2 + \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 \\ \frac{\partial^2 \alpha_\infty}{\partial x_1 \partial x_2} = -2 \frac{\partial \lambda_\infty}{\partial x_1} \frac{\partial \lambda_\infty}{\partial x_2} \end{cases} \quad (\text{IV.21})$$

Since $\nabla A_{k'}$ and $\nabla B_{k'}$ are uniformly bounded in L^1 we can normalize $A_{k'}$ and $B_{k'}$ in such a way that $A_{k'}$ and $B_{k'}$ are uniformly bounded in $L^2(D^2)$. In a similar way, since now $\nabla \alpha_{k'}$ is uniformly bounded in $L^2(D^2)$ we can normalize $\alpha_{k'}$ in such a way that $\alpha_{k'}$ is uniformly bounded in $W^{1,2}(D^2)$. Thus

$$\limsup_{k' \rightarrow +\infty} \left\| \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_2} \llcorner \vec{e}_{2,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 \right\|_{H^{-1}(D^2)} < +\infty \quad , \quad (\text{IV.22})$$

moreover

$$\limsup_{k' \rightarrow +\infty} \left\| \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_1} \llcorner \vec{e}_{1,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 \right\|_{H^{-1}(D^2)} < +\infty \quad , \quad (\text{IV.23})$$

and finally

$$\limsup_{k' \rightarrow +\infty} \left\| \frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \right\|_{H^{-1}(D^2)} < +\infty \quad . \quad (\text{IV.24})$$

Taking this last quantity, we can always extract a subsequence, that we will still denote k' , such that

$$\frac{\partial \lambda_{k'}}{\partial x_1} \frac{\partial \lambda_{k'}}{\partial x_2} \rightharpoonup f \quad \text{weakly in } H^{-1}(D^2) \quad (\text{IV.25})$$

Comparing this convergence with the second line of (III.9) gives

$$f = \partial_{x_1} \lambda_\infty \partial_{x_2} \lambda_\infty + \sum_{j \in \mathbb{N}} d_j \delta_{q_j} \in H^{-1}(D^2) \quad . \quad (\text{IV.26})$$

But, arguing as for $\alpha_{k'}$, we have that $\alpha_\infty \in W^{1,1}(D^2)$ and hence, using the last line of (IV.21), we have that

$$\nu = \sum_{j \in \mathbb{N}} d_j \delta_{q_j} \in H^{-1}(D^2)$$

This implies that this atomic measure is zero,

$$\nu \equiv 0 \quad (\text{IV.27})$$

which is the unique atomic measure included in H^{-1} .

Similarly, from (IV.22) we can extract a subsequence, still denoted k' , such that

$$\left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_2} \llcorner \vec{e}_{2,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 \rightharpoonup g_1 \quad \text{weakly in } H^{-1}(D^2) \quad (\text{IV.28})$$

and

$$\left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_1} \llcorner \vec{e}_{1,k'} \right)^2 + \left| \frac{\partial \lambda_{k'}}{\partial x_2} \right|^2 - \left| \frac{\partial \lambda_{k'}}{\partial x_1} \right|^2 \rightharpoonup g_1 \quad \text{weakly in } H^{-1}(D^2) \quad (\text{IV.29})$$

Comparing these convergences with the first line of (IV.17) gives in one hand

$$h_- := \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 + \sum_{i \in \mathbb{N}} c_i \delta_{p_i} - g_1 \leq 0 \quad (\text{IV.30})$$

and in the other hand

$$h_+ := \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 + \sum_{i \in \mathbb{N}} c_i \delta_{p_i} + g_2 \geq 0 \quad (\text{IV.31})$$

Using the two first lines of (IV.21), we have that

$$\left[\left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 - g_1 \right] \in H^{-1}(D^2) \quad \text{and} \quad \left[\left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 + g_2 \right] \in H^{-1}(D^2)$$

Let χ be a cut off function in $C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ identically equal to 1 on $[0, 1]$, equal to 0 on $(2, +\infty)$ and $0 \leq \chi \leq 1$ on \mathbb{R}^+ . For any $\varepsilon > 0$ we denote $\chi_\varepsilon(t) := \chi(t/\varepsilon)$. For any $i_0 \in \mathbb{N}$, the map $\chi_\varepsilon(|x - p_{i_0}|)$ weakly converge to zero in $W_0^{1,2}(D^2)$ thus

$$0 \geq \lim_{\varepsilon \rightarrow 0} \langle h_-, \chi_\varepsilon(|x - p_{i_0}|) \rangle = \lim_{\varepsilon \rightarrow 0} \left\langle \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 + \sum_{i \in \mathbb{N}} c_i \delta_{p_i} - g_1, \chi_\varepsilon(|x - p_{i_0}|) \right\rangle = c_{i_0} \quad (\text{IV.32})$$

