

The Variations of Yang-Mills Lagrangian.

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Dedicated to Gang Tian on his sixtieth Birthday.

I Introduction

Yang-Mills theory is growing at the interface between high energy physics and mathematics. It is well known that Yang-Mills theory and Gauge theory in general had a profound impact on the development of modern differential and algebraic geometry. One could quote Donaldson invariants in four dimensional differential topology, Hitchin Kobayashi conjecture relating the existence of Hermitian-Einstein metric on holomorphic bundles over Kähler manifolds and Mumford stability in complex geometry or also Gromov Witten invariants in symplectic geometry...etc. While the influence of Gauge theory in geometry is quite notorious, one tends sometimes to forget that Yang-Mills theory has been also at the heart of fundamental progresses in the non-linear analysis of Partial Differential Equations in the last decades. The purpose of this survey is to present the variations of this important lagrangian. We shall raise analysis question such as existence and regularity of Yang-Mills minimizers or such as the compactification of the “moduli space” of critical points to Yang-Mills lagrangian in general.

II The Plateau Problem.

Before to move to the Yang-Mills minimization problem we will first recall some fundamental facts regarding the minimization of the area in the parametric approach and some elements of the resolution of the so called *Plateau problem*.

Let Γ be a simple closed Jordan Curve in \mathbb{R}^3 : there exists $\gamma \in C^0(S^1, \mathbb{R}^3)$ such that $\Gamma = \gamma(S^1)$.

Plateau problem : *Find a C^1 immersion u of the two dimensional disc D^2 which is continuous up to the boundary, whose restriction to ∂D^2 is an homeomorphism and which minimizes the area*

$$Area(u) = \int_{D^2} |\partial_{x_1} u \times \partial_{x_2} u| dx_1 dx_2 \quad .$$

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The area is a fairly degenerated functional :

- i) It has a huge invariance group : $\text{Diff}(D^2)$, the group of diffeomorphism of the disc. Let u_n be a minimizing sequence of the *Plateau problem* above, then the composition of u_n with any sequence of diffeomorphism Ψ_n of D^2 is still a minimizing sequence. The sequence Ψ_n can for instance degenerate so that $u_n \circ \Psi_n$ converges to a point !
- ii) The area of u does not control the image $u(D^2)$ which could be uniformly bounded while $u(D^2)$ becomes dense in \mathbb{R}^3 !

In order to solve the Plateau Problem J.Douglas and independently Radó minimize instead the *Dirichlet energy*

$$\text{Area}(u) \leq E(u) = \frac{1}{2} \int_{D^2} [|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2] dx_1 dx_2 \quad ,$$

the inequality comes from the pointwise inequality

$$2 |\partial_{x_1} u \times \partial_{x_2} u| \leq |\partial_{x_1} u|^2 + |\partial_{x_2} u|^2 \quad ,$$

and equality holds if and only if

$$H(u) := |\partial_{x_1} u|^2 - |\partial_{x_2} u|^2 - 2i \partial_{x_1} u \cdot \partial_{x_2} u = 0 \quad .$$

This condition is satisfied if and only if the differential du preserves angle that is to say u is *conformal*.

The *Dirichlet energy* E has much better properties than the area : it is

- i) it is *coercive* :

$$\forall u_n \quad \text{s.t.} \quad \limsup_{n \rightarrow +\infty} E(u_n) < +\infty \quad \exists u_\infty \rightharpoonup u_\infty \quad \text{in} \quad W^{1,2} \quad ,$$

- ii) *lower semi-continuous*

$$E(u_\infty) \leq \liminf_{n \rightarrow +\infty} E(u_n) \quad ,$$

- iii) it's *invariance group* in the domain is reduced to the 3-dimensional Möbius group $\mathcal{M}(D^2)$

$$\mathcal{M}(D^2) := \left\{ \Psi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \quad \text{s.t.} \quad \theta \in \mathbb{R} \quad |a| < 1 \right\} \quad .$$

Why minimizing E instead of the area has any chance to give a solution to the Plateau problem ?

Let u be an immersion of the disc D^2 and denote g_u the pull back of the standard flat metric $g_{\mathbb{R}^3}$ in \mathbb{R}^3 : $g_u := u^*g_{\mathbb{R}^3}$. The *uniformization theorem* on D^2 gives the existence of a diffeomorphism Ψ of the disc such that

$$\Psi^*g_u = e^{2\lambda} [dx_1^2 + dx_2^2] \quad (\text{i.e. } u \circ \Psi \text{ is conformal})$$

for some function λ from D^2 into \mathbb{R} . So if u minimizes E in the desired class we can replace it by $v := u \circ \Psi$ which is conformal and for which

$$E(v) = \text{Area}(v) = \text{Area}(u) \leq E(u) \quad .$$

Hence u is conformal. Assuming now it is not a minimizer of the area, we would then find w in the desired class such that

$$A(w) < A(u) = E(u) \quad ,$$

and taking again Ψ' s.t. $E(w \circ \Psi') = A(w)$ we would contradict that u minimizes E .

Hence the difficulty is to find a minimizer of E in the class of C^1 immersions sending continuously and monotonically ∂D^2 into Γ .

One introduces

$$\mathcal{F} := \left\{ \begin{array}{l} u \in W^{1,2}(D^2) \quad ; \quad u \in C^0(\partial D^2, \Gamma) \\ \text{and } u \text{ is monotone from } \partial D^2 \simeq S^1 \text{ into } \Gamma \end{array} \right\} \quad .$$

Fixing the images of three distinct points in ∂D^2 in order to kill the action of the remaining *gauge group* $\mathcal{M}(D^2)$ one proves the existence of a minimizer of E in \mathcal{F} which happens to be a solution to the Plateau problem (A thorough analysis is required to prove that the minimizer is indeed a C^1 immersion).

II.1 The conformal parametrization choice as a Coulomb gauge.

As we have seen the conformal parametrizations of immersed discs play a central role in the resolution of the Plateau problem. In the present subsection we establish a one to one correspondence between this choice of conformal parametrization and a *Coulomb gauge* choice.

Let u be a conformal immersion of the disc D^2 (i.e. $H(u) \equiv 0$ on D^2). Let $\lambda \in \mathbb{R}$ such that

$$e^\lambda := |\partial_{x_1} u| = |\partial_{x_2} u| \quad .$$

Introduce the *moving frame* associated to this parametrization

$$\vec{e}_j := e^{-\lambda} \partial_{x_j} u \quad \text{for } j = 1, 2 \quad .$$

The family (\vec{e}_1, \vec{e}_2) realizes an orthonormal basis of the tangent space of $u(D^2)$ at $u(x_1, x_2)$. This can be also interpreted as a section of the *frame bundle* of $u(D^2)$ equipped with the induced metric $g_{\mathbb{R}^3}$.

A simple computation gives

$$\begin{aligned} \operatorname{div}(\vec{e}_1, \nabla \vec{e}_2) &= \partial_{x_1} \left[e^{-2\lambda} \partial_{x_1} u \cdot \partial_{x_1 x_2}^2 u \right] + \partial_{x_2} \left[e^{-2\lambda} \partial_{x_1} u \cdot \partial_{x_2}^2 u \right] \\ &= 2^{-1} \partial_{x_1} \left[e^{-2\lambda} \partial_{x_2} |\partial_{x_1} u|^2 \right] - 2^{-1} \partial_{x_2} \left[e^{-2\lambda} \partial_{x_1} |\partial_{x_2} u|^2 \right] \\ &= \partial_{x_1 x_2}^2 \lambda - \partial_{x_2 x_1}^2 \lambda = 0 \quad . \end{aligned} \tag{II.1}$$

In other words, introducing the 1 form on D^2 given by $A := \vec{e}_1 \cdot d\vec{e}_2$, which is nothing but the *connection form* associated to the *Levi-Civita connection* induced by $g_u = u^* g_{\mathbb{R}^3}$ on the corresponding frame bundle for the trivialization given by (\vec{e}_1, \vec{e}_2) , equation (II.1) becomes

$$d^{*g} A = d^{*g} (\vec{e}_1 \cdot d\vec{e}_2) = 0 \quad , \tag{II.2}$$

where $*_g$ is the *Hodge operator* associated to the induced metric g_u . The equation (II.2) is known as being the *Coulomb condition*. We will see again this condition in the following sections and it is playing a central role in the survey.

Vice versa one proves, see for instance [He] or [40], that for any immersion u , non necessarily conformal, any frame (\vec{e}_1, \vec{e}_2) satisfying the *Coulomb condition* (II.2) corresponds to a conformal parametrization (i.e. $\exists \Psi \in \operatorname{Diff}(D^2)$ s.t. $v := u \circ \Psi$ is conformal and $\vec{e}_j = |\partial_{x_j} v|^{-1} \partial_{x_j} v$). This observation is the basis of the *Hélein method* for constructing isothermal coordinates.

III A Plateau type problem on the lack of integrability

In the rest of the class G denotes an arbitrary compact Lie group. We will sometime restrict to the case where G is a special unitary group $SU(n)$ and we will mention it explicitly. The corresponding Lie algebra will be denoted by \mathcal{G} and the neutral element of G is denoted e .

III.1 Horizontal equivariant plane distributions.

III.1.1 The definition.

Consider the simplest principal fiber structure $P := B^m \times G$ where B^m is the unit m -dimensional ball of \mathbb{R}^m . Denote by π the projection map which to $\xi = (x, h) \in P$ assigns the *base point* x . Denote by $Gr_m(TP)$ to be the *Grassmanian* of m -dimensional subspaces of the tangent bundle to P .

We define the notion of *equivariant horizontal distribution of plane* to be a map

$$H : P = B^m \times G \longrightarrow Gr_m(TP)$$

$$\xi = (x, h) \longrightarrow H_\xi$$

satisfying the following 3 conditions

i) the *bundle condition* :

$$\forall \xi \in P \quad H_\xi \in T_\xi P \quad ,$$

ii) the *horizontality condition*

$$\forall \xi \in P \quad \pi_* H_\xi = T_{\pi(\xi)} B^m \quad ,$$

iii) the *equivariance condition*

$$\forall \xi \in P \quad \forall g \in G \quad (R_g)_* H_\xi = H_{R_g(\xi)}$$

where R_g is the *right multiplication map* by g on P which to any $\xi = (x, h)$ assigns $R_g(\xi) := (x, hg)$.

III.1.2 Characterizations of equivariant horizontal distribution of plane by 1-forms on B^m taking values into \mathcal{G} .

Let H be an equivariant horizontal distribution of plane in $P = B^m \times G$. Clearly the following holds

$$\forall \xi = (x, h) \in P \quad \forall X \in T_x B^m \quad \exists! X^H(\xi) \in T_\xi P \quad \text{s.t.} \quad \pi_* X^H = X \quad .$$

The vector $X^H(\xi)$ is called the *horizontal lifting* of X at ξ .

At the point (x, e) (recall that e denotes the neutral element of G) we identify $T_{(x,e)}P \simeq T_x M \oplus \mathcal{G}$. Using this identification we deduce the existence of $A_x \cdot X$ such that

$$X^H(x, e) = (X, -A_x \cdot X) \quad .$$

The one form A is called *connection 1-form* associated to H . The linearity of A_x with respect to X is a straightforward consequence of the definition of X^H and therefore A defines a 1-form on B^m taking values into \mathcal{G} .

For any element $B \in \mathcal{G} \simeq T_e G$ we denote by B^* the unique vector-field on G satisfying

$$B^*(e) = B \quad \text{and} \quad \forall g \in G \quad B^*(g) := (R_g)_* B \quad ,$$

and by an abuse of notation $B^*(g)$ is simply denoted Bg . Using this notation we have

$$\forall \xi = (x, h) \in P \quad \forall X \in T_x B^m$$

$$X^H(\xi) = (X, -(A_x \cdot X)^*(h)) = (R_h)_* X^H(x, e) \quad .$$

At any point $\xi \in P$ Any vector $Z \in T_\xi P$ admits a decomposition according to H : we denote by Z^V the projection parallel to H_ξ onto the tangent plane to the vertical fiber given by the kernel of π_* :

$$Z^V := Z - (\pi_* Z)^H \quad .$$

III.2 The lack of integrability of equivariant horizontal distribution of planes.

A m -dimensional plane distribution H is said to be *integrable* if it identifies at every point with the tangent space to a m -dimensional *foliation*.

We aim to "measure" the *lack of integrability* of an equivariant horizontal distribution of planes. To that aim we recall the following classical result

Theorem III.1 (Frobenius). *An m -dimensional plane distribution H is integrable if and only if for any pair of vector fields Y and Z contained in H at every point the bracket $[Y, Z]$ is still included in H at every point.* \square

In the particular case of equivariant horizontal distribution of planes in $P = B^m \times G$ we have that H is integrable if and only if

$$\forall X, Y \text{ vector-fields on } B^m \quad [X^H, Y^H]^V \equiv 0 \quad .$$

We shall now compute $[X^H, Y^H]^V$ in terms of the one form A .

We write

$$\begin{aligned} [X^H, Y^H]_{(x,e)} &= [(X, -(A \cdot X)^*), (Y, -(A \cdot Y)^*)]_{(x,e)} \\ &= [(X, 0), (Y, 0)]_{(x,e)} + [(X, 0), (0, -(A \cdot Y)^*)]_{(x,e)} \\ &\quad + [(0, -(A \cdot X)^*), (Y, 0)]_{(x,e)} + [(0, (A \cdot X)^*), (0, (A \cdot Y)^*)]_{(x,e)} \quad . \end{aligned} \tag{III.1}$$

The definition of the Bracket operation on the Lie algebra \mathcal{G} together with the commutation of the vector-field bracket operation with the push-forward operation of the right multiplication map gives that

$$[(A \cdot X)^*, (A \cdot Y)^*] = ([A \cdot X, A \cdot Y])^* \quad , \tag{III.2}$$

where the brackets in the r.h.s. of the identity is the Lie algebra bracket operation. The definition of the Lie bracket of vector fields gives

$$[(X, 0), (0, -(A \cdot Y)^*)]_{(x,e)} = (0, -d(A \cdot Y) \cdot X) \quad . \quad (\text{III.3})$$

Finally we write

$$[(X, 0), (Y, 0)] = ([X, Y], 0) = ([X, Y], -A \cdot [X, Y]) + (0, A \cdot [X, Y]) \quad . \quad (\text{III.4})$$

Combining (III.1), (III.2), (III.3) and (III.4) gives

$$[X^H, Y^H]^V = d(A \cdot X) \cdot Y - d(A \cdot Y) \cdot X + A([X, Y]) + [A \cdot X, A \cdot Y] \quad ,$$

and using Cartan formula on the expression of the exterior derivative of a one form we obtain

$$[X^H, Y^H]^V = dA(X, Y) + [A \cdot X, A \cdot Y] \quad . \quad (\text{III.5})$$

The two form we obtained

$$F_A(X, Y) := dA(X, Y) + [A \cdot X, A \cdot Y] \quad (\text{III.6})$$

is the so called *curvature* of the plane distribution H and "measures" the lack of integrability of H . It will be conventionally denoted

$$F_A = dA + \frac{1}{2}[A \wedge A] \quad \text{or simply} \quad F_A = dA + A \wedge A \quad .$$

The Lie algebra, and hence the compact Lie group, is equipped with the Killing form associated to a finite dimensional representation, hence unitary, for which the form defines a scalar product invariant under adjoint action. For instance $\mathcal{G} = o(n)$ or $\mathcal{G} = u(n)$

$$\langle B, C \rangle = -\text{Tr}(BC) \quad .$$

If the Lie algebra \mathcal{G} is semi-simple : it is a direct sum of Lie algebras with no non trivial ideal, the Lie algebra is equipped with the *Killing scalar product* :

$$\langle B, C \rangle := -\text{Tr}(\text{ad}(B) \text{ad}(C)) \quad ,$$

where $\text{ad}(B)$ is the following endomorphism of \mathcal{G} : $\text{ad}(B)(D) := [B, D]$.

The Lagrangian we are considering for measuring the lack of integrability of the plane distribution H is just the L^2 norm of the curvature

$$\int_{B^m} \sum_{i < j} \left| [\partial_{x_i}^H, \partial_{x_j}^H]^V \right|^2 dx^m = \int_{B^m} |F_A|^2 dx^m \quad ,$$

where dx^m is the canonical volume form on B^m . The L^2 norm of the curvature is also known as being the *Yang-Mills energy* of the connection form A and is denoted $YM(A)$. We can now state one of the main problem these notes are addressing.

Yang-Mills Plateau Problem : *Let η be a 1-form on ∂B^m taking values into a Lie algebra \mathcal{G} of a compact Lie group G does there exists a 1-form A into \mathcal{G} realising*

$$\inf \left\{ YM(A) = \int_{B^m} |dA + A \wedge A|^2 dx^m \quad ; \quad \iota_{\partial B^m}^* A = \eta \right\} ,$$

where $\iota_{\partial B^m}$ is the canonical inclusion of the boundary ∂B^m into \mathbb{R}^m .

In other words we are asking the following question : *given an equivariant horizontal plane distribution over the boundary of the unit ball in \mathbb{R}^m , can one extend it inside the ball in an optimal way with respect to the L^2 norm of the integrability defect.*

In order to study this variational problem we first have to identify it's invariance group.

III.3 The Gauge Invariance.

In this subsection we identify the group of the Yang-Mill Plateau problem corresponding to the diffeomorphism group of the disc for the area in the classical Plateau problem.

Let g be a map from B^m into G . We denote by $L_{g^{-1}}$ the left multiplication by g^{-1} defined as follows

$$L_{g^{-1}} ; P = B^m \times G \longrightarrow P$$

$$\xi = (x, h) \longrightarrow (x, g^{-1} h) .$$

Let H be an *equivariant horizontal distribution of planes* on P we observe that the push-forward by $L_{g^{-1}}$ of H , $(L_{g^{-1}})_* H$, is still an *equivariant horizontal distribution of planes*. We now compute the connection 1-form associated to this new distribution.

Let X be a vector of $T_x B^m$ and $x(t)$ a path in B^m such that $\dot{x}(0) = X$. Let $h(t) \in G$ such that $\xi(t) := (x(t), h(t))$ is the horizontal lifting of $x(t)$ starting at the neutral element e of G (i.e. $\dot{\xi} = (\dot{x})^H(\xi(t))$ and $\xi(0) = (x, e)$). Since $\dot{x}^V = 0$ we have in particular

$$\frac{dh}{dt}(0) = -A \cdot X .$$

The push forward by $L_{g^{-1}}$ of the horizontal vector $X^H(x, e)$ is the horizontal lifting of X at (x, g^{-1}) for the distribution $(L_{g^{-1}})_* H : X^{(L_{g^{-1}})_* H}(x, g^{-1})$. Hence we have

$$\begin{aligned} X^{(L_{g^{-1}})_* H}(x, g^{-1}) &= (L_{g^{-1}})_* X^H(x, e) = (L_{g^{-1}})_*(X, -A \cdot X) \\ &= \frac{d}{dt} (x(t), g^{-1} h(t)) = (X, dg^{-1} \cdot X - g^{-1} A \cdot X) \\ &= (X, - (g^{-1} dg \cdot X + g^{-1} A \cdot X g) g^{-1}) \\ &= (X, - (g^{-1} dg \cdot X + g^{-1} A \cdot X g)^* (g^{-1})) . \end{aligned}$$

Hence we have proved that the horizontal lift at (x, e) for the new plane distribution $(L_{g^{-1}})_*H$ is $(X, -(g^{-1}dg \cdot X + g^{-1}A \cdot Xg))$ and the associated connection one form associated to the distribution $(L_{g^{-1}})_*H$ is

$$A^g = g^{-1}dg \cdot X + g^{-1}A \cdot Xg \quad .$$

The curvature associated to this new distribution is given by

$$F_{A^g}(X, Y) = dA^g(X, Y) + [A^g(X), A^g(Y)] \quad .$$

We have in one hand

$$dA^g(X, Y) = dg^{-1} \wedge dg(X, Y) + [dg^{-1}g \wedge g^{-1}Ag](X, Y) \quad ,$$

and in the other hand

$$[A^g(X), A^g(Y)] = [g^{-1}dg \wedge g^{-1}Ag](X, Y) + g^{-1}[A(X), A(Y)]g \quad .$$

Summing the two last identities and using the fact that $g^{-1}dg + dg^{-1}g = 0$ gives finally

$$F_{A^g} = g^{-1}F_Ag \quad .$$

Since the Killing scalar product on \mathcal{G} is invariant under the adjoint action of G we have

$$YM(A^g) = YM(A) \quad .$$

The action of $L_{g^{-1}}$ on the plane distribution H leaves invariant it's Yang Mills energy and realizes therefore a "huge" invariance group of the *Yang-Mills Plateau Problem* which is called the *Gauge group* of the problem.

Exactly as for the classical Plateau problem we discussed in the first part of the survey the task for solving the Yang-Mills Plateau problem will be to "kill" this gauge invariance and, here again, the Coulomb Gauge choices will be of great help.

III.4 The Coulomb Gauges.

We first start with the simplest group, the abelian group $G = S^1$. The Yang-Mills Plateau problem in this case becomes :

Find a minimizer of

$$\inf \left\{ YM(A) = \int_{B^m} |dA|^2 dx^m \quad ; \quad \iota_{\partial B^m}^* A = \eta \right\} \quad ,$$

where η is some given 1-form on the boundary ∂B^m .

In a reminiscent way to the classical Plateau problem our starting functional is degenerate and we shall replace it by a more coercive one

$$YM(A) \leq E(A) = \int_{B^m} |dA|^2 + |d^*A|^2 dx^m \quad ,$$

with equality if and only if $d^*A = 0$ (i.e. A satisfies the Coulomb condition).

The following coercivity inequality holds

$$\forall A \in W^{1,2}(B^m, \mathcal{G}) \quad \text{s.t.} \quad \iota_{\partial B^m}^* A = \eta$$

$$\int_{B^m} |A|^2 + \sum_{i,j=1}^m |\partial_{x_i} A_j|^2 dx^m \leq C \left[E(A) + \|\eta\|_{H^{1/2}(\partial B^m)}^2 \right] \quad , \quad (\text{III.7})$$

for some fixed constant independent of η and A , where $H^{1/2}(\partial B^m)$ is the fractional trace space of $W^{1,2}(B^m)$. The convexity of E in $W_\eta^{1,2}(\wedge^1 B^m, \mathcal{G})$ together with the previous coercivity inequality implies that the following problem admits a unique minimizer A_0 of

$$\min_{A \in W_\eta^{1,2}(\wedge^1 B^m, \mathcal{G})} E(A) \quad .$$

The Euler Lagrange equation reads

$$\forall \phi \in W_0^{1,2}(\wedge^1 B^m, \mathcal{G}) \quad \int_{B^m} d\phi \wedge *dA + (-1)^m d^*A \wedge d*\phi = 0 \quad . \quad (\text{III.8})$$

This gives in particular that each of the components of A are harmonic. We choose f to be an arbitrary function in $L^2(B^m, \mathcal{G})$ and we solve

$$\begin{cases} -\Delta u = f & \text{in } B^m \\ u = 0 & \text{on } \partial B^m \end{cases} \quad .$$

Classical elliptic theory gives that the 1-form $\phi := du$ is in $W_0^{1,2}(\wedge^1 B^m, \mathcal{G})$. By substituting $\phi = du$ in (III.8) we obtain

$$\forall f \in L^2(B^m, \mathcal{G}) \quad \int_{B^m} d^*A \cdot f dx^m = 0 \quad .$$

Hence we deduce $d^*A = 0$ in B^m and there exists a unique solution to (III.8) which is also the unique solution of the following system

$$\begin{cases} d^*dA_0 = 0 & \text{in } \mathcal{D}'(B^m) \\ d^*A_0 = 0 & \text{in } \mathcal{D}'(B^m) \\ \iota_{\partial B^m}^* A_0 = \eta & . \end{cases}$$

The components of A are harmonic in B^m and are therefore smooth moreover we have

$$YM(A_0) = E(A_0) \quad .$$

Let now B in $W_\eta^{1,2}(\wedge^1 B^m, \mathcal{G})$ we claim that there exists a gauge change g such that $YM(B) = YM(B^g) = E(B^g)$. This can be seen as follows. Let φ be the solution of

$$\begin{cases} -\Delta\varphi = d^*B & \text{in } \mathcal{D}'(B^m) \\ \varphi = 0 & \text{on } \partial B^m \end{cases} .$$

Hence we have

$$\begin{cases} d(B + d\varphi) = dB & \text{in } \mathcal{D}'(B^m) \\ d^*(B + d\varphi) = 0 & \text{in } \mathcal{D}'(B^m) \\ \iota_{\partial B^m}(B + d\varphi) = \eta \end{cases} .$$

Taking $g := \exp(i\varphi)$ we have $YM(B) = YM(B^g) = E(B^g)$.

Hence A_0 realizes

$$\min_{A \in W_\eta^{1,2}(\wedge^1 B^m, \mathcal{G})} YM(A) \quad .$$

Indeed if there would be $B \in W_\eta^{1,2}(\wedge^1 B^m, \mathcal{G})$ such that $YM(B) < YM(A_0)$ we choose g such that $YM(B^g) = E(B^g)$ and we would contradict the fact that A_0 minimizes E .

Taking now a general compact Lie group G we would also like to propose to minimize E instead of YM but we need first ensure that a Coulomb gauge always exists. We have the following lemma which answers positively to this last question

Lemma III.1. *Let $A \in L^2(\wedge^1 B^m, \mathcal{G})$. The following variational problem*

$$\inf_{g \in W_e^{1,2}(B^m, G)} \int_{B^m} |g^{-1}dg + g^{-1}Ag|^2 dx^m$$

is achieved and each minimizer satisfies the Coulomb condition

$$d^*(g^{-1}dg + g^{-1}Ag) = 0 \quad .$$

□

Proof of Lemma III.1. Let g_k be a minimizing sequence. Since the group is compact we have that

$$\limsup_{k \rightarrow +\infty} \int_{B^m} |dg_k|^2 dx^m < +\infty \quad .$$

Hence, using Rellich Kondrachov's theorem there exists a subsequence $g_{k'}$ converging weakly in $W^{1,2}(B^m, G)$ to g_∞ and strongly in every $L^p(B^m, G)$ space for any $p < +\infty$

hence $g_\infty \in W_e^{1,2}(B^m, G)$. The same holds for g_k^{-1} and it's weak limit in $W^{1,2}(B^m, G)$ is the inverse of g_∞ . Hence we have

$$g_k^{-1} dg_k + g_k^{-1} A g_k \rightharpoonup g_\infty^{-1} dg_\infty + g_\infty^{-1} A g_\infty \quad \text{in } \mathcal{D}'(B^m) \quad .$$

The lower semi continuity of the L^2 norm implies that g_∞ is a minimizer of (III.6).

For any $U \in C_0^\infty(B^m, \mathcal{G})$ we introduce

$$g_\infty(t) := g_\infty \exp(tU) \quad .$$

We have

$$\begin{aligned} g_\infty^{-1}(t) dg_\infty^{-1}(t) + g_\infty^{-1}(t) A g_\infty(t) &= \exp(-tU) d \exp(tU) \\ &+ \exp(-tU) [g_\infty^{-1} dg_\infty + g_\infty^{-1} A g_\infty] \exp(tU) \quad . \end{aligned}$$

Hence

$$\frac{d}{dt} [g_\infty^{-1}(t) dg_\infty^{-1}(t) + g_\infty^{-1}(t) A g_\infty(t)] = dU - [U, A^{g_\infty}] \quad .$$

Since g_∞ is a minimizer we have

$$0 = \frac{d}{dt} \int_{B^m} |A^{g_\infty(t)}|^2 dx^m = 2 \int_{B^m} \langle (dU - [U, A^{g_\infty}]) \cdot A^{g_\infty} \rangle dx^m \quad .$$

We use the identity $\langle [U, V], W \rangle = \langle U, [V, W] \rangle$ to deduce that

$$[U, A^{g_\infty}] \cdot A^{g_\infty} = 0$$

and then we have proved that for any $U \in C_0^\infty(B^m, \mathcal{G})$ we have

$$0 = \int_{B^m} dU \cdot A^{g_\infty} dx^m = \int_{B^m} \langle U, d^* A^{g_\infty} \rangle dx^m \quad .$$

This finishes the proof of the lemma. □

Since every connection form posses a Coulomb Gauge representative it is then tempting to minimize E instead of YM following the main lines of the abelian case. However due to the non-linearity $A \wedge A$ in F_A it is not clear whether the E energy controls the $W^{1,2}$ norm of A in a similar way of (III.7) in the general case.

