The Parametric Approach to the Willmore Flow

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28 January 2022

We introduce a parametric framework for the study of Willmore gradient flows which enables to consider a general class of weak, energy-level solutions and opens the possibility to study energy quantization and finite-time singularities. We restrict in this first work to a small-energy regime and prove that, for small-energy weak immersions, the Cauchy problem in this class admits a unique solution.

MSC 49Q10, 53A05, 58E15, 35J35, 35J48, 35K41, 35K91

Contents

1. Introduction

The present paper moves the first steps towards a parametric theory for the Willmore flow that, we believe, will lead to an effective study of singularities, bubbling analysis and energy quantization.

We start by recalling what a Willmore surface is.

1.1. Willmore Surfaces The *Willmore energy* of a surface $\mathcal{S} \subset \mathbb{R}^n$ was initially considered in the work of Poisson [Poi16] and Germain [Ger21] on elastic plates. It was reconsidered in a purely geometric perspective in the works of Thomsen [Tho23] and Blaschke [Bla29], in attempt to merge the study of minimal surfaces with conformal invariance. It was reintroduced in recent times in the works of Willmore [Wil65], and Bryant [Bry84] in pure mathematics and by Canham [Can70] and Helfrich [Hel73] in theoretical biology while modeling the shapes of blood cells and lipid bilayers (and, in fact, it is sometimes referred to as Canham-Helfrich energy). It comes in the following three variants:

$$\mathcal{W}_0(\mathcal{S}) = \frac{1}{2}\int_{\mathcal{S}} |A^\circ|^2 \, d\sigma, \qquad \mathcal{W}_1(\mathcal{S}) = \int_{\mathcal{S}} |H|^2 \, d\sigma, \qquad \mathcal{W}_2(\mathcal{S}) = \frac{1}{4}\int_{\mathcal{S}} |A|^2 \, d\sigma, \tag{1.1}$$

were H the mean curvature, A is the 2nd fundamental form, A° it tracefree part and $d\sigma$ the area element. If K denotes the Gauss curvature of $\mathcal{S}$, there holds

$$\frac{1}{2}|A^\circ|^2 = |H|^2 - K = \frac{1}{4}|A|^2 - \frac{1}{2}K, \tag{1.2}$$

hence, by Gauss-Bonnet theorem, if the topology of $\mathcal{S}$ is fixed, at least in a smooth setting such energies are all variationally equivalent (in particular, they have the same Euler-Lagrange operator). Depending on the context, it may however be more favourable to work with one then another.

The *Willmore operator* is the associated Euler-Lagrange operator:

$$\delta\mathcal{W} = \Delta^\perp H + Q(A^\circ)H,$$

where $\Delta^\perp$ is the Laplace operator on the normal bundle and

$$Q(A^\circ)H = \big\langle A^\circ, \langle H, A^\circ\rangle\big\rangle = g^{\mu\sigma}g^{\nu\tau}\big\langle A^\circ_{\mu\nu}, \langle A^\circ_{\sigma\tau}, H\rangle\big\rangle.$$

Similarly as for the mean curvature, $\delta\mathcal{W}$ is a normal-valued vector field along $\mathcal{S}$. When $n = 3$, the expression simplifies somewhat:

$$\delta\mathcal{W} = \Delta^\perp H + |A^\circ|^2 H = (\Delta H_{\mathrm{sc}} + 2(H_{\mathrm{sc}}^2 - K)H_{\mathrm{sc}})N,$$

where N is the Gauss map of $\mathcal{S}$ and $H_{\mathrm{sc}} = \langle H, N\rangle$ is the scalar mean curvature.

Willmore surfaces are those surfaces with vanishing Willmore operator. Any of the Willmore energies (1.1) is invariant under conformal transformations of $\mathbb{R}^n$ and, in fact, the Lagrangian density $|A^\circ|^2 d\sigma$ is a pointwise conformal invariant, see Chen [Che74]. As a consequence, the Willmore operator and the notion of Willmore surface are also conformal invariants.

1.2. Willmore Flows A *Willmore L^2–gradient flow* in $\mathbb{R}^n$ (*Willmore flow* for short) of a closed, abstract surface Σ is a 1-parameter family of immersions $\Phi(t,\cdot):\Sigma\to\mathbb{R}^n$, $t\in I\subseteq\mathbb{R}$ evolving according to the law

$$\frac{\partial}{\partial t}\Phi=-\delta\mathcal{W}+U \quad \text{in } I\times\Sigma, \tag{1.3}$$

where, for each t, $\delta\mathcal{W}$ is the Willmore operator of $\mathcal{S}_t=\Phi(t,\Sigma)$ and $U=U^\mu\partial_\mu\Phi$ is a tangent tangent vector field, possibly time-dependent.

One good reason to consider Willmore flows is that they satisfy the energy identity, namely if $I=(0,T)$, then

$$\mathcal{W}_0(\Phi(t,\cdot))-\mathcal{W}_0(\Phi(0))=-\int_0^t\int_\Sigma|\delta\mathcal{W}|^2\,d\sigma_g\,d\tau, \quad \text{for } 0\le t<T. \tag{1.4}$$

It may be rephrased by saying that, among all families of immersions whose velocity vector has normal part with L^2-norm equal to $\|\delta\mathcal{W}\|_{L^2(S^2)}$, Willmore flows are those with most rapidly decreasing Willmore energy (in any of the forms given in (1.1)). Thus, at least in principle, they have the potential to converge efficiently to Willmore immersions as $t\to+\infty$.

This is a feature common to gradient flows that makes them particularly worth studying. The first to consider L^2–gradient flows in a geometric context were Eells and Sampson [ES64] in the context of harmonic maps. Since then, the study of parabolic geometric flows has widened to the extent that some of them constitute research areas on their own right, the mean curvature flow and Hamilton's Ricci flow being two of the best-known examples.

It should be noted right away that what is typically called a Willmore flow is a family solving (1.3) with $U=0$, which we will call here a *normal Willmore flow*. Since Σ is closed, and $\delta\mathcal{W}$ is a tensor, it is classical fact that there is a bijective correspondence between tangential components and family of reparametrizations of Σ, see e.g. Mantegazza [Man11, Proposition 1.3.4] for the case, entirely analogous in this regard, of the mean curvature flow. Consequently, if, say, I is a connected interval containing 0, for every family solving (1.3) there is a unique family of diffeomorphisms $\varphi: I\times\Sigma\to\Sigma$ with $\varphi(0,\cdot)=\mathrm{id}_\Sigma$ so that the reparametrized family $\Phi(t,\varphi(t,\cdot))$, $t\in I$ is a normal Willmore flow, and on the other hand, every reparametrization of a normal Willmore flow will be a Willmore flow (1.3) for some U.

Thus, in this sense, similarly as for immersions of surfaces, flows can be regarded as equivalence classes of solutions to (1.3), two of them being equivalent if one can be reparametrized into another. As for surfaces, depending on the situation one may choose one parametrization over another, and in this case this may be done through the choice of the tangential component. This will be a crucial fact in the present work.

The study of Willmore flows was introduced by Kuwert and Schätzle [KS01, KS02] and Simonett [Sim01] and is since then subject of a growing number of works. Particularly useful for us will be the one by Kuwert and Scheuer [KS20] providing asymptotic estimates on the barycenter along the flow.

Our attention here focuses on the following foundational result. Consider the Cauchy problem for the normal Willmore flow:

$$\begin{cases}\dfrac{\partial}{\partial t}\Phi=-\delta\mathcal{W}, & \text{in } (0,T)\times\Sigma,\\ \Phi(0,\cdot)=\Phi_0 & \text{on } \Sigma.\end{cases} \tag{1.5}$$

Theorem ([KS01, KS02]). *There exists* $\varepsilon_0(n) > 0$ *so that, for a smooth immersion* $\Phi_0 : \Sigma \to \mathbb{R}^n$ *satisfying* $\mathcal{W}_0(\Phi_0) = \mathcal{W}_0(\Phi_0(\Sigma)) < \varepsilon_0$*, then* (1.5) *has a unique solution in the smooth category, which furthermore exists for all times and converges to a round sphere.*

It should be said immediately that if $\mathcal{W}_0(\Phi)$ is sufficiently small, Σ must be a sphere. Indeed, as already noticed in [Wil65], it is always $\mathcal{W}_1(\Phi) \geq 4\pi$ and from (1.2), using Gauss-Bonnet one sees that

$$\chi(\Sigma) = \frac{1}{2\pi}\int_\Sigma K\, d\sigma = \frac{1}{2\pi}\Big(\mathcal{W}_1(\Phi) - \mathcal{W}_0(\Phi)\Big) \geq \frac{1}{2\pi}\Big(4\pi - \varepsilon_0\Big) > 1, \tag{1.6}$$

where $\chi(\Sigma)$ is the Euler-Poincaré characteristic of Σ.

Such theorem was concerned with *smooth* solutions. As for other geometric flows however, an effective study of singularities and bubbling analysis requires eventually to work at the *energy level*, namely, to consider appropriate notions of weak solution.

We have in mind as a particular example the classical work on the harmonic map flow done by Struwe [Str85, Str08] and complemented by the works of others, the 2nd author [Riv93], Freire [Fre95b, Fre95a], Chang, Ding and Ye [CDY92], Topping [Top02] Bertsch, Dal Passo and van der Hout [BDvdH02] just to mention a few.

We believe that the framework introduced by the 2nd author in a series of works [Riv08, Riv14, Riv16], which led for instance to an effective energy quantization analysis of Willmore surfaces by Bernard and the 2nd author [BR14] to be, when suitably adapted, the appropriate one. We want to give in the present paper an idea of why this should be true by introducing, under particularly favourable hypotheses, an energy-level class of weak Willmore flow and prove a uniqueness statement for the corresponding Cauchy problem in this class for a broad set of weak initial data, which we believe to be sufficiently close to the largest possible one (among unbranched surfaces).

Let us mention that Lamm and Koch in [KL12] obtained (among other results of geometric interest) an existence and uniqueness result for the Willmore flow for entire graphs in a weak framework with Lipschitz initial datum. Such datum needs to be small in the Lipschitz norm.

1.3. Well-Balanced Conformal Willmore Flows We shall work, in the present paper, always in a low energy regime, namely we shall arrange things so that the Willmore energy of the surfaces in consideration $\mathcal{W}_0(\mathcal{S})$ is as small as needed; furthermore, we shall also work in codimension one, namely $n = 3$. The first major consequence of this is that, as already said above, with (1.6) we may directly assume that the underlying topology is that of the standard sphere S^2. The second one is that we can take advantange of results from the work of De Lellis and Müller [DM05, DM06].

So, from now, it is $\Sigma = S^2$, and the underling reference metric and complex structure are the standard ones.

Central in the theory developed in [Riv08, Riv14, Riv16] and in the present one is the idea of working with *conformal immersions*. The first advantage of doing so is that the Willmore operator becomes uniformly elliptic, with ellipticity constants depending on the conformal factor, and this permits eventually the regularity bootstrap. The second one is that, exploiting conservation laws issuing from the conformal invariance (as explained in Bernard [Ber16]), the Willmore operator of a conformal immersion, which is a 4th order quasilinear elliptic system, can be recast as a 2nd order semilinear system involving Jacobian-type nonlinearities, which allows regularity bootstrap by means of integrability by compensation, similarly as in the work of Hélein [Hél02] on weakly harmonic maps in two dimensions.

The idea is then to consider *Willmore flows in conformal gauge*, where the equation becomes uniformly parabolic, if the conformal factor is uniformly bounded away from zero, and then use a slice-wise in time (elliptic) integrability by compensation arguments to bootstrap the regularity of the equation, which – as is often the case when working with parabolic PDEs in small energy regime – will suffice to get the regularity also in the time variable. This approach was successfully used by the 2nd author in [Riv93] for the case of the harmonic map flow.

Indeed, (1.3) is invariant under reparametrizations and thus it is degenerate parabolic, as is the case for others geometric flows such as the mean curvature flow or the Ricci flow - and this can be a serious source of troubles. The celebrated trick of DeTurck [DeT83], originally devised for the Ricci flow but easily adapted to the present situation, is one way of overcoming this problem. For the sake of compleneteness, we outlined it in Appendix A. Such method has the advantage of working regardless of the topology of Σ, but, as an inspection of the proof reveals, does not seem to be suitable when working with low degrees of smoothness and moreover does not give explicitly a control on the parametrization that is chosen by such gauge.

We are instead going to consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U, & \text{in } (0,T)\times S^2, \\ \Phi(0,\cdot) = \Phi_0 & \text{on } S^2. \end{cases} \tag{1.7}$$

where the tangential vector field U is chosen so that $\Phi(t,\cdot)$ is conformal for every t.

It is a fairly simple matter to find an explicit characterization for U, which – unsurprisingly, given the relationship between complex and conformal structures on surfaces – is best expressed in complex notation. To this aim, we recall that if $\Phi : S^2 \to \mathbb{R}^3$ is a conformal immersion with metric $g = \mathrm{e}^{2\lambda} g_{S^2}$, the second fundamental form may be written in complex notation as

$$A = h_0 + \overline{h}_0 + H \otimes g,$$

where H is the mean curvature and

$$h_0 = A_{zz}\, dz \otimes dz = \frac{1}{4}(A_{11} - A_{22} - 2iA_{12}) dz \otimes dz.$$

Similarly, the tracefree second fundamental form is written as

$$A^\circ = A - Hg = h_0 + \overline{h}_0.$$

With this formalism, we have the following.

Lemma 1.1. *The tangential component of a conformal Willmore flow satisfies*

$$\bar{\partial} U^{(1,0)} = -\langle \delta\mathcal{W}, \overline{h}_0 \rangle^{\sharp_g}, \tag{1.8}$$

that is

$$\partial_{\bar{z}}(U^1 + iU^2)\partial_z \otimes d\bar{z} = 2\mathrm{e}^{-2\lambda}\Big\langle -\delta\mathcal{W}, A_{\bar{z}\bar{z}}\Big\rangle \partial_z \otimes d\bar{z}.$$

Basic facts for the $\overline{\partial}$-operator on vector fields is recalled in Appendix B. It is certainly good news that, at least on the sphere, it defines an uniformly elliptic, zero-cokernel operator.

However, (1.8) do not suffices *per se* to guarantee that (1.7) is uniformly parabolic, not even for short time, since the control on the conformal factor in time still needs to be addressed. Geometrically this is evident: any conformal Willmore flow can be composed with any 1-parameter family of conformal self-maps of S^2 and remain conformal, and $\mathrm{Aut}(S^2)$, the set conformal (i.e. biholomorphic) self-map of S^2 is not compact. One can clearly see this also by noting that (1.8) has a nontrivial kernel given by the conformal Killing (holomorphic, with complex identification) vector fields on S^2 which is 6 dimensional, as $\mathrm{Aut}(S^2)$.

Moreover, (1.8) is an equation satisfied for every fixed t, and does not give any information on the regularity in time of U. Geometrically, this this means that the 1-parameter family of maps in $\mathrm{Aut}(S^2)$ which we may compose a conformal Willmore flow may be taken nonsmooth with respect to t.

Precisely because $\mathrm{Aut}(S^2)$ is 6-dimensional however, a final choice of a 6-dimensional constraint will be enough to tame the action of such gauge group. This will be defined by the following.

Definition 1.2. *An immersion $\Phi : S^2 \to \mathbb{R}^3$ is called* well-balanced *if there holds*

$$\int_{S^2} I d\sigma_g = 0 \quad \textit{and} \quad \int_{S^2} \Phi \times I \, d\sigma = 0, \tag{1.9}$$

where I denotes standard embedding of S^2, $d\sigma$ its area element and $d\sigma_g$ the area element for the induced metric $g = \Phi^\star g_{\mathbb{R}^3}$.

Remark 1.3 Note that being well-balanced is a translation-invariant condition, namely if Φ is well-balanced, so is $\Phi + k$ for every $k \in \mathbb{R}^3$.

Conditions (1.8) and (1.9) together with a good choice for the parametrization of the initial datum , which we shall now discuss, will be sufficient to control the behaviour of the tangential component U in on (1.7). Moreover, they are meaningful also for the notion of weak conformal Willmore flow that we are going to define.

1.4. Chosing an ad-hoc Parametrization for initial Data with Small Energy. From a geometric perspective, both the Cauchy problems (1.5) and (1.7)-(1.8) possess an obvious “gauge invariance” for the initial datum, namely if $\Phi_0(S^2) = \mathcal{S}$ is the immersed sphere representing the initial datum, and φ is any diffeomorphism of S^2, then $\Phi_0 \circ \varphi$ is again a parametrization for the same surface $\mathcal{S}$, and there is no a priori preferred choice – or possibility to distinguish – between Φ_0 and $\Phi_0 \circ \varphi$. This a relevant issue for a parametric theory.

The conformal gauge choice helps to reduce this invariance (Φ_0 has to be conformal, and so φ must belong to $\mathrm{Aut}(S^2)$), but does not break it entirely. To this aim, we shall use the following result contained in the work of De Lellis and Müller [DM05, DM06].

