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The Rates of Growth in a Limit Group

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Abstract

We study the set of growth rates of a limit group with respect to all its finite generating sets. To this end, we examine the asymptotic geometry of hyperbolic groups and, in particular, use the theory of asymptotic cones to define from a sequence of morphisms between free groups, a faithful action of a limit group on a real tree. In addition, we review the theory of limit groups with a topological approach, as developed in [CG05], and show that they are equationally Noetherian without recurring to Rips theory. Finally, we use this fact to prove that the set of growth rates of any limit group is well-ordered and therefore provide a self-contained proof of a result obtained by K. Fujiwara and Z. Sela in [FS20] which does not rely on Rips machines.

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Contents

Introduction

All relevant definitions and results will be introduced rigorously in the subsequent sections.

Motivation and context

The study of growth of a finitely generated group goes back to the works of A.S. Schwarz in [Šva55] and independently by J. Milnor in [M$^+$68] even if some preliminary cases of growth were considered by H.U. Krause in [Kra53] and geometric growth was already studied by V.A. Efremovich in [Efr53]. The growth type essentially measures the "volume" of balls in certain metric spaces and we study its asymptotic behaviour when the radius of the balls tends to infinity. In the case of finitely generated groups, the growth function of a group G with respect to a finite generating set S is the function that associates to any real number t the number $\#B_S(G,t)$ of elements in the open ball of radius t in G about the identy element for the S–word length. Then, one can define an equivalence relation on the set of growth functions of G to define a group invariant which is the growth type of G, see Subsection 4.1. The motivation of Schwarz and Milnor to analyse the growth of groups were of geometric nature. It was observed by Schwarz and Efremovich that the growth of the volumes of Riemannian balls in the universal cover of a closed Riemannian manifold is related to the growth of its fundamental group. At the same time, Milnor and Wolf demonstrated that the growth type of the fundamental group of a compact Riemannian manifold gives important information about the mean, or sectional, curvature of the manifold. Subsequently, it was shown that the study of group growth can serve as a tool for various problems in differential geometry. For example, it has been used for the classification of homogeneous Riemannian manifolds with zero Ricci curvature, see [AK75].

In his seminal work, Milnor formulated a question about the types of growth admissible by a finitely generated group. More precisely, Milnor asked: "Is it true that the growth function of every finitely generated group is necessarily equivalent to a polynomial or to the function 2^n?". This question was answered in the negative by R. Grigorchuk in [Gri83], where the author constructed a group of intermediate growth. Despite the negative character of the answer, the existence of groups of intermediate growth has enriched group theory and the areas of its applications. As example, groups of intermediate growth are related to fractal geometry and the study of dynamical systems, see [BGN03]. Furthermore, one example of dynamical property related to growth of groups was examined by G.M. Adel'son-Vel'skii and Y.A. Shreider in [AVS57]. They showed that groups with subexponential growth are amenable and therefore have strong dynamical properties. Far from being exhaustive, other remarkable applications resulting from the study of growth of finitely generated groups lie in the fields of geometric group theory, dynamical systems or random walks. We refer to [Gri14] for more examples and references.

All these motivations lead to the study of growth of finitely generated groups. In a short period of time, Milnor, Wolf, Hartley, Guivarc'h and Bass discovered that nilpotent groups have polynomial growth of integer degree. The converse result, namely that any finitely generated group with polynomial growth contains a nilpotent subgroup of finite index was proved by M. Gromov in his celebrated paper [Gro81]. This result has stimulated many activities in different areas of mathematics. As an example, one application of this theorem is the quasi-isometric rigidity of finitely generated abelian groups. That is, any group which is quasi-isometric to a finitely generated abelian group contains a free abelian subgroup of finite index. For our case of interest, Gromov's polynomial growth Theorem, together with the fact that any group containing a free subsemigroup on at least two generators has exponential growth, produces numerous examples of finitely generated group with exponential growth and therefore objects of study. Some classes of groups with exponential growth are solvable non-virtually nilpotent groups, non-elementary Gromov hyperbolic groups or non-virtually nilpotent linear groups which are all known to contain a free subsemigroup on two generators. Moreover, note that all groups contained in these three classes are of uniform exponential growth.