In fact the answer to that question is "no" as we can see in the following example. We take $G = SU(2)$ and we identify $su(2)$ with the imaginary quaternions. On \mathbb{R}^4 we identify canonically the point of coordinates (x_0, x_1, x_2, x_3) with the quaternion $\mathbf{x} := x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$. For a quaternion $\mathbf{y} = y_0 + y_1 \mathbf{i} + y_2 \mathbf{j} + y_3 \mathbf{k}$ we denote by $\mathfrak{S}(\mathbf{y})$ the element in $su(2)$ given by $y_1 \sigma_1 + y_2 \sigma_2 + y_3 \sigma_3$ where σ_i are the Pauli matrices to which we identify \mathbf{i}, \mathbf{j} and \mathbf{k}

$$\mathbf{i} \leftrightarrow \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathbf{j} \leftrightarrow \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{k} \leftrightarrow \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

forming an orthogonal basis of $su(2)$ with norms $\sqrt{2}$ for each vector of the basis.

On B^4 , for $\lambda \in \mathbb{R}_+^*$, we consider the family of one forms into $su(2)$ given by

$$A_\lambda := \lambda^2 \frac{\Im(\mathbf{x} d\bar{\mathbf{x}})}{1 + \lambda^2 |\mathbf{x}|^2} .$$

The corresponding curvature is given by

$$F_{A_\lambda} = \lambda^2 \frac{d\mathbf{x} \wedge d\bar{\mathbf{x}}}{(1 + \lambda^2 |\mathbf{x}|^2)^2} .$$

One easily verifies that¹

$$\lim_{\lambda \rightarrow +\infty} \int_{B^4} |F_{A_\lambda}|^2 dx^4 = \int_{\mathbb{R}^4} |F_{A_1}|^2 dx^4 = \int_{\mathbb{R}^4} 48 \frac{dx^4}{(1 + |x|^2)^4} = 8\pi^2 < +\infty$$

but one verifies also that

$$\lim_{\lambda \rightarrow +\infty} \int_{B^4} \sum_{i,j=1^4} |\partial_{x_i}(A_\lambda)_j|^2 dx^4 = +\infty$$

we might then think that by changing the gauge we can avoid this blow up of the $W^{1,2}$ norm of the connection form but, as we see now, this cannot be the case. Consider the *second Chern form* $\text{Tr}(F_{A_\lambda} \wedge F_{A_\lambda})$, it satisfies

$$\lim_{\lambda \rightarrow +\infty} \text{Tr}(F_{A_\lambda} \wedge F_{A_\lambda}) = 8\pi^2 \delta_0 dx^4 . \quad (\text{III.9})$$

The *second Chern form* is invariant under gauge transformation and for any choice of Gauge g this 4-form, which has to be closed, is on B^4 the exterior derivative of the *transgression form* known as the *Chern-Simon 3-form* :

$$\forall g : B^4 \rightarrow SU(2)$$

$$\text{Tr}(F_{A_\lambda} \wedge F_{A_\lambda}) = d \left[\text{Tr} \left(A_\lambda^g \wedge dA_\lambda^g + \frac{1}{3} A_\lambda^g \wedge [A_\lambda^g, A_\lambda^g] \right) \right] .$$

Assume now there would have been a gauge g_λ s.t.

$$\liminf_{\lambda \rightarrow +\infty} \|A_\lambda^{g_\lambda}\|_{W^{1,2}(B^4)} < +\infty .$$

¹The somehow surprising factor 48 comes from the fact that there are 6 curvature coordinates and each curvature coordinate has the form

$$|(F_{A_1})_{ij}|^2 = \frac{8}{(1 + |x|^2)^2}$$

where we have used that the square of the norm of each Pauli matrix is 2.

Then for some sequence $\lambda_k \rightarrow +\infty$, using Rellich Kondrachov theorem $A^k := A_{\lambda_k}^{g_k}$ would weakly converge to some limit A^∞ in $W^{1,2}(\wedge^1 B^4, su(2))$ and strongly in $L^p(\wedge^1 B^4, su(2))$ for any $p < 4$. Hence

$$\mathrm{Tr} \left(A^k \wedge dA^k + \frac{1}{3} A^k \wedge [A^k, A^k] \right) \rightharpoonup \mathrm{Tr} \left(A^\infty \wedge dA^\infty + \frac{1}{3} A^\infty \wedge [A^\infty, A^\infty] \right)$$

in $\mathcal{D}'(B^4)$. Taking now the exterior derivative and using again the gauge invariance of the second Chern form we obtain

$$\mathrm{Tr}(F_{A_{\lambda_k}} \wedge F_{A_{\lambda_k}}) \rightharpoonup \mathrm{Tr}(F_{A^\infty} \wedge F_{A^\infty}) \quad \text{in } \mathcal{D}'(B^4) \quad .$$

Since A^∞ is in $W^{1,2}$ the 4-form $\mathrm{Tr}(F_{A^\infty} \wedge F_{A^\infty})$ is an L^1 function, however, from (III.9) it is equal to the Dirac mass. This gives a contradiction and we have proved the following proposition

Proposition III.1. *There exists $A^k \in W^{1,2}(B^4, su(2))$ such that*

$$\limsup_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 < +\infty \quad ,$$

but

$$\liminf_{k \rightarrow +\infty} \inf \left\{ \int_{B^4} \sum_{i,j=1}^4 |\partial_{x_i}(A^k)_j^g|^2 dx^4 ; g \in W^{2,2}(B^4, SU(2)) \right\} = +\infty \quad .$$

□

Hence by minimizing E instead of YM we don't get enough control on the minimizing sequence A^k in order to extract a converging subsequence to a solution to the Yang-Mills Plateau problem.

The situation would have been much better in dimension less than 4 where a $W^{1,2}$ control of A in terms of $E(A)$ do exist. In dimension equal to 4, despite proposition III.1, there is still a positive result in that line which says roughly that such a control do exist for some gauge provided the *Yang-Mills energy* stays below some positive threshold. The following section is devoted to the proof of this result by K.Uhlenbeck.

IV Uhlenbeck's Coulomb Gauge Extraction Method

IV.1 Uhlenbeck's construction.

We have seen that in dimension 4 - and higher of course- there is no hope to control the $W^{1,2}$ norm of sequences of connection forms from the E energy. The fact that the dimension 4 is critical for this phenomenon comes from the optimal *Sobolev embedding*

$$W^{1,2}(B^4) \hookrightarrow L^4(B^4) \quad ,$$

which does not hold in higher dimension.

Theorem IV.1. *Let $m \leq 4$ and G be a compact Lie group. There exists $\varepsilon_G > 0$ and $C_G > 0$ such that for any $A \in W^{1,2}(B^m, \mathcal{G})$ satisfying*

$$\int_{B^m} |dA + A \wedge A|^2 dx^m < \varepsilon_G \quad ,$$

there exists $g \in W^{2,2}(B^m, G)$ such that

$$\left\{ \begin{array}{l} \int_{B^m} |A^g|^2 + \sum_{i,j=1}^4 |\partial_{x_i} A_j^g|^2 dx^m \leq C_G \int_{B^m} |dA + A \wedge A|^2 dx^m \\ d^* A^g = 0 \quad \text{in } B^m \\ \iota_{\partial B^m}^* (*A^g) = 0 \quad , \end{array} \right. \quad (\text{IV.1})$$

where $A^g = g^{-1}dg + g^{-1}Ag$ and $\iota_{\partial B^m}$ is the canonical inclusion map of the boundary of the unit ball into \mathbb{R}^m . \square

Remark IV.1. *The same result holds in arbitrary dimension replacing the L^2 -norm for the curvature by the $L^{m/2}$ norm and the $W^{1,2}$ norm of the connection by the $W^{1,m/2}$ -norm (see [59]). The proof we are giving below can be transposed word by word in this more general setting. The adaptation requires just a shift of the exponents $4 \rightarrow m$, $2 \rightarrow m/2$. \square*

For $m < 4$ the non linearity $A \wedge A$ is a compact perturbation of dA and the problem is a perturbation to a simple linear one that we solved in the abelian case. We will then restrict the presentation to the case $m = 4$. We assume that the compact Lie group is represented by a subgroup of invertible matrices in R^n for some $n \in \mathbb{N}^*$ which gives an isometric embedding of G in an euclidian space.

We aim to solve the Coulomb equation $d^* A^g = 0$ keeping this time a control of A^g in $W^{1,2}$ (lemma III.1 was only giving an L^2 control). Since we have little energy the hope is to use a fixed point argument close to the zero connection and for g close to the identity. The linearization of the *Coulomb non-linear elliptic PDE*

$$\left\{ \begin{array}{l} d^* [g^{-1} dg] = -d^* [g^{-1} A g] \quad \text{in } B^4 \\ \partial_r g g^{-1} = - \langle A, \partial_r \rangle \quad \text{on } \partial B^4 \quad , \end{array} \right.$$

for $A^t = t\omega$ and $g^t = \exp(tU)$ gives

$$\left\{ \begin{array}{l} \Delta U = -d^* dU = d^* \omega \quad \text{in } B^4 \\ \partial_r U = - \langle \omega, \partial_r \rangle \quad \text{on } \partial B^4 \quad . \end{array} \right.$$

This linearized problem is solvable for any map $\omega \in W^{1,2}(B^4, \mathcal{G})$ and we get a unique $U \in W^{2,2}(B^4, \mathcal{G})$ solving the previous linear equation.

However, in order to be able to apply the implicit function theorem we need the following non-linear mapping to be smooth

$$\begin{aligned} \mathcal{N}^0 &: W^{1,2}(B^4, \mathcal{G}) \times W^{2,2}(B^4, \mathcal{G}) \longrightarrow W^{1,2}(B^4, \mathcal{G}) \times H^{1/2}(\partial B^4, \mathcal{G}) \\ (\omega, U) &\longrightarrow \left(d^* [g_U^{-1} dg_U + g_U^{-1} \omega g_U], \partial_r g_U g_U^{-1} - \langle \omega, \partial_r \rangle \right) \quad , \end{aligned}$$

where $g_U := \exp(U)$. This is however not the case in dimension 4. This is due to the fact that $W^{2,2}$ does not embed in C^0 in 4 dimension but only in the *Vanishing Mean Oscillation* space $VMO(B^4)$ hence simple algebraic operations such as the multiplication of two G valued $W^{2,2}$ maps is not continuous in 4 dimension.

If one replaces however $W^{1,2} \times W^{2,2}$ by a "slightly smaller space" $W^{1,p} \times W^{2,p}$ for any $p > 2$ (p being as close as we want from 2) then the space $W^{2,p}$ embeds continuously (and compactly) in C^0 and the map \mathcal{N}^0 becomes suddenly smooth ! and a fixed point argument is conceivable in this smaller space.

Uhlenbeck's strategy consists in combining a fixed point argument in smaller spaces - in which the problem is invertible - together with a continuity argument.

This method is rather generic in the sense that it can be applied to critical *extensions* or *lifting problems* of maps in the Sobolev space $W^{1,m}(M^m)$ which misses to embed in C^0 but for which however the notion of homotopy class is well defined (see [47] and [61]) and prevents to find global extensions or liftings when the norm of the map is too high. As an illustration we shall give two results.

Theorem IV.2. *For any $m \geq 1$, and any compact Lie group G there exists $\varepsilon_{m,G} > 0$ and $C_{m,G} > 0$ such that for any map $g \in W^{1,m}(S^m, G)$ satisfying*

$$\int_{S^m} |dg|^m dvol_{S^m} < \varepsilon_m \quad ,$$

there exists an extension $\tilde{g} \in W^{1,m+1}(B^{m+1}, G)$ equal to g on ∂B^{m+1} such that

$$\int_{B^{m+1}} |d\tilde{g}|^{m+1} dx^{m+1} \leq C_m \int_{S^m} |dg|^m dvol_{S^m} \quad .$$

□

Remark IV.2. *The existence of such an extension $\tilde{g} \in W^{1,m+1}(B^{m+1}, G)$ is clearly not true for general $g \in W^{1,m}(S^m, S^m)$. Indeed, consider for instance $m = 3$ and $G = SU(2) \simeq S^3$, if such an extension would exist one would have using Stokes theorem.*

$$0 = \int_{B^4} \tilde{g}^* dx^4 = \frac{1}{4} \int_{S^3} g^* dvol_{SU(2)} = \frac{|S^3|}{4} \deg(g) \quad ,$$

where $\deg(g)$ is the topological degree of the map g which is not necessarily zero. However there always exists an extension in the “slightly” larger space $W^{1,(m+1,\infty)}(B^{m+1}, G)$ of maps from B^m into G with one derivative in the Marcinkiewicz weak $L^{m+1}(B^{m+1})$ space (see subsection IV.3 below). \square

Theorem IV.2 is proved in [33] using Uhlenbeck’s method. The second example is the following one

Theorem IV.3. *Let P be a G principal bundle over S^m where G is a compact Lie group and where π is the projection associated to this bundle. There exists $\varepsilon_{m,G} > 0$ and $C_{m,G} > 0$ such that for any $g \in W^{1,m}(S^m, S^m)$ satisfying*

$$\int_{S^m} |dg|^m d\text{vol}_{S^m} < \varepsilon_{m,G} \quad ,$$

there exists $v \in W^{1,m}(S^m, P)$ such that

$$\int_{S^m} |dv|^m d\text{vol}_{S^m} \leq C_{m,G} \int_{S^m} |dg|^m d\text{vol}_{S^m} \quad ,$$

and

$$\pi \circ v = g \quad .$$

\square

Proof of theorem IV.1.

Fix some $2 < p < 4$. For any $\varepsilon > 0$ we introduce

$$\mathcal{U}^\varepsilon := \left\{ A \in W^{1,p}(B^4, \mathcal{G}) \quad \text{s.t.} \quad \int_{B^4} |F_A|^2 dx^4 < \varepsilon \right\} \quad ,$$

and for any $\varepsilon > 0$ and $C > 0$ and we consider

$$\mathcal{V}_C^\varepsilon := \left\{ \begin{array}{l} A \in \mathcal{U}^\varepsilon \quad \text{s.t.} \quad \exists g \in W^{2,p}(B^4, G) \\ \int_{B^4} |dA^g|_{g_{S^4}}^p dx^4 \leq C \int_{B^4} |F_A|^p dx^4 \\ \int_{B^4} |dA^g|_{g_{S^4}}^2 dx^4 \leq C \int_{B^4} |F_A|^2 dx^4 \\ d^* A^g = 0 \quad \text{and} \quad \iota_{\partial B^4} * A^g = 0 \quad . \end{array} \right\}$$

The first goal is to show the following

Claim 1

$$\exists \varepsilon > 0 \quad C > 0 \quad \text{s.t.} \quad \mathcal{V}_C^\varepsilon = \mathcal{U}^\varepsilon \quad .$$

In order to prove the claim we shall establish successively

- 1 - The set \mathcal{U}^ε is path connected.
- 2 - The set $\mathcal{V}_C^\varepsilon$ is closed in \mathcal{U}^ε for the $W^{1,p}$ -topology
- 3 - For $\varepsilon > 0$ chosen small enough and $C > 0$ large enough the set $\mathcal{V}_C^\varepsilon$ is open for the $W^{1,p}$ -topology

Since $\mathcal{V}_C^\varepsilon$ is non empty, this will imply the claim 1 for this choice of ε and C .

Proof of the path connectedness of \mathcal{U}^ε . For A in \mathcal{U}^ε and $t \in [0, 1]$ we define the connection form A^t to be the image of A by the dilation of rate t^{-1} : $A^t = t \sum_{j=1}^4 A_j(tx) dx_j$. We have in particular

$$F_{A^t} = t^2 \sum_{i,j=1}^4 (F_A)_{ij}(x) dx_i dx_j \quad ,$$

hence

$$\int_{B^4} |F_{A^t}|^2 dx^4 = \int_{B_t^4} |F_A|^2 dx^4 < \varepsilon \quad ,$$

and

$$\sum_{i,j=1}^4 \int_{B^4} |\partial_{x_i}(A_t)_{x_j}|^p dx^4 = O(t^{2p-4}) \quad . \quad (\text{IV.2})$$

This shows that in one hand $A^t \in \mathcal{U}^\varepsilon$ for any $t \in [0, 1]$ and that in the other hand $A^t \rightarrow 0$ strongly in $W^{1,p}(B^4)$. Hence A^t is a continuous path contained in \mathcal{U}^ε connecting A and 0 which prove the path connectedness of \mathcal{U}^ε .

Proof of the closeness of $\mathcal{V}_C^\varepsilon$ in \mathcal{U}^ε .

Let $A^k \in \mathcal{V}_C^\varepsilon$ and assume A^k converges strongly in $W^{1,p}$ to some limit $A^\infty \in \mathcal{U}^\varepsilon$. We claim that $A^\infty \in \mathcal{V}_C^\varepsilon$.

Since $A^k \rightarrow A^\infty$ strongly in $W^{1,p}$, $dA^k \rightarrow dA^\infty$ strongly in L^p and, using Sobolev embedding, $A^k \rightarrow A^\infty$ in $L^{4p/4-p}$. Hence, due to the later,

$$A^k \wedge A^k \longrightarrow A^\infty \wedge A^\infty \text{ strongly in } L^{2p/4-p}(B^4) \quad .$$

We have chosen $2 < p < 4$ in such a way that $p < 2p/4 - p$. Hence we deduce that

$$F_{A^k} \longrightarrow F_{A^\infty} \quad \text{strongly in } L^p(B^4) \quad . \quad (\text{IV.3})$$

Let g^k be a sequence such that

$$\int_{B^4} |d(A^k)^{g^k}|_{g_{B^4}}^q d\text{vol}_{S^4} \leq C \int_{B^4} |F_{A^k}|^q d\text{vol}_{S^4} \quad , \quad (\text{IV.4})$$

for $q = 2, p$ and

$$d^*(A^k)^{g^k} = 0 \quad \text{and} \quad \iota_{\partial B^4}^* (A^k)^{g^k} = 0 \quad . \quad (\text{IV.5})$$

Since both $d(A^k)^{g^k}$ and $d^*(A^k)^{g^k}$ are uniformly bounded in L^p and since there is no harmonic 3-form² on B^4 whose restriction on the boundary is zero, classical L^p -Hodge theory (see for instance [21]) infers that $(A^k)^{g^k}$ is uniformly bounded in $W^{1,p}$. So, using Sobolev embedding, it is in particular bounded in $L^{4p/4-p}$ and since $(A^k)^{g^k} = (g^k)^{-1}dg^k + (g^k)^{-1}A^k g^k$, using that A^k is bounded in $W^{1,p}$ and hence in $L^{4p/4-p}$, we deduce that dg^k is bounded also in $L^{4p/4-p}$. Since $4 < 4p/4 - p$, there exists a subsequence $g^{k'}$ converging strongly to some limit g^∞ in C^0 . Going back to the weak convergence of $(A^k)^{g^k}$ in $W^{1,p}$, the strong convergence of A^k in $W^{1,p}$ and the weak convergence of g^k in $W^{1,4p/4-p}$ we deduce that

$$dg^k = g^k (A^k)^{g^k} - A^k g^k$$

is uniformly bounded in $W^{1,p}$ and therefore $g^{k'} \rightharpoonup g^\infty$ weakly in $W^{2,p}(B^4, G)$. Thus we deduce that the following weak convergence in $W^{1,p}(B^4)$ holds

$$(g^{k'})^{-1}dg^{k'} + (g^{k'})^{-1}A^{k'} g^{k'} \rightharpoonup (g^\infty)^{-1}dg^\infty + (g^\infty)^{-1}A^\infty g^\infty \quad .$$

Combining (IV.3) and the latest weak convergence we deduce that

$$\int_{B^4} |d(A^\infty)^{g^\infty}|^q dx^4 \leq C \int_{B^4} |F_{A^\infty}|^q dx^4 \quad , \quad (\text{IV.6})$$

for $q = 2, p$ and, using the following continuous embedding of

$$W^{1,p}(B^4) \hookrightarrow W^{1-1/p,p}(\partial B^4)$$

we have

$$d^*(A^\infty)^{g^\infty} = 0 \quad \text{and} \quad \iota_{\partial B^4}^*(A^\infty)^{g^\infty} = 0 \quad . \quad (\text{IV.7})$$

So we have proved that A^∞ fulfill all the conditions for being in $\mathcal{V}_C^\varepsilon$.

Proof of the openness of $\mathcal{V}_C^\varepsilon$.

Let A be an element of $\mathcal{V}_C^\varepsilon$. It is clear that if we find in $\mathcal{V}_C^\varepsilon$ an open neighborhood for the $W^{1,p}$ -topology of the $W^{1,p}$ Coulomb gauge A^g , then A posses also such a neighborhood. So we can assume right away that $d^*A = 0$ and $\iota_{\partial B^4} * A = 0$.

We are looking for the existence of δ sufficiently small - possibly depending on A - such that for any ω satisfying $\|\omega\|_{W^{1,p}} < \delta$ there exists g close to the identity in $W^{2,p}$ -norm such that

$$d^* [g^{-1}dg + g^{-1}(A + \omega)g] = 0 \quad \text{and} \quad \iota_{\partial B^4} * (A + \omega)^g = 0 \quad .$$

²Assume B is a 3 form satisfying $dB = 0$ and $d^*B = 0$ on B^4 and $\iota_{\partial B^4}^* B = 0$ then $*B = d\varphi$, $B = d\beta$ and $\iota_{\partial B^4}^* \beta = dC$. With these notations we have

$$\int_{B^4} |B|^2 = \int_{B^4} d\beta \wedge d\varphi = \int_{\partial B^4} dC \wedge d\varphi = 0$$

hence $B = 0$.

To that purpose we introduce the map

$$\begin{aligned} \mathcal{N}^A &: W^{1,p}(B^4, \mathcal{G}) \times W^{2,p}(B^4, \mathcal{G}) \longrightarrow L^p(B^4, \mathcal{G}) \times W^{1-1/p,p}(\partial B^4, \mathcal{G}) \\ (\omega, U) &\longrightarrow (d^* [g_U^{-1} dg_U + g_U^{-1}(A + \omega) g_U], \iota_{\partial B^4} * (A + \omega)^{g_U}) \quad , \end{aligned} \quad (\text{IV.8})$$

where $g_U := \exp(U)$. We have seen that this map is smooth.

The derivative of \mathcal{N}^A along the U direction at $(0, 0)$ gives

$$\partial_U \mathcal{N}^A(0, 0) \cdot V = (-\Delta V + [A, dV], \partial_r V) \quad ,$$

where $\Delta = \sum_{k=1}^4 \partial_{x_k}^2$.

Using Calderon Zygmund L^p theory (see for instance [52] or [16]) we have the following a-priori estimate for any V satisfying $\int_{B^4} V dx^4 = 0$

$$\begin{aligned} \|V\|_{W^{2,p}(B^4)} &\leq c [\|\Delta V\|_{L^p(B^4)} + \|\partial_r V\|_{W^{1-1/p,p}(\partial B^4)}] \\ &\leq c [\|\partial_U \mathcal{N}^A(0, 0) \cdot V\|_{\mathcal{F}} + \|[A, dV]\|_{L^p(B^4)}] \\ &\leq c [\|\partial_U \mathcal{N}^A(0, 0) \cdot V\|_{\mathcal{F}} + c \|A\|_{L^4(B^4)} \|dV\|_{L^{4p/4-p}(B^4)}] \quad , \end{aligned}$$

where \mathcal{F} is the hyperplane of $L^p(B^4, \mathcal{G}) \times W^{1-1/p,p}(\partial B^4, \mathcal{G})$ made of couples (f, g) such that

$$\int_{B^4} f(x) dx^4 = - \int_{\partial B^4} g(y) d\text{vol}_{\partial B^4} \quad .$$

From the fact that $A \in \mathcal{V}_C^\varepsilon$ we deduce that $\|A\|_{L^4} \leq C_4 \sqrt{C} \varepsilon$ where C_4 is the Sobolev constant coming from the embedding into $L^4(B^4)$ of closed 3 forms on B^4 with adjoint exterior derivative in L^2 and whose restriction to ∂B^4 is zero. Hence for any V with average zero on B^4 we have

$$\|V\|_{W^{2,p}(B^4)} \leq c [\|\partial_U \mathcal{N}^A(0, 0) \cdot V\|_{\mathcal{F}} + c C_4 \sqrt{C} \varepsilon \|dV\|_{L^{4p/4-p}}] \quad .$$

Using again the embedding of $W^{1,p}(\wedge^1 B^4, \mathcal{G})$ into $L^{4p/4-p}(\wedge^1 B^4, \mathcal{G})$ and denoting C_p the corresponding constant, we have then

$$[1 - c C_4 \sqrt{C} \varepsilon C_p] \|V\|_{W^{2,p}(B^4)} \leq c \|\partial_U \mathcal{N}^A(0, 0) \cdot V\|_{\mathcal{F}} \quad .$$

Having chosen ε such that $c C_4 \sqrt{C} \varepsilon C_p < 1/2$ we have that $\partial_U \mathcal{N}^A(0, 0)$ has zero Kernel. A classical result from Calderon-Zygmund theory (see [16]) asserts that

$$\mathcal{L} : W^{2,p}(B^4, \mathcal{G}) \longrightarrow \mathcal{F}$$

$$V \longrightarrow (-\Delta V, \partial_r V)$$

is invertible and hence has zero index. By continuity of the index the maps

$$\mathcal{L}_t : W^{2,p}(B^4, \mathcal{G}) \longrightarrow \mathcal{F}$$

$$V \longrightarrow (-\Delta V + t[A, dV], \partial_r V)$$

has also zero index and since $\mathcal{L}_1 = \partial_U \mathcal{N}^A(0, 0)$ has trivial kernel it is invertible. So we can apply the implicit function theorem and there exists $\delta > 0$ together with an open neighborhood \mathcal{O} of 0 in the subspace of $W^{2,p}(B^4, \mathcal{G})$ with average 0 on B^4 such that

$$\forall \omega \in W^{1,p}(B^4, \mathcal{G}) \quad \text{satisfying} \quad \|\omega\|_{W^{1,p}(S^4, \mathcal{G})} < \delta$$

$$\exists ! V_\omega \in \mathcal{O} \quad \text{s. t.} \quad \mathcal{N}^A(V_\omega, \omega) = 0 \quad \text{and} \quad \int_{B^4} V_\omega = 0 \quad ,$$

and \mathcal{O} can be taken smaller and smaller as δ tends to zero. We denote $g_\omega := \exp(V_\omega)$.

It remains to establish the control of the L^p norm (resp. L^2 norm) of $d(A + \omega)^{g_\omega}$ in terms of the L^p norm (resp. L^2 norm) of $F_{A+\omega}$.