Theorem ([DM05, DM06])**.** *There exist ε_0, $C > 0$ so that, if $\mathcal{S} \subset \mathbb{R}^3$ is an immersed surface with area $\mathcal{A}(\mathcal{S}) = 4\pi$ and Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon_0$, there exists a conformal parametrization $\Phi : S^2 \to \mathcal{S}$ satisfying*

$$\|\Phi - I - c\|_{W^{2,2}(S^2)} + \|e^\lambda - 1\|_{L^\infty(S^2)} \leq C\sqrt{\mathcal{W}_0(\mathcal{S})}, \tag{1.10}$$

where $I : S^2 \to \mathbb{R}^3$ denotes the standard immersion of S^2 and $c = \fint_{S^2} \Phi \, d\sigma$.

In this theorem the fact that the area of the surface is 4π can be seen to a normalization achievable by scaling. We shall need another one, achievable by translations; to this aim recall that the *barycenter* of an immersed surface $\mathcal{S} \subset \mathbb{R}^3$ is defined as

$$\mathcal{C}(\mathcal{S}) = \int_{\mathcal{S}} \mathrm{id}_{\mathbb{R}^3}\, d\mathcal{H}^2 = \int_{\Sigma} \Phi \, d\sigma_g,$$

where $\Phi : \Sigma \to \mathcal{S}$ is any parametrization of $\mathcal{S}$.

The set of initial data for the conformal Willmore flow will consist *geometrically* of the set of immersed surfaces $\mathcal{S} \subset \mathbb{R}^3$ with Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area 4π and vanishing barycenter $\mathcal{C}(\mathcal{S}) = 0$. *Parametrically* we shall choose a parametrization provided by the above theorem, which is in addition well-balanced as in Definition 1.2. More precisely:

Definition 1.4. *For $\varepsilon > 0$, $\mathscr{D}^\varepsilon(S^2, \mathbb{R}^3)$ is the set of smooth conformal immersions $\Phi : S^2 \to \mathbb{R}^3$ so that the surface $\mathcal{S} = \Phi(S^2)$ has Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon$, area $\mathcal{A}(\mathcal{S}) = 4\pi$, barycenter $\mathcal{C}(\mathcal{S}) = 0$, is well-balanced and so that (1.10) holds for $C > 0$ given by that estimate.*

This is, when restricted to the smooth category, the suitable class of initial data that shall be considered in this work, for sufficiently small $\varepsilon > 0$. It will be enlarged to its weak $W^{2,2}$-closure when considering the extension of the theory to the weak framework, which we discuss below.

We want to stress that the only essential requirement in Definition 1.4 is the control (smallness) of the Willmore energy. All the others can be seen as normalizations. More precisely, the first result of this work, which will be used to prove the main one, is the following extension of the theorem above:

Proposition 1.5. *There are ε_0, δ, $C > 0$ with the following properties:*

(i) Any immersed surface $\mathcal{S} \subset \mathbb{R}^3$ with area 4π and Willmore energy $\mathcal{W}_0(\mathcal{S}) \leq \varepsilon_0$ admits a conformal parametrization satisfying (1.10) which is also well-balanced.

(ii) For any well-balanced conformal parametrization $\Psi : S^2 \to \mathcal{S}$ with conformal factor e^ν and some vector $c \in \mathbb{R}^3$ so that

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)} \leq \delta, \tag{1.11}$$

there holds

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)} \leq C\sqrt{\mathcal{W}_0(\mathcal{S})}.$$

Furthermore, the following local uniqueness *property holds: if Ψ' is another well-balanced conformal immersion satisfying (1.11), and $\psi \in \mathrm{Aut}(S^2)$ is the conformal diffeomorphism so that $\Psi' = \Psi \circ \psi$, there is a neighborhood $\mathcal{O} \subset \mathrm{Aut}(S^2)$ of the identity e (depending only on δ) so that, if $\psi \in \mathcal{O}$, then $\psi = e$.*

1.5. Conformal Weak Flows We now define an energy-level class of maps where one can consider weak conformal Willmore flows. We believe it to the a prototype for future works concerned with Willmore flows at energy level.

Central to the definitions we shall give shortly is that the validity of the energy identity (1.4) (in fact, a slightly weaker version will suffice). This should be, broadly speaking, a requirement to avoid the presence of pathological solutions that invalidate the uniqueness of the solution to the Cauchy problem, as the examples of Topping [Top02] and Bertsch, Dal Passo and van der Hout [BDvdH02] show in the case of the harmonic map flow.

From [Riv08, Riv14, Riv16] we recall the notion of weak $W^{2,2}$-Lipschitz immersion. If $W^{1,\infty}_{\mathrm{imm}}(S^2,\mathbb{R}^3)$ denotes the set of Lipschitz immersions, namely those Lipschitz maps $\Phi : S^2 \to \mathbb{R}^3$ so that there exists $C = C(\Phi) > 0$ with

$$\frac{1}{C} g_{S^2} \leq g = \Phi^\star g_{\mathbb{R}^3} \leq C g_{S^2}$$

almost everywhere in the sense of metrics, we let

$$\mathscr{E}(S^2,\mathbb{R}^3) = W^{1,\infty}_{\mathrm{imm}}(S^2,\mathbb{R}^3) \cap W^{2,2}(S^2).$$

Every map in such set admits a conformal reparametrization and moreover, it is possible to define its Willmore operator in the sense of distributions. Starting from the divergence form of the Willmore operator introduced in [Riv08]:[1]

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H - 2(\nabla H)^{\top_\Phi} - |H|^2 d\Phi\Big) = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big),$$

the Willmore operator of $\Phi \in \mathscr{E}(S^2,\mathbb{R}^3)$ is defined as the distribution-valued two form given by

$$\Big(\delta\mathcal{W} d\sigma_g, \varphi\Big)_{\mathcal{D}'} = \int_{S^2} \Big(\Big\langle H, \Delta_g \varphi\Big\rangle - \Big\langle \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}, \nabla\varphi\Big\rangle_g\Big) d\sigma_g,$$

for every $\varphi \in C^\infty(S^2,\mathbb{R}^3)$.

We can now give the following central definitions.

Definition 1.6 (Weak Initial Data). *For $\varepsilon > 0$, $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ is the closure of $\mathscr{D}^\varepsilon(S^2,\mathbb{R}^3)$ (from Definition 1.4) with respect to the weak $W^{2,2}(S^2)$-topology.*

We will consider $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ only for $\varepsilon > 0$ sufficiently small. As a consequence of the works [Riv08, Riv14, Riv16], $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ is a subset of the space $\mathscr{E}(S^2,\mathbb{R}^3)$, which would be the broadest possible choice for the theory (among nonbranched surfaces, at least). We do not know at present whether $\mathscr{W}^\varepsilon(S^2,\mathbb{R}^3)$ coincides, or strictly contained in, $\mathscr{E}(S^2,\mathbb{R}^3)$.

Definition 1.7 (Well-Balanced Energy Class). *For ε, δ, $T > 0$, $\mathscr{W}^{\varepsilon,\delta}_{[0,T]}(S^2,\mathbb{R}^3)$ is set of locally integrable maps $\Phi : (0,T) \times S^2 \to \mathbb{R}^3$ so that*

(i) For almost every t, $\Phi(t,\cdot)$ is in $\mathscr{E}(S^2,\mathbb{R}^3)$ and conformal,

(ii) There holds

$$\|\Phi - I - c\|_{L^\infty((0,T),W^{2,2}(S^2))} + \|\mathrm{e}^\lambda - 1\|_{L^\infty((0,T)\times S^2)} \leq \delta, \tag{1.12}$$

where I denotes standard embedding of S^2, $\mathrm{e}^\lambda = \mathrm{e}^{\lambda(t,\cdot)}$ is the conformal factor of $\Phi(t,\cdot)$ and $c(t) = \fint_{S^2} \Phi(t,\cdot)\, d\sigma$,

[1] We denote, here and in the sequel:

- $\nabla^{*_g}(Z \otimes \omega) = \frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}\, g^{\mu\nu}\omega_\nu Z\big)$, *minus* the formal L^2-adjoint of the covariant derivative induced on the pull-back bundle $\Phi^\star(T\mathbb{R}^3)$ acting on sections of $\Phi^\star(T\mathbb{R}^3) \otimes T^*S^2$,
- $\langle A, H\rangle^{\sharp_g} = g^{\mu\xi}\langle A_{\xi\nu}, H\rangle \partial_\mu \otimes dx^\nu \simeq g^{\mu\xi}\langle A_{\xi\nu}, H\rangle \partial_\mu\Phi \otimes dx^\nu$ the the 1st-index raising of $\langle A, H\rangle$, and similarly for $\langle A^\circ, H\rangle$.

(iii) *There holds*

$$\delta\mathcal{W} \in L^2((0,T)\times S^2) \quad \text{and} \quad \mathcal{W}_0(\Phi(t,\cdot)) \leq \varepsilon \ \text{ for a.e. } t, \tag{1.13}$$

(iv) *Φ is well-balanced for a.e. t.*

Finally we let, also for $T=+\infty$,

$$\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3) = \bigcap_{\tau\in(0,T)} \mathscr{W}^{\varepsilon,\delta}_{[0,\tau]}(S^2,\mathbb{R}^3).$$

Assumption (1.12) is quite natural if we look at Proposition 1.5. In the energy class, a weak Willmore flow is defined as follows.

Definition 1.8 (Weak Willmore Flow)**.** *$\Phi \in \mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ is a weak solution of the Willmore flow with tangential component $U = U^\mu \partial_\mu \Phi$:*

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + U \quad \text{in } (0,T)\times S^2,$$

if for every $\varphi \in C^\infty_c((0,T)\times S^2,\mathbb{R}^3)$ there holds

$$-\int_0^T \int_{S^2} \left\langle \Phi, \frac{\partial}{\partial t}\varphi \right\rangle d\sigma_g\, dt = -\int_0^T \Big(\delta\mathcal{W} d\sigma_g, \varphi(t,\cdot)\Big)_{\mathcal{D}'} dt + \int_0^T \int_{S^2} \big\langle U, \varphi \big\rangle\, d\sigma_g\, dt.$$

Our main result is the following.

Theorem 1.9. *There exists $\varepsilon_0>0$ so that the Cauchy problem for the conformal Willmore flow (1.7)-(1.8) with initial datum in $\mathscr{W}^{\varepsilon_0}(S^2,\mathbb{R}^3)$ has a weak solution in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ for some $\delta>0$, assuming the initial datum in the sense of traces. Such solution is smooth, exists for all times and smoothly converges to the standard embedding I of S^2 in $\mathbb{R}^3$. Furthermore, if the initial datum is smooth, such weak solution is also unique.*

We can compare this result with the above metioned one of Kuwert and Schätzle [KS01, KS02]. They obtain, in the smooth class, long-time existence, uniqueness and convergence to a round sphere for the Cauchy problem of the normal flow (1.5). A central feature our result is that the uniqueness of this smooth solution is in the broad class of finite energy solutions, and the fact that it converges exactly to the standard embedding.

We expect the solution to be unique also if the initial datum is nonsmooth; we plan to address this question in the future.

The proof of the regularity part of Theorem 1.9 shares evident similarities with the corresponding one for the harmonic map flow obtained in [Riv93]. In that work, the core estimate that was obtained for weak solutions of the harmonic map flow was of the form

$$\|u(t,\cdot)\|_{W^{2,2}} \leq C\Big(\|\partial_t u(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

which could then be squared and integrated in time to yield higher regularity, and eventually smoothness by the classical theory by Struwe [Str85, Str08]. We shall obtain a similar result, namely an inequality of the form

$$\|\Phi(t,\cdot)\|_{W^{4,2}} \leq C\Big(\|e^\lambda \delta\mathcal{W}(t,\cdot)\|_{L^2} + 1\Big) \quad \text{for a.e. } t,$$

for weak solutions of the conformal Willmore flow, and likewise obtain higher regularity from it. The overall procedure shall be however more technical.

1.6. Final Comments We have intentionally decided to work in a small-energy regime in this first paper. We plan to consider more general scenarios in future works, where more technical, localization/energy-concentration arguments will be dealt with.

One has also to take into account that, when the underlying surface is not a sphere, there is more than one conformal class, so to properly work with a conformal Willmore flow, one has to take into account the nontriviality of the corresponding Teichmüller space. The work of Rupflin and Topping [RT16] on the Teichmüller harmonic map flow also faces the difficulty of "following" the conformal class along the flow.

In the future we plan to determine whether the class $\mathscr{W}^{\varepsilon}(S^2,\mathbb{R}^3)$ coincides or not with $\mathscr{E}(S^2,\mathbb{R}^3)$, and whether the solution given by Theorem 1.9 is unique also in the case of weak initial data. We shall also seek to extend the argument for branched weak initial data.

Finally, we plan to carry an accurate study of singularities (blow-up points, degeneration of conformal factor or conformal class...) in forthcoming works. For this we will likely build upon some of the work already done on the subject, Mayer and Simonett [MS02], Blatt [Bla09] and Chill, Fašangová and Schätzle [CFS09] just to mention a few.

One of the questions relative to the parametric approach of the Willmore flow is the following: can the conformal class of a conformal Willmore flow – suitably normalized to remove any obvious gauge invariance – degenerate in finite time?

2. Preliminaries

We state here known results that shall play a key role in the paper.

Theorem 2.1. *([Bet92], [Riv93], [Ge99]) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain, let $a \in W^{1,2}(\Omega)$ and $f \in L^p(\Omega)$ for $1 < p < 2$. Let $u \in W^{1,(2,\infty)}(\Omega)$[2] be a solution to*

$$-\Delta u = \langle \nabla^{\perp} a, \nabla u \rangle + f \quad \text{in } \Omega.$$

Then $u \in W^{2,p}_{loc}(\Omega)$.

Theorem 2.2 ([Hél02], [Riv16]). *There exists $\varepsilon_0 > 0$ so that, if $\Phi \in \mathscr{E}(B_1,\mathbb{R}^n)$ is conformal with conformal factor e^{λ} and so that*

$$\int_{B_1} |A|_g^2 \, d\sigma_g \leq \varepsilon_0,$$

then, if $C_{(2,\infty)} > 0$ is constant so that

$$\|d\lambda\|_{L^{(2,\infty)}(B_1)} \leq C_{(2,\infty)},$$

for any $0 < r < 1$ there holds

$$\|d\lambda\|_{L^2(B_r)} + \|\lambda - \ell\|_{L^\infty(B_r)} \leq C \int_{B_1} |A|_g^2 \, d\sigma_g,$$

for some constants $\ell \in \mathbb{R}$ and $C = C(r, C_{(2,\infty)}) > 0$.

Remark 2.3 By the triangle inequalily, without loss of generality we can take $\ell = \lambda(0)$ in the above estimate.

[2] That is, $u \in W^{1,1}(\Omega)$ and ∇u is in the Lorentz space $L^{(2,\infty)}(\Omega)$.

Theorem 2.4 ([DM05, DM06]). *Let $\mathcal{S} \subset \mathbb{R}^3$ be an immersed surface with area $\mathcal{A}(\mathcal{S}) = 4\pi$ and let g be the induced metric. Then there holds*

$$\int_{\mathcal{S}} |A_{sc} - g|_g^2 \, d\sigma_g \leq C \int_{\mathcal{S}} |A^\circ|_g^2 \, d\sigma_g, \tag{2.1}$$

where $A_{sc}(\cdot\,,\cdot) = \langle A(\cdot\,,\cdot), N\rangle$ is the scalar second fundamental form of $\mathcal{S}$. Furthermore, there is $\varepsilon_0 > 0$ so that if

$$\mathcal{W}_0(\mathcal{S}) = \frac{1}{2}\int_{\mathcal{S}} |A^\circ|_g^2 \, d\sigma_g \leq \varepsilon_0,$$

there exists a conformal parametrization $\Phi : S^2 \to \mathbb{R}^3$ satisfying

$$\|\Phi - (c_{\mathcal{S}} + I)\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\lambda - 1\|_{L^\infty(S^2)} \leq C\sqrt{\mathcal{W}_0(\mathcal{S})}, \tag{2.2}$$

for some vector $c_{\mathcal{S}} \in \mathbb{R}^3$ and for an absolute constant $C > 0$, where e^λ is the conformal factor of the induced metric and I is the standard immersion of S^2 into $\mathbb{R}^3$.

Remark 2.5 From the minimality property of the average:

$$\left\| f - \fint_{S^2} f \, d\sigma \right\|_{L^2(S^2)} = \inf_{c\in\mathbb{R}} \|f - c\|_{L^2(S^2)},$$

and since $\int_{S^2} I \, d\sigma = 0$, we may suppose $c_{\mathcal{S}} = \fint_{S^2} \Phi \, d\sigma_{S^2}$ in (2.2).