Finally, our work falls within a broader framework. On the one hand, we notice that the notion of growth can be defined for many algebraic and combinatorial objects, especially for semigroups, associative and Lie algebras, graphs and discrete metric spaces, see [dlHGCS99]. On the other hand, the theory of growth of groups is part of a larger area of mathematics which studies the coarse asymptotic properties of various algebraic and geometric objects. As R. Grigorchuk points out in [Gri14], the second period of studies of group growth begins in the eighties and splits into three directions: the study of analytic properties of growth series, the study of groups of intermediate growth, and the study around Gromov's problem on the existence of groups of exponential but not uniformly exponential growth, see [GLP81]. Even if L. Bartholdi and J. Wilson prove, respectively in [Bar03] and [Wil04], the existence of finitely generated groups of exponential growth but not of uniform exponential growth, this master's thesis focuses on the third point and studies groups with uniform exponential growth. More specifically, we study the possible growth rates of groups of uniform exponential growth.

Present work

The present paper aims to prove the well-ordering of the set of growth rates, which is an invariant of groups, of any limit group. That is, groups that are limits of free groups in a compact space of marked groups, see Section 3. We are therefore not interested in the growth rate with respect to a particular finite generating set but rather in the countable set of exponential growth rates with respect to all possible finite generating sets of a given limit group. For the rest of this introduction, we denote the growth rate of a group G with respect to a finite generating set S:

$$e\,(G,S) := \lim_{t\to\infty} \#B_S\,(G,t)^{\frac{1}{t}}\ .$$

Then, the group G is of exponential growth if $e\,(G,S) > 1$. Moreover, we denote the set of growth rates of G with respect to any finite generating set by

$$\xi(G) := \{\, e\,(G,S) \ \mid S \text{ is a finite generating set of } G\,\},$$

and we say that G is of uniform exponential growth if $\inf \xi(G) > 1$. In one hand, if the group G is of polynomial growth, then $\xi(G) = \{\,1\,\}$ and thus is well-ordered. On the other hand, if G is of non-uniform exponential growth, then $\xi(G)$ is not well-ordered. This master's thesis is based on the work done by K. Fujiwara and Z. Sela in [FS20], which prove that $\xi(G)$ is well-ordered for any hyperbolic group, and is intended to present a self-contained proof of their result. Their motivation comes from the work of W.P. Thurston on the volumes of hyperbolic 3–manifolds, see [Thu79]. Analysing the volumes of Dehn fillings of finite volume hyperbolic 3–manifolds, Thurston proves that the set of volumes of hyperbolic 3–manifolds is well-ordered. Similarly, Fujiwara and Sela examine the set of growth rates in a hyperbolic group which are groups with strong geometric properties. In particular, their asymptotic geometries are well understood.

It is a well-known result in geometric group theory that any non-elementary hyperbolic group contains a non-abelian free subgroup, and therefore has exponential growth. In fact, any non-elementary hyperbolic group has uniform exponential growth, see [Kou98]. For example, a free group has uniform exponential growth and, moreover, it is known that it reaches its minimum growth rate when considering its basis as finite generating set, see [dlH00, page 194]. Other examples of groups that achieve their minimum growth rate are the Baumslag-Solitar group $BS(1,p)$ and the lamlighter group $\mathcal{L}_p$ for $p \geq 3$, see [BT17]. Aside from these examples, and a few other related cases, there is no known classes of groups for which the exponential growth rate achieves its infimum on a known generating set. For the fundamental group of an orientable surface of genus greater than two, only a lower bound for its set of growth rates is known, see [dlH00, page 195]. Also, A. Sambusetti shows in [Sam99] that the infimum of $\xi(G)$ is not reached for any finite generating set when G is a free product $G = G_1 * G_2$ with G_1 non-Hopfian and G_2 non-trivial. So, there exist groups of uniform exponential growth whose set of growth rates does not admit a minimum. However, there exist lower bounds for $\xi(G)$ that are uniform in G, see [BT16]. In addition, G.N. Arzhantseva and I.G. Lysenok give in [AL06] a linear lower bound on the exponential growth rate of any hyperbolic group. We are not able to recover the announced linearity of this bound. However, a slight modification on the actual lower bound of the main result of this article, which is sufficient for our use is the following