The Coulomb gauge $(A + \omega)^{g_\omega}$ satisfies for $q = 2$ and $q = p$

$$\|d(A + \omega)^{g_\omega}\|_{L^q} \leq \|F_{A+\omega}\|_{L^p} + \|(A + \omega)^{g_\omega} \wedge (A + \omega)^{g_\omega}\|_{L^p} \quad .$$

We have

$$\|(A + \omega)^{g_\omega}\|_{L^4} \leq C_4 \|dA\|_{L^2} + \|\omega\|_{L^4} + \|dg^\omega\|_{L^4} \quad .$$

Using the fact that A is the Coulomb gauge whose $W^{1,2}$ norm is controlled by the L^2 norm of F_A - which is assumed itself to be less than ε - by taking ε small enough - independently of A - by taking δ small enough - depending possibly on A - which ensures in particular that $\|dg_\omega\|_{L^4}$ is sufficiently small, and by using the embedding of closed forms with L^q exterior co-derivative and whose restriction to ∂B^4 is zero into $L^{4q/4-q}$ since $d^*(A + \omega)^{g_\omega} = 0$ and $\iota_{\partial B^4}^*(A + \omega)^g = 0$ we have established that

$$\|d(A + \omega)^{g_\omega}\|_{L^q} \leq \|F_{A+\omega}\|_{L^q} + 2^{-1} \|d(A + \omega)^{g_\omega}\|_{L^q} \quad .$$

This implies that $A+\omega$ fulfills the conditions for being in $\mathcal{V}_C^\varepsilon$ for and ω satisfying $\|\omega\|_{W^{1,p}} < \delta$ where δ has also been taken small enough in such a way that $\|F_{A+\omega}\| < \varepsilon$. This concludes the proof of the openness of $\mathcal{V}_C^\varepsilon$ with respect to the $W^{1,p}$ -topology for well chosen constants $\varepsilon > 0$ and $C > 0$ and this concludes the proof of the **claim 1**.

End of the proof of theorem IV.1.

With the claim 1 at hand now, we are going to conclude the proof of theorem IV.1

Let $A \in W^{1,2}(\wedge^1 B^4, \mathcal{G})$ such that $\int_{B^4} |F_A|^2 d\text{vol}_{B^4} < \varepsilon$. Since C^∞ is dense in $W^{1,2}$ there exists A^t a family of smooth 1-form on B^4 into \mathcal{G} converging strongly to A in $W^{1,2}$

as t goes to zero. Using again the embedding of $W^{1,2}$ into L^4 we have the existence of $t_0 > 0$ such that

$$\forall t < t_0 \quad \int_{B^4} |F_{A^t}|^2 \, d\text{vol}_{B^4} < \varepsilon \quad .$$

Thus A^t is in \mathcal{U}^ε and, due to the claim 1, it is also in $\mathcal{V}_C^\varepsilon$. Let g^t such that $d^*(A^t)^{g^t} = 0$ with

$$\int_{B^4} \left| d \left((A^t)^{g^t} \right) \right|^2 \, dx^4 \leq C \int_{B^4} |F_{A^t}|^2 \, dx^4 \quad .$$

Again, since there is no non-trivial closed and co-closed 3-form on B^4 the previous identity implies that $(A^t)^{g^t}$ is bounded in $W^{1,2}$ and then in L^4 too. The approximating connection one forms A^t are converging strongly to A in $W^{1,2}$ and hence in L^4 thus $d(g^t)$ is bounded in L^4 . We then deduce the existence of a sequence $t_k \rightarrow 0$ such that g^{t_k} converges weakly in $W^{1,4}(B^4, G)$ to some limit g^0 . Using Rellich Kondrachov compactness theorem g^{t_k} converges strongly to g^0 in $L^p(B^4)$ for any $p < +\infty$ and hence g^0 is also taking values in G since we have almost everywhere convergence of the sequence. Using the previous convergences we deduce first that

$$(g^{t_k})^{-1} dg^{t_k} + (g^{t_k})^{-1} A g^{t_k} \rightharpoonup (g^0)^{-1} dg^0 + (g^0)^{-1} A g^0 \quad \text{in } \mathcal{D}'(B^4) \quad .$$

This implies that $d^*((A)^{g^0}) = 0$. Since both A^t and $(A^t)^{g^t}$ are bounded in $W^{1,2}$ and since g^t is bounded in L^4 , using

$$dg^t = g^t (A^t)^{g^t} - A^t g^t \quad ,$$

we deduce that g^t is bounded in $W^{2,2}$ and hence the trace of g^{t_k} weakly converges to the trace of g^0 in $H^{3/2}(\partial B^4, G)$. So we can pass to the limit in the equation $\iota_{\partial B^4}^* (A^{t_k})^{g^{t_k}} = 0$ and we obtain

$$\iota_{\partial B^4}^* (A^0)^{g^0} = 0 \quad .$$

Finally since F_{A^t} is strongly converging to F_A in L^2 , using also the lower semi-continuity of the L^2 norm together with the weak convergence of $(A^{t_k})^{g^{t_k}}$ towards A^{g^0} we have

$$\int_{B^4} |d(A^{g^0})| \, dx^4 \leq C \int_{B^4} |F_A|^2 \, dx^4 \quad .$$

This concludes the proof of theorem IV.1.

IV.2 A refinement of Uhlenbeck's Coulomb Gauge extraction theorem.

We have seen that Uhlenbeck's result is optimal in the sense that without assuming anything about the smallness of the Yang-Mills energy there is no hope to obtain a gauge of $W^{1,2}$ controlled energy. One might wonder however if the smallness of the L^2 norm of

the curvature is the ultimate criterium for ensuring the existence of controlled Coulomb Gauges. The answer is "no" and one can very slightly reduce this requirement. Recall the notion of weak L^2 quasi-norm. We say that a measurable function f on B^m is in the *weak L^2 space* if

$$|f|_{2,\infty} := \left[\sup_{\alpha>0} \alpha^2 |\{x \in B^m ; |f(x)| > \alpha\}| \right]^{1/2} < +\infty \quad ,$$

where $|\cdot|$ denotes the Lebesgue measure on B^m . This quantity defines a quasi-norm which is equivalent to a norm (see for instance [17]) that we denote $\|\cdot\|_{2,\infty}$. The *weak L^2 space* equipped with $\|\cdot\|_{2,\infty}$ is complete and define then a *Banach space* denoted $L^{2,\infty}$ called also *Marcinkiewicz weak L^2 space* or also *Lorentz weak L^2 space*. It is larger than L^2 . Indeed, for any function $f \in L^2$ we have

$$\|f\|_{2,\infty} \leq \sup_{\alpha>0} \int_{x : |f|(x)>\alpha} |f|^2(x) dx \leq \int |f|^2(x) dx = \|f\|_2^2 \quad .$$

It is strictly larger than L^2 : the function $f(x) := |x|^{-m/2}$ is in $L^{2,\infty}(B^m)$ but not in $L^2(B^m)$. It is also not difficult to see that $L^{2,\infty}(B^m) \hookrightarrow L^p(B^m)$ for any $1 \leq p < 2$. More generally we define the $L^{q,\infty}$ space of measurable functions f satisfying

$$|f|_{q,\infty} := \left[\sup_{\alpha>0} \alpha^q |\{x \in B^m ; |f(x)| > \alpha\}| \right]^{1/q} < +\infty \quad .$$

This defines again a quasi-norm equivalent to a norm³ if $q > 1$. So it is a space which "sits" between $L^q(B^m)$ and all the $L^p(B^m)$ spaces for any $p < q$. This is a space which has the same scaling properties as L^q but has however the big advantage of containing the Riesz functions $|x|^{-m/q}$ which play a central role in the theory of elliptic PDE. As we will see later the space $L^{q,\infty}(B^m)$ has also the advantage of being the dual of a Banach space, the *Lorentz space* $L^{q',1}(B^m)$ of measurable functions f satisfying

$$\int_0^{+\infty} |\{x ; |f|(x) > \alpha\}|^{1/q'} d\alpha < +\infty \quad , \tag{IV.9}$$

where $1/q' = 1 - 1/q$ (see [17]). This later space has very interesting "geometric" properties that will be useful for the analysis of Yang-Mills Lagrangian as we will see below.

We have the following theorem

³This is not true for $q = 1$, the space L^1 -weak cannot be made equivalent to a normed space - unfortunately, otherwise the analysis could make the economy of Calderon Zygmund theory and a major part of harmonic analysis that would suddenly become trivial....!

Theorem IV.4. *Let $m \leq 4$ and G be a compact Lie group. There exists $\varepsilon_G > 0$ and $C_G > 0$ such that for any $A \in W^{1,2}(B^m, \mathcal{G})$ satisfying*

$$\sup_{\alpha > 0} \alpha^2 |\{x \in B^m ; |F_A(x)| > \alpha\}| < \varepsilon_G \quad , \quad (\text{IV.10})$$

there exists $g \in W^{2,2}(B^m, G)$ such that

$$\left\{ \begin{array}{l} \int_{B^m} |A^g|^2 + \sum_{i,j=1}^4 |\partial_{x_i} A_j^g|^2 dx^m \leq C_G \int_{B^m} |dA + A \wedge A|^2 dx^m \\ d^* A^g = 0 \quad \text{in } B^m \\ \iota_{\partial B^m}^* (*A^g) = 0 \quad , \end{array} \right. \quad (\text{IV.11})$$

where $A^g = g^{-1}dg + g^{-1}Ag$ and $\iota_{\partial B^m}$ is the canonical inclusion map of the boundary of the unit ball into \mathbb{R}^m . Moreover we have also

$$\sum_{i,j=1}^4 \|\partial_{x_i} A_j^g\|_{2,\infty}^2 \leq C_G \|F_A\|_{2,\infty}^2 \quad .$$

□

The weakening of the smallness criterium by replacing *small* L^2 by the less restrictive *small* $L^{2,\infty}$ condition for the existence of a controlled Coulomb gauge has been first observed in [5]. This was a very precious observation for the control of the loss of energies in so called *neck annular regions* in the study of conformally invariant problems such as *Willmore surfaces* or also *Yang-Mills Fields* as we will see below. The estimate (IV.10) comes naturally from the ϵ -regularity property which holds in neck regions.

Proof of theorem IV.4.

It follows exactly the same scheme as the proof of theorem IV.4 but we will need to use interpolation spaces between $L^{q,\infty}$ and $L^{q,1}$, the *Lorentz spaces* $L^{q,s}$ and some of their properties.

Let $2 < p < 4$ and

$$\hat{\mathcal{U}}^\varepsilon := \left\{ A \in W^{1,p}(B^4, \mathcal{G}) \quad \text{s.t.} \quad |F_A|_{L^{2,\infty}(B^4)}^2 < \varepsilon \right\} \quad ,$$

and for any $\varepsilon > 0$ and $C > 0$ and we consider

$$\hat{\mathcal{V}}_C^\varepsilon := \left\{ \begin{array}{l} A \in \hat{\mathcal{U}}^\varepsilon \quad \text{s. t.} \quad \exists g \in W^{2,p}(B^4, G) \\ \int_{B^4} |dA^g|^p dx^4 \leq C \int_{B^4} |F_A|^p dx^4 \\ \int_{B^4} |dA^g|^2 dx^4 \leq C \int_{B^4} |F_A|^2 dx^4 \\ \|dA^g\|_{L^{2,\infty}(B^4)}^2 \leq C \|F_A\|_{L^{2,\infty}(B^4)}^2 \\ d^* A^g = 0 \quad \text{and} \quad \iota_{\partial B^4} * A^g = 0 \quad . \end{array} \right.$$

The first goal is to show the following

Claim

$$\exists \varepsilon > 0 \quad C > 0 \quad \text{s.t.} \quad \hat{\mathcal{V}}_C^\varepsilon = \hat{\mathcal{U}}^\varepsilon \quad .$$

The proof of the claim is again divided in 3 steps.

Proof of the path connectedness of $\hat{\mathcal{U}}^\varepsilon$. For A in \mathcal{U}^ε and $t \in [0, 1]$ we define the connection form A^t to be the image of A by the dilation of rate t^{-1} : $A^t = t \sum_{j=1}^4 A_j(t x) dx_j$. We have in particular

$$F_{A^t} = t^2 \sum_{i,j=1}^4 (F_A)_{ij}(x) dx_i dx_j \quad ,$$

hence $|F_{A^t}|(x) = t^2 |F_A|(t x)$ and

$$|F_{A^t}|_{L^{2,\infty}(B^4)} = |F_A|_{L^{2,\infty}(B_t^4)} \leq |F_A|_{L^{2,\infty}(B^4)} < \varepsilon$$

and ⁴ this path connects A to 0 in the $W^{1,p}$ topology due to (IV.2). Hence this concludes the proof of the path connectedness of $\hat{\mathcal{U}}^\varepsilon$.

The *proof of the closeness of $\hat{\mathcal{V}}_C^\varepsilon$ in $\hat{\mathcal{U}}^\varepsilon$* is identical to the proof of the closeness of $\mathcal{V}_C^\varepsilon$ in \mathcal{U}^ε .

Proof of the openness of $\hat{\mathcal{V}}_C^\varepsilon$ in $\hat{\mathcal{U}}^\varepsilon$.

We consider the map \mathcal{N}^A defined by (IV.8). We recall⁵ the definition of the space $L^{q,s}(B^m)$ where $1 < q < \infty$ and $1 \leq s < +\infty$. A measurable function f on B^m belongs to $L^{q,s}(B^m)$ if

$$|f|_{q,s} := \left[\int_0^\infty t^{\frac{s}{q}} f^*(t) \frac{dt}{t} \right]^{1/s} < +\infty \quad , \quad (\text{IV.12})$$

⁴This last inequality illustrates what we meant at the beginning of this subsection by $L^{2,\infty}$ has the same scaling properties as L^2 .

⁵For a more thorough presentation of the Lorentz spaces and it's interaction with Calderon Zygmund theory in particular the reader is invited to consult the first chapter of [17] as well as [53] or [55].

where $f^*(t)$ is the decreasing rearrangement function associated to f , defined on \mathbb{R}_+ , and satisfying

$$\forall \alpha > 0 \quad |\{t > 0 ; f^*(t) > \alpha\}| = |\{x \in B^m ; |f|(x) > \alpha\}| \quad .$$

This defines again a quasi-norm equivalent to a norm for which the space is complete (see [17]). One verifies that the space $L^{q,1}(B^m)$ defined by (IV.9) coincides with the space given by (IV.12) for $s = 1$. One verifies also that $L^{q,q}(B^m) = L^q(B^m)$ and that for any $q \in (1, +\infty)$ and any $1 \leq s < \sigma \leq +\infty$ we have $L^{q,s}(B^m) \hookrightarrow L^{q,\sigma}(B^m)$. We have also that $\forall q < r$ and $\forall t, s \in [1, \infty]$ the following continuous embedding holds $L^{p,s}(B^m) \hookrightarrow L^{q,t}(B^m)$. The following multiplication rules holds and are continuous bilinear mappings in the corresponding spaces with the corresponding estimates

$$L^{p,s} \cdot L^{q,t} \hookrightarrow L^{r,\sigma} \quad , \quad (\text{IV.13})$$

where $r^{-1} = p^{-1} + q^{-1}$ and $s^{-1} + t^{-1} = \sigma^{-1}$ and where $1 < p, q < +\infty$ such that $r \geq 1$ and $1 \leq s, t \leq \infty$ such that $1 \leq \sigma \leq +\infty$. In particular we have for any $2 \leq p < 4$

$$L^{4,\infty} \cdot L^{\frac{4p}{4-p},p} \hookrightarrow L^p \quad . \quad (\text{IV.14})$$

Before to move on with the proof of theorem IV.4 we shall need a last tool from function theory : the *improved Sobolev embeddings* (see [55]). For $1 \leq p < m$ the following embedding is continuous

$$W^{1,p}(B^m) \hookrightarrow L^{\frac{mp}{m-p},p}(B^m) \quad , \quad (\text{IV.15})$$

and more generally for any $t \in [1, +\infty]$

$$W^{1,(p,t)}(B^m) \hookrightarrow L^{\frac{mp}{m-p},t}(B^m) \quad , \quad (\text{IV.16})$$

where $W^{1,(p,t)}(B^m)$ denotes the space of measurable functions on B^m with distributional derivative in the Lorentz space $L^{p,t}(B^m)$.

Proof of openness of $\hat{\mathcal{V}}_C^\varepsilon$ continued. Using Calderon Zygmund $L^{q,t}$ theory we have the following bound

$$\begin{aligned} \|V\|_{W^{2,p}(B^4)} &\leq c \left[\|\Delta V\|_{L^p(B^4)} + \|\partial_r V\|_{W^{1-1/p,p}(\partial B^4)} \right] \\ &\leq c \left[\|\partial_U \mathcal{N}^A(0,0) \cdot V\|_{\mathcal{F}} + \|[A, dV]\|_{L^p(B^4)} \right] \\ &\leq c \left[\|\partial_U \mathcal{N}^A(0,0) \cdot V\|_{\mathcal{F}} + c \|A\|_{L^{4,\infty}(B^4)} \|dV\|_{L^{4p/4-p,p}(B^4)} \right] \quad , \end{aligned}$$

where $\mathcal{F} := W^{1,p}(B^4, \mathcal{G}) \times W^{1-1/p,p}(\partial B^4, \mathcal{G})$. From the fact that $A \in \mathcal{V}_C^\varepsilon$ we deduce that $\|A\|_{L^{4,\infty}} \leq C_4 \sqrt{C} \varepsilon$ where C_4 is the Sobolev constant coming from the embedding into

$L^4(B^4)$ of closed 3 forms on B^4 with adjoint exterior derivative in L^2 and whose restriction to ∂B^4 is zero. Hence for any V with average zero on B^4 we have

$$\|V\|_{W^{2,p}(B^4)} \leq c [\|\partial_U \mathcal{N}^A(0,0) \cdot V\|_{\mathcal{F}} + c C_4 \sqrt{C} \varepsilon \|dV\|_{L^{4p/4-p,p}}] .$$

Using again the embedding (IV.15) and denoting C_p the corresponding constant, we have then

$$[1 - c C_4 \sqrt{C} \varepsilon C_p] \|V\|_{W^{2,p}(B^4)} \leq c \|\partial_U \mathcal{N}^A(0,0) \cdot V\|_{\mathcal{F}} .$$

Having chosen ε such that $c C_4 \sqrt{C} \varepsilon C_p < 1/2$ we have that $\partial_U \mathcal{N}^A(0,0)$, which is again a Fredholm operator of index zero, has a trivial kernel and is hence invertible.

The rest of the proof is completed by easily transposing to our present setting each argument of the case of small L^2 Yang-Mills energy which was detailed in the previous subsection.

IV.3 Controlled gauges without small energy assumption.

One might wonder why a $W^{1,2}$ control is wished and why one could not give up a bit our requirements and look for some control of a "weaker norm". This is indeed possible, together with Mircea Petrache [33], the author proved the existence of global gauges A^g whose $L^{4,\infty}$ norm is controlled by the Yang-Mills energy which is not necessarily small. Precisely we have.

Theorem IV.5. *Let (M^4, g) be a riemannian 4-manifold. There exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties.*

Let ∇ be a $W^{1,2}$ connection over an $SU(2)$ -bundle over M . Then there exists a global $W^{1,(4,\infty)}$ section of the bundle (possibly allowing singularities) over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the following bound.

$$\|A\|_{L^{(4,\infty)}(M^4)} \leq f (\|F_\nabla\|_{L^2(M^4)}) \quad , \quad (\text{IV.17})$$

where F_∇ is the curvature form of ∇ . □

The following question is still unsolved.

Open Problem : *Under the same assumptions as the ones of theorem IV.5, find $A \in L^{4,\infty}(M^4)$ such that (IV.17) holds and*

$$d^* A \equiv 0 \quad .$$

□

Observe that $F_A \in L^2(M^4)$, $A \in L^{4,\infty}(M^4)$ together with $d^* A = 0$ imply $A \in W^{1,(2\infty)}(M^4)$.

Having the existence a global representative with controlled $W^{1,(2,\infty)}$ -norm and satisfying the Coulomb condition could be useful for studying the Yang-Mills flow in 4 dimension. A partial positive answer to the above open problem is given by Yu Wang in [60].

The proof of theorem IV.5 is deduced from the following existence result which is a global counterpart of theorem IV.2.

Theorem IV.6. [33] *Let G be a compact lie group and u be a map in $W^{1,n}(S^n, G)$ then there exists an extension \tilde{u} of u in $W^{1,(n+1,\infty)}(B^{n+1}, G)$ and satisfying*

$$\sup_{\alpha>0} \alpha^{n+1} |\{x \in B^{n+1} \ ; \ |\nabla\tilde{u}| > \alpha\}| \leq \gamma_n \left(\int_{S^n} |\nabla u|^n \, d\text{vol}_{S^n} \right) \ ,$$

where γ_n is a universal function. □

Here again, there are counterexample to the existence of an extension in the “slightly” smaller space $W^{1,n+1}(B^{n+1}, G)$. In fact it is proved in [36] that the previous result does not require a Lie group structure in the target to be true : it extends to general closed target riemannian manifold in general.

V The resolution of the Yang-Mills Plateau problem in the critical dimension.

V.1 The small energy case.

We first present the resolution of the *Yang-Mills Plateau* problem in the case where the given connection at the boundary has a small trace norm. Precisely we shall prove the following result.

Theorem V.1. *Let G be a compact Lie group and $m \leq 4$. There exists $\delta_G > 0$ such that for any 1-form $\eta \in H^{1/2}(\wedge^1 \partial B^m, \mathcal{G})$ satisfying*

$$\|\eta\|_{H^{1/2}(\partial B^m)} < \delta_G \ , \tag{V.1}$$

then the following minimization problem is achieved by a 1-form $A^0 \in W^{1,2}(\wedge^1 B^m, \mathcal{G})$

$$\inf \left\{ YM(A) = \int_{B^m} |dA + A \wedge A|^2 \, dx^m \ ; \ \iota_{\partial B^m}^* A = \eta \right\} \ .$$

□

The previous theorem is a corollary of the following weak closure theorem

Theorem V.2. *Let G be a compact Lie group and $m \leq 4$. There exists $\delta_G > 0$ such that for any 1-form $\eta \in H^{1/2}(\wedge^1 \partial B^m, \mathcal{G})$ satisfying*

$$\|\eta\|_{H^{1/2}(\partial B^m)} < \delta_G \quad , \quad (\text{V.2})$$

then for any $A^k \in W^{1,2}(\wedge^1 B^m, \mathcal{G})$ satisfying

$$\limsup_{k \rightarrow +\infty} YM(A^k) = \int_{B^m} |dA^k + A^k \wedge A^k|^2 dx^m < +\infty \quad \text{and} \quad \iota_{\partial B^m}^* A^k = \eta \quad ,$$

there exists a subsequence $A^{k'}$ and a Sobolev connection $A^\infty \in W^{1,2}(\wedge^1 B^m, \mathcal{G})$ such that

$$D(A^{k'}, A^\infty) := \inf_{g \in W^{2,2}(B^4, \mathcal{G})} \int_{B^m} |A^{k'} - (A^\infty)^g|^2 dx^m \longrightarrow 0 \quad ,$$

moreover

$$YM(A^\infty) \leq \liminf_{k' \rightarrow 0} YM(A^{k'}) \quad \text{and} \quad \iota_{\partial B^m}^* A^\infty = \eta \quad .$$

□

Proof of theorem V.2. We present the proof in the critical case $m = 4$. The case $m < 4$ being almost like the abelian linear case treated . Let B be the minimizer of E in $W_\eta^{1,2}(\wedge^1 B^4, \mathcal{G})$ and using (III.7) and the Sobolev embedding $W^{1,2}(B^4)$ into $L^4(B^4)$ we have

$$\left[\int_{B^4} |B|^4 dx^4 \right]^{\frac{1}{2}} + \sum_{i,j=1}^m \int_{B^4} |\partial_{x_i} B_j|^2 dx^4 \leq C [E(B) + \|\eta\|_{H^{1/2}}^2] \quad . \quad (\text{V.3})$$

The one form B is the harmonic extension of η and classical elliptic estimate gives

$$E(B) \leq C \|\eta\|_{H^{1/2}(\partial B^4)}^2 \quad . \quad (\text{V.4})$$

Combining (V.3) and (V.4) we obtain the existence of a constant C independent of η such that

$$\int_{B^4} |F_B|^2 dx^4 \leq C \left[\|\eta\|_{H^{1/2}(\partial B^4)}^2 + \|\eta\|_{H^{1/2}(\partial B^4)}^4 \right] \quad .$$

We choose first $\delta_G > 0$ such that $C [\delta_G^2 + \delta_G^4] < \varepsilon_G$ in such a way that we can apply theorem IV.1 and we have the existence of a minimizing sequence A^k of YM in $W_\eta^{1,2}(\wedge^1 B^4, \mathcal{G})$ with a Coulomb gauge $(A^k)^{g^k}$ controlled in $W^{1,2}$:

$$\|(A^k)^{g^k}\|_{W^{1,2}(B^4)} \leq C \|F_{A^k}\|_{L^2(B^4)} \leq C [\delta_G + \delta_G^2] \quad . \quad (\text{V.5})$$

Without loss of generality we can assume that

$$(A^k)^{g^k} \rightharpoonup \hat{A}^\infty \quad \text{weakly in } W^{1,2}(\wedge^1 B^4, \mathcal{G}) \quad ,$$

for some 1-form \hat{A}^∞ which satisfies the Coulomb condition $d^*\hat{A}^\infty = 0$ and for which

$$\int_{B^4} |F_{\hat{A}^\infty}|^2 dx^4 \leq \liminf_{k \rightarrow +\infty} \int_{B^4} |F_{(A^k)^{g^k}}|^2 dx^4 = \liminf_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 \quad . \quad (\text{V.6})$$

We claim that the restriction of \hat{A}^∞ to ∂B^4 is gauge equivalent to η . Because of the weak convergence of $(A^k)^{g^k}$ to \hat{A}^∞ weakly in $W^{1,2}$, by continuity of the trace operation from $W^{1,2}$ into $H^{1/2}$ we have

$$\iota_{\partial B^4}^*(A^k)^{g^k} = (g^k)^{-1} \iota_{\partial B^4}^* dg^k + (g^k)^{-1} \eta g^k \rightharpoonup \iota_{\partial B^4}^* \hat{A}^\infty \quad (\text{V.7})$$

weakly in $H^{1/2}(\wedge^1 \partial B^4, \mathcal{G})$. Using the continuous embedding

$$H^{1/2}(\partial B^4) \hookrightarrow L^3(\partial B^4) \quad ,$$

we have that the restriction of g^k to ∂B^4 converges weakly to some limit g^∞ in $W^{1,3}(\partial B^4)$ and we have, using (V.5),

$$\begin{aligned} \|dg^\infty\|_{L^3(\partial B^4)} &\leq \liminf_{k \rightarrow +\infty} \|dg^k\|_{L^3(\partial B^4)} \\ &\leq C \left[\|\eta\|_{H^{1/2}(\partial B^4)} + \liminf_{k \rightarrow +\infty} \|(A^k)^{g^k}\|_{H^{1/2}(\partial B^4)} \right] \\ &\leq C [\delta_G + \delta_G^2] \quad . \end{aligned} \quad (\text{V.8})$$

Using now Rellich Kondrachov theorem (see for instance [7]), this convergence is strong in L^q for any $q < +\infty$ which implies that g^∞ takes values almost everywhere in G and $g^\infty \in W^{1,3}(\partial B^4, G)$. We have moreover

$$(g^\infty)^{-1} dg^\infty + (g^\infty)^{-1} \eta g^\infty = \iota_{\partial B^4}^* \hat{A}^\infty \quad .$$

Using the continuous embedding

$$L^\infty \cap W^{1,3}(\partial B^4) \cdot H^{1/2}(\partial B^4) \hookrightarrow H^{1/2}(\partial B^4)$$

(the proof of this continuous embedding is also similar to the one of Lemma B1 in [33]), we have that

$$\begin{aligned} &\|dg^\infty\|_{H^{1/2}(\partial B^4)} \\ &\leq C \left[\|g^\infty\|_\infty + \|g^\infty\|_{W^{1,3}(\partial B^4)} \right] \left[\|\eta\|_{H^{1/2}(\partial B^4)} + \|\iota_{\partial B^4}^* \hat{A}^\infty\|_{H^{1/2}(\partial B^4)} \right] \\ &\leq C [\delta_G + \delta_G^2] \quad . \end{aligned} \quad (\text{V.9})$$

We shall now make use of the following theorem which, as for theorem IV.2 can be proved following Uhlenbeck's Coulomb gauge extraction method.