Theorem 2.6 ([Riv08], [Riv16], [Ber16]). *The Willmore operator of an immersion of a surface $\Phi : \Sigma \to \mathbb{R}^n$,*

$$\delta\mathcal{W} = \Delta_g^\perp H + Q(A^\circ)H, \tag{2.3}$$

where $\Delta_g^\perp$ denotes the Laplace operator on the normal bundle of $\Phi(\Sigma)$ and

$$Q(A^\circ)H = \langle A^\circ, \langle H, A^\circ\rangle\rangle_g = g^{\mu\sigma}g^{\nu\tau}\langle A^\circ_{\mu\nu}, \langle A^\circ_{\sigma\tau}, H\rangle\rangle,$$

may be written equivalently in divergence form as

$$\delta\mathcal{W} = \nabla^{*_g}\Big(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big) = \Delta_g H + \nabla^{*_g}\Big(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\Big). \tag{2.4}$$

When $n = 3$, if $w \in \Gamma(\Phi^\star(T\mathbb{R}^3) \otimes T^*\Sigma)$ denotes the vector-valued form along Φ given by

$$w = \nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} = \nabla H - 2(\nabla H)^\top - |H|^2 d\Phi. \tag{2.5}$$

Then the following formulas hold true:

$$\delta\mathcal{W} = \nabla^{*_g} w, \tag{2.6}$$

$$\Phi \times \delta\mathcal{W} = \nabla^{*_g}\Big(-d\Phi \times H + \Phi \times w\Big), \tag{2.7}$$

$$\Big\langle \Phi, \delta\mathcal{W}\Big\rangle = d^{*_g}\Big\langle \Phi, w\Big\rangle, \tag{2.8}$$

$$\Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\Big\langle \Phi, \delta\mathcal{W}\Big\rangle + 4H = \nabla^{*_g}\Big(-2\Phi \times (d\Phi \times H) + \Phi \times (\Phi \times w) + \Phi\langle \Phi, w\rangle\Big). \tag{2.9}$$

Remark 2.7 (i) Equations (2.6), (2.7), (2.8) and (2.9) follow from Noether's theorem and the invariance of the Willmore energy under translations, rotations, dilations and inversions respectively. To obtain (2.9) (since inversions do not form *per se* a 1–parameter family of transformations), one applies Noether's theorem to the local family $F(s,y) = \mathcal{I} \circ \tau_{sv} \circ \mathcal{I}(y)$, where $\mathcal{I}(y) = \frac{y}{|y|^2}$ denotes the inversion with respect to the unit sphere and $\tau_v(y) = y + v$ is the translation of vector $v \in \mathbb{R}^3$. This transformation is generated by the vector field $\mathcal{X}(y) = v|y|^2 - 2y\langle y, v\rangle$ and gives rise to the law (2.9).

(ii) The equivalence between the first and second expressions for w in (2.5) is obtained by noting that, since H is a normal vector, there holds

$$\begin{aligned}(\nabla H)^\top &= g^{\mu\nu}\langle \partial_\alpha H, \partial_\mu \Phi\rangle \partial_\nu \Phi \otimes dx^\alpha \\ &= -g^{\mu\nu}\langle H, \partial^2_{\alpha\mu}\Phi\rangle \partial_\nu \Phi \otimes dx^\alpha \\ &= -g^{\mu\nu}\langle H, A_{\alpha\mu}\rangle \partial_\nu \Phi \otimes dx^\alpha \\ &= -\langle H, A\rangle^{\sharp_g},\end{aligned}$$

and since one similarly has $|H|^2 d\Phi = \langle H, Hg\rangle^{\sharp_g}$, there holds

$$2(\nabla H)^\top + |H|^2 d\Phi = -2\langle H, A\rangle^{\sharp_g} + \langle H, Hg\rangle^{\sharp_g} = -\langle H, A\rangle^{\sharp_g} - \langle H, A^\circ\rangle^{\sharp_g}.$$

Theorem 2.8 ([KS01, KS02, KS20]). *There exists an $\varepsilon_0 = \varepsilon_0(n) > 0$ so that, if $\Phi : [0,T) \times S^2 \to \mathbb{R}^n$ is a smooth normal Willmore flow*

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W},$$

with smooth initial datum $\Phi(0,\cdot) = \Phi_0$ and $\mathcal{W}_0(\Phi_0) \le \varepsilon_0$, then:

(i) its area satisfies

$$|\mathcal{A}(\Phi(t,\cdot)) - \mathcal{A}(\Phi_0)| \le C\mathcal{A}_0(\Phi_0)\mathcal{W}_0(\mathcal{S}),$$

for a constant $C = C(n) > 0$.

(ii) Its barycenter $\mathcal{C}(\Phi) = \fint_{S^2} \Phi \, d\sigma_g$ satisfies

$$|\mathcal{C}(\Phi(t,\cdot)) - \mathcal{C}(\Phi_0)| \le C\mathcal{W}_0(\mathcal{S})$$

for a constant $C = C(n) > 0$.

(iii) Φ exists for all times and smoothly converges to a round sphere $t \to \infty$.

3. Consequences of DeLellis-Müller's Theorem

The proof of Proposition 1.5 will follow from two lemmas.

Lemma 3.1. *The function $F : \mathrm{Aut}(S^2) \to \mathbb{R}^6$ given by*

$$F(\psi) = (F_1(\psi), F_2(\psi)) = \left(\int_{S^2} I\, \psi^\star d\sigma, \int_{S^2} (I \circ \psi) \times I \, d\sigma\right), \tag{3.1}$$

where $\psi^\star d\sigma$ denotes the pullback of area element $d\sigma$ via ψ, is differentiable and $dF(e)$ is an isomorphism.

Proof. Differentiability follows since F is composition of smooth functions and operations. We shall now use the language of differential forms and so along this proof it is convenient to temporarily change our notation for the area element from $d\sigma$ to ω_{S^2}.

Recall that $\omega_{S^2} = \omega_{\mathbb{R}^3} \llcorner N = \omega_{\mathbb{R}^3}(N, \cdot, \cdot)$, where N Gauss map of S^2. More explicitly, since $N(y) = y$, we have the formula

$$\omega_{S^2} = (dy^1 \wedge dy^2 \wedge dy^3)(y, \cdot, \cdot) = y^1 dy^2 \wedge dy^3 - y^2 dy^1 \wedge dy^3 + y^3 dy^1 \wedge dy^2.$$

Now $\mathrm{Aut}(S^2)$ has dimension 6 as a manifold and we consider the basis for the tangent space $T_e \,\mathrm{Aut}(S^2)$ given by the vector fields generating, respectively, rotations and "spherical dilations" about the coordinate axes:

$$Z_1(y) = (0, -y^3, y^2), \qquad Z_2(y) = (y^3, 0, -y^1), \qquad Z_3(y) = (-y^2, y^1, 0),$$
$$Z_4(y) = e_1 - y^1 y, \qquad Z_5(y) = e_2 - y^2 y, \qquad Z_6(y) = e_3 - y^3 y.$$

We shall prove that

$$\left(\frac{\partial F^j}{\partial Z_a}(e)\right)_{1 \le a, j \le 6} = -\frac{8\pi}{3} \mathbf{1}_{6\times 6}. \tag{3.2}$$

To compute $\partial_X F(e)$, if Φ^X denotes the local flow of the vector field X, we have to evaluate

$$\frac{\partial}{\partial X} F(e) = \frac{d}{dt} F(\Phi^X(t, \cdot))\Big|_{t=0}.$$

Let us look at F_1. By Cartan's formula, since $d\omega_{S^2} = 0$, it is

$$\frac{\partial}{\partial t} \Phi^X(t, \cdot)\omega_{S^2}\Big|_{t=0} = \mathcal{L}_X \omega_{S^2} = d(X \llcorner \omega_{S^2}),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Since Z_a's for $a = 1, 2, 3$ generate isometries, $\mathcal{L}_{Z_a} \omega_{S^2} = 0$ and hence *a fortiori*

$$\frac{d}{dt} F_1\big(\Phi^{Z_a}(t, \cdot)\big)\Big|_{t=0} = \int_{S^2} I \,\mathcal{L}_{Z_a} \omega_{S^2} = 0 \qquad \text{for } a = 1, 2, 3.$$

As for the Z_a's for $a = 4, 5, 6$, we see that

$$Z_4 \llcorner \omega_{S^2} = -y^2 dy^3 + y^3 dy^2, \qquad Z_5 \llcorner \omega_{S^2} = y^1 dy^3 - y^3 dy^1, \qquad Z_6 \llcorner \omega_{S^2} = -y^1 dy^2 + y^2 dy^1,$$
$$d(Z_4 \llcorner \omega_{S^2}) = -2 dy^2 \wedge dy^3, \qquad d(Z_5 \llcorner \omega_{S^2}) = 2 dy^1 \wedge dy^3, \qquad d(Z_6 \llcorner \omega_{S^2}) = -2 dy^1 \wedge dy^2,$$

and hence with Stokes' theorem we get

$$\begin{aligned}
\frac{d}{dt} F_1(\Phi^{Z_4}(t, \cdot))\Big|_{t=0} &= \int_{S^2} I \mathcal{L}_{Z_4} \omega_{S^2} \\
&= \Big(\int_{S^2} y^1(-2dy^2 \wedge dy^3), \int_{S^2} y^2(-2dy^1 \wedge dy^3), \int_{S^2} y^3(-2dy^1 \wedge dy^2)\Big) \\
&= -\frac{8\pi}{3}(1, 0, 0),
\end{aligned}$$

and similarly

$$\frac{d}{dt}F_1\Big(\Phi^{Z_5}(t,\cdot)\Big)\Big|_{t=0} = -\frac{8\pi}{3}(0,1,0), \qquad \frac{d}{dt}F_1\Big(\Phi^{Z_6}(t,\cdot)\Big)\Big|_{t=0} = -\frac{8\pi}{3}(0,0,1).$$

Now we consider F_2. Since we can write

$$F_2\Big(\Phi^X(t,\cdot)\Big) = \int_{S^2} \Phi^X(t,\cdot)\times I\,d\sigma,$$

it is

$$\frac{d}{dt}F_2\Big(\Phi^X(t,\cdot)\Big)\Big|_{t=0} = \int_{S^2} X\times I\,d\sigma,$$

and thus one directly computes that for $a=1,2,3$ it is

$$\int_{S^2} Z_a\times I\,d\sigma = -\frac{8\pi}{3}e_a, \qquad \int_{S^2} Z_{a+3}\times I\,d\sigma = 0.$$

Putting together all these computations yields (3.2) and hence the thesis. □

Lemma 3.2. *There exists $E>0$ with the following property. For any $\eta_1\le E$ there exist $\eta_2>0$ so that, if $\mathcal{S}\subset\mathbb{R}^3$ is an immersed surface with area $\mathcal{A}(\mathcal{S})=4\pi$ and $\Phi: S^2\to\mathcal{S}$ is a conformal immersion with conformal factor e^λ and $c\in\mathbb{R}^3$ is a vector so that*

$$\|\Phi - I - c\|_{W^{2,2}(S^2)} + \|e^\lambda - 1\|_{L^\infty(S^2)} \le \eta_1, \tag{3.3}$$

then there exists a conformal self-map $\psi\in\mathrm{Aut}(S^2)$ so that $\Psi=\Phi\circ\psi$ is well-balanced and ψ is the unique self-map with such property in the Riemannian ball $B_{\eta_2}(e)\subset\mathrm{Aut}(S^2)$. In addition, if e^ν denotes the conformal factor of Ψ, there holds

$$\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)} \le 2\eta_1. \tag{3.4}$$

Proof. Let $\mathcal{F}: W^{2,2}(S^2,\mathbb{R}^3)\times\mathrm{Aut}(S^2)\to\mathbb{R}^6$ be given by

$$\mathcal{F}(f,\psi) = \Big(\int_{S^2} I\,\frac{1}{2}|d(f\circ\psi)|^2 d\sigma, \int_{S^2}(f\circ\psi)\times I\,d\sigma\Big).$$

Note that this definition makes sense for every $f\in W^{2,2}(S^2)$ and, if Φ is a conformal immersion $\mathcal{F}(\Phi,e)=0$ means that Φ is well-balanced as in Definition 1.2. Moreover $\mathcal{F}(I,\cdot)=0$ coincides with F given in (3.1). Finally $\mathcal{F}$ is invariant by translations in its first component: $\mathcal{F}(f,\cdot)=\mathcal{F}(f+k,\cdot)$ for every $k\in\mathbb{R}^3$.

As a consequence of Lemma 3.1, $d_\psi\mathcal{F}(I,e)=d_\psi\mathcal{F}(I,\cdot)(e)$ is an isomorphism, and hence by the implicit function theorem, there exists $E,\eta_2>0$ so that if (3.3) holds for $\eta_1\le E$, there is a unique $\psi=\psi_\Phi$ in the Riemannian ball $B_{\eta_2}(e)\subset\mathrm{Aut}(S^2)$ so that $\mathcal{F}(\Phi-c,\psi)=\mathcal{F}(\Phi,\psi)=0$, i.e. so that $\Psi=\Phi\circ\psi$ is well-balanced (recall also Remark 1.3).

Finally, since ψ is biholomorphic, $\forall N\in\mathbb{N}$ we can estimate $\sum_{k=1}^N \mathrm{dist}(\nabla^k\psi,\nabla^k e)\le C_N\,\mathrm{dist}(\psi,e)$ for some $C_N>0$ independent of ψ. So (3.3) holds, by the triangle inequality and the continuity of the Lebesgue integral,

$$\begin{aligned}
&\|\Psi - I - c\|_{W^{2,2}(S^2)} + \|\mathrm{e}^\nu - 1\|_{L^\infty(S^2)}\\
&= \|\Phi\circ\psi - I - c\|_{W^{2,2}(S^2)} + \|\tfrac{1}{\sqrt{2}}|d\psi|\mathrm{e}^{\lambda\circ\psi} - 1\|_{L^\infty(S^2)}\\
&= \|(\Phi - c)\circ\psi - I\|_{W^{2,2}(S^2)} + \|\tfrac{1}{\sqrt{2}}|d\psi|\mathrm{e}^{\lambda\circ\psi} - 1\|_{L^\infty(S^2)}\\
&= \eta_1 + o(1) \quad \text{as } \mathrm{dist}(\psi,e)\to 0,
\end{aligned}$$

and, since $\|\Psi - c\|_{L^2(S^2)}$ and $\|e^\lambda\|_{L^\infty(S^2)}$ are uniformly bounded, the remainder $o(1)$ can be taken uniform in Φ, Ψ, ψ, c and hence, choosing η_2 sufficiently small we obtain to (3.4). □

Proof of Proposition 1.5. It suffices to prove the thesis for $\mathcal{W}_0(\mathcal{S}) \le \varepsilon_0$ sufficiently small. For part (i), combine Theorem 2.4 and Lemma 3.2.

For part (ii), Let $\varepsilon_0 > 0$ be sufficiently small so that

$$C\varepsilon_0 \le \frac{1}{2}E,$$

where E is as in Lemma 3.2 and C is the constant of Theorem 2.4, and let $\Phi : S^2 \to \mathcal{S}$ be the conformal parametrization given by that theorem. By Lemma 3.2 there exists a unique choice of $\alpha = \alpha_\Phi$ in $B_{\eta_2}(e) \subset \mathrm{Aut}(S^2)$ with $\mathcal{F}(\Phi, \alpha) = 0$ i.e. $\Phi' = \Phi \circ \alpha$ is well-balanced and

$$\|\Phi' - I - c\|_{W^{2,2}(S^2)} + \|e^{\lambda'} - 1\|_{L^\infty(S^2)} \le 2C\sqrt{\mathcal{W}_0(\Phi')} \le \sqrt{E}.$$

So now if δ is taken so that

$$\delta \le \frac{1}{2}\sqrt{E},$$

since Ψ is already well-balanced, by uniqueness it must be $\Phi' = \Psi$, and the thesis follows also for the local uniqueness part, with $\mathcal{O} = B_{\eta_2}(e)$. □

The following simple consequence of Theorem 2.4 will also be needed later.

Lemma 3.3. *If $\Phi : S^2 \to \mathbb{R}^3$ is a conformal immersion with conformal factor e^λ and $B_r(x_0) \subset S^2$ is a disk of radius r (in the standard metric of S^2), there holds*

$$\int_{B_r(x_0)} |A|_g^2 \, d\sigma_g \le C\Big(\int_{S^2} |A^\circ|_g^2 \, d\sigma_g + \mathrm{e}^{4C_0} r^2 \Big), \tag{3.5}$$

where $C_0 = \|\lambda\|_{L^\infty(S^2)}$ and $C > 0$ is an absolute constant.

Proof. It is a consequence of (2.1) applied to the immersed surface $\mathcal{S} = a\,\Phi(S^2)$, where $a = \sqrt{\frac{4\pi}{\mathcal{A}(\Phi)}}$ and $\mathcal{A}(\Phi) = \int_{S^2} \mathrm{e}^{2\lambda} \, d\sigma$ is the area of $\Phi(S^2)$. Indeed, since

$$4\pi \mathrm{e}^{-2C_0} \le \mathcal{A}(\Phi) \le 4\pi \mathrm{e}^{2C_0},$$

it follows that $e^{-C_0} \le a \le \mathrm{e}^{C_0}$ and we can estimate

$$\int_{B_r(x_0)} |a\, g|_g^2 d\sigma_g = a^2 \int_{B_r(x_0)} 2\mathrm{e}^{2\lambda} d\sigma \le C\mathrm{e}^{4C_0} r^2.$$

and thus

$$\begin{aligned}\int_{B_r(x_0)} |A|_g^2 \, d\sigma_g &\le 2\int_{B_r(x_0)} |A_{\mathrm{sc}} - ag|_g^2 \, d\sigma_g + 2\int_{B_r(x_0)} |a\, g|_g^2 \, d\sigma_g \\ &\le C\int_{B_r(x_0)} |A^\circ|_g^2 \, d\sigma_g + C\mathrm{e}^{4C_0} r^2,\end{aligned}$$

which proves (3.5). □

4. Estimates of Elliptic Type

This section is dedicated to prove the following.