In a similar way we have

$$0 \leq \lim_{\varepsilon \rightarrow 0} \langle h_+, \chi_\varepsilon(|x - p_{i_0}|) \rangle = \lim_{\varepsilon \rightarrow 0} \left\langle \left| \frac{\partial \lambda_\infty}{\partial x_1} \right|^2 - \left| \frac{\partial \lambda_\infty}{\partial x_2} \right|^2 + \sum_{i \in \mathbb{N}} c_i \delta_{p_i} + g_2, \chi_\varepsilon(|x - p_{i_0}|) \right\rangle = c_{i_0} \quad (\text{IV.33})$$

Comparing (IV.32) and (IV.33) gives for any i_0 $c_{i_0} = 0$ and then we have proved that

$$\mu \equiv 0 \quad (\text{IV.34})$$

Combining (IV.17), (IV.27) and (IV.33) implies then

$$\begin{cases} |\partial_{x_1} \lambda_{k'}|^2 - |\partial_{x_2} \lambda_{k'}|^2 \rightharpoonup |\partial_{x_1} \lambda_\infty|^2 - |\partial_{x_2} \lambda_\infty|^2 & \text{in } \mathcal{D}'(D^2) \\ \partial_{x_1} \lambda_{k'} \partial_{x_2} \lambda_{k'} \rightharpoonup \partial_{x_1} \lambda_\infty \partial_{x_2} \lambda_\infty & \text{in } \mathcal{D}'(D^2) \end{cases} \quad (\text{IV.35})$$

Translating this information in terms of $\alpha_{k'}$ and α_∞ gives

$$\left\{ \begin{array}{l} -\frac{\partial^2 \alpha_{k'}}{\partial x_2 \partial x_2} + \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_2} \llcorner \vec{e}_{2,k'} \right)^2 \rightharpoonup -\frac{\partial^2 \alpha_\infty}{\partial x_2 \partial x_2} + \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_2} \llcorner \vec{e}_{2,\infty} \right)^2 \quad \text{weakly in } H^{-1}(D^2) \\ -\frac{\partial^2 \alpha_{k'}}{\partial x_1 \partial x_1} + \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_1} \llcorner \vec{e}_{1,k'} \right)^2 \rightharpoonup -\frac{\partial^2 \alpha_\infty}{\partial x_1 \partial x_1} + \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_1} \llcorner \vec{e}_{1,\infty} \right)^2 \quad \text{weakly in } H^{-1}(D^2) \\ \frac{\partial^2 \alpha_{k'}}{\partial x_1 \partial x_2} \rightharpoonup \frac{\partial^2 \alpha_\infty}{\partial x_1 \partial x_2} \quad \text{weakly in } H^{-1}(D^2) \end{array} \right. \quad (\text{IV.36})$$

Denote by $\hat{\alpha}_\infty$ the weak limit (modulo extraction of a subsequence) of $\alpha_{k'}$ in $W^{1,2}$ and let $\beta_\infty := \hat{\alpha}_\infty - \alpha_\infty$. We have

$$\left\{ \begin{array}{l} \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_1} \llcorner \vec{e}_{1,k'} \right)^2 + \left(\frac{\partial \vec{n}_{\xi_{k'}}}{\partial x_2} \llcorner \vec{e}_{2,k'} \right)^2 \rightharpoonup \Delta \beta_\infty + \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_1} \llcorner \vec{e}_{1,\infty} \right)^2 + \left(\frac{\partial \vec{n}_{\xi_\infty}}{\partial x_2} \llcorner \vec{e}_{2,\infty} \right)^2 \\ \frac{\partial^2 \beta_\infty}{\partial x_1 \partial x_2} = 0 \quad \text{in } \mathcal{D}'(D^2). \end{array} \right. \quad (\text{IV.37})$$

Or in other words, since (II.8) holds,

$$\left\{ \begin{array}{l} |\nabla \vec{n}_{\xi_{k'}}|^2 dx_1 dx_2 \rightharpoonup |\nabla \vec{n}_{\xi_\infty}|^2 dx_1 dx_2 + \Delta \beta_\infty dx_1 dx_2 \\ \frac{\partial^2 \beta_\infty}{\partial x_1 \partial x_2} = 0 \quad \text{in } \mathcal{D}'(D^2). \end{array} \right. \quad (\text{IV.38})$$

The defect measure is then given by the laplacian of an $W^{1,2}(D^2)$ function β_∞ whose distributional cross derivative $\partial^2 \beta_\infty / \partial x_1 \partial x_2$ is zero. This implies (I.39) and theorem I.3 is proved. \square

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