Theorem V.3. *Let G be a compact Lie group. There exists $\varepsilon_G > 0$ such that for any $g \in H^{3/2}(\partial B^3, G)$ satisfying*

$$\|g\|_{H^{3/2}(\partial B^4, G)} < \varepsilon_G \quad ,$$

there exists an extension $\tilde{g} \in W^{2,2}(B^4, G)$ of g satisfying

$$\|\tilde{g}\|_{W^{2,2}(B^4, G)} \leq C \|g\|_{H^{3/2}(\partial B^4, G)} \quad .$$

□

End of the proof of theorem V.2. We choose δ_G small enough such that the r.h.s. of (V.9) $C [\delta_G + \delta_G^2]$ is smaller than ε_G given by the previous theorem. Let $\tilde{g}^\infty \in W^{2,2}(B^4, G)$ be an extension of g^∞ given by theorem V.3. Then

$$A^\infty := (\hat{A}^\infty)^{(\tilde{g}^\infty)^{-1}} \in W_\eta^{1,2}(\wedge^1 B^4, \mathcal{G})$$

and we have using (V.6)

$$\int_{B^4} |F_{A^\infty}|^2 dx^4 \leq \liminf_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 \quad .$$

Since A^k is a minimizing sequence of the Yang-Mills Plateau problem in $W_\eta^{1,2}(\wedge^1 B^4, \mathcal{G})$, the connection form A^∞ is a solution to this problem and theorem V.2 is proved. □

V.2 The general case and the point removability result for $W^{1,2}$ Sobolev connections.

The theorem V.2 as it is stated does not hold without the small norm assumption (V.1) this is due to the fact that the theorem V.3 and similar results such as theorem IV.2 do not hold for general data without smallness assumption (see again remark IV.2). We shall instead prove the following result where the boundary condition is relaxed to a constrained trace modulo gauge action.

Theorem V.4. *Let G be a compact Lie group and $m \leq 4$. For any 1-form $\eta \in H^{1/2}(\wedge^1 \partial B^m, \mathcal{G})$ the following minimization problem*

$$\inf \left\{ \int_{B^m} |F_A|^2 dx^m ; \iota_{\partial B^m}^* A = \eta^g \quad \text{for some } g \in H^{3/2}(\partial B^4, G) \right\} \quad (\text{V.10})$$

is achieved by a 1-form $A^0 \in W^{1,2}(\wedge^1 B^m, \mathcal{G})$. □

In fact theorem V.4 is a corollary of a general closure result.

Theorem V.5. For any compact Lie group G and any dimension $m \leq 4$, the space of Sobolev connections

$$\mathfrak{A}_\eta(B^m) := \{A \in W^{1,2}(B^4, \mathcal{G}) ; \iota_{\partial B^m}^* A = \eta^g \quad \text{for some } g \in H^{3/2}(\partial B^4, G)\}$$

is weakly sequentially closed for sequences of controlled Yang-Mills energy. Precisely, for any $A^k \in \mathfrak{A}_\eta(B^m)$ satisfying

$$\limsup_{k \rightarrow +\infty} YM(A^k) = \int_{B^m} |dA^k + A^k \wedge A^k|^2 dx^m < +\infty \quad ,$$

there exists a subsequence $A^{k'}$ and a Sobolev connection $A^\infty \in \mathfrak{A}_\eta(B^m)$ such that

$$d(A^{k'}, A^\infty) := \inf_{g \in W^{1,2}(B^4, G)} \int_{B^m} |A^{k'} - (A^\infty)^g|^2 dx^m \longrightarrow 0 \quad ,$$

moreover

$$YM(A^\infty) \leq \liminf_{k' \rightarrow 0} YM(A^{k'}) \quad .$$

□

Proof of theorem V.5. Here again we restrict to the most delicate case : $m = 4$.

Let A^k be a sequence of \mathcal{G} -valued 1-forms and denote by ε_G the positive constant in Uhlenbeck's theorem IV.1⁶. A straightforward covering argument combined by some induction procedure gives the existence of a subsequence that we still denote A^k and N points $p_1 \cdots p_N$ in $\overline{B^4}$ such that

$$\forall \delta > 0 \quad \exists \rho_\delta > 0$$

$$\sup_{k \in \mathbb{N}} \sup \left\{ \int_{B_{\rho_\delta}(y) \cap B^4} |F_{A^k}|^2 dx^4 ; y \in B^4 \setminus \cup_{l=1}^N B_\delta(p_l) \right\} < \varepsilon_G \quad .$$

The case without concentration : $\{p_1 \cdots p_N\} = \emptyset$.

Let $\rho > 0$ such that

$$\sup_{k \in \mathbb{N}} \sup_{y \in B^4} \left\{ \int_{B_\rho(y)} |F_{A^k}|^2 dx^4 \right\} < \varepsilon_G \quad .$$

We fix a finite *good covering*⁷ of B^4 by balls of radius $\rho/2$. Denote $\{B_{\rho/2}(x_i)\}_{i \in I}$ this covering. On each of the rice larger ball $B_\rho(x_i)$ for any $k \in \mathbb{N}$ we take a controlled

⁶We choose in fact ε_G small enough for the controlled Gauge Uhlenbeck theorem to be valid for this constant when the domain is any intersection of B^4 with a ball $B_\rho(y)$ for $y \in \overline{B^4}$ and $0 < \rho < 1$

⁷The word "good" means that any intersections of elements of the covering is either empty or diffeomorphic to B^4 (see [6]).

Coulomb gauge $(A^k)^{g_i^k}$ such that

$$\begin{aligned} & \left[\int_{B_\rho^4(x_i)} |(A^k)^{g_i^k}|^4 dx^4 \right]^{\frac{1}{2}} + \sum_{l,j=1}^m \int_{B_\rho^4(x_i)} |\partial_{x_l}((A)^{g_i^k})_j|^2 dx^4 \\ & \leq C \int_{B_\rho(x_i)} |F_{A^k}|^2 dx^4 \quad , \end{aligned} \quad (\text{V.11})$$

and

$$d^*(A^k)^{g_i^k} = 0 \quad . \quad (\text{V.12})$$

For any pair $i \neq j$ in I such that $B_\rho(x_i) \cap B_\rho(x_j) \neq \emptyset$ we denote

$$g_{ij}^k := g_i^k (g_j^k)^{-1} \in W^{2,2}(B_\rho(x_i) \cap B_\rho(x_j), G) \quad ,$$

and we have in particular

$$(A^k)^{g_j^k} = (g_{ij}^k)^{-1} dg_{ij}^k + (g_{ij}^k)^{-1} (A^k)^{g_i^k} g_{ij}^k \quad . \quad (\text{V.13})$$

Observe that for any triplet $i \neq j, j \neq l$ and $i \neq l$ such that $B_\rho(x_i) \cap B_\rho(x_j) \cap B_\rho(x_l) \neq \emptyset$ we have the co-cycle condition

$$\forall k \in \mathbb{N} \quad g_{ij}^k g_{jl}^k = g_{il}^k \quad . \quad (\text{V.14})$$

Combining (V.11) and (V.13) together with the improved Sobolev embedding $W^{1,2}(B^4) \hookrightarrow L^{4,2}(B^4)$ where $L^{4,2}$ is the Lorentz interpolation space given by (IV.12) we obtain that for any pair $i \neq j$ such that $B_\rho(x_i) \cap B_\rho(x_j) \neq \emptyset$

$$\|dg_{ij}^k\|_{L^{4,2}(B_\rho(x_i) \cap B_\rho(x_j))}^2 \leq C \int_{B_\rho(x_i) \cup B_\rho(x_j)} |F_{A^k}|^2 dx^4 \quad . \quad (\text{V.15})$$

From (V.13) we have

$$-\Delta g_{ij}^k = (A^k)^{g_i^k} \cdot dg_{ij}^k - dg_{ij}^k \cdot (A^k)^{g_j^k} \quad . \quad (\text{V.16})$$

Using again the improved Sobolev embedding $W^{1,2}(B^4) \hookrightarrow L^{4,2}(B^4)$, inequalities (V.11) and (V.15) together with the continuous embedding

$$L^{4,2} \cdot L^{4,2} \hookrightarrow L^{2,1} \quad ,$$

we obtain

$$\|\Delta g_{ij}^k\|_{L^{2,1}(B_\rho(x_i) \cap B_\rho(x_j))} \leq C \int_{B_\rho(x_i) \cup B_\rho(x_j)} |F_{A^k}|^2 dx^4 \quad . \quad (\text{V.17})$$

Using Calderon Zygmund theory in Lorentz interpolation spaces (see [53]) we obtain that $g_{ij}^k \in W_{loc}^{2,(2,1)}(B_\rho(x_i) \cap B_\rho(x_j))$ where $W^{2,(2,1)}$ denotes the space of functions with two derivatives in $L^{2,1}$ and using (V.15) together with (V.17) we obtain the following estimate

$$\|\nabla^2 g_{ij}^k\|_{L^{2,1}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j))} \leq C \left[\int_{B_\rho(x_i) \cup B_\rho(x_j)} |F_{A^k}|^2 dx^4 \right]^{1/2} . \quad (\text{V.18})$$

We can then extract a subsequence such that

$$\begin{cases} \forall i \in I & (A^k)g_i^k \rightharpoonup A^{i,\infty} \quad \text{weakly in } W^{1,2}(B_\rho(x_i)) \\ \forall i \neq j & g_{ij}^k \rightharpoonup g_{ij}^\infty \quad \text{weakly in } W^{2,(2,1)}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j)) \end{cases} ,$$

moreover $A^{i,\infty}$ and g_{ij}^∞ satisfy the following identities

$$\begin{cases} \forall i \neq j & A^{j,\infty} = (g_{ij}^\infty)^{-1} dg_{ij}^\infty + (g_{ij}^\infty)^{-1} A^{i,\infty} g_{ij}^\infty \\ \forall i, j, l & g_{ij}^\infty g_{jl}^\infty = g_{il}^\infty \end{cases} , \quad (\text{V.19})$$

and we have the following estimate

$$\|\nabla^2 g_{ij}^\infty\|_{L^{2,1}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j))}^2 \leq C \liminf_{k \rightarrow +\infty} \int_{B_\rho(x_i) \cup B_\rho(x_j)} |F_{A^k}|^2 dx^4 . \quad (\text{V.20})$$

It is proved⁸ in [39] that

$$W^{2,(2,1)}(B^4) \hookrightarrow C^0(B^4) ,$$

hence we deduce that $g_{ij}^\infty \in C^0 \cap W^{2,(2,1)}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j))$ and for any $i \neq j$ there exists $\overline{g}_{ij}^\infty \in G$ such that

$$\begin{aligned} & \|g_{ij}^\infty - \overline{g}_{ij}^\infty\|_{L^\infty(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j))}^2 \\ & \leq C \liminf_{k \rightarrow +\infty} \int_{B_\rho(x_i) \cup B_\rho(x_j)} |F_{A^k}|^2 dx^4 < 2 C \varepsilon_G . \end{aligned} \quad (\text{V.21})$$

⁸This embedding can be proved on \mathbb{R}^4 as follows. Let u be a *Schwartz function* in $\mathcal{S}(\mathbb{R}^4)$. Then the following representation formula holds

$$u(x) = C |x|^{-2} \star \Delta u ,$$

where $C|x|^{-2}$ is the Green Kernel of the *Laplace operator*. From this identity one deduce the following inequality

$$\|u\|_{L^\infty} \leq C \| |x|^{-2} \|_{L^{2,\infty}} \|\Delta u\|_{L^{2,1}} ,$$

and the embedding $W^{2,(2,1)}(\mathbb{R}^4) \hookrightarrow C^0(\mathbb{R}^4)$ follows from a density argument.

Taking ε_G small enough there exists a unique lifting

$$U_{ij}^\infty \in W^{2,(2,1)}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j))$$

such that

$$\forall i \neq j \quad g_{ij}^\infty = \overline{g_{ij}^\infty} \exp(U_{ij}^\infty) \quad ,$$

and

$$\|U_{ij}^\infty\|_\infty \leq C \varepsilon_G \quad ,$$

for some constant C depending only on G . Following an induction argument⁹ such as the one followed in [28] for the proof of theorem II.11, we can smooth the U_{ij}^∞ in order to produce a sequence

$$g_{ij}^\infty(t) \in C^\infty(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j), G)$$

satisfying

$$g_{ij}^\infty(t) \longrightarrow g_{ij}^\infty \quad \text{strongly in } W^{2,(2,1)}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j)) \quad \text{as } t \rightarrow 0 \quad ,$$

and

$$\forall t \quad \forall i, j, l \quad g_{ij}^\infty(t) g_{jl}^\infty(t) = g_{il}^\infty(t) \quad .$$

Since the ball B^4 is topologically trivial, the previous *cocycle condition* defines a trivial Čech smooth co-chain for the presheaf of G -valued smooth functions (see for instance [6] section 10 chapter II) and for any $i \in I$ and any $t > 0$ there exists $\rho_i(t) \in C^\infty(B_{3\rho/4}(x_i), G)$ such that

$$g_{ij}^\infty(t) = \rho_i(t) \rho_j(t)^{-1} \quad . \quad (\text{V.22})$$

We shall now make use of the following technical lemma which is proved in [59].

Lemma V.1. *Let G be a compact Lie group and $\{U_i\}_{i \in I}$ be a good covering of B^4 . There exists $\delta > 0$ such that for any pair of co-chains*

$$\forall i \neq j \quad h_{ij}, g_{ij} \in W^{2,2} \cap C^0(U_i \cap U_j, G) \quad ,$$

satisfying

$$\forall i, j, l \quad g_{ij} g_{jl} = g_{il} \quad \text{and} \quad h_{ij} h_{jl} = h_{il} \quad \text{in } U_i \cap U_j \cap U_l \quad .$$

Assume

$$\forall i \neq j \quad \|g_{ij}^{-1} h_{ij} - e\|_{L^\infty(U_i \cap U_j)} < \delta \quad ,$$

⁹A co-cycle smoothing argument by induction argument is also proposed in [19] under the weaker hypothesis that the co-cycles g_{ij}^∞ are $W^{1,4}$ in 4 dimension. This is made possible due to the fact that $C^\infty(B^4, G)$ is dense in $W^{1,4}(B^4, G)$ (see [47]). The works of Takeshi Isobe [19], [20] are proposing a framework for studying the analysis of gauge theory in conformal and super-critical dimension.

where e is the constant map equal to the neutral element of G , then, for any strictly smaller good covering of $B^4 \setminus \{V_i\}_{i \in I}$ satisfying $\bar{V}_i \subset U_i$, there exists a family of maps $\sigma_i \in W^{2,2} \cap C^0(U_i \cap U_j, G)$ such that

$$\forall i \neq j \quad h_{ij} = (\sigma_i)^{-1} g_{ij} \sigma_j \quad \text{in } V_i \cap V_j \quad .$$

□

We apply the previous lemma to $h_{ij} := g_{ij}^\infty(t)$ and $g_{ij} := g_{ij}^\infty$ for t small enough and we deduce the existence of

$$\sigma_i(t) \in W^{2,2} \cap C^0(B_{\rho/2}(x_i) \cap B_{\rho/2}(x_j)) \quad ,$$

such that

$$\forall i \neq j \quad g_{ij}^\infty(t) = \sigma_i(t)^{-1} g_{ij}^\infty \sigma_j(t) \quad \text{in } B_{\rho/2}(x_i) \cap B_{\rho/2}(x_j) \quad . \quad (\text{V.23})$$

Combining (V.22) and (V.23) we have

$$\forall i \neq j \quad g_{ij}^\infty = \sigma_i \rho_i (\sigma_j \rho_j)^{-1} \quad \text{in } B_{\rho/2}(x_i) \cap B_{\rho/2}(x_j) \quad .$$

Combining this identity with (V.19) we set

$$A^0 := (\sigma_i \rho_i)^{-1} d(\sigma_i \rho_i) + (\sigma_i \rho_i)^{-1} A^{i,\infty} (\sigma_i \rho_i) \quad \text{in } B_{\rho/2}(x_i) \quad .$$

Clearly A^0 extends to a $W^{1,2}$ \mathcal{G} -valued 1-form in B^4 , moreover, following the arguments in the proof of theorem V.2, the restriction of A^0 to ∂B^4 is gauge equivalent to η . This concludes the proof of theorem V.5 in the absence of concentration points.

The general case with possible concentration : $\{p_1 \cdots p_N\} \neq \emptyset$.

Following the arguments in the previous case, for any $\delta > 0$ we exhibit a subsequence A^k , a covering by balls $B_{\rho_\delta}(x_i)$ of $B^4 \setminus \cup_{l=1}^N B_\delta^4(p_l)$ and a family of gauge changes g_i^k such that

$$\begin{cases} \forall i \in I & (A^k)^{g_i^k} \rightharpoonup A^{i,\infty} \quad \text{weakly in } W^{1,2}(B_\rho(x_i)) \\ \forall i \neq j & g_{ij}^k \rightharpoonup g_{ij}^\infty \quad \text{weakly in } W^{2,(2,1)}(B_{3\rho/4}(x_i) \cap B_{3\rho/4}(x_j)) \end{cases} .$$

The family g_{ij}^∞ defines again a $W^{2,(2,1)}$ -co-chain that we can approximate in $C^0 \cap W^{2,2}$ by a smooth one $g_{ij}^\infty(t)$. Using the fact that the second homotopy group of the compact Lie group is trivial $\pi_2(G) = 0$ (see for instance [8] chapter V proposition 7.5) we deduce that the co-chain $g_{ij}^\infty(t)$, defined on a covering of $B^4 \setminus \cup_{l=1}^N B_\delta^4(p_l)$. is trivial for the Čech cohomology for the co-chains on the pre-sheaf of smooth G -valued functions. Following each step of the above argument we construct a $W^{1,2}$ \mathcal{G} -valued 1-form A^0 in $B^4 \setminus \cup_{l=1}^N B_\delta^4(p_l)$

which is gauge equivalent to $A^{i,\infty}$ in $B_{\rho/2}^4(x_i)$ for each $i \in I$ and whose restriction on $\partial B^4 \setminus \cup_{l=1}^N B_\delta^4(p_l)$ is also gauge equivalent to η . Moreover we have

$$\int_{B^4 \setminus \cup_{l=1}^N B_\delta^4(p_l)} |F_{A^0}|^2 dx^4 \leq \liminf_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 \quad .$$

Using a diagonal argument with $\delta \rightarrow 0$ we can extend A^0 as a \mathcal{G} -valued 1-form in $W_{loc}^{1,2}(B^4 \setminus \{p_1 \cdots p_N\})$ and still satisfying

$$\int_{B^4} |F_{A^0}|^2 dx^4 \leq \liminf_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 \quad . \quad (\text{V.24})$$

We conclude the proof of theorem V.5 by changing the gauge of A^0 in the neighborhood of each blow up point p_l making use of the following theorem V.7, known as *point removability theorem*, which gives the existence of a change of gauge g in order to extend our connection 1-form $(A^0)^g$ as a $W^{1,2}$ \mathcal{G} valued 1-form in the neighborhood of each p_l . We then paste together these $W^{1,2}$ -gauges by using the same technique as the one we used in the case without blow up points in order to get a global $W^{1,2}$ representative of A^0 on B^4 gauge equivalent to η on ∂B^4 and satisfying (V.24). This concludes the proof of theorem V.5. \square

Theorem V.6. [Point removability] *Let G be a compact Lie group and $A \in W_{loc}^{1,2}(\wedge^1 \overline{B^4} \setminus \{0\}, \mathcal{G})$ such that*

$$\int_{B^4} |dA + A \wedge A|^2 dx^4 < +\infty \quad ,$$

then there exists a gauge change $g \in W_{loc}^{2,2}(\overline{B^4} \setminus \{0\}, G)$ such that

$$A^g \in W^{1,2}(\wedge^1 B^4, \mathcal{G}) \quad .$$

\square

Remark V.1. *Point removability results play an important rôle in the analysis of conformally invariant variational problems. This is a natural consequence of due to the existence of point concentration which is inherent to the conformal invariance. These results are often formulated for the critical points of conformally invariant lagrangians and in the present case it has been first proved by K.Uhlenbeck for Yang-Mills fields (see [58]). Observe that here we are not assuming that A is satisfying a particular equation.* \square

Remark V.2. *Beyond geometric analysis, point removability results play also an important rôle in complex geometry. One could for instance quote the work of Bando [3] about the possibility to extend an hermitian holomorphic structure $F_A^{0,2} = 0$ with L^2 bounded curvature on a the punctured ball $B^4 \setminus \{0\}$ as a smooth holomorphic bundle throughout the origin. Beside the holomorphicity condition $F_A^{0,2} = 0$ no further "equation" is assumed and in particular the Einstein equation $\omega \cdot F_A^{1,1} = cI$ is not assumed and the connection form is not necessarily a Yang-Mills field.* \square

Remark V.3. *A similar point removability result can be established at the boundary. It suffices to extend carefully the gauge about the “singular point” at the boundary in order to reduce to the interior case. Details are left to the reader. \square*

Proof of theorem V.6. Without loss of generality we can assume that

$$\int_{B^4} |F_A|^2 dx^4 < \delta \quad ,$$

where $\delta > 0$ will be fixed later on in the proof. Denote for $i \geq 2$

$$T_i := B_{2^{-i+2}}^4(0) \setminus B_{2^{-i-2}}^4(0) \quad .$$

From theorem IV.1 there exists $\delta > 0$ such that, on each annulus T_i there exists a change of gauge g_i such that there exists $g_i \in W^{2,2}(T_i, G)$ such that

$$\left\{ \begin{array}{l} \int_{T_i} 2^{2i} |A^{g_i}|^2 + \sum_{k,l=1}^4 |\partial_{x_k} A_l^{g_i}|^2 dx^m \leq C_G \int_{T_i} |dA + A \wedge A|^2 dx^4 \\ d^* A^{g_i} = 0 \quad \text{in } T_i \\ \iota_{\partial T_i}^* (*A^{g_i}) = 0 \quad . \end{array} \right. \quad (\text{V.25})$$

On $T_i \cap T_{i+1} = B_{2^{-i+1}}^4 \setminus B_{2^{-i-2}}^4$ the transition function $g_{i+1} = g_i(g_{i+1})^{-1}$ satisfy

$$A^{g_{i+1}} = (g_{i+1})^{-1} dg_{i+1} + (g_{i+1})^{-1} A^{g_i} g_{i+1} \quad . \quad (\text{V.26})$$

Hence (V.25) imply

$$2^{2i} \int_{B_{2^{-i+1}}^4 \setminus B_{2^{-i-2}}^4} |dg_{i+1}|^2 dx^4 \leq C \int_{B_{2^{-i+2}}^4 \setminus B_{2^{-i-3}}^4} |dA + A \wedge A|^2 dx^4 \quad . \quad (\text{V.27})$$

Taking the adjoint of the covariant derivative of equation (V.26)

$$-\Delta g_{i+1} = A^{g_i} \cdot dg_{i+1} - dg_{i+1} \cdot A^{g_{i+1}} \quad ,$$

and, arguing as in the first part of the proof of theorem V.5, we deduce the existence of $\overline{g_{i+1}} \in G$ such that

$$\begin{aligned} & \|g_{i+1} - \overline{g_{i+1}}\|_{L^\infty(T_i \cap T_{i+1})} \\ & \leq C 2^i \|dg_{i+1}\|_{L^2(T_i \cap T_{i+1})} + C \sum_{k,l=1}^4 \|\partial_{x_k x_l}^2 g_{i+1}\|_{L^{2,1}(T_i \cap T_{i+1})} \\ & \leq C \left[\int_{T_i \cup T_{i+1}} |dA + A \wedge A|^2 dx^4 \right]^{1/2} \leq C \sqrt{\delta} \quad . \end{aligned} \quad (\text{V.28})$$

We now modify the gauge change g_i as follows. Precisely, for any $i \in \mathbb{N}$, we denote

$$\bar{\sigma}_i := \overline{g_{12}} \overline{g_{23}} \cdots \overline{g_{i-1i}} \in G \quad .$$

Observe that

$$A^{g_i \bar{\sigma}_i^{-1}} = \bar{\sigma}_i A^{g_i} \bar{\sigma}_i^{-1} \quad . \quad (\text{V.29})$$

Hence $A^{g_i \bar{\sigma}_i^{-1}}$ is still a Coulomb gauge satisfying

$$\left\{ \begin{array}{l} \int_{T_i} 2^{2i} |A^{g_i \bar{\sigma}_i^{-1}}|^2 + \sum_{k,l=1}^4 |\partial_{x_k} A_l^{g_i \bar{\sigma}_i^{-1}}|^2 dx^m \leq C_G \int_{T_i} |dA + A \wedge A|^2 dx^4 \\ d^* A^{g_i \bar{\sigma}_i^{-1}} = 0 \quad \text{in } T_i \\ \iota_{\partial B^m}^* (*A^{g_i \bar{\sigma}_i^{-1}}) = 0 \quad . \end{array} \right. \quad (\text{V.30})$$

Denote $h_i := g_i \bar{\sigma}_i^{-1}$ the transition functions on $T_i \cap T_{i+1}$ for these new gauges are given by

$$h_{i+1} := g_i \bar{\sigma}_i^{-1} \overline{\sigma_{i+1}} (g_{i+1})^{-1} = g_i \overline{g_{i+1}} (g_{i+1})^{-1} \quad .$$

Using (V.28) we have

$$\|h_{i+1} - e\|_{L^\infty(T_i \cap T_{i+1})} \leq C \left[\int_{T_i \cup T_{i+1}} |dA + A \wedge A|^2 dx^4 \right]^{1/2} \leq C \sqrt{\delta} \quad . \quad (\text{V.31})$$

Exactly as for g_i , using the identity

$$A^{h_{i+1}} = (h_{i+1})^{-1} dh_{i+1} + (h_{i+1})^{-1} A^{h_i} h_{i+1} \quad , \quad (\text{V.32})$$

together with (V.29) and (V.30) we obtain

$$\begin{aligned} & 2^i \|dh_{i+1}\|_{L^2(T_i \cap T_{i+1})} + \sum_{k,l=1}^4 \|\partial_{x_k x_l}^2 h_{i+1}\|_{L^{2,1}(T_i \cap T_{i+1})} \\ & \leq C \left[\int_{T_i \cup T_{i+1}} |dA + A \wedge A|^2 dx^4 \right]^{1/2} \leq C \sqrt{\delta} \quad , \end{aligned} \quad (\text{V.33})$$

where e is the content function on $T_i \cap T_{i+1}$ equal to the neutral element of G . Having chosen δ small enough we ensure that the transition functions of this new set of trivialisation are contained in a neighborhood of the neutral element into which the exponential map defines a diffeomorphism and there exist U_{i+1} such that $h_{i+1} = \exp(U_{i+1})$ and

$$\begin{aligned} & \|U_{i+1}\|_{L^\infty(T_i \cap T_{i+1})} + 2^i \|dU_{i+1}\|_{L^2(T_i \cap T_{i+1})} + \sum_{k,l=1}^4 \|\partial_{x_k x_l}^2 U_{i+1}\|_{L^{2,1}(T_i \cap T_{i+1})} \\ & \leq C \left[\int_{T_i \cup T_{i+1}} |dA + A \wedge A|^2 dx^4 \right]^{1/2} \leq C \sqrt{\delta} \quad . \end{aligned} \quad (\text{V.34})$$

Let ρ be a smooth function on \mathbb{R}_+ identically equal to 1 between 0 and $\sqrt{2}$ and compactly supported in $[0, 2]$. On B^4 we define

$$\rho_i(x) := \rho(|x|2^i) \quad V_i := B_{2^{-i+3/2}} \setminus B_{2^{-i}} \quad \text{and} \quad \tau_i := \exp(\rho_i U_{i+1}) \quad .$$

With these notations we have

$$\text{on} \quad V_{i+1} \cap V_i = B_{2^{-i+1/2}} \setminus B_{2^{-i}} \quad \text{we have} \quad \tau_i = h_{i+1} \quad \text{and} \quad \tau_{i+1} = e \quad .$$

Hence on $V_{i+1} \cap V_i$ we have

$$\begin{aligned} A^{h_i \tau_i} &= \tau_i^{-1} d\tau_i + \tau_i^{-1} A^{h_i} \tau_i \\ &= (h_{i+1})^{-1} dh_{i+1} + (h_{i+1})^{-1} A^{h_i} h_{i+1} = A^{h_{i+1}} = A^{h_{i+1} \tau_{i+1}} \quad , \end{aligned} \tag{V.35}$$

and the 1-form \hat{A} equal to $A^{h_i \tau_i}$ on each annulus V_i defines a global $W_{loc}^{1,2}$ connection 1-form on $B^4 \setminus \{0\}$ gauge equivalent to A . Clearly, for $k = 1, 2$, we have the pointwise estimate

$$|d^k \tau_i| \leq C \sum_{l=0^k} 2^{il} |d^{k-l} U_{i+1}| \quad \text{on} \quad V_i \quad .$$

Combining this fact together with (V.30), (V.34) and (V.35) we obtain

$$\int_{V_i} 2^{2i} |A^{h_i \tau_i}|^2 + \sum_{k,l=1}^4 |\partial_{x_k} A_l^{h_i \tau_i}|^2 dx^4 \leq C_G \int_{T_i \cup T_{i+1}} |dA + A \wedge A|^2 dx^4 \quad .$$

Summing over i gives

$$\int_{B^4} |x|^{-2} |\hat{A}|^2 + \sum_{k,l=1}^4 |\partial_{x_k} \hat{A}_l|^2 dx^4 \leq C_G \int_{B^4} |dA + A \wedge A|^2 dx^4 \quad ,$$

\hat{A} is then in $W^{1,2}(\wedge^1 B^4, \mathcal{G})$ and this concludes the proof of theorem V.6. \square

VI The Yang-Mills equation in sub-critical and critical dimensions.