Theorem 4.1. *Let* $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ *be conformal with conformal factor* e^λ *and Willmore operator* $\delta\mathcal{W}$ *in* $L^2(B_1)$*. Then* $\Phi \in W^{4,2}_{loc}(B_1)$*, and furthermore if* $C_{(2,\infty)} > 0$ *is constant so that*

$$\|d\lambda\|_{L^{(2,\infty)}(B_1)} \le C_{(2,\infty)},$$

there exists an $\varepsilon_0 > 0$ *depending only on* $C_{(2,\infty)}$ *so that if*

$$\int_{B_1} |A|_g^2 \, d\sigma_g \le \varepsilon_0,$$

then the following estimate holds:

$$\|d\Phi\|_{W^{3,2}(B_{1/2})} \le C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|\mathrm{e}^\lambda\|_{L^2(B_1)}\Big), \tag{4.1}$$

where $C = C(C_{(2,\infty)}) > 0$.

Remark 4.2 The fact that the estimate (4.1) does not include $\|\Phi\|_{L^2(B_{1/2})}$ on the left hand-side is motivated by the translation invariance of all the quantities on the right-hand side.

4.1. Qualitative Estimates

Proposition 4.3. *Let* $1 < q < \infty$ *and let* $\Phi \in \mathscr{E}(B_1, \mathbb{R}^n)$ *be a conformal with Willmore operator* $\delta\mathcal{W}$ *in* $L^q(B_1)$*. It suffices to know that* $H \in L^p(B_1)$ *for some* $p > 2$ *to deduce that* $\Phi \in W^{4,q}_{loc}(B_1)$*.*

Proof. We may certainly assume $2 < p < 4$. Since

$$\Delta\Phi = 2\mathrm{e}^{2\lambda} H \in L^p,$$

elliptic regularity theory gives that $\Phi \in W^{2,p}_{\mathrm{loc}}$, and this in turn implies

$$A = (\nabla^2\Phi)^\perp \in L^p_{\mathrm{loc}}.$$

Looking at (2.4):

$$-\Delta H = \nabla^*(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}) - \mathrm{e}^{2\lambda}\delta\mathcal{W},$$

we see that

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} = \mathrm{e}^{-2\lambda}\Big(\langle A^\circ, H\rangle^{\sharp} + \langle A, H\rangle^{\sharp}\Big) \in L^{\frac{p}{2}}_{\mathrm{loc}},$$

and, since $q > 1$, we have in particular that $\delta\mathcal{W} \in W^{-1,2}$; thus $\Delta H \in W^{-1,\frac{p}{2}}_{\mathrm{loc}}$, whence elliptic regularity and Sobolev embedding give

$$H \in W^{1,\frac{p}{2}}_{\mathrm{loc}} \hookrightarrow L^{\left(\frac{p}{2}\right)^*}_{\mathrm{loc}},$$

and from this, it follows that $A \in L^{\left(\frac{p}{2}\right)^*}_{\mathrm{loc}}$. Then

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} \in L^{\frac{1}{2}\left(\frac{p}{2}\right)^*}_{\mathrm{loc}},$$

whence $\Delta H \in W^{-1,\frac{1}{2}\left(\frac{p}{2}\right)^*}_{\mathrm{loc}}$, so by elliptic regularity

$$H \in W^{1,\frac{1}{2}\left(\frac{p}{2}\right)^*}_{\mathrm{loc}}.$$

This process can be iterated, and since the sequence p, $\left(\frac{p}{2}\right)^*$, $\left(\frac{1}{2}\left(\frac{p}{2}\right)^*\right)^*$, $\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{p}{2}\right)^*\right)^*\right)^*$,... is strictly monotone increasing and unbounded, after a finite number (depending on p) of steps we get that

$$\left(\frac{1}{2}\left(\frac{1}{2}\left(\cdots\left(\frac{p}{2}\right)^*\cdots\right)^*\right)^*\right)^* \geq 2.$$

We then deduce that $-\Delta H \in W^{-1,2}_{\mathrm{loc}}$, and thus that $H \in W^{1,2}_{\mathrm{loc}}$. By Sobolev embedding, this yields $H \in L^r_{\mathrm{loc}}$ for every $r < \infty$, and in turn elliptic estimates give

$$\Phi \in W^{2,r}_{\mathrm{loc}} \quad \forall\, r < \infty,$$

hence also $A \in L^r_{\mathrm{loc}}$ for every $r < \infty$. From Liouville equation

$$-\Delta\lambda = \mathrm{e}^{2\lambda} K,$$

since $|K| \leq C|A|^2$, we have $\Delta\lambda \in L^r_{\mathrm{loc}}$ and hence $\lambda \in W^{2,r}_{\mathrm{loc}}$ for every $r < \infty$. With this we infer that in fact $A, H \in W^{1,r}_{\mathrm{loc}}$ and so

$$\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g} \in W^{1,r}_{\mathrm{loc}},$$

Thus $-\Delta H \in L^q_{\mathrm{loc}}$, which implies $H \in W^{2,q}_{\mathrm{loc}}$ and hence (again since $\lambda \in W^{2,r}_{\mathrm{loc}}$) that $\Phi \in W^{4,q}_{\mathrm{loc}}$. □

Lemma 4.4. *With the same hypotesis as in Theorem 2.6, the vector-valued form (2.5) satisfies* $\langle w, d\Phi\rangle_g = 0$.

Proof. Since $|d\Phi|^2_g = 2$ and H is a normal vector field, we have

$$\begin{aligned}\langle w, d\Phi\rangle_g &= \langle \nabla H, d\Phi\rangle_g - 2\left\langle (\nabla H)^\top, d\Phi\right\rangle_g - |H|^2|d\Phi|^2_g \\ &= -\langle \nabla H, d\Phi\rangle_g - 2|H|^2 \\ &= -\frac{1}{2}\langle \nabla\Delta_g\Phi, d\Phi\rangle_g - 2|H|^2 \\ &= \frac{1}{2}\langle \Delta_g\Phi, \nabla^* d\Phi\rangle_g - 2|H|^2 \\ &= \frac{1}{2}|\Delta_g\Phi|^2 - \frac{1}{2}|\Delta_g\Phi|^2 = 0.\end{aligned}$$

□

Proposition 4.5. $\Phi : B_1 \to \mathbb{R}^3$ *be an immersion. Consider the Hodge decomposition*

$$w = d\mathscr{L} + *dL, \tag{4.2}$$

and consider further the Hodge decompositions

$$-d\Phi \times H - (*d\Phi)\times L = d\mathscr{R} + *dR, \tag{4.3}$$

$$-\langle *d\Phi, L\rangle = d\mathscr{S} + *dS. \tag{4.4}$$

Then the following relations hold:

$$\Delta_g\Phi = d\Phi \times \Big(d\mathscr{R} + *dR\Big) + \Big\langle d\Phi, d\mathscr{S} + *dS\Big\rangle \tag{4.5}$$
$$\Delta_g R = dN \times (d\mathscr{R} + *dR) - \langle dN, d\mathscr{S} + *dS\rangle + (*d\Phi) \times d\mathscr{L}, \tag{4.6}$$
$$\Delta_g S = \langle dN, d\mathscr{R} + *dR\rangle + \langle *d\Phi, d\mathscr{L}\rangle, \tag{4.7}$$

where N is the Gauss map of Φ.

Proof. Note first that, if the Hodge decompositions (4.2), (4.3), (4.4) hold, we necessarily have

$$\Delta_g\mathscr{L} = d^*w = \delta\mathcal{W},$$
$$\Delta_g\mathscr{R} = -d\Phi \times d\mathscr{L}, \tag{4.8}$$
$$\Delta_g\mathscr{S} = -\langle d\Phi, d\mathscr{L}\rangle. \tag{4.9}$$

With (2.9) and the identity[3]

$$d^*(-\Phi \times d\Phi \times H) = 2H - \Phi \times d\Phi \times dH = 2H - \Phi \times d\Phi \times w,$$

we see that there holds

$$\begin{aligned}
&\Phi \times (\Phi \times \delta\mathcal{W}) + \Phi\langle\Phi, \delta\mathcal{W}\rangle + 4H \\
&= d^*\Big(- 2\Phi \times (d\Phi \times H) + \Phi \times (\Phi \times w) + \Phi\langle\Phi, w\rangle\Big) \\
&= d^*\Big(- \Phi \times (d\Phi \times H)\Big) \\
&\quad + d^*\Big(\Phi \times \Big(- d\Phi \times H + \Phi \times (d\mathscr{L} + *dL)\Big) + \Phi\Big\langle\Phi, d\mathscr{L} + *dL\Big\rangle\Big) \\
&= 2H - \Phi \times (d\Phi \times w) \\
&\quad + d^*\Big(\Phi \times \Big(- d\Phi \times H + \Phi \times d\mathscr{L} + *d(\Phi \times L) - (*d\Phi) \times L\Big)\Big) \\
&\quad + d^*\Big(\Phi\Big(\langle\Phi, d\mathscr{L}\rangle + *d\langle\Phi, L\rangle - \langle *d\Phi, L\rangle\Big)\Big) \\
&= 2H - \Phi \times (d\Phi \times w) \\
&\quad + d^*\Big(\Phi \times \Big(\Phi \times d\mathscr{L} + d\mathscr{R} + *dR + *d(\Phi \times L)\Big)\Big) \\
&\quad + d^*\Big(\Phi\Big(\langle\Phi, d\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big)\Big).
\end{aligned}$$

Now, on the one hand we have, from (4.8),

$$\begin{aligned}
&d^*\Big(\Phi \times \Big(\Phi \times d\mathscr{L} + d\mathscr{R} + *dR + *d(\Phi \times L)\Big)\Big) \\
&= d\Phi \times \Big(\Phi \times d\mathscr{L} + d\mathscr{R} + *dR + *d(\Phi \times L)\Big) + \Phi \times \Big(d\Phi \times d\mathscr{L} + \Phi \times \Delta_g\mathscr{L} + \Delta_g\mathscr{R}\Big) \\
&= d\Phi \times \Big(\Phi \times d\mathscr{L} + d\mathscr{R} + *dR + *d(\Phi \times L)\Big) + \Phi \times (\Phi \times \delta\mathcal{W})
\end{aligned}$$

[3] here, the notation is: $d\Phi \times dH = g^{\mu\nu}\partial_\mu\Phi \times \partial_\nu H$, so that $\Phi \times d\Phi \times dH = g^{\mu\nu}(\Phi \times \partial_\mu\Phi \times \partial_\nu H)$. Similarly for $w \times dH$ and $\Phi \times d\Phi \times w$.

on the other hand from (4.9) it follows that

$$
\begin{aligned}
& d^*\Big(\Phi\big(\langle\Phi, d\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\big)\Big) \\
&= \Big\langle d\Phi, \langle\Phi, d\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle + \Phi\Big(\langle d\Phi, d\mathscr{L}\rangle + \langle\Phi, \Delta_g\mathscr{L}\rangle + \Delta_g\mathscr{S}\Big) \\
&= \Big\langle d\Phi, \langle\Phi, d\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle + \Phi\langle\Phi, \delta\mathcal{W}\rangle
\end{aligned}
$$

thus we deduce

$$
\begin{aligned}
2H &= -\Phi\times(d\Phi\times w) \\
&\quad + d\Phi\times\Big(\Phi\times d\mathscr{L} + d\mathscr{R} + *dR + *d(\Phi\times L)\Big) \\
&\quad + \Big\langle d\Phi, \langle\Phi, d\mathscr{L}\rangle + d\mathscr{S} + *dS + *d\langle\Phi, L\rangle\Big\rangle \\
&= -\Phi\times(d\Phi\times w) + d\Phi\times\Big(d\mathscr{R} + *dR\Big) + \Big\langle d\Phi, d\mathscr{S} + *dS\Big\rangle \\
&\quad + d\Phi\times\Big(\Phi\times d\mathscr{L} + *d(\Phi\times L)\Big) + \Big\langle d\Phi, \langle\Phi, d\mathscr{L}\rangle + *\langle\Phi, L\rangle\Big\rangle
\end{aligned}
$$

By definition of $\mathscr{L}$ and L, with Lemma 4.4 the last line in the above expression is

$$
\begin{aligned}
& d\Phi\times\Big(\Phi\times d\mathscr{L} + *d(\Phi\times L)\Big) + \Big\langle d\Phi, \langle\Phi, d\mathscr{L}\rangle + *\langle\Phi, L\rangle\Big\rangle \\
&= d\Phi\times\Big(\Phi\times(d\mathscr{L} + *dL) + (*d\Phi)\times L\Big) + \Big\langle d\Phi, \langle\Phi, d\mathscr{L} + *dL\rangle + \langle *d\Phi, L\rangle\Big\rangle \\
&= d\Phi\times\Big(\Phi\times w + (*d\Phi)\times L\Big) + \Big\langle d\Phi, \langle\Phi, w\rangle + \langle *d\Phi, L\rangle\Big\rangle \\
&= d\Phi\times\Big(\Phi\times w\Big) + \Big\langle d\Phi, \langle\Phi, w\rangle\Big\rangle + d\Phi\times\Big((*d\Phi)\times L\Big) + \Big\langle d\Phi, \langle *d\Phi, L\rangle\Big\rangle \\
&= \Phi\times(d\Phi\times w),
\end{aligned}
$$

and this yields (4.5). Next, since $\langle d\Phi\times H, N\rangle = 0$, we see that

$$
\begin{aligned}
\Big\langle d\mathscr{R} + *dR, N\Big\rangle &= \Big\langle -d\Phi\times H - (*d\Phi)\times L, N\Big\rangle \\
&= -\Big\langle N\times(*d\Phi), L\Big\rangle \\
&= -\Big\langle d\Phi, L\Big\rangle \\
&= -*d\mathscr{S} + dS
\end{aligned}
$$

and similarly, using the rules of the vector product, we have

$$
\begin{aligned}
\Big(d\mathscr{R} + *dR\Big)\times N &= (-d\Phi\times H - (*d\Phi)\times L\Big)\times N \\
&= N\times\Big(d\Phi\times H\Big) + N\times\Big((*d\Phi)\times L\Big) \\
&= -H\times(N\times d\Phi) - d\Phi\times(H\times N) \\
&\quad - L\times\Big(N\times(*d\Phi)\Big) - (*d\Phi)\times(L\times N) \\
&= H\times(*d\Phi) - L\times d\Phi - (*d\Phi)\times(L\times N) \\
&= -(*d\Phi)\times H + d\Phi\times L + N\langle *d\Phi, L\rangle \\
&= *d\mathscr{R} - dR - N(d\mathscr{S} + *dS),
\end{aligned}
$$

and thus, codifferentiating these identities we have

$$\Delta_g \mathscr{R} \times N + \left(d\mathscr{R} + *dR\right) \times dN = -\Delta_g R - \left\langle dN, d\mathscr{S} + *dS\right\rangle - N\Delta_g \mathscr{S},$$
$$\left\langle \Delta_g \mathscr{R}, N\right\rangle + \left\langle d\mathscr{R} + *dR, dN\right\rangle = \Delta_g S.$$

It now suffices to notice that, by definition of $\mathscr{R}$ and $\mathscr{S}$ it is

$$\begin{aligned} N\Delta_g \mathscr{S} &= -N\langle d\Phi, d\mathscr{L}\rangle \\ \Delta_g \mathscr{R} \times N &= -(d\Phi \times d\mathscr{L}) \times N = N \times (d\Phi \times d\mathscr{L}) = \left\langle d\Phi, \langle N, d\mathscr{L}\rangle\right\rangle, \\ \left\langle \Delta_g \mathscr{R}, N\right\rangle &= -\langle d\Phi \times d\mathscr{L}, N\rangle = -\langle N \times d\Phi, d\mathscr{L}\rangle = \langle *d\Phi, d\mathscr{L}\rangle \end{aligned}$$

and in particular

$$\begin{aligned} \Delta_g \mathscr{R} \times N + N\Delta_g \mathscr{S} &= \left\langle d\Phi, \langle N, d\mathscr{L}\rangle\right\rangle - N\langle d\Phi, d\mathscr{L}\rangle \\ &= d\mathscr{L} \times (d\Phi \times N) \\ &= d\mathscr{L} \times (*d\Phi). \end{aligned}$$

Substituting these relations in the ones above then gives (4.6) and (4.7). □

Proposition 4.6. *Let $\Phi \in \mathscr{E}(B_1, \mathbb{R}^3)$ be conformal with conformal factor e^{λ} and so that $\delta\mathcal{W} \in L^p(B_1)$ for some $p > 1$. Then $H \in L^r_{loc}(B_1)$ for every $r < \infty$.*

Proof. We may certainly assume $1 < p < 2$.

Step 1: there exists $\mathscr{L}$ and L realizing the Hodge decomposition (4.2) with $\mathscr{L} \in W^{2,p}(B_1)$ and $L \in L^{(2,\infty)}_{loc}(B_1)$. Indeed, we let $\mathscr{L}$ solve

$$\begin{cases} \Delta\mathscr{L} = e^{2\lambda}\delta\mathcal{W} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Elliptic regularity theory gives then that $\mathscr{L} \in W^{2,p}$. By construction we have

$$d^*(w - d\mathscr{L}) = 0 \quad \text{in } B_1,$$

thus L exists as a distribution in B_1 thanks to Poincaré's lemma and it is determined up to an additive constant. Note now that

$$\begin{aligned} \Delta L &= d^*(*d\mathscr{L} - *w) \\ &= d^*\Big(* d\mathscr{L} - *(\nabla H + \langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big) \\ &= -d^*\Big(* (\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big), \end{aligned}$$

so if we let

$$L_0(x) = -\int_{B_1} \Big\langle dK(x-y), *(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g})\Big\rangle\, dy,$$

where $K(x) = -\frac{1}{2\pi}\log|x|$ is the fundamental solution of the Laplace operator, given that

$$|dK(x-y)| \le \frac{C}{|x-y|} \quad \Longrightarrow \quad \sup_{x\in B_1} \|dK(x-\cdot)\|_{L^{2,\infty}(B_1)} \le C,$$

we see that $L_0 \in L^{(2,\infty)}$, and hence, since $L - L_0$ is harmonic, that $L \in L^{(2,\infty)}_{\mathrm{loc}}$.