VI.1 Yang-Mills fields.

Until now we have produced solutions to the Yang-Mills Plateau problem in dimension less or equal to 4 but we have not addressed issues related to the special properties that should be satisfied by these solutions. Maybe one of the first question that should be looked at is whether these minima define smooth *equivariant horizontal plane distributions* or not.

In order to study the regularity of solutions to the Yang-Mills Plateau problem we have first to produce the *Euler Lagrange equation* attached to this variational problem. This is the so called *Yang-Mills equation*.

Definition VI.1. Let G be a compact Lie group and A be an L^2 connection 1-form on B^m into the Lie Algebra \mathcal{G} of G . Assume that

$$\int_{B^m} |dA + A \wedge A|^2 dx^m < +\infty \quad ,$$

we say that A is a Yang-Mills field if

$$\forall \xi \in C_0^\infty(\wedge^1 B^m, \mathcal{G})$$

$$\frac{d}{dt} \int_{B^m} |d(A + t\xi) + (A + t\xi) \wedge (A + t\xi)|^2 dx^m |_{t=0} = 0 \quad .$$

□

Observe that this definition makes sense for any $A \in L^2$ such that $F_A \in L^2$, indeed we have for any ξ in $C_0^\infty(\wedge^1 B^m, \mathcal{G})$

$$F_{A+t\xi} = F_A + t (d\xi + A \wedge \xi + \xi \wedge A) + t^2 \xi \wedge \wedge \xi \in L^2(\wedge^2 B^m, \mathcal{G}) \quad .$$

For such a $A \in L^2$ and for any ξ in $C_0^\infty(\wedge^1 B^m, \mathcal{G})$ we denote by $d_A \xi$ the following 2-form

$$d_A \xi(X, Y) := d\xi(X, Y) + [A(X), \xi(Y)] + [\xi(X), A(Y)] \quad .$$

So we have for instance

$$d_A \xi(\partial_{x_i}, \partial_{x_j}) = \partial_{x_i} \xi_j - \partial_{x_j} \xi_i + [A_i, \xi_j] + [\xi_i, A_j] \quad .$$

We have then the following proposition.

Proposition VI.1. [Yang-Mills Equation] Let $A \in L^2(\wedge^1 B^m, \mathcal{G})$ such that $F_A \in L^2(\wedge^2 B^m, \mathcal{G})$. The connection 1-form A is a Yang-Mills field if

$$\begin{aligned} \forall \xi \in C_0^\infty(\wedge^1 B^m, \mathcal{G}) \\ \int_{B^4} d_A \xi \cdot F_A = 0 \quad , \end{aligned} \tag{VI.36}$$

which is equivalent to

$$d_A^* F_A = 0 \quad \text{in } \mathcal{D}'(B^m) \quad . \tag{VI.37}$$

In coordinates this reads

$$\forall i = 1 \cdots m \quad \sum_{j=1}^m \partial_{x_j} (F_A)_{ij} + [A_j, (F_A)_{ij}] = 0 \quad . \tag{VI.38}$$

□

The Yang-Mills equation (VI.36) is also written symbolically as follows

$$d^* F_A + [A, \lrcorner F_A] = 0 \quad ,$$

where \lrcorner is referring to the contraction operation between tensors with respect to the flat metric on B^m . The proof of the last statement of the proposition goes as follows (VI.36) in coordinates is equivalent to

$$\sum_{i,j=1}^m \int_{B^m} \langle \partial_{x_i} \xi_j - \partial_{x_j} \xi_i + [A_i, \xi_j] + [\xi_i, A_j], (F_A)_{ij} \rangle dx^m = 0 \quad ,$$

using integration by parts and the fact that the Killing metric, invariant under adjoint action, satisfies $\langle U, [V, W] \rangle = \langle W, [U, V] \rangle$ we obtain

$$\sum_{i,j=1}^m \int_{B^m} \langle -\partial_{x_i} (F_A)_{ij} + [(F_A)_{ij}, A_i], \xi_j \rangle + \langle \partial_{x_j} (F_A)_{ij} + [A_j, (F_A)_{ij}], \xi_i \rangle dx^m = 0 \quad ,$$

which implies (VI.38).

The gauge invariance of the integrant of Yang-Mills lagrangian implies that (VI.38) is solved for A if and only if it is solved for any gauge transformation A^g of A . More generally we have the following

$$\begin{cases} \forall A \in W^{1,2}(\wedge^1 B^m, \mathcal{G}) & \forall g \in W^{2,2}(B^m, G) \\ d_{(A^g)}^* F_{A^g} = g^{-1} d_A^* F_A g \quad , \end{cases} \quad (\text{VI.39})$$

where we recall that $A^g := g^{-1} dg + g^{-1} A g$.

The Yang-Mills equation (VI.37) has to be compared with the *Bianchi identity* to which it is a kind of "dual equation". This is a *structure equation* which holds for any connection 1-form.

Proposition VI.2. [Bianchi identity] *For any $A \in L^2(\wedge^1 B^m, \mathcal{G})$ such that $F_A \in L^2(\wedge^2 B^m, \mathcal{G})$ the following identity holds*

$$d_A F_A = 0 \quad ,$$

where $d_A F_A$ is the 3-form given by

$$\begin{aligned} d_A F_A(X, Y, Z) &:= dF_A(X, Y, Z) \\ &+ [A(X), F_A(Y, Z)] + [A(Y), F_A(Z, X)] + [A(Z), F_A(X, Y)] \quad . \end{aligned}$$

□

The proof of *Bianchi identity* goes as follows. We have

$$\begin{aligned}
dF_A(X, Y, Z) &= d(A \wedge A)(X, Y, Z) \\
&= [dA(X, Y), A(Z)] + [dA(Y, Z), A(X)] + [dA(Z, X), A(Y)] \\
&= [F_A(X, Y), A(Z)] + [F_A(Y, Z), A(X)] + [F_A(Z, X), A(Y)] \quad ,
\end{aligned}$$

where we have used the *Jacobi identity*

$$[[A(X), A(Y)], A(Z)] + [[A(Y), A(Z)], A(X)] + [[A(Z), A(X)], A(Y)] = 0 \quad .$$

This concludes the proof of *Bianchi identity*.

In the particular case where G is **abelian**, Yang-Mills equation together with Bianchi identity is equivalent to the harmonic map form system

$$\begin{cases} d^*F_A = 0 & \text{Yang-Mills} \\ dF_A = 0 & \text{Bianchi} \end{cases} \quad ,$$

whose solutions are known to be analytic in every dimension. We are now asking about the same regularity issue in the non abelian case.

VI.2 The regularity of $W^{1,2}$ Yang-Mills Fields in sub-critical and critical dimensions.

Due to the huge gauge invariance, the “intrinsic” Yang-Mills equation (VI.38) as such cannot generate any kind of improved regularity for an arbitrary solution¹⁰. Nevertheless the “breaking this invariance” by taking the Coulomb gauge will make Yang-Mills PDE elliptic and will generate regularity under the ad-hoc assumptions. Precisely we have the following result.

Theorem VI.7. *Let G be a compact Lie group and $m \geq 3$. Let $A \in W^{1,m/2}(\wedge^1 B^m, \mathcal{G})$ be a solution of the Yang-Mills equation (VI.37) satisfying the Coulomb condition $d^*A = 0$ then A is C^∞ in B^m . \square*

Proof of theorem VI.7.

We assume that A is Coulomb and satisfy the Yang-Mills equation (VI.37). Hence we have

$$d^*dA + d^*(A \wedge A) + [A, \mathcal{L}dA] + [A, \mathcal{L}A \wedge A] = 0 \quad .$$

¹⁰Assuming one solution A would be smooth and taking any arbitrary other “non smooth” gauge g the new expression of the connection A^g is not smooth though it still solves Yang-Mills

Using the fact that $d^*A = 0$ the Yang-Mills equation in this Coulomb gauge reads then

$$\Delta A = d^*(A \wedge A) + [A, \mathcal{L}dA] + [A, \mathcal{L}(A \wedge A)] \quad , \quad (\text{VI.40})$$

and theorem VI.7 is now the direct consequence of the following result. \square

Theorem VI.8. *Let $m > 2$ and $N \in \mathbb{N}^*$. Let $f \in C^\infty(\mathbb{R}^N \times (\mathbb{R}^m \otimes \mathbb{R}^N), \mathbb{R}^N)$ and let $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that there exists $C > 0$ satisfying*

$$|f(\xi, \Xi)| \leq C |\xi| |\Xi| \quad \text{and} \quad |g(\xi)| \leq C |\xi|^3 \quad . \quad (\text{VI.41})$$

Let $u \in W^{1,m/2}(B^m, \mathbb{R}^N)$ satisfying

$$\Delta u = f(u, \nabla u) + g(u) \quad , \quad (\text{VI.42})$$

then u is C^∞ . \square

Proof of theorem VI.8.

We start with the following embedding

$$W^{1,m/2}(B^m) \hookrightarrow L^m(B^m) \quad .$$

We claim that there exists $\alpha > 0$ such that

$$\sup_{x_0 \in B_{1/2}^m(0) ; 0 < \rho < 1/4} \rho^{-4\alpha} \int_{B_\rho(x_0)} |u|^m(x) dx^m < +\infty \quad . \quad (\text{VI.43})$$

Let $\epsilon > 0$ to be fixed later. There exists $\rho_0 > 0$ such that

$$\sup_{x_0 \in B_{1/2}^m(0) ; 0 < \rho < \rho_0} \int_{B_\rho(x_0)} [|u|^m(x) + |\nabla u|^{m/2}(x)] dx^m < \epsilon \quad . \quad (\text{VI.44})$$

Let now $x_0 \in B_{1/2}^m(0)$ and $\rho < \rho_0$ arbitrary. On $B_\rho(x_0)$ we consider φ to be the solution of

$$\begin{cases} \Delta \varphi = f(u, \nabla u) + g(u) & \text{in } B_\rho^m(x_0) \\ \varphi = 0 & \text{on } \partial B_\rho^m(x_0) \end{cases} \quad . \quad (\text{VI.45})$$

Classical elliptic estimates (see [16]) give the existence of a constant independent of ρ such that

$$\begin{aligned} \|\varphi\|_{L^m(B_\rho^m(x_0))} &\leq C \|f(u, \nabla u) + g(u)\|_{L^{m/3}(B_\rho^m(x_0))} \\ &\leq C \|u\|_{L^m(B_\rho^m(x_0))} \|\nabla u\|_{L^{m/2}(B_\rho^m(x_0))} + C \|u\|_{L^m(B_\rho^m(x_0))}^3 \quad . \end{aligned} \quad (\text{VI.46})$$

The difference $v := u - \varphi$ is harmonic on $B_\rho^m(x_0)$. Hence $|v|^m$ is subharmonic

$$\Delta|v|^m = m|v|^{m-2}|\nabla v|^2 + \frac{m}{2} \left(\frac{m}{2} - 1 \right) |v|^{m-4} |\nabla|v|^2|^2 \geq 0 \quad .$$

This gives that

$$\forall r < \rho \quad \int_{\partial B_r(x_0)} \frac{\partial|v|^m}{\partial r} \geq 0 \quad ,$$

which implies that

$$\frac{d}{dr} \left[\frac{1}{r^m} \int_{B_r^m(x_0)} |v|^m(x) dx^m \right] \geq 0 \quad .$$

So we have in particular

$$\int_{B_{\rho/4}(x_0)} |v|^m(x) dx^m \leq 4^{-m} \int_{B_\rho(x_0)} |v|^m(x) dx^m \quad . \quad (\text{VI.47})$$

From this inequality we deduce

$$\begin{aligned} \int_{B_{\rho/4}(x_0)} |u|^m(x) dx^m &\leq 2^{m-1} \int_{B_{\rho/4}(x_0)} [|v|^m + |\varphi|^m] dx^m \\ &\leq 2^{-m-1} \int_{B_\rho(x_0)} |v|^m(x) dx^m + 2^{m-1} \int_{B_\rho(x_0)} |\varphi|^m dx^m \\ &\leq 2^{-2} \int_{B_\rho(x_0)} |u|^m(x) dx^m + 2^m \int_{B_\rho(x_0)} |\varphi|^m dx^m \quad . \end{aligned} \quad (\text{VI.48})$$

Combining (VI.46) and (VI.48) we then have

$$\begin{aligned} \int_{B_{\rho/4}(x_0)} |u|^m(x) dx^m \\ \leq \left[2^{-2} + C_0 \left[\|\nabla u\|_{L^{m/2}(B_\rho)}^m + \|u\|_{L^m(B_\rho)}^{2m} \right] \right] \int_{B_\rho(x_0)} |u|^m(x) dx^m \quad . \end{aligned} \quad (\text{VI.49})$$

We choose $\varepsilon > 0$ such that $C_0\varepsilon^2 \leq 2^{-1}$ and we have then established that for any $\rho < \rho_0$

$$\int_{B_{\rho/4}(x_0)} |u|^m(x) dx^m \leq \frac{1}{2} \int_{B_\rho(x_0)} |u|^m(x) dx^m \quad . \quad (\text{VI.50})$$

Iterating this inequality gives (VI.43). Inserting the *Morrey bound* (VI.43) into the equation (VI.42) gives

$$\sup_{x_0 \in B_{1/2}^m(0) ; 0 < \rho < 1/4} \rho^{-m\alpha/3} \int_{B_\rho(x_0)} |\Delta u|^{m/3}(x) dx^m < +\infty \quad . \quad (\text{VI.51})$$

The *Adams-Sobolev* embeddings (see [1]) give then the existence of $p > m/2$ such that $\nabla u \in L^p_{loc}(B^m_{1/2}(0))$. It is easy then to see that the PDE (VI.42) becomes sub-critical for $W^{1,p}$ (with $p > m/2$) in m dimension and we can apply a similar bootstrap arguments to obtain the desired regularity for u . This concludes the proof of theorem VI.8. \square

Remark VI.4. *The proof of the regularity of Yang-Mills fields in the critical 4 dimension is "soft" in comparison with the proof of the regularity of the "cousin problem" : the harmonic maps between a surface and a manifold. Both equations are critical respectively in 4 and 2 dimensions but the analysis of the harmonic map equation is made more delicate by the fact that the non-linearity in the harmonic map equation is in L^1 which is a space which does not behave "nicely" with respect to Calderon-Zygmund operations. There is no such a difficulty for Yang-Mills. What is delicate however is to construct a "good gauge" in which Yang-Mills equation becomes elliptic. In a somewhat parallel way the difficulty posed by the harmonic maps equation was overcome by the author by solving a gauge problem (see [41]).*

One consequence of the previous regularity result and the point removability result V.6 is the following *point removability* theorem for Yang-Mills fields in 4 dimension

Theorem VI.9. [Point Removability for Yang-Mills in conformal dimension.]
Let A be a weak solution in $W^{1,2}_{loc}(\wedge^1 B^4, \mathcal{G})$ to Yang-Mills equation

$$d^*_A F_A = d^* F_A + [A, \lrcorner F_A] = 0 \quad \text{in } \mathcal{D}'(B^4 \setminus \{0\}) \quad .$$

Assume

$$\int_{B^4} |dA + A \wedge A|^2 dx^4 < +\infty \quad ,$$

then there exists $g \in W^{2,2}_{loc}(B^4 \setminus \{0\})$ such that

$$A^g \in C^\infty(B^4) \quad ,$$

and A^g solves the Yang-Mills equation strongly in the whole ball B^4 . \square

Proof of theorem VI.9. The point removability result V.4 gives the existence of a $W^{2,2}_{loc}$ -gauge such that $A^g \in W^{1,2}(\wedge^1 B^4, \mathcal{G})$. Using Uhlenbeck's Coulomb gauge IV.1 extraction theorem we can assume that A^g satisfies the Coulomb condition $d^* A^g = 0$ in $\mathcal{D}'(B^4)$. So A^g is a $W^{1,2}$ -solution of a system of the form

$$\Delta u = f(u, \nabla u) + g(u) \quad \text{in } \mathcal{D}'(B^4 \setminus \{0\}) \quad ,$$

where f and g are smooth maps satisfying (VI.41). The distribution $\Delta u - f(u, \nabla u) - g(u)$ is supported in $\{0\}$. Hence by a classical result in distribution theory this distribution is a finite linear combination of derivatives of Dirac masses :

$$-\Delta u + f(u, \nabla u) + g(u) = \sum_{|\alpha| \leq N} C_\alpha \partial_\alpha \delta_0 \quad , \quad (\text{VI.52})$$

where $N \in \mathbb{N}$, $\alpha = (\alpha_1 \cdots \alpha_4) \in \mathbb{N}^4$, $|\alpha| := |\alpha_1| + \cdots + |\alpha_4|$, $C_\alpha \in \mathbb{R}^4$ and ∂_α denotes the partial derivative

$$\partial_\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} \frac{\partial^{\alpha_4}}{\partial x_4^{\alpha_4}} \quad .$$

Let χ be an arbitrary smooth compactly supported function in $B_1^4(0)$. Denote $\chi_\varepsilon(x) := \chi(x/\varepsilon)$, multiply equation (VI.52) by this function and integrate over B^4 gives

$$\sum_{|\alpha| \leq N} C_\alpha \frac{\partial^\alpha \chi(0)}{\varepsilon^{|\alpha|}} = \frac{1}{\varepsilon} \int_{B_\varepsilon^4} \nabla \chi(x/\varepsilon) \cdot \nabla u + \int_{B_\varepsilon^4} \chi_\varepsilon [f(u, \nabla u) + g(u)] \quad .$$

Hence, using $\nabla u \in L^2$ and $u \in L^4$ we have

$$\left| \sum_{|\alpha| \leq N} C_\alpha \frac{\partial^\alpha \chi(0)}{\varepsilon^{|\alpha|}} \right| = o(\varepsilon) \quad .$$

Since $\partial^\alpha \chi(0)$ are arbitrary, this implies that $C_\alpha = 0$ for any α . So the equation $-\Delta u + f(u, \nabla u) + g(u) = 0$ holds on the whole ball and we can apply theorem VI.8 to $u = A^g$ and obtain that it is C^∞ which concludes the proof of theorem VI.9. \square

It is clear that the solutions to the Yang-Mills Plateau problems satisfy the Yang-Mills equation and hence we have the following corollary.

Corollary VI.1. *Let G be a compact Lie group and $m \leq 4$. For any 1-form $\eta \in H^{1/2}(\wedge^1 \partial B^m, \mathcal{G})$ the following minimization problem*

$$\inf \left\{ \int_{B^m} |F_A|^2 dx^m ; \iota_{\partial B^m}^* A = \eta^g \text{ for some } g \in H^{3/2}(\partial B^m, G) \right\} \quad (\text{VI.53})$$

is achieved by a 1-form $A^0 \in W^{1,2}(\wedge^1 B^m, \mathcal{G})$ which is C^∞ in any local $W^{1,2}$ -Coulomb gauge inside the ball B^m . \square

VII Concentration compactness and energy quantization for Yang-Mills Fields in critical dimension.

The goal of this section is to study establish the behavior of sequences of Yang-Mills fields of uniformly bounded energy in critical dimension 4. There are three main problematics attached to this study

- Modulo extraction of subsequence, do we have strong converge to a limiting Yang-Mills ?

- If the strong convergence does not hold where is located the lack of strong convergence in the base ?
- How much Yang-Mills energy is lost at the limit ?

We have already several tools and results at hand that we established in the previous sections in order to provide a relatively precise answer to these three questions. The proof of our main result in this section is based in particular on the following "quantitative reformulation" of the regularity theorem VI.7 which belongs to the family of the so called ϵ -regularity results for *conformally invariant problems*.

Theorem VII.1. [ϵ -regularity for Sobolev solutions to Yang-Mills in conformal dimension] *Let G be a compact Lie group, there exists $\varepsilon_{G,4} > 0$ such that for any \mathcal{G} -valued 1-forms A in $W^{1,2}(\wedge^1 B_1^4(0), \mathcal{G})$ satisfying the Yang-Mills equation*

$$d_A^* F_A = d^* F_A + [A, \lrcorner F_A] = 0 \quad \text{in} \quad \mathcal{D}'(B_1^4(0)) \quad .$$

and the small energy condition

$$\int_{B_1^4(0)} |F_A|^2 dx^4 < \varepsilon_{G,4} \quad ,$$

then there exists a gauge g in which the following estimates holds : for any $l \in \mathbb{N}$ there exists $C_l > 0$ such that

$$\|\nabla^l(A)^g\|_{L^\infty(B_{1/2}(0))}^2 \leq C_l \int_{B_1^m(0)} |F_A|^2 dx^4 \quad . \quad (\text{VII.1})$$

□

Proof of theorem VII.1. We choose $\varepsilon_{G,4} > 0$ that will be definitively fixed a bit later in the proof to be at least smaller than the $\varepsilon_G > 0$ of the *Coulomb gauge* extraction result theorem th-III.2. We now work in this *Coulomb gauge* and we omit to mention the superscript g . So, from now on until the end of the proof, we are then assuming that we have

$$\int_{B_1^4} |A|^2 dx^4 + \sum_{i,j=1}^4 |\partial_{x_i} A_j|^2 dx^4 \leq C_G \int_{B_1^4} |F_A|^2 dx^4 < \varepsilon_{G,4} \quad , \quad (\text{VII.2})$$

from which we deduce in particular

$$\int_{B_1^4} |A|^4 dx^4 \leq C_0 \left[\int_{B_1^4} |F_A|^2 dx^4 \right]^2 \quad , \quad (\text{VII.3})$$

for some constant $C_0 > 0$. Recall that in the Coulomb gauge we are choosing, the connection form A satisfies the elliptic system (VI.40) to which we can apply the arguments

of the proof of theorem VI.8 that we are going to follow closely keeping track this time of each estimate. In particular, having chosen $\varepsilon_{G,4}$ small enough we have inequality (VI.50) which holds for $u = A$ and for any $B_\rho(x_0) \subset B_1(0)$ and we deduce

$$\begin{aligned} \forall x_0 \in B_{3/4}(0) \quad \forall \rho < 1/4 \\ \int_{B_\rho(x_0)} |A|^4 dx^4 \leq 2 \rho^\alpha \int_{B_1^4(0)} |A|^4 dx^4 \\ \leq 2 C_0 \rho^\alpha \left[\int_{B_1^4(0)} |F_A|^2 dx^4 \right]^2, \end{aligned} \quad (\text{VII.4})$$

where $\alpha = \log 2 / \log 4$. Inserting this inequality in the Yang-Mills PDE in Coulomb gauge (VI.40) we obtain the existence of a constant $C_1 > 0$ such that

$$\begin{aligned} \forall x_0 \in B_{3/4}(0) \quad \forall \rho < 1/4 \\ \int_{B_\rho(x_0)} |\Delta A|^{4/3} dx^4 \leq C_1 \rho^{\alpha/3} \left[\int_{B_1^4(0)} |F_A|^2 dx^4 \right]^{4/3}. \end{aligned} \quad (\text{VII.5})$$

Combining (VII.4) and (VII.5) we deduce from *Adams-Morrey inequalities* (see [1])

$$\begin{aligned} \|\nabla A\|_{L^p(B_{3/5}(0))} \leq C \sup_{x_0 \in B_{3/4}(0); \rho < 1/4} \left[\rho^{-\alpha/3} \int_{B_\rho(x_0)} |\Delta A|^{4/3} dx^4 \right]^{3/4} \\ + C \|A\|_{L^2(B_1(0))}, \end{aligned} \quad (\text{VII.6})$$

where

$$p = \frac{16 - 4\alpha/3}{8 - \alpha} > 2.$$

Hence we have for this $p > 2$

$$\|\nabla A\|_{L^p(B_{3/5}(0))} \leq C \left[\int_{B_1^4(0)} |F_A|^2 dx^4 \right]^{1/2}.$$

Since $p > 2$ the non-linear elliptic system (VI.40) becomes sub-critical in 4 dimensions and a standard bootstrap argument gives (VII.1). This concludes the proof of theorem VII.1. \square The previous ε -regularity result is the main step for proving the following *concentration compactness* theorem for sequences of Yang-Mills fields.

Theorem VII.2. [Concentration compactness for Yang-Mills Fields in conformal dimension] *Let (M^4, h) be a closed 4 dimensional riemannian manifold and P a*

principal smooth G bundle over M^4 . Let ∇^k be a sequence of Yang-Mills connections satisfying

$$\limsup_{k \rightarrow +\infty} \int_{M^4} |F_{\nabla^k}|_h^2 dvol_h < +\infty \quad .$$

Then there exists a subsequence $\nabla^{k'}$, a smooth G -bundle P^∞ over (M^4, h) , a smooth Yang-Mills connection ∇^∞ of P^∞ and finitely many points $\{p_1 \cdots p_N\}$ in M^4 such that for any contractible open set $D^4 \subset M^4 \setminus \{p_1 \cdots p_N\}$ with compact closure, there exists a sequence of trivialization of P over D^4 for which

$$A^{k'} \rightarrow A^\infty \quad \text{strongly in } C^l(D^4) \quad \forall l \in \mathbb{N} \quad ,$$

where $A^{k'}$ (resp. A^∞) is the connection 1-form associated to $\nabla^{k'}$ (resp. ∇^∞) in this sequence of local trivializations of P (resp. P^∞) over D^4 . Moreover we have the following weak convergence in Radon measure

$$\mu^{k'} := |F_{\nabla^{k'}}|_h^2 dvol_h \rightharpoonup \mu^\infty := |F_{\nabla^\infty}|_h^2 dvol_h + \nu \quad , \quad (\text{VII.7})$$

where ν is a non negative atomic measure supported by the points p_j

$$\nu := \sum_{j=1}^N f_j \delta_{p_j} \quad . \quad (\text{VII.8})$$

□

Proof of theorem VII.2. We follow step by step the proof of theorem V.5 replacing for the choice of the covering the Uhlenbeck Coulomb Gauge threshold $\varepsilon_G(M^4, h)$ by the smaller positive constant $\varepsilon_{G,4}(M^4, h)$ given by the ϵ -regularity result VII.1 on the manifold (M^4, h) . Observe that, because of the epsilon regularity result, the Coulomb gauges $(A^k)^{g_i^k}$ are pre-compact for any C^l -topology on each ball $B_\rho(x_i)$. This gives also the pre-compactness of the transition functions g_{ij}^k in any of the C^l topologies. Hence, the co-cycles g_{ij}^k converge in any of these topology to the limiting (now smooth) co-cycle g_{ij}^∞ which defines a smooth G -bundle P^∞ over $M^4 \setminus \{p_1 \cdots p_N\}$. Moreover the limiting collection of 1-forms $A^{j,\infty}$ defines a connection ∇^∞ on P^∞ satisfying also the Yang-Mills equation which passes obviously to the limit under C^∞ convergence. The gauge invariant quantities such as $|F_{\nabla^k}|^2$ converge also to the corresponding limiting quantities and we have then, modulo extraction of a further subsequence, the existence of a limiting radon measure ν supported on the points p_j exclusively such that (VII.9) holds. Finally applying the point removability theorem and once again the ϵ -regularity we extend ∇^∞ globally on M^4 as a Yang-Mills smooth connection of the bundle P^∞ which also obviously extends throughout the p_j as a smooth bundle. This concludes the proof of theorem VII.2. □

Finally, we identify the *concentration atomic measure* by proving that the weights f_j in front of the Dirac masses δ_{p_j} are the sums of Yang-Mills energies of Yang-Mills fields

over S^4 , the so called "bubbles". Precisely we have the following *energy identity result* which was first established for instantons in [56] and for Yang-Mills fields in general in [42]. The proof we present below is using the *interpolation Lorentz spaces* following a technic introduced in [24] and [25].