Step 2: There exist $\mathscr{R}$, R, $\mathscr{S}$ and S realizing the Hodge decompositions (4.3), (4.4) with $\mathscr{R}, \mathscr{S} \in W^{2,p^*}(B_1)$ and $R, S \in W^{1,(2,\infty)}_{loc}(B_1)$. Indeed, define $\mathscr{R}$ and $\mathscr{S}$ by

$$\begin{cases} \Delta\mathscr{R} = -d\Phi \times d\mathscr{L} & \text{in } B_1, \\ \mathscr{R} = 0 & \text{on } \partial B_1, \end{cases} \qquad \begin{cases} \Delta\mathscr{S} = -\langle d\Phi, d\mathscr{L}\rangle & \text{in } B_1, \\ \mathscr{S} = 0 & \text{on } \partial B_1. \end{cases}$$

Since by Sobolev embedding $d\mathscr{L} \in W^{1,p} \hookrightarrow L^{p^*}$, so elliptic regularity gives $\mathscr{R}, \mathscr{S} \in W^{2,p^*}$. By construction we then have

$$d^*\Big(- d\Phi \times H - (*d\Phi) \times L - d\mathscr{R}\Big) = 0,$$
$$d^*\Big(- \langle *d\Phi, L\rangle - d\mathscr{S}\Big) = 0,$$

and thus R and S exist as distributions by Poincaré lemma and are determined up to additive constants, and, since L is in $L^{(2,\infty)}_{\mathrm{loc}}$, so are dR and dS.

Step 3: conclusion. From relations (4.6), (4.7), we see that R and S satisfy a system Jacobians plus some extra terms, namely

$$\begin{cases} \Delta R = dN \times (*dR) - \langle dN, *dS\rangle + f_R, \\ \Delta S = \langle dN, *dR\rangle + f_S, \end{cases}$$

where, since $d\mathscr{L} \in L^{p^*}$ and $d\mathscr{R}, d\mathscr{S} \in W^{1,p^*} \hookrightarrow L^\infty$, we have

$$f_R = dN \times d\mathscr{R} - \langle dN, d\mathscr{S}\rangle + (*d\Phi) \times d\mathscr{L} \qquad \in L^2,$$
$$f_S = \langle dN, d\mathscr{R}\rangle + \langle *d\Phi, d\mathscr{L}\rangle \qquad \in L^2.$$

Thanks to Theorem 2.1, we get that $R, S \in W^{2,q}_{\mathrm{loc}}$ for every $q < 2$ and hence that

$$dR, dS \in W^{1,r}_{\mathrm{loc}} \quad \text{for every } r < \infty.$$

Inserting thin information in (4.5) gives that $\Delta\Phi$, and so H, is in L^r_{loc} for every $r < \infty$. □

4.2. Quantitative Estimates

In the computations that follow, we shall make use of various Gagliardo-Nirenberg inequalities, namely, of multiplicative Sobolev inequalities such as

$$\|u\|_{L^4(\Omega)} \leq C\|u\|^{1/2}_{L^2(\Omega)}\|u\|^{1/2}_{W^{1,2}(\Omega)},$$

for $u \in W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$ bounded, regular domain. We refer for instance to [Nir59].

Proposition 4.7. *Let $\Omega \subset \mathbb{R}^2$ is a bounded, regular domain with $0 \in \Omega$ and $\Phi : \Omega \to \mathbb{R}^n$ is a conformal immersion of class $W^{4,2}$ with conformal factor e^λ. Let $E > 0$ and $C_\infty > 0$ be constants so that*

$$\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)} \leq E \quad \text{and} \quad \|\lambda - \Lambda\|_{L^\infty(\Omega)} \leq C_\infty,$$

where $\Lambda = \lambda(0)$. Then, the following estimate holds:

$$\|\Delta^2\Phi\|_{L^2(\Omega)} \leq 2\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(\Omega)} + C\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)}\|\nabla^2\Phi\|_{W^{2,2}(\Omega)}, \tag{4.10}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

First we point out a few elementary facts.

Lemma 4.8. *For an immersion* $\Phi : B_1 \to \mathbb{R}^n$ *there holds*

$$\frac{1}{2}\Delta_g^2\Phi = \Delta_g H = \Delta_g^{\perp} H - \langle A, \langle H, A\rangle\rangle_g - 2\langle \nabla H, A\rangle^{\sharp_g} - \langle \nabla H, H\rangle^{\sharp_g}, \tag{4.11}$$

and, if Φ *is conformal with conformal factor* e^{λ}*, there holds*

$$\Delta^2\Phi = \mathrm{e}^{4\lambda}\Big(\Delta_g^2\Phi + 2\langle d\lambda, \nabla H\rangle_g + (2|d\lambda|_g^2 + \Delta_g\lambda)H\Big). \tag{4.12}$$

The proof is elementary.

Lemma 4.9. *The following pointwise estimates hold for an absolute constant* $C > 0$ *and* $k = 0, 1, 2$*:*

$$\frac{1}{C}|\mathrm{e}^{-\lambda}\nabla^2\Phi| \le |\mathrm{e}^{-\lambda}A| + |d\lambda| \le C\,\mathrm{e}^{-\lambda}|\nabla^2\Phi|, \tag{4.13}$$

$$|\nabla^k(\mathrm{e}^{2\lambda}H)| \le C|\nabla^{k+2}\Phi|, \tag{4.14}$$

$$|\nabla^k(\mathrm{e}^{2\lambda}H)| \le C|\nabla^k A|, \tag{4.15}$$

and the following estimate holds:

$$\|A\|_{W^{k,2}(\Omega)} \le C\|\nabla^2\Phi\|_{W^{k,2}(\Omega)}, \tag{4.16}$$

for $C = C(\Omega, E, C_\infty) > 0$ *and* $k = 0, 1, 2$.

Proof. Since

$$A_{\mu\nu} = \partial^2_{\mu\nu}\Phi - \Gamma^{\sigma}_{\mu\nu}\,\partial_\sigma\Phi \quad \text{and} \quad e^{2\lambda} = \frac{1}{2}|d\Phi|^2,$$

and

$$\begin{aligned}
&\Gamma^1_{11} = \partial_1\lambda, && \Gamma^1_{12} = \Gamma^1_{21} = \partial_2\lambda, && \Gamma^1_{22} = -\partial_1\lambda,\\
&\Gamma^2_{11} = -\partial_2\lambda, && \Gamma^2_{12} = \Gamma^1_{21} = \partial_1\lambda, && \Gamma^2_{22} = \partial_2\lambda,
\end{aligned}$$

estimates (4.13) and consequently (4.16) for $k = 0$ follow. Next, since

$$\mathrm{e}^{2\lambda}H = \frac{1}{2}\Delta\Phi = \frac{1}{2}(\Delta\Phi)^{\perp} = \frac{1}{2}(A_{11} + A_{22}),$$

estimates (4.14) and (4.15) follow. Now differentiating of the above identities:

$$\begin{aligned}
\partial_\xi A_{\mu\nu} &= \partial^3_{\xi\mu\nu}\Phi - \partial_\xi\Gamma^{\sigma}_{\mu\nu}\,\partial_\sigma\Phi - \Gamma^{\sigma}_{\mu\nu}\,\partial^2_{\xi\sigma}\Phi,\\
\partial^2_{\zeta\xi}A_{\mu\nu} &= \partial^4_{\zeta\xi\mu\nu}\Phi - \partial^2_{\zeta\xi}\Gamma^{\sigma}_{\mu\nu}\,\partial_\sigma\Phi - \partial_\xi\Gamma^{\sigma}_{\mu\nu}\,\partial^2_{\zeta\sigma}\Phi - \partial_\zeta\Gamma^{\sigma}_{\mu\nu}\,\partial^2_{\xi\sigma}\Phi - \Gamma^{\sigma}_{\mu\nu}\,\partial^3_{\zeta\xi\sigma}\Phi,
\end{aligned}$$

and

$$\begin{aligned}
2d\lambda\mathrm{e}^{2\lambda} &= \langle\nabla^2\Phi, d\Phi\rangle,\\
(2\nabla^2\lambda + 4d\lambda\otimes d\lambda)\mathrm{e}^{2\lambda} &= \langle\nabla^3\Phi, d\Phi\rangle + \nabla^2\Phi\langle\dot\otimes\rangle\nabla^2\Phi,\\
(2\nabla^3\lambda + 8\nabla^2\lambda\otimes d\lambda + 8d\lambda\otimes d\lambda\otimes d\lambda)\mathrm{e}^{2\lambda} &= \langle\nabla^4\Phi, d\Phi\rangle + 2\nabla^3\Phi\langle\dot\otimes\rangle\nabla^2\Phi,
\end{aligned}$$

(where $\langle\dot{\otimes}\rangle$ means scalar product in the vector part, inner product in one of the covariant entries and tensor product in the remaining ones) yields the estimates

$$|\Gamma^\sigma_{\mu\nu}| \leq C|d\lambda|, \qquad |\partial_\xi\Gamma^\sigma_{\mu\nu}| \leq C|\nabla^2\lambda|, \qquad |\partial^2_{\zeta\xi}\Gamma^\sigma_{\mu\nu}| \leq C|\nabla^3\lambda|,$$

and

$$\begin{aligned}
|d\lambda| &\leq C\mathrm{e}^{-\lambda}|\nabla^2\Phi|,\\
|\nabla^2\lambda| &\leq C\Big(|d\lambda|^2 + \mathrm{e}^{-\lambda}|\nabla^3\Phi| + \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2\Big)\\
&\leq C\Big(\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^2 + \mathrm{e}^{-\lambda}|\nabla^3\Phi|\Big),\\
|\nabla^3\lambda| &\leq C\Big(|\nabla^2\lambda||d\lambda| + |d\lambda|^3 + \mathrm{e}^{-\lambda}|\nabla^4\Phi| + \mathrm{e}^{-2\lambda}|\nabla^3\Phi||\nabla^2\Phi|\Big)\\
&\leq C\Big(\mathrm{e}^{-2\lambda}|\nabla^3\Phi||\nabla^2\Phi| + \mathrm{e}^{-3\lambda}|\nabla^2\Phi|^3 + \mathrm{e}^{-\lambda}|\nabla^4\Phi|\Big),
\end{aligned}$$

thanks to which we estimate in turn

$$\begin{aligned}
|\nabla A| &\leq C\Big(|\nabla^3\Phi| + \mathrm{e}^{\lambda}|\nabla^2\lambda| + |d\lambda||\nabla^2\Phi|\Big)\\
&\leq C\Big(|\nabla^3\Phi| + \mathrm{e}^{-\lambda}|\nabla^2\Phi|^2\Big),\\
|\nabla^2 A| &\leq C\Big(|\nabla^4\Phi| + \mathrm{e}^{\lambda}|\nabla^3\lambda| + |\nabla\lambda||\nabla^2\Phi| + |d\lambda||\nabla^3\Phi|\Big)\\
&\leq C\Big(|\nabla^4\Phi| + \mathrm{e}^{-\lambda}|\nabla^3\Phi||\nabla^2\Phi| + \mathrm{e}^{-2\lambda}|\nabla^2\Phi|^3\Big).
\end{aligned}$$

Thus, with the help of Gagliardo-Nirenberg we can estimate

$$\begin{aligned}
\|\nabla A\|_{L^2} &\leq C\Big(\|\nabla^3\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}|\nabla^2\Phi|^2\|_{L^2}\Big)\\
&\leq C\Big(\|\nabla^3\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|^2_{L^4}\Big)\\
&\leq C\Big(\|\nabla^3\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{1,2}}\Big)\\
&\leq C\Big(\|\nabla^3\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{1,2}}\Big),
\end{aligned}$$

yielding (4.16) for $k = 1$. Similarly, since

$$\begin{aligned}
\|\nabla^2 A\|_{L^2} &\leq C\Big(\|\nabla^4\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}|\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} + \|\mathrm{e}^{-2\lambda}|\nabla^2\Phi|^3\|_{L^2}\Big)\\
&\leq C\Big(\|\nabla^4\Phi\|_{L^2} + \mathrm{e}^{-\Lambda}\||\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|^3_{L^6}\Big),
\end{aligned}$$

with Hölder and Gagliardo-Nirenberg we estimate

$$\begin{aligned}
\||\nabla^3\Phi||\nabla^2\Phi|\|_{L^2} &\leq \|\nabla^3\Phi\|_{L^4}\|\nabla^2\Phi\|_{L^4}\\
&\leq C\|\nabla^2\Phi\|^{\frac14}_{L^2}\|\nabla^2\Phi\|^{\frac34}_{W^{2,2}}\|\nabla^2\Phi\|^{\frac34}_{L^2}\|\nabla^2\Phi\|^{\frac14}_{W^{2,2}}\\
&\leq C\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}},
\end{aligned}$$

and

$$\|\nabla^2\Phi\|^3_{L^6} \leq C\|\nabla^2\Phi\|^2_{L^2}\|\nabla^2\Phi\|_{W^{2,2}},$$

so to obtain

$$\|\nabla^2 A\|_{L^2} \leq C\Big(\|\nabla^4\Phi\|_{L^2} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|^2_{L^2}\|\nabla^2\Phi\|_{W^{2,2}}\Big),$$

yielding (4.16) also for $k = 2$. □

Proof of Proposition 4.7. It can be deduced from the following three lemmas and (4.16).

Lemma 4.10. *There holds*

$$\|e^{4\lambda}\Delta_g^{\perp}H\|_{L^2(\Omega)} \le C\|e^{-\lambda}A\|_{L^2(\Omega)}\|A\|_{W^{2,2}(\Omega)} + \|e^{4\lambda}\delta\mathcal{W}\|_{L^2(\Omega)}, \tag{4.17}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Lemma 4.11. *There holds*

$$\|e^{4\lambda}\Delta_g H\|_{L^2(\Omega)} \le \|e^{4\lambda}\Delta_g^{\perp}H\|_{L^2(\Omega)} + C\|e^{-\lambda}A\|_{L^2(\Omega)}\|A\|_{W^{2,2}(\Omega)}, \tag{4.18}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Lemma 4.12. *There holds*

$$\|\Delta^2\Phi\|_{L^2(\Omega)} \le 2\|e^{4\lambda}\Delta_g H\|_{L^2(\Omega)} + C\|e^{-\lambda}\nabla^2\Phi\|_{L^2(\Omega)}\|\nabla^2\Phi\|_{W^{2,2}(\Omega)}, \tag{4.19}$$

for a constant $C = C(\Omega, E, C_\infty) > 0$.

Proof of Lemma 4.10. With the classical form of the Willmore operator (2.3) and the pointwise estimate (4.15), by means of Hölder's and Gagliardo-Nirenberg inequalities we estimate

$$\begin{aligned}
\|\langle A^\circ, \langle H, A^\circ\rangle\rangle_g\|_{L^2} &\le C\|e^{-4\lambda}\langle A^\circ, \langle H, A^\circ\rangle\rangle\|_{L^2} \\
&\le C\|e^{-4\lambda}|A^\circ|^2 H\|_{L^2} \\
&\le C\|e^{-6\lambda}|A|^3\|_{L^2} \\
&\le Ce^{-6\Lambda}\|A\|_{L^6}^3 \\
&\le Ce^{-6\Lambda}\|A\|_{L^2}^2\|A\|_{W^{2,2}} \\
&\le Ce^{-4\Lambda}\|e^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}},
\end{aligned}$$

which yields (4.17). □

Proof of Lemma 4.11. From formula (4.11), we have

$$\|\Delta_g H\|_{L^2} \le \|\Delta_g^{\perp}H\|_{L^2} + C\Big(\|\langle A, \langle H, A\rangle\rangle_g\|_{L^2} + \|\langle\nabla H, A\rangle_g^{\sharp_g}\|_{L^2} + \|\langle\nabla H, H\rangle^{\sharp_g}\|_{L^2}\Big).$$

Similarly as in the proof of Lemma (4.10), we have

$$\|\langle A, \langle H, A\rangle\rangle_g\|_{L^2} \le Ce^{-4\Lambda}\|e^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}}.$$

Next, since

$$\nabla H = e^{-2\lambda}\nabla(e^{2\lambda}H) - 2H \otimes d\lambda,$$

with (4.15) we may pointwise estimate

$$|\nabla H| \le Ce^{-2\lambda}\Big(|\nabla A| + |A||d\lambda|\Big),$$

thus allowing to deduce

$$\begin{aligned}
\|\langle\nabla H, H\rangle^{\sharp_g}\|_{L^2} &\le C\|e^{-\lambda}|\nabla H||H|\|_{L^2} \\
&\le C\|e^{-3\lambda}(|\nabla A| + |A||d\lambda|)|H|\|_{L^2} \\
&\le C\|e^{-5\lambda}(|\nabla A| + |A||d\lambda|)|A|\|_{L^2};
\end{aligned}$$

now with Hölder and Gagliardo-Nirenberg we see that, on the one hand,

$$\begin{aligned}\|\mathrm{e}^{-5\lambda}|\nabla A||A|\|_{L^2} &\le C\mathrm{e}^{-5\Lambda}\||\nabla A||A|\|_{L^2}\\ &\le C\mathrm{e}^{-5\Lambda}\|\nabla A\|_{L^4}\|A\|_{L^4}\\ &\le C\mathrm{e}^{-5\Lambda}\|A\|_{L^2}\|A\|_{W^{2,2}}\\ &\le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},\end{aligned}$$

and on the other hand

$$\begin{aligned}\|\mathrm{e}^{-5\lambda}|A|^2|d\lambda|\|_{L^2} &\le C\mathrm{e}^{-5\Lambda}\|d\lambda\|_{L^2}\|A\|_{L^\infty}^2\\ &\le C\mathrm{e}^{-5\Lambda}\|d\lambda\|_{L^2}\|A\|_{L^2}\|A\|_{W^{2,2}}\\ &\le C\mathrm{e}^{-4\Lambda}\|d\lambda\|_{L^2}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},\end{aligned}$$

so we deduce

$$\|\langle\nabla H, H\rangle^{\sharp_g}\|_{L^2} \le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}}.$$

Similarly, we see that

$$\begin{aligned}\|\langle\nabla H, A\rangle_g^{\sharp_g}\|_{L^2} &\le C\|\mathrm{e}^{-3\lambda}\langle\nabla H, A\rangle^{\sharp}\|_{L^2}\\ &\le C\|\mathrm{e}^{-3\lambda}|\nabla H||A|\|_{L^2}\\ &\le C\|\mathrm{e}^{-5\lambda}(|\nabla A| + |A||d\lambda|)|A|\|_{L^2},\end{aligned}$$

and so similarly as before we deduce

$$\|\langle\nabla H, A\rangle_g^{\sharp_g}\|_{L^2} \le C\mathrm{e}^{-4\Lambda}\|\mathrm{e}^{-\lambda}A\|_{L^2}\|A\|_{W^{2,2}},$$

yielding (4.18). □

Proof of Lemma 4.12. From formula (4.12), it follows that

$$\|\Delta\Phi\|_{L^2} \le 2\|\mathrm{e}^{4\lambda}\Delta_g H\|_{L^2} + C\Big(\|\mathrm{e}^{4\lambda}\langle d\lambda, \nabla H\rangle_g\|_{L^2} + \|\mathrm{e}^{4\lambda}|d\lambda|_g^2 H\|_{L^2} + \|\mathrm{e}^{4\lambda}\Delta_g\lambda\, H\|_{L^2}\Big).$$

Now with (4.13), (4.14), and (again) the identity

$$\nabla H = \mathrm{e}^{-2\lambda}\nabla(\mathrm{e}^{2\lambda}H) - 2H\otimes d\lambda,$$

we estimate with Hölder and Galgliardo-Nirenberg

$$\begin{aligned}\|\mathrm{e}^{4\lambda}\langle d\lambda, \nabla H\rangle_g\|_{L^2} &\le C\|\mathrm{e}^{2\lambda}\langle d\lambda, \nabla H\rangle\|_{L^2}\\ &\le C\|\mathrm{e}^{2\lambda}|d\lambda||\nabla H|\|_{L^2}\\ &\le C\Big\|\mathrm{e}^{2\lambda}\Big(\mathrm{e}^{-\lambda}|\nabla^2\Phi|\Big)\Big(\mathrm{e}^{-2\lambda}|\nabla^3\Phi| + \mathrm{e}^{-3\lambda}|\nabla^2\Phi|^2\Big)\Big\|_{L^2}\\ &\le C\Big(\mathrm{e}^{-\Lambda}\||\nabla^3\Phi|\nabla^2\Phi|\|_{L^2} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^6}^3\Big)\\ &\le C\Big(\mathrm{e}^{-\Lambda}\|\nabla^3\Phi\|_{L^4}\|\nabla^2\Phi\|_{L^4} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^6}^3\Big)\\ &\le C\Big(\mathrm{e}^{-\Lambda}\|\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}} + \mathrm{e}^{-2\Lambda}\|\nabla^2\Phi\|_{L^2}^2\|\nabla^2\Phi\|_{W^{2,2}}\Big)\\ &\le C\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}}.\end{aligned}$$

Similarly, again with (4.14) and Gagliardo-Nirenberg we estimate

$$\begin{aligned}\|e^{4\lambda}|d\lambda|_g^2 H\|_{L^2} &\le C\|e^{2\lambda}|d\lambda|^2 H\|_{L^2}\\ &\le C\|e^{2\lambda}(e^{-2\lambda}|\nabla^2\Phi|^2)(e^{-2\lambda}|\nabla^2\Phi|)\|_{L^2}\\ &\le C\|e^{-2\lambda}|\nabla^2\Phi|^3\|_{L^2}\\ &\le C\|e^{-\lambda}\nabla^2\Phi\|_{L^2}\|\nabla^2\Phi\|_{W^{2,2}}.\end{aligned}$$

Finally, with Liouville's equation

$$-\Delta\lambda = e^{2\lambda}K,$$

the pointwise estimate

$$|K| \le |A|_g^2 \le e^{-4\lambda}|A|^2,$$

we can estimate with Gagliardo-Nirenberg and (4.16):

$$\begin{aligned}\|e^{4\lambda}\Delta_g\lambda H\|_{L^2} &\le C\||A|^2 H\|_{L^2}\\ &\le C\|e^{-2\lambda}|A|^3\|_{L^2}\\ &\le Ce^{-2\Lambda}\|A\|_{L^2}^2\|A\|_{W^{2,2}}\\ &\le C\|e^{-\lambda}A\|_{L^2}^2\|A\|_{W^{2,2}}\\ &\le C\|e^{-\lambda}A\|_{L^2}^2\|\nabla^2\Phi\|_{W^{2,2}}.\end{aligned}$$

All these estimates together yield (4.19). □

The combination of lemmas 4.10, 4.11 and 4.12 immediately gives (4.10), and concludes the proof of Proposition 4.7. □

Proof of Theorem 4.1. The qualitative statement, namely that $\Phi \in W^{4,2}_{\text{loc}}$, is an immediate consequence of Propositions 4.3 and 4.6. We can therefore apply Proposition 4.7, so that (4.10) jointly with elliptic estimates for the bilaplacian give

$$\|d\Phi\|_{W^{3,2}(B_{1/2})} \le C\Big(\|e^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|e^{-\lambda}\nabla^2\Phi\|_{L^2(B_1)}\|\nabla^2\Phi\|_{W^{2,2}(B_1)} + \|d\Phi\|_{L^2(B_1)}\Big) \tag{4.20}$$

for $C = C(E, C_\infty) > 0$. Now we consider rescalings. For $0 < r < 1$ we let

$$\widetilde{\Phi}(x) = \Phi(rx), \quad x \in B_1,$$

and we denote with a tilde all the quantities pertaining to $\widetilde{\Phi}$. From for $k \in \mathbb{N}$ it follows in particular that for $\Omega \subseteq B_1$ we have

$$\begin{aligned}\nabla^k\widetilde{\Phi}(x) &= r^k\nabla^k\Phi(rx),\\ \|\nabla^k\widetilde{\Phi}\|_{L^2(\Omega)} &= r^{k-1}\|\nabla^k\Phi\|_{L^2(r\Omega)},\\ e^{\widetilde{\lambda}(x)} &= re^{\lambda(rx)},\\ \widetilde{\lambda}(x) - \widetilde{\lambda}(0) &= \lambda(rx) - \lambda(0),\\ \widetilde{\delta\mathcal{W}}(x) &= \delta\mathcal{W}(rx),\end{aligned}$$

(the last equality follows either by direct inspection or at once recalling that $\delta\mathcal{W}$ is a vector field) and from these relations, we deduce in particular that

$$\begin{aligned} \|\mathrm{e}^{4\widetilde{\lambda}}\widetilde{\delta\mathcal{W}}\|_{L^2(B_1)} &= r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)}, \\ \|\mathrm{e}^{-2\widetilde{\lambda}}\nabla^2\widetilde{\Phi}\|_{L^2(B_1)} &= \|\mathrm{e}^{-2\lambda}\nabla^2\Phi\|_{L^2(B_r)}, \\ \|\widetilde{\lambda}-\widetilde{\Lambda}\|_{L^\infty(B_1)} &= \|\lambda-\Lambda\|_{L^\infty(B_r)}, \end{aligned}$$

and the last two relations in particular give that $\widetilde{E} \le E$ and $\widetilde{C_\infty} \le C_\infty$. Consequently, applying (4.20) to $\widetilde{\Phi}$ gives

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \le C\big(r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)} + \|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(B_r)}\|\nabla^2\Phi\|'_{W^{2,2}(B_r)} + \|d\Phi\|_{L^2(B_r)}\big), \tag{4.21}$$

where for $k=1,2$ we denoted $\|\Phi\|'_{W^{k,2}(B_\rho)} = \left(\sum_{h=1}^k \rho^{k-1}\|\Phi\|^2_{W^{h,2}(B_\rho)}\right)^{1/2}$ the scale-invariant version of the Sobolev norms where, as we said abefore $C = C(E, C_\infty) > 0$.

Recall now that from (4.13) it is

$$\|\mathrm{e}^{-\lambda}\nabla^2\Phi\|_{L^2(B_{1/2})} \le C\Big(\|\mathrm{e}^{-\lambda}A\|_{L^2(B_{1/2})} + \|d\lambda\|_{L^2(B_{1/2})}\Big),$$

so by Theorem 2.2 we let ε_0 be sufficiently small so to have

$$\|d\lambda\|_{L^2(B_{1/2})} + \|\lambda-\lambda(0)\|_{L^\infty(B_{1/2})} \le C\|\mathrm{e}^{-\lambda}A\|_{L^2(B_1)} \le C\varepsilon_0,$$

for $C = C(C_{(2,\infty)}) > 0$, whence (4.21) can be improved to

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \le C\Big(r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_r)} + \varepsilon_0\|\nabla^2\Phi\|'_{W^{2,2}(B_r)} + \|d\Phi\|_{L^2(B_r)}\Big),$$

for $C = C(C_{(2,\infty)}) > 0$.

Choose finally $\varepsilon_0 > 0$ be sufficiently small so that $C\varepsilon_0 \le \frac{1}{2}$ to obtain

$$\|d\Phi\|'_{W^{3,2}(B_{r/2})} \le \frac{1}{2}\|d\Phi\|'_{W^{3,2}(B_r)} + C\Big(r^3\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(B_1)} + \|d\Phi\|_{L^2(B_1)}\Big),$$

for every $0 < r \le 1/2$. A classical iteration/interpolation argument applied to $\phi(r) = \|d\Phi\|'_{W^{3,2}(B_r)}$ and a covering argument yield to (4.1). □

5. Conformal Willmore Flows

Proof of Lemma 1.1. A metric g is conformal if and only if its Hopf differential (computed with respect to the background complex structure of S^2) vanishes identically. In our case it is

$$\mathrm{Hopf}(g) = g_{zz}dz\otimes dz = \langle\partial_z\Phi, \partial_z\Phi\rangle dz\otimes dz.$$

Since $\delta\mathcal{W}$ is a normal vector field, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\partial_z\Phi,\partial_z\Phi\rangle = \langle\partial_z\partial_t\Phi,\partial_z\Phi\rangle = \langle-\partial_z\delta\mathcal{W}+\partial_z U,\partial_z\Phi\rangle = \langle\delta\mathcal{W},\partial^2_{zz}\Phi\rangle + \langle\partial_z U,\partial_z\Phi\rangle.$$

Since $U = U^z\partial_z\Phi + U^{\bar z}\partial_{\bar z}\Phi$ with $U^z = U^1 + iU^2$ and $U^{\bar z} = U^1 - iU^2$ we have

$$\begin{aligned}\langle \partial_z U, \partial_z\Phi\rangle &= \Big\langle \partial_z U^z\partial_z\Phi + U^z\partial^2_{zz}\Phi + \partial_z U^{\bar z}\partial_{\bar z}\Phi + U^{\bar z}\partial^2_{z\bar z}\Phi, \partial_z\Phi\Big\rangle\\ &= \partial_z U^z g_{zz} + U^z\Big\langle\partial^2_{zz}\Phi, \partial_z\Phi\Big\rangle + \partial_z U^{\bar z} g_{\bar z z} + U^{\bar z}\Big\langle \partial^2_{z\bar z}\Phi, \partial_z\Phi\Big\rangle\\ &= \partial_z U^z g_{zz} + \partial_z U^{\bar z} g_{\bar z z} + \frac{1}{2}\Big(U^z\partial_z g_{zz} + U^{\bar z}\partial_{\bar z} g_{zz}\Big),\end{aligned}$$

thus we have

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\partial_z\Phi, \partial_z\Phi\rangle = \partial_z U^z g_{zz} + \partial_z U^{\bar z} g_{\bar z z} + \frac{1}{2}\Big(U^z\partial_z g_{zz} + U^{\bar z}\partial_{\bar z} g_{zz}\Big) + \Big\langle \delta\mathcal{W}, \partial^2_{zz}\Phi\Big\rangle.$$

If the flow Φ is conformal, then $\partial_t \operatorname{Hopf}(g) \equiv 0$ and $g_{zz}, g_{\bar z\bar z}$ vanish identically. Since moreover $\delta\mathcal{W}$ is a normal vector we may replace $\partial^2_{zz}\Phi$ with A_{zz} and thus obtain, after conjugation,

$$g_{\bar z z}\partial_z U^{\bar z} = -\langle\delta\mathcal{W}, A_{zz}\rangle,$$

and so since $\frac{1}{2}e^{2\lambda} = g_{\bar z z}$, this yields (1.8). □

The proof of the following proposition follows directly by direct inspection of the proof of the original theorems; for the proof of (ii) one uses additionally that conformal Willmore flows are $W^{4,2}$ for almost every time, as proved in Proposition 5.2 below.

Proposition 5.1. *(i) If the surface $\mathcal{S}$ is immersed through a $W^{4,2}$-map, then the same conclusion of Theorem 2.4 and all its consequences obtained in Section 3 still hold.*

(ii) For conformal Willmore flows in $\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ the area and barycenter bounds in Theorem 2.8 still hold if ε is chosen sufficiently small.

A first regularity improvement for conformal Willmore flows is the following.

Proposition 5.2. *For any ε_0, $\delta > 0$, any conformal Willmore flow $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ is in $W^{4,2}(S^2)$ for a.e. $t \in (0,T)$. Futhermore there exist ε_0, δ, $C > 0$ independent of Φ so that $\Phi \in L^2((0,T), W^{4,2}(S^2))$ with*

$$\|d\Phi\|_{L^2((0,T),W^{3,2}(S^2))} \le C(\sqrt{T} + \|e^\lambda\delta\mathcal{W}\|_{L^2((0,T)\times S^2)}), \tag{5.1}$$

for a constant $C > 0$.

Proof. Since $\delta\mathcal{W} \in L^2((0,T)\times S^2)$, Fubini's theorem implies $\delta\mathcal{W}(t,\cdot) \in L^2(S^2)$ for a.e. t. Consequently, by Propositions 4.3 and 4.6 it follows that $\Phi(t,\cdot) \in W^{4,2}(S^2)$.

From Liouville's equation

$$\Delta_{S^2}\lambda = e^{2\lambda}K - 1,$$

the pointwise inequality $|K| \le |A|^2_g = e^{-4\lambda}|A|^2$ and elliptic estimates

$$\|d\lambda\|_{L^\infty((0,T),L^{(2,\infty)}(S^2))} \le C. \tag{5.2}$$

Choose now $\varepsilon_1 > 0$ sufficiently small so that, for a.e. $t \in (0,T)$ so that $\Phi(t,\cdot)$ is $W^{4,2}$, we can fix a value $r > 0$ which satisfies, according to Lemma 3.3,

$$\int_{B_r(x_0)} |A|^2 \, d\sigma \le C\Big(\varepsilon_1 + \mathrm{e}^C r^2\Big) \le \varepsilon_0, \tag{5.3}$$

for every $x_0 \in S^2$, where ε_0 is as in Theorem 4.1. With the estimates (5.2) and (5.3), an application of Theorem 4.1 to a covering S^2 with balls of radius $r/2$ gives, for a.e. $t \in (0,T)$,

$$\|d\Phi\|_{W^{3,2}(S^2)} \le C\Big(\|\mathrm{e}^{4\lambda}\delta\mathcal{W}\|_{L^2(S^2)} + \|\mathrm{e}^{\lambda}\|_{L^2(S^2)}\Big) \le C\Big(\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)} + 1\Big),$$

where the last inequality follows recalling that by assumption the bound (1.12) holds in the class $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$. Now if we square and integrate in t such inequality, recalling that, since $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}$, (1.13) holds by assumption, we obtain (5.1). □

One can see that, in fact, along the proof of Proposition 5.2, well-balanced condition (iv) in Definition 1.7 was not needed, an in fact the proof is completely independent of the tangential component U. Such condition is however essential to prove the following.