Theorem VII.3. [Energy quantization for Yang-Mills Fields in conformal dimension.] *Let (M^4, h) be a closed 4 dimensional riemannian manifold and P a principal smooth G bundle over M^4 . Let ∇^k be a sequence of Yang-Mills connections of uniformly bounded Yang-Mills energy converging strongly away from finitely many points $\{p_1 \cdots p_N\}$ to a limiting Yang-Mills connection ∇^∞ as described in theorem VII.2. Let ν be the atomic concentration measure*

$$\nu := \sum_{j=1}^N f_j \delta_{p_j} \quad ,$$

satisfying

$$\mu^{k'} := |F_{\nabla^{k'}}|_h^2 dvol_h \rightharpoonup \mu^\infty := |F_{\nabla^\infty}|_h^2 dvol_h + \nu \quad .$$

Then for each $j = 1 \cdots N$ there exists finitely many G -Yang-Mills connections $(D_j^i)_{i=1 \cdots N_j}$ over S^4 such that

$$\forall j = 1 \cdots N \quad f_j = \sum_{i=1}^{N_j} \int_{S^4} |F_{D_j^i}|^2 dvol_{S^4} \quad . \quad (\text{VII.9})$$

□

Remark VII.1. *The energy quantization result can be established by direct geometric arguments in the special case of instantons in 4 dimensions (see [15]).*

Proof of theorem VII.3. Since the result is local, the metric in the domain does not play much role and we will present the proof for $M^4 = B_1^4(0)$ equipped with the flat metric and assuming moreover that there is exactly one limiting blow-up point, $N = 1$, which coincide with the origin, $p_1 = 0$. We also express the connection ∇^k in s

Recall that we denote by $\varepsilon_{G,4}$ the positive constant given by the ϵ -regularity theorem VII.1. We detect the "most concentrated bubble" precisely let

$$\rho^k := \inf \left\{ \rho ; \exists x \in B_1^4(0) \quad \text{s. t.} \quad \int_{B_\rho(x)} |F_{A^k}|^2 dx^4 = \frac{\varepsilon_{G,4}}{2} \right\} \quad .$$

Since we are assuming that blow-up is happening exactly at the origin we have that

$$\left\{ \begin{array}{l} \rho^k \rightarrow 0 \quad \text{and} \\ \exists x^k \rightarrow 0 \quad \text{s. t.} \quad \int_{B_{\rho^k}(x^k)} |F_{A^k}|^2 dx^4 = \frac{\varepsilon_{G,4}}{2} \quad . \end{array} \right.$$

We choose a sequence x^k that we call *center of the first bubble* and ρ^k is called the *critical radius of the first bubble*. Let

$$\hat{A}^k(y) := \rho^k \sum_{i=1}^4 A_i^k(\rho^k y + x^k) dy_i \quad .$$

Due to the scaling invariance of the Yang-Mills lagrangian in 4-dimensions, \hat{A}^k , which is the pull-back of A^k by the dilation map $D^k(y) := \rho^k y + x^k$, is a Yang-Mills fields moreover

$$\max_{y \in B_{1/(2\rho^k)}(0)} \int_{B_1^4(y)} |F_{\hat{A}^k}|^2 dy^4 = \int_{B_1^4(0)} |F_{\hat{A}^k}|^2 dy^4 = \frac{\varepsilon_{G,4}}{2} \quad . \quad (\text{VII.10})$$

Applying the ϵ -regularity theorem VII.1 we deduce that on *Uhlenbeck's Coulomb Gauges* g which exists on each unit ball $B_1(y)$ since the $\epsilon_{G,4}$ has been taken smaller than ϵ_G from theorem IV.1

$$\forall l \in \mathbb{N} \quad \sup_{y \in B_{1/(2\rho^k)}(0)} \|\nabla^l(\hat{A})^g\|_{L^\infty(B_{1/2})} \leq C_l \quad ,$$

where C_l is independent of k . Hence, locally in Coulomb gauge, modulo extraction of a subsequence, the sequence \hat{A}^k converges strongly in any C_{loc}^l topology on \mathbb{R}^4 to a limiting Yang-Mills connection \hat{A}^∞ satisfying

$$\int_{B_1^4(0)} |F_{\hat{A}^k}|^2 dy^4 = \frac{\varepsilon_{G,4}}{2}$$

and which is therefore non trivial. Let now π be the stereographic projection with respect to the north pole, due to the *conformal invariance* of Yang-Mills energy $\tilde{A} := \pi^* \hat{A}^\infty$ is a non trivial Yang-Mills Field on $S^4 \setminus \{\text{south p\^ole}\}$. Since \tilde{A} is a smooth \mathcal{G} -valued 1-form with finite Yang-Mills energy and satisfying the Yang-Mills equation we can apply the *point removability* result for Yang-Mills fields, theorem VI.9 and conclude that \tilde{A} extends to a global smooth Yang-Mills G -connection D_1^1 over the whole S^4 which is our *first bubble* and using again the conformal invariance of Yang-Mills energy we have

$$\frac{\varepsilon_{G,4}}{2} \leq \int_{S^4} |F_{D_1^1}|^2 dvol_{S^4} = \lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R\rho^k}(x^k)} |F_{A^k}|^2 dx^4 \quad . \quad (\text{VII.11})$$

We have now to study the loss of Yang Mills energy in the so called *neck region* between the *first bubble* and the *macroscopic solution* A^∞ to which A^k converges away from zero. Precisely we are studying

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 - \int_{B^4} |F_{A^\infty}|^2 dx^4 - \int_{S^4} |F_{D_1^1}|^2 dvol_{S^4} \\ &= \lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R-1}(x^k) \setminus B_{R\rho^k}(x^k)} |F_{A^k}|^2 dx^4 \quad . \end{aligned} \quad (\text{VII.12})$$

For any G there is a minimal Yang-Mills energy among all non-trivial Yang-Mills Fields. This can be proved easily observing that if the energy is less than the $\varepsilon_{G,4}$ threshold, the connections can be represented by a global smooth Yang-Mills 1-form on S^4 to which the C^l estimates of theorem VII.1 apply. Hence since A satisfies globally on S^4 the PDE (VI.40) and for small enough Yang-Mills energy this implies that A is an harmonic one form on S^4 which gives that it is a trivial Yang-Mills fields. Denote

$$\text{YM}(G, S^4) = \min \left\{ \int_{S^4} |F_D|^2 \, d\text{vol}_{S^4} ; D \text{ is a non zero } G \text{ Yang-Mills Field} \right\} .$$

To simplify the presentation we assume that

$$\lim_{k \rightarrow +\infty} \int_{B^4} |F_{A^k}|^2 dx^4 - \int_{B^4} |F_{A^\infty}|^2 dx^4 - \int_{S^4} |F_{D_1}|^2 \, d\text{vol}_{S^4} < \text{YM}(G, S^4) \quad , \quad (\text{VII.13})$$

or in other words

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R-1}(x^k) \setminus B_{R\rho^k}(x^k)} |F_{A^k}|^2 \, dx^4 < \text{YM}(G, S^4) \quad . \quad (\text{VII.14})$$

Alternatively we would have to go through some standard and fastidious induction procedure to remove all the bubbles one by one - each of them taking at least an amount of $\text{YM}(G, S^4)$ Yang-Mills energy - and we would be anyway reduced at the end to study the loss of energy in annuli region where (VII.14) holds (such a procedure is described for instance in [5] proposition III.1 in the framework of *Willmore surfaces*). Under the assumption (VII.14) the goal is ultimately to prove

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R-1}(x^k) \setminus B_{R\rho^k}(x^k)} |F_{A^k}|^2 \, dx^4 = 0 \quad ,$$

that will finish the proof of the theorem. We are now going to prove the following claim.

Claim 1 :

$$\forall \delta > 0 \quad \exists R_\delta > 1 \quad \text{s. t.} \quad \forall r \in [R_\delta \rho^k, R_\delta^{-1}]$$

$$\limsup_{k \rightarrow +\infty} \int_{B_{2r}(x^k) \setminus B_r(x^k)} |F_{A^k}|^2 \, dx^4 < \delta \quad .$$

Proof of claim 1. We argue by contradiction. Assume there exists $\delta_0 > 0$ such that for all $R > 1$ there exists $r^k \in [R \rho^k, R^{-1}]$

$$\limsup_{k \rightarrow +\infty} \int_{B_{2r^k}(x^k) \setminus B_{r^k}(x^k)} |F_{A^k}|^2 \, dx^4 > \delta_0 \quad .$$

Since we can find a sequence r_k for any $R > 1$, using a diagonal argument and the extraction of a subsequence we can assume that

$$\frac{r^k}{\rho^k} \rightarrow +\infty \quad \text{and} \quad r^k \rightarrow 0 \quad .$$

Now we introduce

$$s^k := \inf \left\{ \begin{array}{l} s ; \exists x \in B_{2r^k}(x^k) \setminus B_{r^k}(x^k) \quad \text{s. t.} \\ \int_{B_s(x)} |F_{A^k}|^2 dx^4 = \min \left\{ \frac{\varepsilon_{G,4}}{2}, \frac{\delta_0}{16} \right\} \end{array} \right\} .$$

Let \tilde{x}^k be a point in the dyadic annulus $B_{2r^k}(x^k) \setminus B_{r^k}(x^k)$ where this infimum is achieved. We clearly have

$$B_{s^k}(\tilde{x}^k) \subset B_{R^{-1}}(x^k) \setminus B_{R\rho^k}(x^k)$$

for any $R > 1$ and k large enough. Dilating the Yang-Mills connection A^k about \tilde{x}^k at a rate $(s^k)^{-1}$ we again obtain a limiting non trivial Yang-Mills field, a second *bubble*, \tilde{A}^∞ either on \mathbb{R}^4 or on $\mathbb{R}^4 \setminus \{0\}$ depending whether s^k/r^k tends to zero or not. In any case we have

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R^{-1}}(x^k) \setminus B_{R\rho^k}(x^k)} |F_{A^k}|^2 dx^4 \geq \int_{\mathbb{R}^4} |F_{\tilde{A}^\infty}|^2 dx^4 \geq \min \left\{ \frac{\varepsilon_{G,4}}{2}, \frac{\delta_0}{16} \right\} ,$$

and the *point removability* theorem VI.9 for Yang-Mills fields implies that \tilde{A}^∞ extends to a non-trivial Yang-Mills Field on S^4

$$\int_{\mathbb{R}^4} |F_{\tilde{A}^\infty}|^2 dx^4 \geq \text{YM}(G, S^4) ,$$

which contradicts the fact that we are working under the assumption that there is only one *bubble* (i.e. assumption (VII.14)). So we have proved claim 1.

Combining claim 1 and the ϵ -regularity theorem VII.1 we obtain

$$\forall \delta > 0 \quad \exists R_\delta > 1 \quad \text{s. t.}$$

$$\forall x \in B_{R_\delta^{-1}} \setminus B_{R_\delta \rho^k}(x^k) \quad |x|^2 |F_{A^k}|^2(x) < \delta .$$

Consider an Uhlenbeck Coulomb gauge $(A^k)^{g^k}$ in the annulus $B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)$. Introduce χ to be a cut-off function such that

$$\begin{cases} \chi(x) \equiv 1 & \text{in } R^4 \setminus B_2^4(0) \\ \chi(x) \equiv 0 & \text{in } B_1^4(0) \end{cases} ,$$

and let $\chi^k(x) := \chi(R_\delta \rho^k (x - x^k))$. Extend $(A^k)^{g^k}$ in $B_{2R_\delta \rho^k}(x^k)$ by taking

$$\tilde{A}^k := \chi^k (A^k)^{g^k} .$$

Using again the C^l estimates (VII.1) of ϵ -regularity theorem VII.1 for the Uhlenbeck Coulomb gauge $(A^k)^g$ in the annulus $B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)$ we obtain for any $x \in B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)$

$$\begin{aligned} & |x| |\tilde{A}^k|(x) + |x|^2 |\nabla \tilde{A}^k|(x) \\ & \leq C \left[\int_{B_{4R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k/2}(x^k)} |F_{A^k}|^2 dx^4 \right]^{1/2} . \end{aligned} \quad (\text{VII.15})$$

We have then produced an extension \tilde{A}^k of A^k inside the ball $B_{2R_\delta \rho^k}(x^k)$ equal to A^k in $B_{R_\delta^{-1}} \setminus B_{R_\delta \rho^k}(x^k)$ and satisfying

$$\| |x|^2 |F_{\tilde{A}^k}| \|_{L^\infty(B_{R_\delta^{-1}}(x^k))} < \sqrt{\delta} .$$

This implies in particular

$$\| F_{\tilde{A}^k} \|_{L^{2,\infty}(B_{R_\delta^{-1}}(x^k))} \leq C \sqrt{\delta} , \quad (\text{VII.16})$$

where $C > 0$ is a constant independent of δ and k . Taking δ small enough we can apply theorem IV.4 and find a gauge that we denote simply \bar{A}^k and which satisfies

$$\left\{ \begin{array}{l} \int_{B_{R_\delta^{-1}}(x^k)} |\bar{A}^k|^2 + \sum_{i,j=1}^4 |\partial_{x_i} \bar{A}_j^k|^2 dx^m \leq C_G \int_{B_{R_\delta^{-1}}(x^k)} |F_{\bar{A}^k}|^2 dx^4 \\ d^* \bar{A}^k = 0 \quad \text{in } B_{R_\delta^{-1}}(x^k) \\ \iota_{\partial B_{R_\delta^{-1}}(x^k)}^* (*\bar{A}^k) = 0 . \end{array} \right. \quad (\text{VII.17})$$

Using the gauge invariance of Yang-Mills integrant (VI.39) together with the C^l estimates (VII.1) of ϵ -regularity theorem VII.1 applied to the Uhlenbeck Coulomb gauge $(A^k)^g$ in the annulus $B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)$ we obtain on $B_{R_\delta^{-1}}(x^k)$

$$\begin{aligned} |d_{\bar{A}^k}^* F_{\bar{A}^k}| & \leq C |\nabla \chi^k| |(A^k)^{g^k}|^2 + C |\chi^k| |(A^k)^{g^k}|^3 \\ & \leq C (R_\delta \rho^k)^{-3} \mathbf{1}_{k,\delta} \delta \end{aligned} \quad (\text{VII.18})$$

where $\mathbf{1}_{k,\delta}$ is the characteristic function of $B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)$. This implies the following estimate

$$\| d_{\bar{A}^k}^* F_{\bar{A}^k} \|_{L^{(4/3,1)}(B_{R_\delta^{-1}}(x^k))} \leq C \int_{B_{2R_\delta \rho^k}(x^k) \setminus B_{R_\delta \rho^k}(x^k)} |F_{\bar{A}^k}|^2 dx^4 . \quad (\text{VII.19})$$

where we recall that $L^{(4/3,1)}$ is the Lorentz space whose dual is the *Marcinkiewicz weak* L^4 space : $L^{4,\infty}$. Using the embedding (IV.15) for $p = 2$ and $m = 4$

$$W^{1,2}(B^4) \hookrightarrow L^{4,2}(B^4) \quad ,$$

we obtain from (VII.17) the estimate

$$\|\bar{A}^k\|_{L^{4,2}(B_{R_\delta^{-1}}(x^k))}^2 \leq C_G \int_{B_{R_\delta^{-1}}(x^k)} |F_{\bar{A}^k}|^2 dx^4 \quad . \quad (\text{VII.20})$$

Using now one of the embeddings (IV.13) :

$$L^2(B^4) \cdot L^{4,2}(B^4) \hookrightarrow L^{4/3,1}(B^4) \quad ,$$

we obtain

$$\|d^* d \bar{A}^k\|_{L^{4/3,1}(B_{R_\delta^{-1}}(x^k))} \leq C_G \int_{B_{R_\delta^{-1}}(x^k)} |F_{\bar{A}^k}|^2 dx^4 \quad . \quad (\text{VII.21})$$

Combining this fact with the three lines of (VII.17) together with classical elliptic estimates in Lorentz spaces (see [53]) gives

$$\|F_{\bar{A}^k}\|_{L^{2,1}(B_{R_\delta^{-1}}(x^k))}^2 \leq C_G \int_{B_{R_\delta^{-1}}(x^k)} |F_{\bar{A}^k}|^2 dx^4 \quad .$$

Combining this inequality with the estimate (VII.16) of the curvature in the dual space $L^{2,\infty}$ in the *neck region* we obtain

$$\forall \delta > 0 \quad \exists R_\delta > 1 \quad \text{s. t.}$$

$$\limsup_{k \rightarrow +\infty} \int_{B_{R_\delta^{-1}}(x^k) \setminus B_{2R_\delta \rho^k}(x^k)} |F_{A^k}|^2 dx^4 \leq C \sqrt{\delta}$$

from which we deduce

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{B_{R^{-1}}(x^k) \setminus B_{R \rho^k}(x^k)} |F_{A^k}|^2 dx^4 = 0 \quad .$$

This implies

$$|F_{A^k}|^2 dx^4 \rightharpoonup \mu^\infty := |F_{A^\infty}|^2 dx^4 + \int_{S^4} |F_{D_1^1}|^2 d\text{vol}_{S^4} \delta_0 \quad .$$

This completes the proof of the theorem VII.3. □

VIII The resolution of the Yang-Mills Plateau problem in super-critical dimensions.

VIII.1 The absence of $W^{1,2}$ local gauges.

We can reformulate the sequential weak closure of $W^{1,2}$ connections we proved in the previous sections for the dimensions up to 4 in the following way. Let G be a compact Lie group and (M^m, h) a compact riemanian manifold. Introduce the space of so called *Sobolev connections* defined by

$$\mathfrak{A}_G(M^m) := \left\{ \begin{array}{l} A \in L^2(\wedge^1 M^m, \mathcal{G}) ; \int_{M^m} |dA + A \wedge A|_h^2 dvol_h < +\infty \\ \text{locally } \exists g \in W^{1,2} \quad \text{s.t.} \quad A^g \in W^{1,2} \quad . \end{array} \right\}$$

It is clear that any Sobolev $W^{1,2}$ -connection of a smooth G -bundle in the classical sense of [15] defines an element in $\mathfrak{A}_G(M^m)$ for $m \leq 4$. This follows from the existence of a global representative in L^p ($p < 4$) given by theorem IV.5.

We now prove the converse : that any element $A \in \mathfrak{A}_G(M^m)$ for $m \leq 4$ define a smooth bundle and a $W^{1,2}$ -Sobolev connection in this bundle in the classical sense given in [15]. Precisely we have the following result.

Proposition VIII.1. *Let G be a compact Lie group and (M^m, h) be a riemannian manifold of dimension $m \leq 4$. Let $A \in \mathfrak{A}_G(M^m)$ then there exists a smooth bundle E over M^m and there exists a $W^{1,2}$ -connection ∇ of E such that A is locally gauge equivalent to ∇ . \square*

Proof of proposition VIII.1. We cover M^m by a locally finite family of open balls $B_{\rho_i}(x_i)$ realizing a good covering (In the sense of [6]) such that on each of these sets A has a $W^{1,2}$ representative and $\int_{B_{\rho_i}(x_i)} |F_A|_h^2 dvol_h < \varepsilon$ where ε is positive number small enough in order to ensure the existence of a Coulomb $W^{1,2}$ -representative. We denote by g_i the gauge change such that $A^{g_i} = g_i^{-1} d g_i + g_i^{-1} A g_i$ is such a Coulomb $W^{1,2}$ -representative. On $B_{\rho_i}(x_i) \cap B_{\rho_j}(x_j)$ we denote $g_{ij} := g_i^{-1} g_j$. Following word by word the arguments of the proof of theorem V.5 we have that $g_{ij} \in C^0 \cap W^{1,2}(B_{\rho_i}(x_i) \cap B_{\rho_j}(x_j), G)$ and the family satisfy moreover obviously the co-cycle condition $g_{ij} = g_{ik} g_{kj}$.

Having taken $\varepsilon > 0$ small enough we can ensure that the C^0 norms of the g_{ij} - possibly on the intersection of slightly smaller balls - are small and there exists an equivalent smooth cocycle (h_{ij}) (i.e. $h_{ij} = (\sigma_i)^{-1} g_{ij} \sigma_j$ where $\sigma_i \in C^0 \cap W^{1,2}(B_{\rho_i}(x_i), G)$ and $h_{ij} \in C^\infty$.) The cocycle (h_{ij}) associated to the good covering $B_{\rho_i}(x_i)$ defines a smooth bundle E over M^m . We have

$$A^{g_i \sigma_i} \in W^{1,2}(\wedge^1 B_{\rho_i}(x_i), \mathcal{G}) \quad \text{and} \quad A^{g_j \sigma_j} = h_{ij}^{-1} d h_{ij} + h_{ij}^{-1} A^{g_i \sigma_i} h_{ij} \quad .$$

Hence by definition $(A^{g_i \sigma_i})$ defines a $W^{1,2}$ -connection on E . It is obviously locally gauge equivalent to A . This concludes the proof of the proposition. \square

Using the same ingredients¹¹ as the one used to prove theorem V.2 one establishes the following result which is the boundary-free counterpart of theorem V.2.

Theorem VIII.1. *For $m \leq 4$ the space $\mathfrak{A}_G(M^m)$ is weakly sequentially closed below any given Yang-Mills energy level : precisely For any $A^k \in \mathfrak{A}_G(M^m)$ satisfying*

$$\limsup_{k \rightarrow +\infty} YM(A^k) = \int_{M^m} |dA^k + A^k \wedge A^k|_h^2 dvol_h < +\infty \quad ,$$

there exists a subsequence $A^{k'}$ and a Sobolev connection $A^\infty \in \mathfrak{A}_G(M^m)$ such that

$$d(A^{k'}, A^\infty) := \inf_{g \in W^{1,2}(M^m, G)} \int_{M^m} |A^{k'} - (A^\infty)^g|_h^2 dvol_h \longrightarrow 0 \quad ,$$

moreover

$$YM(A^\infty) \leq \liminf_{k' \rightarrow 0} YM(A^{k'}) \quad .$$

□

Remark VIII.1. *Observe that the space $\mathfrak{A}_G(M^m)$ contains for instance global L^2 one forms taking values into the Lie algebra \mathfrak{G} that correspond to smooth connections of some principal G -bundle over M^m . If the Yang-Mills energy of a sequence of such smooth connections is uniformly bounded, we can extract a subsequence converging weakly to a Sobolev connection and corresponding possibly to another G -bundle. This possibility of "jumping" from one bundle to another, as predicted for instance in the concentration compactness result theorem VII.2, is encoded in the definition of $\mathfrak{A}_G(M^m)$. □*

Because of this weak closure property the space $\mathfrak{A}_G(M^m)$ is the ad-hoc space for minimizing Yang-Mills energy in dimension less or equal than 4. This is however not the case in higher dimension. We have the following proposition.

Proposition VIII.2. *For $m > 4$ the space $\mathfrak{A}_{SU(2)}(M^m)$ is not weakly sequentially closed below a given Yang-Mills energy level : precisely there exists $A^k \in \mathfrak{A}_{SU(2)}(M^m)$ satisfying*

$$\limsup_{k \rightarrow +\infty} YM(A^k) = \int_{M^m} |dA^k + A^k \wedge A^k|^2 dx^m < +\infty \quad ,$$

and a $su(2)$ -valued form $A^\infty \in L^2$ such that

$$d(A^{k'}, A^\infty) := \inf_{g \in W^{1,2}(M^m, SU(2))} \int_{M^m} |A^{k'} - (A^\infty)^g|_h^2 dvol_h \longrightarrow 0 \quad ,$$

but in every neighborhood U of every point of M^m there is no g such that $(A^\infty)^g \in W^{1,2}(U)$ (i.e. $A^\infty \notin \mathfrak{A}_{SU(2)}(M^m)$). □

¹¹Observe that the topology of the underlying bundles defined by each A^k and given by proposition VIII.1 plays no role in the proof and that the arguments of theorem V.2 carry over in a straightforward way. Again our approach is to work with analytical objects, i.e. global L^2 -one forms, ignoring the geometric structures that each of this one form is inducing.

The proposition is not difficult to prove but we prefer to illustrate this fact by a small cartoon. We consider a sequence A^k of smooth 1-forms on B^5 , the unit 5 dimensional ball, into $su(2)$ such that $\limsup_{k \rightarrow +\infty} YM(A^k) < +\infty$. The drawing is representing the flow lines of the divergence free vector field associated to the closed Chern 4-form : $Tr(F_{A^k} \wedge F_{A^k})$. In this cartoon A^k is weakly converging in L^2 to some limit $su(2)$ -valued 1-form A^∞ such that $F_{A^\infty} \in L^2$ but this 1-form satisfies

$$d(Tr(F_{A^\infty} \wedge F_{A^\infty})) = 8\pi^2 [\delta_P - \delta_N] \neq 0 \quad ,$$

where P and N are two distinct points of B^5 - the red dots in the figure 3 cartoon. Hence for almost every small radii $r > 0$ we have for instance

$$\int_{\partial B_r^5(P)} Tr(F_{A^\infty} \wedge F_{A^\infty}) = 8\pi^2 \quad .$$

Assume there would exist $g \in W^{1,2}$ such that $(A^\infty)^g \in W^{1,2}$ in the neighborhood of P . The gauge invariance of the Chern form gives that

$$\int_{\partial B_r^5(P)} Tr(F_{(A^\infty)^g} \wedge F_{(A^\infty)^g}) = 8\pi^2 \quad .$$

However we have seen in section III that the fact that $\iota_{\partial B_r^5}^*(A^\infty)^g$ is in $W^{1,2}(\wedge^1 \partial B_r^5, su(2))$ for almost every r - due to Fubini theorem - imposes

$$\int_{\partial B_r^5(P)} Tr(F_{(A^\infty)^g} \wedge F_{(A^\infty)^g}) = 0 \quad ,$$

which is a contradiction.

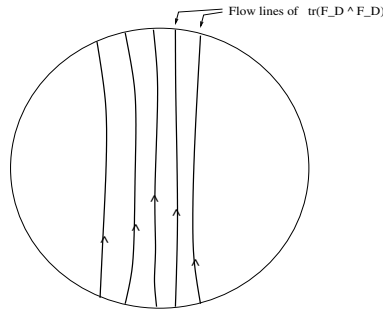


Fig. 1: Sequence of smooth connections time 1

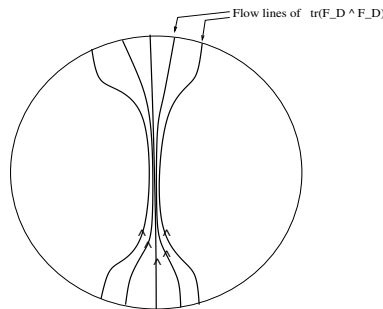


Fig. 2: Sequence of smooth connections time 2

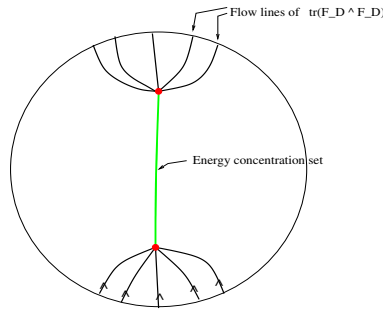


Fig. 3: Sequence of smooth connections - the limit.