Proposition 5.3. *For every $p < 2$ there exist $\delta > 0$ with the following property. Let $\Phi \in \mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ be a weak Willmore flow. Then its tangential component is in $L^2((0,T),L^p(S^2))$, with*

$$\|U\|_{L^2((0,T),L^p(S^2))} \le C\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2((0,T)\times S^2))}, \tag{5.4}$$

for a constant $C = C(p) > 0$.

Proof. We may certainly assume $p > 1$. In this proof we find it convenient to clearly distinguish between the non-immersed (pulled-back) tangential component $U = U^\mu\partial_\mu$, the immersed one $d\Phi(U) = U^\mu\partial_\mu\Phi$, and the associated tangent vector field on S^2, $dI(U) = U^\mu\partial_\mu I$, where $I : S^2 \to \mathbb{R}^3$ denotes is the standard immersion. With this notation Φ satisfies weakly

$$\frac{\partial}{\partial t}\Phi = -\delta\mathcal{W} + d\Phi(U) \quad \text{in } (0,T)\times S^2.$$

In what follows, it will be implicitly understood that all the slice-wise operations are valid for a.e. fixed t. Finally we shall make use of the notation and concepts recalled in Appendix B.

From Lemma 1.1, we deduce that U is given by

$$U^{(1,0)} = -\overline{\partial}^{-1}(\langle\delta\mathcal{W},\overline{h}_0\rangle^{\sharp_g}) + \Omega$$

for some time-dependent holomorphic vector field $\Omega = \Omega(t,\cdot) \in \mathfrak{X}^\omega(S^2)$. Classical elliptic estimates and Hölder's inequality permit to estimate

$$\begin{aligned} \|\overline{\partial}^{-1}(\langle\delta\mathcal{W},\overline{h}_0\rangle^{\sharp_g})\|_{L^p(S^2)} &\le C_p\|\langle\delta\mathcal{W},\overline{h}_0\rangle^{\sharp}_g\|_{L^1(S^2)} \\ &\le C_p\||\delta\mathcal{W}|\mathrm{e}^{-2\lambda}|A^\circ|\|_{L^1(S^2)} \\ &\le C_p\|\mathrm{e}^{-\lambda}|A^\circ|\|_{L^2(S^2)}\|\mathrm{e}^{-\lambda}\delta\mathcal{W}\|_{L^2(S^2)} \\ &\le C_p\sqrt{\mathcal{W}_0(\Phi)}\,\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)} \\ &\le C_p\|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)}, \end{aligned} \tag{5.5}$$

where we estimated $\|\mathrm{e}^{-\lambda}\delta\mathcal{W}\|_{L^2(S^2)} \leq \|\mathrm{e}^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)}$ thanks to the simple inequality

$$1 = \mathrm{e}^{-\lambda}\mathrm{e}^{\lambda} \leq \sup_{S^2}(\mathrm{e}^{-\lambda})\mathrm{e}^{\lambda} \leq C(1+\delta)\mathrm{e}^{\lambda}$$

issuing from property (1.12). We now examine Ω. It will be more practical to look at the associated (time-dependent) conformal Killing vector field i.e. generating conformal transformations:

$$V = \Omega + \overline{\Omega},$$

Similarly as in the proof of Lemma 3.1, a basis for the vector space of conformal Killing fields $T_e\,\mathrm{Aut}(S^2)$ is given, in its immersed representative, is given by

$$\begin{aligned} &dI(Z_1)(y) = (0, -y^3, y^2), \quad &&dI(Z_2)(y) = (y^3, 0, -y^1), \quad &&dI(Z_3)(y) = (-y^2, y^1, 0), \\ &dI(Z_4)(y) = e_1 - y^1 y, &&dI(Z_5)(y) = e_2 - y^2 y, &&dI(Z_6)(y) = e_3 - y^3 y. \end{aligned}$$

One checks that this basis is orthogonal with respect the L^2–scalar product and each element has the same length. Thus, we may write

$$V = \sum_{a=1}^{6} V^a Z_a = C\sum_{a=1}^{6}(V, Z_a)_{L^2(S^2)} Z_a = C\sum_{a=1}^{6}(U, Z_a)_{L^2(S^2)} Z_a, \tag{5.6}$$

where the last inequality is a consequence of the fact that, by construction, the normal solution of the $\overline{\partial}$-operator is L^2-orthogonal to the space of holomorphic vector fields. So, to estimate V it suffices to estimate the (time-dependent) coefficients

$$(V, Z_a)_{L^2} = \int_{S^2}\langle U, Z_a\rangle\, d\sigma \quad \text{for } a = 1, \dots, 6.$$

Now, we may write the integrand as

$$\langle U, Z_a\rangle = \langle dI(U), dI(Z_a)\rangle = \langle d\Phi(U), dI(Z_a)\rangle + \langle dI(U) - d\Phi(U), dI(Z_a)\rangle$$

where the second term can be estimated as

$$\Big|\langle dI(U) - d\Phi(U), dI(Z_a)\rangle\Big| \leq C|dI - d\Phi||U|$$

and thus, upon integration, Hölder's inequality and property (1.12) of the set $\mathscr{W}^{\varepsilon,\delta}_{[0,T)}(S^2, \mathbb{R}^3)$, we see that the estimate

$$\begin{aligned} \left|\int_{S^2}\langle dI(U) - d\Phi(U), dI(Z_a)\rangle d\sigma\right| &\leq C\int_{S^2}|dI - d\Phi||U|\, d\sigma \\ &\leq C\|dI - d\Phi\|_{L^{p'}(S^2)}\|U\|_{L^p(S^2)} \\ &\leq C\|\Phi - I - c\|_{W^{2,2}(S^2)}\|U\|_{L^p(S^2)} \\ &\leq C_p\,\delta\,\|U\|_{L^p(S^2)}, \end{aligned} \tag{5.7}$$

holds, which will suit our purposes. We are left to estimate the terms

$$\int_{S^2}\langle d\Phi(U), dI(Z_a)\rangle\, d\sigma \quad \text{for } a = 1, \dots, 6.$$

Differentiating the well-balanced conditions (1.9) gives,

$$0 = \frac{d}{dt}\int_{S^2} I \times \Phi \, d\sigma = \int_{S^2} I \times (-\delta\mathcal{W} + d\Phi(U))\, d\sigma,$$
$$0 = \frac{d}{dt}\int_{S^2} I d\sigma_g = \int_{S^2} I\big(\langle 2H, \delta\mathcal{W}\rangle + \mathrm{div}_g(U)\big) d\sigma_g,$$

and thus that

$$\int_{S^2} I \times d\Phi(U)\, d\sigma = \int_{S^2} I \times \delta\mathcal{W}\, d\sigma, \tag{5.8}$$

and, upon integration by parts, that

$$\int_{S^2} dI(U)\, d\sigma_g = \int_{S^2} I\langle 2H, \delta\mathcal{W}\rangle\, d\sigma_g.$$

On the other hand, a direct calculation shows that

$$\begin{pmatrix} \langle d\Phi(U), dI(Z_1)\rangle \\ \langle d\Phi(U), dI(Z_2)\rangle \\ \langle d\Phi(U), dI(Z_3)\rangle \end{pmatrix} = I \times d\Phi(U), \tag{5.9}$$

and similarly that

$$\begin{pmatrix} \langle d\Phi(U), dI(Z_4)\rangle \\ \langle d\Phi(U), dI(Z_5)\rangle \\ \langle d\Phi(U), dI(Z_6)\rangle \end{pmatrix} = d\Phi(U) - \langle d\Phi(U), I\rangle I = (d\Phi(U))^{\top}, \tag{5.10}$$

where $(\cdot)^{\top}$ denotes the orthogonal projection onto the tangent space of S^2 in the standard immersion. Integrating (5.9) and using (5.8) we can estimate for $a = 1, 2, 3$

$$\Big|(d\Phi(U), dI(Z_a))_{L^2}\Big| \leq C\Big|\int_{S^2} I \times \delta\mathcal{W}\, d\sigma\Big| \leq C\|e^{\lambda}\delta\mathcal{W}\|_{L^2(S^2)}. \tag{5.11}$$

Integrating (5.10) we get instead

$$\sum_{a=1}^{3} C(d\Phi(U), dI(Z_a))_{L^2} e_a = \int_{S^2} (d\Phi(U))^{\top} d\sigma,$$

and if we write the integrand as

$$(d\Phi(U))^{\top} = (dI(U))^{\top} + (d\Phi(U) - dI(U))^{\top}$$

and notice that, similarly as for (5.7) we can estimate

$$\begin{aligned} \Big|\int_{S^2} (d\Phi(U) - dI(U))^{\top} d\sigma\Big| &\leq C\int_{S^2} |d\Phi(U) - dI(U)|\, d\sigma \\ &\leq C\|d\Phi - dI\|_{L^{p'}(S^2)}\|U\|_{L^p(S^2)} \\ &\leq C_p\, \delta\, \|U\|_{L^p(S^2)}, \end{aligned}$$

using (5.8) (since $dI(U) = dI(U)^\top$), we have for $a = 4, 5, 6$

$$
\begin{aligned}
|(d\Phi(U), dI(Z_a))_{L^2}| &\le C\Big|\int_{S^2} dI(U) d\sigma\Big| + C_p\,\delta\,\|U\|_{L^p(S^2)} \\
&\le C\Big|\int_{S^2} I\langle 2H, \delta\mathcal{W}\rangle\, d\sigma_g\Big| + C_p\,\delta\,\|U\|_{L^p(S^2)} \\
&\le C\|H\mathrm{e}^\lambda\|_{L^2(S^2)}\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + C_p\,\delta\,\|U\|_{L^p(S^2)} \\
&\le C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big),
\end{aligned}
\tag{5.12}
$$

where we also used that $\|H\mathrm{e}^\lambda\|_{L^2(S^2)} = C\mathcal{W}_1(\Phi)$ is bounded uniformly in t.

Estimates (5.7), (5.11) and (5.12) inserted in (5.6) yield

$$\|V\|_{L^\infty(S^2)} \le C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big).$$

and so, in conjunction with (5.5), we get

$$\|U\|_{L^p(S^2)} \le C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2(S^2)} + \delta\,\|U\|_{L^p(S^2)}\Big).$$

Taking the L^2-norm in time of such inequality gives

$$\|U\|_{L^2((0,T),L^p(S^2))} \le C_p\Big(\|\mathrm{e}^\lambda\delta\mathcal{W}\|_{L^2((0,T),L^2(S^2))} + \delta\,\|U\|_{L^2((0,T),L^p(S^2))}\Big),$$

and so, if δ is chosen sufficiently small, we reach (5.4). □

Remark 5.4 The δ of Proposition 5.3 may be smaller than that given by Proposition 1.5. An inspection of the proof however shows however that we may equivalently have taken the same δ at the price of choosing $\varepsilon > 0$ sufficiently small, because we may apply instead Propositions 5.1 and 1.5 for a.e. t so that $\Phi(t,\cdot)$ is $W^{4,2}$, and argue in the end similarly as above. This variant would have been equally fine for our purposes.

Corollary 5.5. *There exists $\varepsilon_0, \delta > 0$ so that any Willmore flow in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ is in $C^\infty((0,T]\times S^2)$, and, if it is a solution to the Cauchy problem (1.7) for smooth initial datum Φ_0, then it is in $C^\infty([0,T]\times S^2)$.*

Proof. By definition and by Proposition 5.2, $d\Phi$ is in $L^\infty((0,T),W^{1,2}(S^2))\cap L^2((0,T),W^{3,2}(S^2))$, we have sufficient regularity to expand the Willmore operator in the flow equation by means of formulas (2.4) and (4.11):

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi + \frac{1}{2}\mathrm{e}^{-4\lambda}\Delta^2\Phi = \frac{1}{2}\mathrm{e}^{-4\lambda}&\Big(2\langle d\lambda, \nabla H\rangle_g + (2|d\lambda|_g^2 + \Delta_g\lambda)H\Big) \\
&- \nabla^{*_g}\Big(\langle A, H\rangle^{\sharp_g} + \langle A^\circ, H\rangle^{\sharp_g}\Big) + U;
\end{aligned}
$$

since e^λ is by assumption uniformly bounded, the equation is uniformly parabolic, and by Proposition 5.3, $U \in L^2(0,T), L^p(S^2))$ for any $p < 2$.

This is enough to start a boostrapping procedure using first L^p-L^q and then Schauder parabolic estimates in a fashion similar to the elliptic case discussed in Proposition 4.3. To bootstrap the regularity of the tangential component U, one uses higher-regularity variants of Proposition 5.3, whose proofs are similar to the basic case. □

Proof of Theorem 1.9. *Case of smooth initial datum.* By Corollary 5.5, it suffices to prove that there exists a unique smooth solution in $\mathscr{W}^{\varepsilon_0,\delta}_{[0,T)}(S^2,\mathbb{R}^3)$ with the required properties.

An application DeTurck's trick (see Appendix A) yields existence and uniqueness of a smooth solution to the Cauchy problem for the normal Willmore flow:

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi^0 = -\delta\mathcal{W} & \text{in } (0,T)\times S^2, \\ \Phi^0(0,\cdot) = \Phi_0 & \text{on } S^2, \end{cases}$$

and if $\varepsilon_0 > 0$ is small enough, by Theorem 2.8 Φ^0 exists for all and smoothly converges to a round sphere.[4] We conformalize such flow composing it with the family $(\phi(t,\cdot))_{t\in[0,+\infty)}$ of canonical quasi-conformal mappings associated to the family of metrics $g^0(t,\cdot) = \Phi^0(t,\cdot)^\star g_{\mathbb{R}^3}$, see [AB60]. The fact that it is $\phi(0,\cdot) = e$ that such family is smooth both in the space and in time follows from the theory of quasi-conformal mappings. Then $\Phi^1(t,\cdot) = \Phi^0(t,\phi(t,\cdot))$ is a conformal Willmore flow defined for all times and converging to a conformal parametrization of a round sphere.

Let

$$a(t) = \sqrt{\frac{4\pi}{\mathcal{A}(\mathcal{S}_t)}}$$

the normalizing function of time so that $a(t)\mathcal{S}_t$ has always area 4π and let $\varepsilon_0 > 0$ be sufficiently small as in Theorem 2.4 and also so that $C\varepsilon_0 \leq \delta$ where C is as in (2.2) and δ is sufficiently small as in Propositions 1.5 and 5.3.[5] In this way, there exists a family of conformal diffeomorphisms $(\psi(t,\cdot))_{t\in[0,+\infty)} \subset \mathrm{Aut}(S^2)$ so that $\Phi(t,\cdot) = \Phi^1(t,\psi(t,\cdot))$ is conformal, well-balanced and

$$\|a(t)\Phi(t,\cdot) - I - c(t)\|_{W^{2,2}(S^2)} + \|a(t)\mathrm{e}^{\lambda(t,\cdot)} - 1\|_{L^\infty(S^2)} \leq C\sqrt{\mathcal{W}_0(\mathcal{S}_t)}, \tag{5.13}$$

where $c(t) = ⨍_{S^2} \Phi(t,\cdot)\,d\sigma$ (see Remark 2.5) and moreover there exist a neighborhood of the identity $\mathcal{O} \subset \mathrm{Aut}(S^2)$ where such choice is unique.

As for $a(t)$, thanks the area control of Theorem 2.8 (recall that $\mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathcal{S}_0) = 4\pi$) it is

$$|\mathcal{A}(\mathcal{S}_t) - 4\pi| \leq C\mathcal{W}_0(\mathcal{S}_t) = o(1) \quad \text{as } t \to +\infty,$$

and hence

$$|a(t) - 1| \leq C\mathcal{W}_0(\mathcal{S}_t) = o(1) \quad \text{as } t \to +\infty,$$

which means that we may remove $a(t)$ from the estimate (5.13).

As for $c(t)$, we may write

$$\begin{aligned} c(t) = ⨍_{S^2} \Phi(t,\cdot)\,d\sigma &= ⨍_{S^2} \Phi(t,\cdot)\mathrm{e}^{2\lambda(t,\cdot)}\,d\sigma + ⨍_{S^2} \Phi(t,\cdot)(1-\mathrm{e}^{2\lambda(t,\cdot)})d\sigma \\ &= \mathcal{C}(\mathcal{S}_t) + ⨍_{S^2} \Phi(t,\cdot)(1-\mathrm{e}^{2\lambda(t,\cdot)})d\sigma, \end{aligned}$$

[4] Note carefully: the convergence is to *some* smooth parametrization, of a round sphere of *some* center and radius, not to I modulo dilation and translation. For instance, if Φ_0 is *any* smooth parametrization of a round sphere, then Φ^0 trivially converges to Φ_0.

[5] See in this regard Remark 5.4.

where $\mathcal{C}(\mathcal{S}_t)$ denotes the barycenter as in Theorem 2.8. Thus, from: the barycenter control of Theorem 2.8, the control on the conformal factor issuing from (5.13), the fact that $\psi(t,\cdot)\in\mathcal{O}$, smooth convergence and $\mathcal{C}(\mathcal{S})=0$, we can estimate

$$|c(t)|\le C\mathcal{W}_0(\mathcal{S}_t),$$

and hence also $c(t)$ can be removed from estimate (5.13) as well and deduce that

$$\|\Phi(t,\cdot)-I\|_{W^{2,2}(S^2)}+\|\mathrm{e}^{\lambda(t,\cdot)}-1\|_{L^\infty(S^2)}\le C\sqrt{\mathcal{W}_0(\mathcal{S}_t)}. \tag{5.14}$$

Since $\mathcal{W}_0(\mathcal{S}_t)=o(1)$ as $t\to+\infty$, we obtain that $\Phi(t,\cdot)$ converges to I in $W^{2,2}$ and that its conformal factor converges uniformly to 1. Convergence of higher order derivatives to I then follow from this and the smooth convergence of Φ^1.