In the above cartoon¹² the limiting 1-form A^∞ is a 1-form of a smooth connection but on a $SU(2)$ -bundle which is only defined over $B^5 \setminus \{P\} \cup \{N\}$.

The conclusions we can draw at this point are the following. In dimension $m \leq 3$, the Yang Mills energy is controlling the bundle in the sense that from a sequence of connections of a given bundle with uniformly bounded L^2 -curvature one can extract a weakly converging sequence to a limiting Sobolev connection of the same bundle. In dimension 4, under the same assumptions one can extract a weakly converging sequence to a limiting Sobolev connection but possibly from *another bundle*. The last example is illustrating the fact that, starting in 5 dimension, the Yang-Mills energy is not coercive enough to control the bundle structure : The Yang-Mills energy control does not prevent the corresponding bundle to degenerate and to have local twists at the limit. In fact, instructed from what happens with the Dirichlet energy of maps from B^3 into S^2 one could even construct a sequence of smooth connections whose weak limit is given by a form satisfying

$$\text{supp} [d(\text{Tr}(F_{A^\infty} \wedge F_{A^\infty}))] = B^5 \quad !$$

This means that the limiting bundle has become everywhere singular on B^5 .

In order to find an ad-hoc weakly sequentially closed space of connections below any Yang-Mills energy level, we have then to weaken the notion of *Sobolev connections*. Sobolev connections are singular but the underlying bundle was smooth. We see that the

¹²A reader interested in having a rigorous implementation of this cartoon could take for instance a sequence of axially symmetric smooth maps u_k from B^5 into S^4 with uniformly bounded $W^{1,4}$ - energy and weakly converging to a map u^∞ , smooth away from P and N and such that

$$d(u^* \omega_{S^4}) = |S^4| [\delta_P - \delta_N]$$

where ω_{S^4} is the volume form on S^4 . If ∇_0 is a smooth connection of a non trivial $SU(2)$ -bundle E_0 over S^4 , the cartoon is realised by A^k a one form representing the pull back of the connection by u^k over the trivial bundle $(u^k)^{-1}E_0$

L^2 control of the curvature does not preserve the smoothness of the bundle at the limit in dimension $m > 4$. Since we do not impose anything more than having bounded Yang-Mills energy, we have to relax the notion of bundle, the carrier of the connection, and allow the bundle to have singularities. Contrary to classical differential geometry where the bundle precedes the connection, inspired by the “philosophy” of proposition VIII.1, we shall work with one-forms and “generate” singular bundles from these one forms.

The new variational objects will be called *Weak connections*. The quest for finding the right space is similar to the one at the origin of *Geometric Measure Theory* when the class of *integer rectifiable currents* - i.e. sub manifolds with singularities in a way - have been produced in order to solve the *Classical Plateau Problem* in super critical dimension $m > 2$. We are looking for a **geometric measure theoretic version of bundles and connections**.

VIII.2 Tian’s results on the compactification of the space of smooth Yang-Mills Fields in high dimensions.

The need of developing a *Geometric measure theoretic* version of bundle and connections beyond the too small class of *Sobolev connections* on smooth bundles has been already encountered in the study of the compactification of the *moduli space* of smooth Yang-Mills fields by G.Tian in [57].

Theorem VIII.2. *Let G be a compact Lie group and A^k be a sequence of \mathcal{G} -valued 1-forms in B^m . Assume A^k are all smooth solutions to the Yang-Mills equation*

$$d_{A^k}^* F_{A^k} = 0 \quad .$$

Assume

$$\limsup_{k \rightarrow +\infty} \int_{B^m} |dA^k + A^k \wedge A^k|^2 dx^m < +\infty \quad .$$

Then there exists¹³ a subsequence $A^{k'}$ and a limiting \mathcal{G} -valued 1-form A^∞

$$d(A^{k'}, A^\infty) := \inf_{g \in W^{1,2}(M^m, G)} \int_{M^m} |A^{k'} - (A^\infty)^g|_h^2 dvol_h \longrightarrow 0 \quad . \quad (\text{VIII.1})$$

Moreover there exists a $m-4$ rectifiable closed subset of B^m , K , of finite $m-4$ Hausdorff measure, $\mathcal{H}^{m-4}(K) < +\infty$ such that in

$$\forall B_r(x_0) \subset B^m \setminus K \quad \exists g \in W^{1,2}(B_r(x_0), G) \quad s. t.$$

$$(A^\infty)^g \text{ is a smooth solution of Yang-Mills equation in } B_r(x_0) \quad .$$

¹³The weak compactness of the sequence of Yang-Mills connection as it is presented in Tian’s work [57] does not involve the gauge invariant L^2 -distance (VIII.1). Nevertheless by combining the ϵ regularity result VIII.5 with the L^2 -control of some gauge by the Yang-Mills energy given in [35] one can recast the weak convergence mentioned in [57] into (VIII.1).

and the following weak convergence as Radon measure holds

$$|F_{A^{k'}}|^2 dx^m \rightharpoonup |F_{A^\infty}|^2 dx^m + f \mathcal{H}^{m-4} \llcorner K \quad , \quad (\text{VIII.2})$$

where $\mathcal{H}^{m-4} \llcorner K$ is the restriction to K of the $m - 4$ Hausdorff measure and f is a $\mathcal{H}^{m-4} \llcorner K$ measurable function $f(x) \in L^\infty(K)$. \square

This result is very close to a similar result proved in [23] by F.H.Lin for harmonic maps in super-critical dimension $m \geq 3$.

Proof of theorem VIII.2. The starting point of the proof of theorem VIII.2 is the following monotonicity formula computed first by P. Price in [38].

Proposition VIII.3. [Monotonicity formula] *Let $m \geq 4$ and A be a \mathcal{G} -valued 1-forms in $B_1^m(0)$ assume that A is a smooth solution to the Yang-Mills equation*

$$d_A^* F_A = 0 \quad \text{in } B_1^m(0) \quad .$$

then the following monotonicity formula holds for any point p in $B_1^m(0)$ and any $r < \text{dist}(p, \partial B_1^m(0))$

$$\frac{d}{dr} \left[\frac{1}{r^{m-4}} \int_{B_r^m(p)} |F_A|^2 dx^m \right] = \frac{4}{r^{m-4}} \int_{\partial B_r^m(p)} |F_A \llcorner \partial_r|^2 d\text{vol}_{\partial B_r^m} \quad , \quad (\text{VIII.3})$$

where $\partial_r = r^{-1} \sum_{i=1}^m x_i \partial_{x_i}$ and

$$F_A \llcorner \partial_r := r^{-1} \sum_{i,j=1}^m (F_A)_{ij} x_i dx_j \quad .$$

\square

The *monotonicity formula* is a direct consequence of the *stationarity condition* which is satisfied by any smooth critical point of the Yang-Mills lagrangian. Precisely the *stationarity condition* says

$$\forall X \in C_0^\infty(B^m, \mathbb{R}^m) \quad \frac{d}{dt} \int_{B^m} |\Psi_t^* F_A|^2 \Big|_{t=0} = 0 \quad , \quad (\text{VIII.4})$$

where Ψ_t is the flow of X .

This condition is equivalent to the following *stationary equation* :

$$\int_{B^m} |F_A|^2 \text{div } X - 4 \sum_{ijk=1}^m \langle (F_A)_{ij}, (F_A)_{kj} \rangle \partial_{x_i} X^k dx^m \quad . \quad (\text{VIII.5})$$

The monotonicity formula is obtained by applying (VIII.4) to the each vector-field X of the following form : On the ball $B_r(p)$ the vector-field X is equal to the radial one, $X = \partial_r$

for canonical coordinates centered at p which generates the dilations centred at p , and it realizes a smooth interpolation to 0 outside $B_{r+\delta}(p)$ for any $\delta > 0$ (see [38]). Once such a vector field is chosen one computes (VIII.4) and make δ tend to zero. This computation gives then (VIII.3).

The second ingredient of the proof is the extension of the *Coulomb gauge extraction* in dimension larger than 4 to the framework of the so called Morrey spaces where the $m - 4$ densities of Yang-Mills energy are assumed to be small everywhere and at any scale. The following result has been established first in [26] by Yves Meyer and the author of the present survey and independently by Terence Tao and Gang Tian in [54].

Theorem VIII.3. [Coulomb Gauge extraction] *Let $m \geq 4$ and G be a compact Lie group, there exists $\varepsilon_{m,G} > 0$ such that for any smooth \mathcal{G} -valued 1-forms A in $B_1^m(0)$ satisfying the small Morrey energy condition*

$$\|F_A\|_{M_{2,4}^0(B_1^m(0))}^2 := \sup_{p \in B_1^m(0), r > 0} \frac{1}{r^{m-4}} \int_{B_r^m(p) \cap B_1^m(0)} |F_A|^2 dx^m < \varepsilon_{m,G} \quad ,$$

then there exists a gauge $g \in W^{2,2}(B_{1/2}^m(0), G)$ such that

$$\sup_{p \in B_1^m(0), r > 0} \frac{1}{r^{m-4}} \int_{B_r^m(p) \cap B_1^m(0)} \sum_{i,j=1}^m |\partial_{x_i}(A^g)_j|^2 \leq C \|F_A\|_{M_{2,4}^0(B_1^m(0))}^2 \quad ,$$

and

$$\begin{cases} d^*(A)^g = 0 & \text{in } B_1^m(0) \\ \iota_{\partial B_1^m(0)}^* (A)^g = 0 \quad , \end{cases}$$

where the constant C only depends on m and G . □

In this Coulomb gauge any Yang-Mills smooth connection one form satisfies

$$\Delta A^g = d^*(A^g \wedge A^g) + [A^g, \lrcorner dA^g] + [A^g, \lrcorner (A^g \wedge A^g)] \quad . \quad (\text{VIII.6})$$

We shall make now use of the following generalisation of theorem VI.8 to Morrey spaces.

Theorem VIII.4. *Let $m \geq 4$ and $N \in \mathbb{N}^*$. Let $f \in C^\infty(\mathbb{R}^N \times (\mathbb{R}^m \otimes \mathbb{R}^N), \mathbb{R}^N)$ and let $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that there exists $C > 0$ satisfying*

$$|f(\xi, \Xi)| \leq C |\xi| |\Xi| \quad \text{and} \quad |g(\xi)| \leq C |\xi|^3 \quad . \quad (\text{VIII.7})$$

There exists $\varepsilon > 0$ such that for any u in $L^4 \cap W^{1,2}(B^m, \mathbb{R}^N)$ satisfying

$$\sup_{p \in B_1^m(0), r > 0} \frac{1}{r^{m-4}} \int_{B_r^m(p) \cap B_1^m(0)} |\nabla u|^2 dx^m < \varepsilon_{m,G} \quad , \quad (\text{VIII.8})$$

and

$$\Delta u = f(u, \nabla u) + g(u) \quad , \quad (\text{VIII.9})$$

then we have for any $l \in \mathbb{N}$ the existence of $C_l > 0$ such that

$$\|\nabla^l u\|_{L^\infty(B_{1/2}^m(0))}^2 \leq C_l \sup_{p \in B_1^m(0), r > 0} \frac{1}{r^{m-4}} \int_{B_r^m(p) \cap B_1^m(0)} |\nabla u|^2 dx^m \quad . \quad (\text{VIII.10})$$

□

The proof of this theorem is more or less identical to the one of theorem VI.8 replacing the different spaces in 4 dimensions by their *Morrey counterparts* in higher dimension, bearing in mind that Calderon-Zygmund theory extends with the natural exponents to these spaces (see [27]).

Combining the *monotonicity formula*, theorem VIII.3 and theorem VIII.4 applied to the PDE (VIII.6), adapting the arguments we followed for proving the corresponding result - theorem VII.1 - from the 4-D counterparts of theorem VIII.4 in the conformal dimension 4, we obtain the following ϵ -regularity result ¹⁴

Theorem VIII.5. [ϵ -regularity for smooth Yang-Mills] *Let $m \geq 4$ and G be a compact Lie group, there exists $\epsilon_{m,G} > 0$ such that for any smooth \mathcal{G} -valued 1-forms A in $B_1^m(0)$ satisfying the Yang-Mills equation*

$$d_A^* F_A = 0 \quad \text{in } B_1^m(0) \quad .$$

and the small energy condition

$$\int_{B_1^m(0)} |F_A|^2 dx^m < \epsilon_{m,G} \quad ,$$

then there exists a gauge g in which the following estimates holds : for any $l \in \mathbb{N}$ there exists $C_l > 0$ such that

$$\|\nabla^l (A)^g\|_{L^\infty(B_{1/2}(0))}^2 \leq C_l \int_{B_1^m(0)} |F_A|^2 dx^m \quad .$$

□

¹⁴An ϵ -regularity theorem for smooth Yang-Mills fields has first been obtained by H.Nakajima (see [30]). It is however a "gauge invariant result" which gives only an L^∞ bound on the curvature under the small energy assumption but is not providing any control of the connection in some gauge. The proof of Nakajima ϵ -regularity for smooth Yang-Mills fields is following the arguments originally introduced by R. Schoen in [46] for proving the corresponding result for smooth harmonic map. It is using the *Bochner Formula* as a starting point together with the *maximum principle* and the *Moser iteration technique*.

Proof of theorem VIII.2 continued.

Let

$$E_r^k := \left\{ p \in B^m ; \frac{1}{r^{m-4}} \int_{B_r(p)} |F_{A^k}| dx^m \geq \varepsilon_{m,G} \right\} ,$$

where $\varepsilon_{m,G}$ is the epsilon in the ϵ -regularity theorem VIII.5. The *monotonicity formula* implies

$$\forall k \in \mathbb{N} \quad \forall r < \rho \quad E_r^k \subset E_\rho^k .$$

Hence, by a standard diagonal argument we can extract a subsequence such that $E_{k',2^{-j}}$ converges to a limiting closed set $E_{\infty,2^{-j}}$ which of course satisfy

$$E_{2^{-j-1}}^\infty \subset E_{2^{-j}}^\infty .$$

Let

$$K := \bigcap_{j \in \mathbb{N}} E_{2^{-j}}^\infty$$

A classical Federer-Zimmer covering argument gives

$$\mathcal{H}^{m-4}(K) < +\infty .$$

With the ϵ -regularity theorem at hand, extracting possibly a further subsequence following a diagonal argument, we can ensure that $A^{k'}$ converges locally away from K in every C^l -norm in the Coulomb gauges constructed in theorem VIII.3 and we have

$$\mu^{k'} := |F_{A^{k'}}|^2 dx^m \rightharpoonup \mu^\infty := |F_{A^\infty}|^2 dx^m + \nu ,$$

where ν is a Radon measure supported in the closed set K . Because of the Radon measure convergence, the monotonicity formula (VIII.3) satisfied by A^k can be transferred to the measure μ^∞

$$\forall p \in B_1^m(0) \quad \forall B_r^m(p) \subset B_1^m(0) \quad \frac{d}{dr} \left[\frac{\mu^\infty(B_r^m(p))}{r^{m-4}} dx^m \right] \geq 0 ,$$

from which we deduce

$$\theta^{m-4}(\mu^\infty, p) := \lim_{r \rightarrow 0} \frac{\mu^\infty(B_r^m(p))}{r^{m-4}} \geq 0 \quad \text{exists for every } p \in B_1^m(0) .$$

Observe that

$$K = \{ p \in B_1^m(0) ; \theta^{m-4}(\mu^\infty, p) > 0 \} . \quad (\text{VIII.11})$$

Using the monotonicity we have that for any $\rho \in (0, 1)$ and any $p \in K \cap B_\rho(0)$

$$\nu(B_r^m(p)) \leq \frac{r^{m-4}}{(1-\rho)^{m-4}} \lim_{k' \rightarrow +\infty} \int_{B_1^m(0)} |F_{A^{k'}}|^2 dx^m .$$

We deduce from this inequality that ν is *absolutely continuous* with respect to $\mathcal{H}^{m-4} \llcorner K$, the restriction to K of the $m - 4$ -Hausdorff measure. Let $\delta > 0$, define

$$G_\delta := \left\{ p \in B_1^m(0) ; \delta < \limsup_{r \rightarrow 0} \frac{1}{r^{m-4}} \int_{B_r^m(p)} |F_{A^\infty}|^2 dx^m \right\} .$$

Considering for any $\delta > 0$ for any $\eta > 0$ and any $p \in G_\delta$ a radius $0 < r_p^\eta < \eta$ such that

$$\int_{B_{r_p^\eta}^m(p)} |F_{A^\infty}|^2 dx^m \geq \frac{\delta}{2} (r_p^\eta)^{m-4}$$

For any $\eta > 0$ we extract from the covering $(B_{r_p^\eta}^m(p))_{p \in G_\delta}$ a Besicovitch subcovering $(B_{r_i}^m(p))_{i \in I}$ of G_δ in such a way that there exists an integer $N_m > 0$ depending only on m such that each point of B^m is covered by at most N_m balls of this sub-covering. We then have

$$\begin{aligned} \mathcal{H}^{m-4}(G_\delta) &\leq \frac{2}{\delta} \limsup_{\eta \rightarrow 0} \sum_{i \in I} \int_{B_{r_i}^m(p)} |F_{A^\infty}|^2 dx^m \\ &\leq \frac{2N_m}{\delta} \limsup_{\eta \rightarrow 0} \int_{\text{dist}(x, K) < \eta} |F_{A^\infty}|^2 dx^m . \end{aligned}$$

Since K is closed and has Lebesgue measure zero we deduce that

$$\mathcal{H}^{m-4} \left(\bigcap_{\delta > 0} G_\delta \right) = 0 ,$$

or in other words

$$\text{for } \mathcal{H}^{m-4} \text{ almost every } p \in B^m \quad \limsup_{r \rightarrow 0} \frac{1}{r^{m-4}} \int_{B_r^m(p)} |F_{A^\infty}|^2 dx^m = 0 .$$

Using the characterization of K given by (VIII.11) and the fact that ν is *absolutely continuous* with respect to $\mathcal{H}^{m-4} \llcorner K$ we deduce from the previous fact that

$$\text{for } \nu \text{ almost every } p \in B^m \quad \lim_{r \rightarrow 0} \frac{\nu(B_r(p))}{r^{m-4}} \text{ exists and is positive} .$$

The following result, which is an important contribution to *Geometric measure theory* was proved by D.Preiss in [37] and answered positively to a conjecture posed by Besicovitch. It permits to conclude the proof of theorem VIII.2. \square

Theorem VIII.6. *Let ν be a Borel non-negative measure in B^m . Assume that ν almost everywhere the n -dimensional density of ν exists and is positive then ν is supported by a n -dimensional rectifiable subset in B^m . \square*

In order to have a more complete description of the *compactification of the moduli space* of smooth Yang-Mills fields in supercritical dimension three main questions are left open in this theorem

- i) What are the special geometric properties satisfied by the set K ?
- ii) What is the energy defect $f(x)$?
- iii) What is the regularity of A^∞ - modulo gauge - throughout K ?

In subcritical dimension $m < 4$, due to the analysis we have exposed in the previous sections, we have that $K = \emptyset$, $f = 0$ and modulo gauge transformation the limiting connection extends to a smooth Yang-Mills field over the whole ball.

In critical 4 dimensions these questions are answered by the point removability theorem VI.9 : the set K is made of isolated points, the function f is the sum of the Yang-Mills energies of Yang-Mills fields over S^4 (see [42]) and, modulo gauge transformation, the limiting connection extends to a smooth Yang-Mills field over the whole ball.

In super-critical dimension it is expected that K with the multiplicity f defines a so called *stationary varifold* (see [50]). This belief comes from the fact that smooth Yang-Mills fields satisfy the *stationarity condition* (VIII.4) and it is expected that this condition should still be satisfied by the weak limit A^∞ itself, and hence, due to the Radon measure convergence (VIII.2) , it would be "transferred" to the measure $f \mathcal{H}^{m-4} \llcorner K$. This last fact is equivalent to the stationarity of (K, f) .

Regarding the regularity of A^∞ a result of T.Tao and G.Tian [54] asserts that A^∞ is a smooth Yang-Mills connection of a smooth bundle defined away of a closed subset $L \subset K$ satisfying $\mathcal{H}^{m-4}(L) = 0$. This partial regularity result is probably not optimal but this optimality or non-optimality is an open problem (a similar open question exists for *stationary harmonic maps* - see [43])

The author has proved in [42] that a uniform L^1 control of the *Hessian* of the curvature leads to the energy quantisation along K . Precisely the following theorem holds.

Theorem VIII.7. [42] *Let A^k be a sequence of smooth Yang-Mills Fields on B^m satisfying*

$$\limsup_{k \rightarrow +\infty} \int_{B^m} |\nabla^2 F_{A^k}| + |F_{A^k}|^2 dx^m < +\infty \quad .$$

Then, modulo extraction of a subsequence, there exists a $m - 4$ rectifiable $K \subset B^m$ and a $\mathcal{H}^{m-4} \llcorner K$ measurable function $f \in L^\infty(K)$. such that

$$|F_{A^k}|^2 dx^m \rightharpoonup |F_{A^\infty}|^2 dx^m + f d\mathcal{H}^{m-4} \llcorner K \quad ,$$

where F_{A^∞} is the curvature of a smooth Yang-Mills field of a G -bundle over B^m . Moreover

For \mathcal{H}^{m-4} -a. e. $x \in K \quad \exists N_x \in \mathbb{N} \quad \nabla_1 \cdots \nabla_{N_x}$ and

$$f(x) = \sum_{j=1}^{N_x} \int_{S^4} |F_{\nabla_j}|^2 \, dvol_{S^4} \quad ,$$

where ∇_j are Yang-Mills connections of some smooth G -bundles over S^4 .

Recently, Aaron Naber and Daniele Valtorta in [29] have been able to prove the following uniform L^1 control of the *Hessian* which had been conjectured by the author in [42].

Theorem VIII.8. [29] *Let A be a smooth Yang-Mills Fields of a smooth G -bundle over a closed manifold (M^m, h) such that*

$$\int_{M^m} |F_A|^2 \, dvol_h \leq \Lambda \quad ,$$

then

$$\int_{M^m} |\nabla^2 F_A| \, dvol_h \leq C(\Lambda, \dim(G), (M^m, h)) \quad .$$

□

Combining theorem VIII.7 and theorem VIII.8 implies the energy quantisation along K (i.e. the conclusion of theorem VIII.7) for sequences of smooth Yang-Mills connections under Yang-Mills energy control only.

VIII.3 The Ω -anti-self-dual instantons.

In 4 dimension a special class of solutions to the Yang-Mills equation, the *anti-self-dual instantons* have been considered by S.Donaldson in the early 80's to produce invariants of differential 4 dimensional manifolds. On a 4-dimensionnal riemannian manifold (M^4, h) for some given $SU(n)$ -bundle over M^m we consider connections A solutions to the equation

$$*_h F_A = -F_A \quad . \tag{VIII.12}$$

This equation is issued from an elliptic complex (see [13]) and is the natural generalization in 4 dimension of the *flat connection equation* $F_A = 0$ on riemann surfaces considered to classify holomorphic complex structures over such a surface. It defines special solutions to Yang-Mills equation. Indeed taking the covariant exterior derivative with respect to A , using the Bianchi identity $d_A F_A = 0$ one obtains $d^{*h} F_A = 0$. The *anti-self-dual equation* (VIII.12) is generalized in higher dimension

$$*_h F_A = -\Omega \wedge F_A \quad , \tag{VIII.13}$$

where Ω is a closed $m - 4$ form. Again, due to *Bianchi identity*, by taking the covariant exterior derivative with respect to the connection A , using *Leibnitz identity* on 1-derivations and the fact that Ω is closed, we obtain the Yang-Mills equation $d^{*h} F_A = 0$. The Ω -anti-self-dual equation is not elliptic in general. There are however special situations of geometric interest when the base manifold has a restricted holonomy.

- *Hermitian Yang-Mills fields.*

Let (M^{2n}, h) be an even dimensional riemannian manifold. We assume that M^{2n} is equipped with an integrable complex structure J_M - i.e. the brackets operation leaves the space of $1 - 0$ vector fields of $TM \otimes \mathbb{C}$ invariant

$$\forall X, Y \text{ vector fields} \quad J[X - iJX, Y - iJY] = i[X - iJX, Y - iJY] \quad .$$

Finally we assume that (M^{2m}, h, J_M) is Kähler : $\omega(\cdot, \cdot) := g(\cdot, J_M \cdot)$ is a closed 2-form. It defines a non degenerate 2-form and $\omega^m/m! = \text{dvol}_g$. Let

$$\Omega := \omega^{m-2}/(m-2)!$$

Consider an hermitian vector bundle E associated to a principal $SU(n)$ bundle over M^m with projection map $\pi : E \rightarrow M^{2m}$. A connection ∇ is Ω anti self-dual if and only if it satisfies the Hermitian Yang-Mills equations :

$$*F_{\nabla} = -\Omega \wedge F_{\nabla} \iff \begin{cases} F_{\nabla}^{0,2} = 0 \\ \omega \cdot F_{\nabla}^{1,1} = 0 \end{cases}$$

where $F_{\nabla}^{0,2}$ (resp. $F_{\nabla}^{1,1}$) is the $0-2$ (resp. $1-1$) part of the curvature (the space $T^*M \otimes \mathbb{C}$ is decomposed according to the eigen-spaces of J_M for the eigenvalues i and $-i$) so the Hermitian Yang-Mills equation

$$\begin{cases} F_{\nabla}^{0,2} = 0 & \Leftrightarrow \quad \forall X, Y \quad F_{\nabla}(X - iJX, Y - iJY) = 0 \\ \omega \cdot F_{\nabla}^{1,1} = 0 & \Leftrightarrow \quad \sum_{l=1}^m (F_{\nabla})_{\epsilon_l, J\epsilon_l} = 0 \quad \text{where } \omega = \sum_{l=1}^m \epsilon_l \wedge J\epsilon_l \end{cases}$$

and ϵ_l denotes an orthonormal basis of (T^*M^{2m}, h) .

The *Hermitian Yang-Mills* equation can be interpreted as follows. The *equivariant horizontal distribution of plane* H associated to ∇ defines an almost complex structure J_{∇} on E in the following way

$$\forall \xi \in E \quad \forall X \in T_{\xi}E \quad J_{\nabla}(X) := J_E(X^V) + (J_M(\pi_*X))^H \quad ,$$

we recall that X^V is the projection onto the tangent vertical space (the kernel of the projection π_*) with respect to the horizontal plane H , and J_E is the complex structure

on the tangent vertical space defined by the $SU(n)$ structure group of the bundle. The first part of the equation *Hermitian Yang-Mills* can be reformulated as follows

$$\forall X, Y \text{ vector fields in } M^{2m} \quad F^{0,2}(X, Y) = 0 \iff$$

$$J_{\nabla} [(X - i J_M X)^H, (Y - i J_M Y)^H] = i [(X - i J_M X)^H, (Y - i J_M Y)^H] \quad ,$$

which is equivalent to say that J_{∇} is integrable and that the hermitian bundle is holomorphic.