Case of weak initial datum. Let $\Phi_{0,j}\in\mathscr{D}^{\varepsilon_0}(S^2,\mathbb{R}^3)$ be a sequence appoximating Φ_0 in the weak $W^{2,2}$-topology, i.e.

$$\Phi_{0,j}\rightharpoonup\Phi_0\quad\text{in } W^{2,2}(S^2).$$

If ε_0 is taken sufficiently small, it follows from the analysis in [Riv08, Riv14, Riv16] (see for instance the proof of Theorem 3.36 in [Riv16]) that also

$$\mathrm{e}^{\lambda_j}\overset{*}{\rightharpoonup}\mathrm{e}^{\lambda}\quad\text{in } L^\infty(S^2).$$

For each j, we let $\Phi_j\in\mathscr{W}^{\varepsilon_0,\delta}_{[0,+\infty)}(S^2,\mathbb{R}^3)$ be the well-balanced conformal Willmore flow given by Theorem 1.9 with initial datum $\Phi_{0,j}$, which we know to be smooth. By estimate (5.14) we have that for every $t>0$ there holds

$$\|\Phi_j(t,\cdot)-I\|_{W^{2,2}(S^2)}+\|\mathrm{e}^{\lambda_j(t,\cdot)}-1\|_{L^\infty(S^2)}\le C\sqrt{\mathcal{W}_0(\mathcal{S}_t)}\le\delta,$$

and by Proposition 5.2 it follows that for every fixed choice of $T>0$ there holds

$$\|\Phi_j\|_{L^2((0,T),W^{4,2}(S^2))}\le C,$$

and finally by Proposition 5.3 we also have, for a fixed choice of $1<p<2$,

$$\|\partial_t\Phi_j\|_{L^2((0,T),L^p(S^2))}=\|-\delta\mathcal{W}_j+U_j\|_{L^2((0,1),L^p(S^2))}\le C,$$

where C does not depend on j.

Weak sequential compactness properties of Sobolev spaces then imply that, up to the extraction of a subsequence, there exists measurable functions $\Phi:(0,+\infty)\times S^2\to\mathbb{R}^3$ and $\lambda:(0,+\infty)\times S^2\to\mathbb{R}$ so that, for every fixed $T>0$, as $j\to\infty$,

$$\Phi_j\to\Phi\quad\text{a.e. in } (0,+\infty)\times S^2, \tag{5.15}$$
$$\Phi_j\rightharpoonup\Phi\quad\text{in } W^{1,p}((0,T)\times S^2), \tag{5.16}$$
$$\Phi_j\rightharpoonup\Phi\quad\text{in } L^2((0,T),W^{4,2}(S^2)),$$
$$\Phi_j\overset{*}{\rightharpoonup}\Phi\quad\text{in } L^\infty((0,T),W^{2,2}(S^2)),$$
$$\mathrm{e}^{\lambda_j}\overset{*}{\rightharpoonup}\mathrm{e}^{\lambda}\quad\text{in } L^\infty((0,T)\times S^2). \tag{5.17}$$

Thus, for every $T>0$, Φ satisfies the conditions (i), (ii), (iv) of Definition 1.2 (and e^{λ} is its conformal factor), its Willmore operator is in $L^2((0,T)\times S^2)$ and finally since also

$$\mathcal{W}_0(\Phi(t,\cdot)) \leq \liminf_{j\to\infty} \mathcal{W}_0(\Phi_j) \leq \varepsilon_0,$$

which means that (ii) is satisfied as well and so $\Phi \in \mathscr{W}^{\varepsilon_0,\delta}_{[0,+\infty)}(S^2,\mathbb{R}^3)$. Moreover, since $p>1$ by (5.16) Φ has a trace at initial time, which, by uniqueness and continuity of the trace operator, must coincide with Φ_0.

Finally, the convergence properties (5.15)-(5.17) are enough to pass to the limit as $j\to\infty$ in Definition 1.8 and thus deduce that Φ is also a weak Willmore flow. By Corollary 5.5 Φ is smooth on $(0,+\infty)\times S^2$ as well. □

A. DeTurck's Trick for the Willmore Flow

We outline here one way to obtain, in the smooth category, short-time existence for the Cauchy's problem (1.5) adapting an idea originally devised by DeTurck [DeT83] in the context of the Ricci flow. There are other possibilities, such as the graph Ansatz [HP99] (in codimension 1) or through the Nash-Moser Implicit Function Theorem [Ham82a, Ham82b].

The key idea is as follows. From the divergence form of the Willmore operator (2.4), we write

$$\delta\mathcal{W} = \Delta_g H + \nabla^{*_g}\left(\langle A^\circ, H\rangle^{\sharp_g} + \langle A, H\rangle^{\sharp_g}\right) = \frac{1}{2}\Delta_g^2\Phi + F\left(d\Phi, \nabla^2\Phi, \nabla^3\Phi\right),$$

where F is a smooth function and the covariant derivatives are with respect to a fixed smooth reference metric on Σ. Now in local coordinates we may write

$$\Delta_g\Phi = g^{\mu\nu}\partial^2_{\mu\nu}\Phi - g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma\Phi,$$

and so

$$\begin{aligned}\Delta_g^2\Phi &= \Delta_g(\Delta_g\Phi)\\ &= g^{\mu\nu}\partial^2_{\mu\nu}(\Delta_g\Phi) - g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma(\Delta_g\Phi)\\ &= g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}\partial^2_{\alpha\beta}\Phi - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma\Phi\right) + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right),\end{aligned}$$

for some smooth function f. Now, because also the metric depends on Φ: $g_{\mu\nu} = \langle\partial_\mu\Phi, \partial_\nu\Phi\rangle$, the term $g^{\alpha\beta}\left(\partial^2_{\mu\nu}\Gamma^\gamma_{\alpha\beta}\right)\partial_\gamma\Phi$ also contains derivative of order 4 in Φ. This causes the Willmore operator to be degenerate elliptic, and the corresponding flow to be degenerate parabolic.

However, writing:

$$\Delta_g^2\Phi = g^{\mu\nu}g^{\alpha\beta}\partial^4_{\mu\nu\alpha\beta}\Phi - g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\right)\partial_\gamma\Phi + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right),$$

one guesses that it may be possible to add a tangent vector field to the Willmore operator which for which the corresponding flow is a uniformly parabolic one. This motivates the following.

Definition A.1. *Let Σ be a closed, orientable surface and let $\Phi_0 : \Sigma \to \mathbb{R}^3$ be a smooth immersion. If $\Phi : \Sigma \to \mathbb{R}^3$ is another smooth immersion,* DeTurck's vector field for the Willmore flow *(for Φ relative to Φ_0) is the vector field tangent to Φ given by*

$$V = V(\Phi_0,\Phi) = -\frac{1}{2}\Delta_g W = -\frac{1}{2}(\Delta_g W)^\gamma\partial_\gamma\Phi,$$

where $W = W^\gamma \partial_\gamma$ is the vector field on Σ given by

$$W^\gamma = g^{\alpha\beta}\left(\Gamma^\gamma_{\alpha\beta} - \breve{\Gamma}^\gamma_{\alpha\beta}\right),$$

where $\Gamma^\gamma_{\alpha\beta}$ and $\breve{\Gamma}^\gamma_{\alpha\beta}$ denote respectively the Christoffel symbols of Φ and Φ_0.

Note that this definition makes sense, since it is a well-known fact in differential geometry that, although the Christoffel symbols are themselves not tensor, the expression $\Gamma^\gamma_{\alpha\beta} - \breve{\Gamma}^\gamma_{\alpha\beta}$ (and consequently its trace W) is.

Proposition A.2 (Short-Time Existence for the Smooth DeTurck-Willmore Flow)**.** *Let Σ be a closed, orientable surface and let $\Phi_0 : \Sigma \to \mathbb{R}^3$ be a smooth immersion. There exists some $T = T(\Phi_0) > 0$ so that the Cauchy problem*

$$\begin{cases} \partial_t \Phi = -\delta\mathcal{W} + V & \text{in } (0,T)\times\Sigma, \\ \Phi(0,\cdot) = \Phi_0, & \text{on } \Sigma, \end{cases} \tag{A.1}$$

has a unique solution in the class $C^\infty([0,T]\times\Sigma, \mathbb{R}^3)$, where $V = V(\Phi_0, \Phi)$ is DeTurck's vector field for the Willmore flow.

Proof. It is sufficent to prove that (A.1) defines a uniformly parabolic system for Φ over Σ. The existence, uniqueness and smoothness of a solution follows then from the general theory for such systems in Hölder spaces [Sol65] (transl. English [MR067]), [LSU68]. We have

$$\Delta_g W = \operatorname{tr}_g\left(\nabla^{(2)} W\right) = g^{\mu\nu}\left(\nabla^g_{\partial_\mu}\nabla^g_{\partial_\nu} W - \nabla^g_{\nabla^g_{\partial_c}} W\right) = g^{\mu\nu}\left(\nabla^g_{\partial_\mu}\nabla^g_{\partial_\nu} W - \Gamma^\sigma_{\mu\nu}\nabla^g_{\partial_\sigma} W\right),$$

so computing directly we see that

$$\begin{aligned}
\Delta_g W &= g^{\mu\nu}\Big(\partial^2_{\mu\nu} W^\xi + \partial_\nu W^\sigma \Gamma^\xi_{\mu\sigma} + \partial_\mu W^\sigma \Gamma^\xi_{\nu\sigma} \\
&\qquad - \Gamma^\tau_{\mu\nu}\partial_\tau W^\xi + W^\sigma \partial_\mu \Gamma^\xi_{\nu\sigma} + W^\sigma \Gamma^\tau_{\nu\sigma}\Gamma^\xi_{\mu\tau} - \Gamma^\sigma_{\mu\nu}\Gamma^\xi_{\tau\sigma} W^\sigma\Big)\partial_\xi \\
&= g^{\mu\nu}\left(\partial^2_{\mu\nu} W^\xi + f(W, \partial_x W, \partial_x\Phi, \partial^2_{xx}\Phi)\right)\partial_\xi \\
&= g^{\mu\nu}\left(\partial^2_{\mu\nu} W^\xi + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right)\right)\partial_\xi,
\end{aligned}$$

thus we have, in *every* choice of local coordinates, that

$$\begin{aligned}
-\delta\mathcal{W} + V &= -\frac{1}{2}\Delta^2_g \Phi + V + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right) \\
&= -\frac{1}{2}\Big(g^{\mu\nu} g^{\alpha\beta}\partial^4_{\mu\nu\alpha\beta}\Phi - g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\right)\partial_\gamma\Phi \\
&\qquad + g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \breve{\Gamma}^\gamma_{\alpha\beta})\right)\partial_\gamma\vec{\Phi}\Big) + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right) \\
&= -\frac{1}{2}\left(g^{\mu\nu} g^{\alpha\beta}\partial^4_{\mu\nu\alpha\beta}\Phi - g^{\mu\nu}\partial^2_{\mu\nu}\left(g^{\alpha\beta}(\breve{\Gamma}^\gamma_{\alpha\beta})\right)\partial_\gamma\vec{\Phi}\right) + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right) \\
&= -\frac{1}{2}\left(g^{\mu\nu} g^{\alpha\beta}\partial^4_{\mu\nu\alpha\beta}\Phi\right) + f\left(\partial_x\Phi, \partial^2_{xx}\Phi, \partial^3_{xxx}\Phi\right),
\end{aligned}$$

so that (A.1) defines, for T sufficiently small and in the smooth category, a uniformly parabolic system of fourth order in Φ. □

One now obtains the analogue short-time existence and uniqueness statement for the Cauchy problem (1.5) combining Proposition A.2 and the the fact that there is a bijective correspondence between tangential components an reparametrizations (see the analogous discussion in [Man11]).

B. The $\overline{\partial}$-operator on Vector Fields

Let Σ be a Riemann surface. the complexified tangent bundle $T^{\mathbb{C}}\Sigma = T\Sigma + iT\Sigma$ splits in two sub-bundles:

$$T^{\mathbb{C}}\Sigma = T\Sigma^{(1,0)} \oplus T\Sigma^{(0,1)},$$

whose sections are respectively $(1,0)$ and $(0,1)$-vector fields:

$$\begin{aligned}\mathfrak{X}^{(1,0)}(\Sigma) = \Gamma(T\Sigma^{(1,0)}) \quad &\ni \quad V = (V^1 + iV^2)\partial_z,\\ \mathfrak{X}^{(0,1)}(\Sigma) = \Gamma(T\Sigma^{(0,1)}) \quad &\ni \quad W = (W^1 + iW^2)\partial_{\bar{z}}.\end{aligned}$$

One can identify $(1,0)$- and real vector fields by means of conjugation and $(1,0)$-projection:

$$\mathfrak{X}^{(1,0)}(\Sigma) \simeq \mathfrak{X}(\Sigma) \quad : \quad V \to V + \overline{V} \quad \text{and} \quad W^{(1,0)} \leftarrow W.$$

With this identification, we can consider holomorphic vector fields as real vector fields, whose flows consist precisely of families of $\mathrm{Aut}(\Sigma)$, the space of conformal self-maps (Möbius transformations) of Σ. Calling $B = \Gamma(T\Sigma^{(1,0)} \otimes T^*\Sigma^{(0,1)})$, the $\overline{\partial}$-operator over $(1,0)$-vector fields is

$$\overline{\partial} : \mathfrak{X}^{(1,0)}(\Sigma) \to B, \quad \overline{\partial}V = \frac{1}{2}\partial_{\bar{z}}(V^1 + iV^2)\partial_z \otimes d\bar{z},$$

whose kernel is the space of holomorphic vector fields $\ker(\overline{\partial}) = \mathfrak{X}^{\omega}(\Sigma)$. Fix now a conformal metric over Σ, $g = \mathrm{e}^{2\lambda}|dz|^2$. The formal L^2-adjoint of $\overline{\partial}$, defined through the formula $(\overline{\partial}V, F)_{L^2} = (V, \overline{\partial}^*F)_{L^2}$ is, if $F = f\partial_z \otimes d\bar{z}$,

$$\overline{\partial}^* : B \to \mathfrak{X}^{(1,0)}(\Sigma), \quad \overline{\partial}^*F = -2\mathrm{e}^{-4\lambda}\partial_z(\mathrm{e}^{2\lambda}f)\partial_z.$$

We deduce that $\ker(\overline{\partial}^*)$ consists of those tensors $F = f\partial_z \otimes d\bar{z}$ so that $\mathrm{e}^{2\lambda}f$ is antiholomorphic. If we lower the fist index of F:

$$F^\flat = f g_{\bar{z}z}\, d\bar{z} \otimes d\bar{z} = \frac{1}{2}e^{2\lambda}f\, d\bar{z} \otimes d\bar{z},$$

then $F^\flat$ is an antiholomorphic quadratic differential, and so its conjugate is a holomorphic quadratic differential. With these identifications, we have

$$\ker(\overline{\partial}^*) \simeq Q^{\omega}(\Sigma).$$

In particular, we note that even though $\overline{\partial}^*$ does depend on the chosen metric, $Q^{\omega}(\Sigma)$ does not.

For given $F \in B$, we consider the equation

$$\overline{\partial}V = F \quad \text{on } \Sigma. \tag{B.1}$$

Then, (B.1) has a solution if and only if

$$F \in \overline{\partial}(\mathfrak{X}^{(1,0)}(\Sigma)) = \ker(\overline{\partial}^*)^{\perp}$$

In such case, if V_0 is one such solution, every other one is of the form $V = V_0 + v$ for $v \in \mathfrak{X}^{\omega}(\Sigma)$.

The *normal solution* to (B.1) is the only one in $\ker(\overline{\partial})^{\perp}$, and we denote it by $\overline{\partial}^{-1}F$. Normal solutions satisfy the typical elliptic estimates, such as for instance

$$\|\overline{\partial}^{-1}F\|_{W^{1,2}(\Sigma)} \leq C\|F\|_{L^2(\Sigma)} \quad \forall\, F \in \ker(\overline{\partial}^*)^{\perp}.$$

for a constant $C = C(\Sigma, g) > 0$.

As a consequence of the Riemann-Roch formula, if γ is the genus of Σ, we have

$$\dim_{\mathbb{C}} Q^{\omega}(\Sigma) = \begin{cases} 0 & \text{if } \gamma = 0 \\ 1 & \text{if } \gamma = 1, \\ 3\gamma - 3 & \text{if } \gamma \geq 2, \end{cases} \quad \text{and} \quad \dim_{\mathbb{C}} \mathfrak{X}^{\omega}(\Sigma) = \begin{cases} 3 & \text{if } \gamma = 0, \\ 1 & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma \geq 2. \end{cases}$$

In particular, (B.1) can be solved for any F when $\Sigma = S^2$.

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Keywords Willmore surfaces, Willmore gradient flow, Geometric flows, Conformally invariant problems, Conformal gauge, Weak solutions, Critical non linear 4th order elliptic and parabolic equations.