If A is a $su(n)$ -valued 1-form representing ∇ in an orthonormal trivialisation, this integrability condition implies, by switching to a local holomorphic trivialisation, that there is a gauge change g (non-unitary anymore but taking values into $Gl(n, \mathbb{C})$) such that locally

$$g^{-1} \bar{\partial} g + g^{-1} A^{0,1} g = (A^g)^{0,1} = 0$$

and the 0 – 1 part of the connection ∇ in this holomorphic trivialization coincide with $\bar{\partial}$. Since A is taking values into $su(n)$, we have that $A^{1,0} = -\overline{A^{0,1}{}^t}$. Thus we obtain the fact that

$$(A^{1,0})^g = h^{-1} \partial h \quad ,$$

where $h := \bar{g}^t g$ is taking values into invertible self-dual matrices. So the so called "Einstein part" of the equation $\omega \cdot F_{\nabla}^{1,1} = 0$ becomes equivalent to the non-linear elliptic equation¹⁵

$$\omega \cdot \bar{\partial}[h^{-1} \partial h] = 0 \quad .$$

- *SU(4)-instantons in Calaby-Yau 4-folds.*

In high energy physics and later in geometry (see the PhD thesis of C.Lewis , [14], [57]) the following generalisation of *instantons* has been introduced. Consider an $SU(n)$ principal bundle P over (M^8, g) a *Calabi-Yau manifold* of complex dimension 4 - i.e. (M^8, g) has holonomy $SU(4)$. Such a manifold posses an integrable complex structure J_M for which (M^8, g, J_M) is Kähler and it posses in addition a global holomorphic 4 – 0 form θ of unit norm - unique modulo unit complex number multiplication - and satisfying

$$\theta \wedge \bar{\theta} = \frac{\omega^4}{4!} \quad .$$

The 4–form Ω defined by

$$\Omega := 4 \Re(\theta) + \frac{\omega^2}{2}$$

¹⁵Recall that in a Kähler manifold $\omega \cdot \bar{\partial}\partial = \Delta$

is closed, parallel and of unit *co-mass* (see [57] lemma 4.4.1) : $\forall x \in M$

$$1 = \|\Omega\|_*(x) = \sup \left\{ \frac{\langle \Omega, u_1 \wedge u_2 \wedge u_3 \wedge u_4 \rangle}{\prod_{i=1}^4 |u_i|} \quad ; \quad u_i \in T_x M \setminus \{0\} \right\} .$$

The holomorphic 4 – 0 form θ defines an isometry of the space of 0 – 2 forms on M^8 such that

$$\forall \alpha, \beta \in \wedge^{0,2} M^8 \quad \alpha \wedge *_\theta \beta = \alpha \cdot \beta \bar{\theta} \quad .$$

Some basic computation gives the the Ω –anti self dual equation in this case is equivalent to

$$*_F \nabla = - \Omega \wedge F_\nabla \quad \iff \quad \begin{cases} (1 + *_\theta) F_\nabla^{0,2} = 0 \\ \omega \cdot F_\nabla^{1,1} = 0 \quad , \end{cases}$$

which is also known as the $SU(4)$ *instanton equation*.

VIII.4 Tian’s regularity conjecture on Ω –anti-self-dual instantons.

Tian’s result in the case of Ω anti-self dual instantons for a closed $m – 4$ –form Ω of co-mass less than 1 is the following.

Theorem VIII.9. *Let (M^m, h) be a compact riemannian manifold. Let Ω be a smooth $m – 4$ closed form in M^m . Assume Ω has co-mass less than 1*

$$\|\|\Omega\|_*\|_{L^\infty(M^m)} \leq 1 \quad ,$$

where

$$|\Omega|_*(x) := \sup \left\{ \frac{\langle \Omega, u_1 \wedge \cdots \wedge u_{m-4} \rangle}{\prod_{i=1}^{m-4} |u_i|} \quad ; \quad u_i \in T_x M \setminus \{0\} \right\} .$$

Let E be an hermitian vector bundle associated to an $SU(n)$ -bundle over M^m and let ∇^k be a sequence of smooth $SU(n)$ –connections satisfying the Ω anti self dual instantons equation

$$*_h F_{\nabla^k} = - \Omega \wedge F_{\nabla^k} \quad \text{in } M^m \quad .$$

Then, modulo extraction of a subsequence, there exists a $m – 4$ rectifiable closed subset of M^m , K , of finite $m – 4$ Hausdorff measure, $\mathcal{H}^{m-4}(K) < +\infty$, an hermitian bundle E_0 defined over $M^m \setminus K$ and a smooth $SU(n)$ connection ∇^∞ of E_0 such that

$$*_h F_{\nabla^\infty} = - \Omega \wedge F_{\nabla^\infty} \quad \text{in } M^m \setminus K \quad , \quad (\text{VIII.14})$$

moreover

$$\forall B_r(x_0) \subset M^m \setminus K \quad \exists \text{ a sequence of trivializations s. t.}$$

$$A^{k'} \rightarrow A^\infty \quad \text{strongly in } C^l(B^m) \quad \forall l \in \mathbb{N} \quad ,$$

where $\nabla^{k'} \simeq d + A^{k'}$ in these trivializations and $\nabla^\infty \simeq d + A^\infty$ in a trivialization of E_0 over B^m . Moreover there exists an integer rectifiable current C such that for any smooth $m - 4$ form φ on M^m

$$\int_{M^m} \text{Tr}(F_{\nabla^{k'}} \wedge F_{\nabla^{k'}}) \wedge \varphi \rightarrow \int_{M^m} \text{Tr}(F_{\nabla^\infty} \wedge F_{\nabla^\infty}) \wedge \varphi + 8\pi^2 C(\varphi) \quad ,$$

and the current C is calibrated by Ω

$$C(\Omega) = \mathbf{M}(C) = \sup\{C(\varphi) ; \|\varphi\|_{L^\infty(M^m)} \leq 1\} \quad ,$$

where \mathbf{M} is the mass of the current C . Finally the following convergence holds weakly as Radon measures

$$|F_{\nabla^{k'}}|_h^2 \, d\text{vol}_h \rightarrow |F_{\nabla^\infty}|_h^2 \, d\text{vol}_h + 8\pi^2 \Theta(C) \mathcal{H}^{m-4} \llcorner K \quad ,$$

where $\Theta(C)$ is the integer valued L^1 function with respect to the restriction of the $m - 4$ Hausdorff measure to K and which is giving the multiplicity of the current C at each point. \square

Observe that no bound is a-priori needed for the Yang-Mills energy of the sequence. We have indeed

$$\begin{aligned} YM(\nabla^k) &= - \int_{M^m} \text{Tr}(F_{\nabla^k} \wedge *_h F_{\nabla^k}) \, d\text{vol}_h \\ &= \int_{M^m} \text{Tr}(F_{\nabla^k} \wedge F_{\nabla^k}) \wedge \Omega \quad . \end{aligned}$$

Since Ω is closed this integral only depends on the cohomology class of the other closed form $\text{Tr}(F_{\nabla^k} \wedge F_{\nabla^k})$ which is the second Chern class of E and which is independent of ∇^k .

Remark VIII.2. Regarding the limiting bundle E_0 , it is important to insist on the fact that there is no reason for E_0 to extend through K over the whole manifold M^m as a smooth bundle. Hence there is a-priori no meaning to give to ∇^∞ over the whole manifold and therefore the Ω -anti-self dual equation (VIII.14) cannot even hold in a distributional sense throughout K if we do not relax the notion of bundle and connections. \square

An important question directly related to the regularity of the limiting configuration (E_0, ∇^∞, C) is the following open problem.

Open problem. *Show that the limiting current C has no boundary :*

$$\partial C = 0 \quad .$$

□

The resolution of this conjecture would imply that the limiting connection (though defined on $M^m \setminus \text{supp}(C)$ only) satisfies globally the stationary equation VIII.5 (i.e. more precisely the version of (VIII.5) on (M^m, h)). One proves that the open question about proving that $\partial C = 0$ is equivalent to

$$d(\text{Tr}(F_{\nabla^\infty} \wedge F_{\nabla^\infty})) = 0 \quad .$$

This last identity should be equivalent to the following

strong approximation property : Does there exist a sequence D^k of smooth $SU(n)$ -connection of smooth hermitian bundles over M^m such that

$$\lim_{k \rightarrow +\infty} \inf_{g \in W^{1,2}} \int_{M^m} |\nabla^\infty - (D^k)^g|_h^2 + |F_{\nabla^\infty} - g^{-1}F_{D^k}g|_h^2 \, d\text{vol}_h = 0 \quad ?$$

If we would know that $\partial C = 0$ then C defines a *calibrated integral cycle*. The optimal regularity for *calibrated* and *semi-calibrated* integral cycles of dimension 2 has been proved respectively in [44] and [12]. Such cycles have at most isolated point singularities. More generally, the calibrated condition implies that such a cycle is homologically *mass minimizing* (see [18]). From this later fact, using Almgren regularity result [2] proved also by C.De Lellis and E.Spadaro ([9], [10] and [11]), we would obtain that C is the integration along a rectifiable set which is a smooth dimension $m - 4$ sub-manifold away from a co-dimension 2 singular set with smooth integer multiplicity away from that set. This result is optimal : integration along holomorphic curves in $\mathbb{C}P^n$ is a calibrated integral cycle for the Fubini Study Kähler form and can have isolated singularities which are of co-dimension 2 within the curve.

In his paper Tian made the following conjecture.

Tian regularity conjecture *Let (E_0, ∇^∞) be the weak limit of smooth Ω -anti-self dual instantons on a bundle E . Then the limiting bundle E_0 and the limiting connections ∇^∞ extend to smooth bundle, resp. smooth connection, away from a closed co-dimension 6 set L in M^m .* □.

There is one case which has been completely settled and where the conjecture has been proved. This is the case of *Hermitian Yang-Mills* fields. In that case the currents defined by

$$\varphi \longrightarrow \int_{M^{2m}} \text{Tr}(F_{\nabla^k} \wedge F_{\nabla^k}) \wedge \varphi$$

is a $(m-2) - (m-2)$ positive cycle (i.e. calibrated by $\omega^{m-2}/(m-2)!$), This condition is of course preserved at the limit and then

$$\varphi \longrightarrow \int_{M^m} \text{Tr}(F_{\nabla^\infty} \wedge F_{\nabla^\infty}) \wedge \varphi + 8\pi^2 C(\varphi)$$

is also a $(m-2) - (m-2)$ positive cycle (i.e. calibrated by $\omega^{m-2}/(m-2)!$). The points of non zero density correspond to the support of C and using an important result by Y.T. Siu [51] we obtain that C is the integration along an holomorphic sub-variety of complex co-dimension 2. Using now a result by S.Bando and Y.T.Siu [4] we obtain that E_0 extends to an analytic reflexive sheaf over the whole M^{2m} it is then locally free (i.e. it realizes a smooth bundle) away from a closed complex co-dimension 3 subset of M^{2m} which is included in K (see for instance [22]). They also prove a point removability asserting that ∇^∞ defines a smooth connection on the part where the sheaf is free which proves the *Tian regularity conjecture* in the special case of *Hermitian Yang-Mills* fields.

VIII.5 The space of weak connections.

As we saw in the previous section, in dimension larger than 4, "bundles with singularities" arise naturally as "carriers" of limits of smooth Yang-Mills fields with uniformly bounded energy over smooth bundles. If now we remove the assumption to be Yang-Mills and just follow sequences of connections with uniformly bounded Yang-Mills energy over smooth bundles we have seen in the beginning of this section that the limiting carrying bundle can have twists everywhere on the base ! Similarly, taking a sequence of closed sub-manifolds with uniformly bounded volume the limit "escape" from the space of smooth sub-manifold and can be singular. The main achievement of the work of Federer and Fleming has been to introduce a class of objects, the *integral cycles* which complete the space of closed sub-manifold with uniformly bounded volume and which was suitable to solve the *Plateau problem* in a general framework. The purpose of the work [34] is to define a class of *weak bundles* and *weak connections* satisfying a closure property under uniformly bounded Yang-Mills energy and suitable to solve the *Yang-Mills Plateau problem*.

We introduce the following stratified definition.

Definition VIII.2. *Let G be a compact Lie group and (M^m, h) a compact riemanian manifold. For $m \leq 4$ the space of weak connections $\mathfrak{A}_G(M^m)$ is defined to coincide with the space of Sobolev connections defined by*

$$\mathfrak{A}_G(M^m) := \left\{ \begin{array}{l} A \in L^2(\wedge^1 M^m, \mathcal{G}) ; \int_{M^m} |dA + A \wedge A|_h^2 \, d\text{vol}_h < +\infty \\ \text{locally } \exists g \in W^{1,2} \quad \text{s.t.} \quad A^g \in W^{1,2} \quad . \end{array} \right\}$$

For $m > 4$ we define the space of weak connections $\mathcal{A}_G(M^m)$ to be

$$\mathcal{A}(M^m) := \left\{ \begin{array}{l} A \in L^2(\wedge^1 M^m, \mathcal{G}) ; \int_{M^m} |dA + A \wedge A|_h^2 \, d\text{vol}_h < +\infty \\ \forall p \in M^m \quad \text{for a.e. } r > 0 \quad \iota_{\partial B_r(p)}^* A \in \mathcal{A}_G(\partial B_r^m(p)) \end{array} \right\} ,$$

where $B_r^m(p)$ denotes the geodesic ball of center p and radius $r > 0$ and $\iota_{\partial B_r(p)}^* A$ is the restriction of the 1-form A on the boundary of $B_r^m(p)$. \square

As an illustration of this space, It is not difficult to check that the limiting connections ∇^∞ from Tian's closure theorem are in \mathcal{A}_G . We have the following result which justifies the previous definition.

Theorem VIII.10. *For any $m \in \mathbb{N}^*$ the space of weak connection $\mathcal{A}_G(B^m)$ is weakly sequentially closed bellow any Yang-Mills energy level. Precisely, let $A^k \in \mathcal{A}_G(B^m)$ such that*

$$\limsup_{k \rightarrow +\infty} \int_{B^m} |dA^k + A^k \wedge A^k|^2 \, dx^m < +\infty \quad ,$$

then there exists a subsequence k' and $A^\infty \in \mathcal{A}_G(B^m)$ such that

$$\delta(A^{k'}, A^\infty) := \sup_{f \in \text{Lip}_1(B^m, \mathbb{R}^{m-4})} \inf_{g \in W^{1,2}(B^m, G)} \int_{B^m} |A^{k'} - (A^\infty)^g \wedge f^* \omega|^2 \frac{dx^m}{|f^* \omega|} \longrightarrow 0 \quad ,$$

where $\text{Lip}_1(B^m)$ is the space of lipschitz function on B^m with norm bounded by 1 and $\omega := dx_1 \wedge \cdots \wedge dx_{m-4}$. \square

Remark VIII.3. *In the statement of theorem VIII.10 δ can be replaced by the more coercive gauge invariant L^2 distance (VIII.1) when $G = SU(2)$. It is expected this fact to be true at least for $G = SU(n)$ and n arbitrary. It is related to an interesting problem in multilinear algebra (see [35]).* \square

Remark VIII.4. *Theorem VIII.10 can be seen as a Rellich Kondrachov type result for weak connections with L^2 -bounded curvatures.* \square

A proof of theorem VIII.10 is given in [34] and [35]. It is using the following strong approximation theorem whose proof is rather technical.

Theorem VIII.11. *Let $A \in \mathcal{A}_G(B^m)$ then there exists A^k which are the connection \mathcal{G} 1-forms on B^m associated to smooth connections of a sequence of smooth bundles over B^m minus polyhedral chains of codimension 5 such that*

$$\lim_{k \rightarrow +\infty} \inf_{g \in W^{1,2}(B^m, G)} \int_{B^m} |A - (A^k)^g|_h^2 + |F_A - g^{-1} F_{A^k} g|^2 \, dx^m = 0 \quad .$$

\square

These two last results take their roots in the following weak sequential closure, respectively strong approximation, results proved in [32].

Theorem VIII.12. *Let $p > 1$ and let F_k be a sequence of L^p two forms in B^3 such that*

$$\forall p \in B^3 \quad \text{for a.e. } B_r(p) \subset B^3 \quad \int_{\partial B_r(p)} F_k \in 2\pi \mathbb{Z} \quad .$$

Assume $F_k \rightharpoonup F_\infty$ weakly in L^p then

$$\forall p \in B^3 \quad \text{for a.e. } B_r(p) \subset B^3 \quad \int_{\partial B_r(p)} F_\infty \in 2\pi \mathbb{Z} \quad .$$

□

The strong approximation result says the following.

Theorem VIII.13. *Let $p > 1$ and let F be an L^p two forms in B^3 such that*

$$\forall p \in B^3 \quad \text{for a.e. } B_r(p) \subset B^3 \quad \int_{\partial B_r(p)} F \in 2\pi \mathbb{Z} \quad .$$

then there exists a sequence A^k of smooth connections of a sequence of smooth $U(1)$ bundles over B^3 minus finitely many points such that

$$\lim_{k \rightarrow +\infty} \int_{B^3} |F_{A^k} - F|^p dx^3 = 0 \quad .$$

□

VIII.6 The resolution of the Yang-Mills Plateau problem in 5 dimensions.

In the present section we are solving the ‘‘Schoen-Uhlenbeck type’’ question introduced for the first time in [48] for harmonic maps. We call it in our framework ‘‘Yang-Mills-Plateau Problem’’. We present the result in 5 dimension which has appeared in [34] while the higher dimensional case is under preparation.

In order to solve the Yang-Mills Plateau problem in the 5-dimensional ball B^5 , we introduce a sub-space of $\mathcal{A}_{SU(n)}(B^5)$ of weak connections admitting a trace.

Let η be a weak connection 1-form of $\mathcal{A}_G(S^4)$ we denote by $\mathcal{A}_{SU(n)}^\eta(B^5)$ the subspace of $\mathcal{A}_{SU(n)}(B^5)$ made of weak connection 1-forms A such that

$$\lim_{r \rightarrow 1^-} \inf_{g \in W^{1,2}(S^4, G)} \int_{S^4} |D_r^* A - g^{-1} dg - g^{-1} \eta g|^2 dvol_{S^4} \quad ,$$

where $D_r^* A$ is the pull-back on $S^4 \simeq \partial B_1^5(0)$ of the restriction of A to $\partial B_r^5(0)$ by the dilation of ratio r . It is proved in [34] that for any $\eta \in \mathcal{A}_G$ the space \mathcal{A}_G^η is weakly sequentially closed under any Yang-Mills energy level. Hence we have the following theorem.

Theorem VIII.14. [Existence for YM minimizers] *Let G be a compact Lie group and let η be an arbitrary Sobolev connection one form in $\mathcal{A}_G(S^4)$ then the following minimization problem is achieved*

$$\inf \left\{ \int_{B^5} |dA + A \wedge A|^2 dx^5 ; A \in \mathcal{A}_G^\eta(B^5) \right\} . \quad (\text{VIII.15})$$

□

Solutions to this minimisation problem will be called *solutions to the Plateau problem* for the boundary connection η .

Once the existence of the minimiser is established it is legitimate to ask about the regularity for these solutions to the *Yang-Mills Plateau problem*. Before to give the optimal regularity we give an intermediate result which holds in the general class of *stationary Yang-Mills* in the space of *weak connections* $\mathcal{A}_G(B^5)$.

Theorem VIII.15. [ϵ -regularity for weak stationary YM fields in 5D] *Let G be a compact Lie group, there exists $\varepsilon_G > 0$ such that for any weak connection 1-forms A in $\mathcal{A}_G(B^5)$ satisfying weakly the Yang-Mills equation*

$$d^* F_A + [A, \lrcorner F_A] = 0 \quad \text{in } \mathcal{D}'(B_1^5(0)) .$$

assume also that it satisfies the stationarity condition

$$\forall X \in C_0^\infty(B^5, \mathbb{R}^5) \quad \left. \frac{d}{dt} \int_{B^5} |\Psi_t^* F_A|^2 \right|_{t=0} = 0 ,$$

where Ψ_t is the flow of X , and the small energy condition

$$\int_{B_1^5(0)} |F_A|^2 dx^5 < \varepsilon_G ,$$

then there exists a gauge $g \in W^{1,2}(B_{1/2}^5(0), G)$ in which the following estimates hold : for any $l \in \mathbb{N}$ there exists $C_l > 0$ such that

$$\|\nabla^l(A)^g\|_{L^\infty(B_{1/2}^5(0))}^2 \leq C_l \int_{B_1^5(0)} |F_A|^2 dx^5 .$$

□

This theorem is a consequence the *monotonicity formula* deduced from the *stationarity condition* and a weak version of the *Coulomb gauge extraction theorem VIII.3* which generalizes the work of T.Tao- G.Tian [54] for *admissible connections* or the work of Y.Meyer and the author [26] for *approximable connections* to the general framework of *weak connections* in $\mathcal{A}_G(B^5)$.

With the ϵ -regularity result at hand, using a *Luckhaus lemma* together with a *Federer dimension reduction method* following the proof of the regularity result of R.Schoen and K.Uhlenbeck for minimizing harmonic map [48] we obtain the following theorem.

Theorem VIII.16. [Regularity for minimizers of the Yang-Mills Plateau problem.] *Let G be a compact Lie group, and let η be the connection 1-form associated to a smooth connection of some G -bundle over ∂B^5 then the minimizers of the Yang-Mills Plateau problem (VIII.15) are smooth connections ∇ of a smooth bundle defined on B^5 minus finitely many points. For $G = SU(2)$ we have*

$$d(\text{tr}(F_\nabla \wedge F_\nabla)) = \sum_i \varepsilon_i \delta_{a_i} \quad \text{with} \quad \varepsilon_i \in \{-1, +1\} \quad ,$$

where a_i are the point singularities. □

It is proved in [34] that the theorem is optimal in the sense that there exists a smooth $su(2)$ -valued 1-form η - defining then a connection of the trivial bundle over S^4 - such that any solution to the Plateau problem (VIII.15) has isolated singularities.

VIII.7 Weak Holomorphic Structures over Complex Manifolds.

With the definition of weak connection at hands one can reformulate a more general version of **Tian's regularity conjecture** as follows.

Conjecture. *Let (M^m, h) be a compact riemannian manifold and Ω a closed $m - 4$ form of co-mass less than one. Let A be a **weak connection** in $\mathcal{A}(M^m)$ for the Lie group G and solving*

$$*_h F_A = -\Omega \wedge F_A \quad \text{in} \quad \mathcal{D}'(M^m) \quad .$$

Does A define a smooth connection of some smooth G -bundle over the whole manifold minus a closed subset of co-dimension 6.

A case of special interest for instance is given by the space of *Weak Hermitian Yang-Mills* $SU(n)$ -connections over a Kähler manifold (M^{2p}, J, ω) : the elements in $\mathcal{A}_{SU(n)}(M^{2p})$ satisfying

$$\begin{cases} F_A^{0,2} = 0 \\ \omega \cdot F_A^{1,1} = 0 \end{cases} \quad \text{in} \quad \mathcal{D}'(M^{2p}) \quad . \quad \text{(VIII.16)}$$

In order to study this question it is of interest to first concentrate on density properties within the following space of *weak holomorphic structures*

$$\mathcal{A}_{SU(n)}^{1,1}(M^{2p}) := \{A \in \mathcal{A}_{SU(n)}(M^{2p}) \quad ; \quad F_A^{0,2} = 0 \quad \text{a. e.}\} \quad .$$

The following result asserts that weak holomorphic structures in 2 complex dimension (the dimension for which the Yang-Mills energy is critical) are defining classical smooth holomorphic structures and are strongly approximable by smooth ones.

Theorem VIII.17. [31] Let (M^4, J, ω) be a Kähler 2 dimensional manifold and $A \in \mathcal{A}_{SU(n)}^{1,1}(M^{2p})$ and denote by E the smooth $SU(n)$ -bundle defined by A and by ∇ the associated Sobolev connection (see proposition VIII.1). Then there exists a smooth holomorphic structure \mathcal{E} on E such that

$$\nabla = \partial_0 + h^{-1}\partial_0h + \bar{\partial}_{\mathcal{E}} \quad ,$$

where h and h^{-1} are $W^{2,p}$ -sections of $GL(E)$ for any $p < 2$, ∂_0 and $\bar{\partial}_{\mathcal{E}}$ are respectively the $1 - 0$ and the $0 - 1$ parts of the Chern connection associated to \mathcal{E} and the choice of some smooth Hermitian structure on the bundle. Moreover there exists a sequence of smooth connections ∇^k satisfying

$$F_{\nabla^k}^{0,2} = 0 \quad ,$$

and

$$\lim_{k \rightarrow +\infty} \inf_{g \in W^{1,2}(M^4, G)} \int_{M^4} |A - (\nabla^k)^g|_h^2 + |F_A - g^{-1}F_{\nabla^k}g|_h^2 \, dvol_h = 0 \quad ,$$

where $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$. □

Remark VIII.5. It is natural to expect the above theorem VIII.17 to be extended in higher dimension but allowing singularities and replacing in the result the smooth holomorphic bundle \mathcal{E} by a the more general notion of reflexive sheaf. □

Finally we are closing this section by mentioning 2 open problems relating weak $1 - 1$ curvatures and the strong approximability property.

Open Problem 1 Let F be a $1 - 1$ real 2-form in \mathbb{C}^2 such that $F \in L_{loc}^p(\mathbb{C}^2)$ for $1 \leq p$ and satisfying for almost every two spheres S in \mathbb{C}^2

$$\int_S F \in \mathbb{Z} \quad . \tag{VIII.17}$$

Prove that F is a closed form¹⁶

The result should still hold in the almost complex framework. As a matter of illustration we mention the following open question. In [45], Tian and the author of these notes establish the optimal partial regularity result for $W^{1,2}$ -pseudo-holomorphic maps from an almost Kähler¹⁷ four dimensional real manifold (M^4, J, ω) into a projective space $\mathbb{C}P^n$

¹⁶If $p \geq 4/3$, using Hölder inequality, one proves that under these assumptions $\int_S F$ is equal to zero for almost every 2-spheres and then F is closed.

¹⁷The form ω , compatible with J , is closed but J is not necessarily integrable.

assuming $u^*\omega_{\mathbb{C}P^n}$ is closed¹⁸ where $\omega_{\mathbb{C}P^n}$ is the Fubini Study Kähler 2-form on $\mathbb{C}P^n$.

A priori only the condition (VIII.17) is satisfied for $F := u^*\omega$ but it is natural to ask whether $d(u^*\omega) = 0$ always hold when u is pseudo-holomorphic and if the closdness assumption made in [45] is redundant or not.

Open Problem 2 *Let (M^{2p}, J, ω) be a p -dimensional complex manifold equiped with a Kähler structure and let $A \in \mathcal{A}_{SU(n)}^{1,1}(M^{2p})$. Does the following identity¹⁹ hold true*

$$d(\text{tr}(F_A \wedge F_A)) = 0 \quad . \quad (\text{VIII.18})$$

□

If (VIII.18) would always hold true under the assumption that $A \in \mathcal{A}_{SU(n)}^{1,1}(M^{2p})$, we could then deduce from such an approximation property the Tian's regularity conjecture in the particular case of *Weak Hermitian Yang-Mills $SU(n)$ connections* over a closed p -Kähler manifold in it's full generality²⁰ .

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¹⁸The condition

$$d(u^*\omega) = 0$$

is obviously always satisfied for any smooth maps. It is equivalent to the local strong approximability by smooth maps for a map in $W^{1,2}(M^4, \mathbb{C}P^n)$. Observe that it is not satisfied by every map in $W^{1,2}(M^4, \mathbb{C}P^n)$. For the particular case of $W^{1,2}$ -pseudo-holomorphic maps the local strong approximability is proven in [45] to be equivalent to the stationarity condition. This is very special to the pseudo-holomorphic case. In general there exists minimizing harmonic maps from B^3 into $\mathbb{C}P^1$, and hence stationary, where $d(u^*\omega_{\mathbb{C}P^1}) \neq 0$.

¹⁹This identity is equivalent to the following strong approximation question : does there exists a sequence of smooth $SU(n)$ -connections ∇^k satisfying

$$\lim_{k \rightarrow +\infty} \inf_{g \in W^{1,2}(M^4, G)} \int_{M^4} |A - (\nabla^k)^g|_h^2 + |F_A - g^{-1}F_{\nabla^k}g|_h^2 \, d\text{vol}_h = 0 \quad ?$$

²⁰The case of "admissible" *Weak Hermitian Yang-Mills $SU(n)$ connections* which are weak limits of smooth ones is already solved by the mean of Siu's results in [51].

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