Theorem (Theorem 4.1.17)**.** *If Γ is a finitely generated δ–hyperbolic group, then there exist constants α, β that depend only on δ so that, for any finite generating set S'_Γ of Γ, the following inequality holds*

$$e\big(\Gamma, S'_\Gamma\big) \geq \big(2\alpha \# S'_\Gamma - 1\big)^{\beta}.$$

Thus, there exists a number N depending only on the hyperbolic constant of the studied group so that the uniform exponential growth of the group is achieved on a generating set of cardinality less than N. Fujiwara and Sela use this last result to prove the well-ordering of the set of growth rates of any hyperbolic groups. In particular, the set of growth rates of any hyperbolic groups admits a minimum strengthening Koubi's Theorem on uniform exponential growth of hyperbolic groups. Our study mainly uses the theory of asymptotic geometry, more precisely the well-developed theory of groups acting on real trees introduced by R.C. Lyndon and I.M. Chiswell in [Lyn63] and [Chi76] respectively and the theory of limit groups initiated by Z. Sela in [Sel01]. Limit groups were defined by Sela, using the Gromov–Hausdorff convergence, as groups acting faithfully on certain specific trees. Originally, these groups were introduced in order to understand the structure of varieties and first order formulas over certain classes of groups, see [Sel01]. A notable result obtained with these objects is a solution to Tarski's problem: do finitely generated non-abelian free groups have the same elementary theory? However, they provide a natural and powerful tool to study variational problems over groups. In our case, we review the theory of limit groups with a topological approach, using marked groups, as developed in [CG05] and use them to obtain knowledge about the asymptotic geometry of a limit group. This permits us to prove the culmination result of this document, that is, the existence of a minimum for the set of growth rates of any limit group.

Theorem (Theorem 4.2.9)**.** *If L is a limit group, then the set $\xi(L)$ is well-ordered.*

It should be noted that marked groups had already been analysed to study the growth of groups. Especially to construct examples of groups with non-uniform exponential growth, see [Nek10]. The key step to obtain the conclusion of Theorem 4.2.9 is to show that the set of growth rates of a free group F_2 on two elements is well-ordered. To prove this last result in Section 4.2, we assume by contradiction that there exists a sequence of finite generating sets (S_i) of F_2 such that $(e\left(F_2, S_i\right))$ is a subset of $\xi(F_2)$ strictly decreasing and verifying $e\left(F_2, S_i\right) \geq 1$ for every $i \in \mathbb{N}$. As previously announced, group growth is an asymptotic property. Thus, it is interesting to consider only the reccuring features of the group. To do so, we build an action of F_2 on one of its asymptotic cones which is a topological space encoding the asymptotic geometry of F_2. We show in Section 4.2 that the given sequence of finite generating sets of F_2 allows us to define an action of a free group on a real tree, one of its asymptotic cones, using the following theorem.

Theorem (Theorem 2.2.11)**.** *Consider a finitely generated group G with finite generating set S_G and (X, x_i, ρ_i) a sequence of G–spaces where X is hyperbolic. If $(|\rho_i|_{x_i})$ diverges towards infinity, then G acts by isometry on the asymptotic cone $T_\mathfrak{u} := \mathrm{Cone}_\mathfrak{u}\left(X, (x_i), \left(|\rho_i|_{x_i}^{-1}\right)\right)$ for any non-principal ultrafilter $\mathfrak{u}$. In addition, if*

$$|\rho_i|_y \geq |\rho_i|_{x_i} \text{ for every } y \in X \text{ and every } i \in \mathbb{N},$$

then $T_\mathfrak{u}$ has no point fixed by all of G and admits a minimal G–invariant subtree.

In addition, using the theory of limit groups discussed in Section 3, we do not only have an action of a free group on a real tree, but, using the upcoming theorem, a faithful action of a limit group L on that real tree.

Theorem (Theorem 3.3.8)**.** *With the above notation, the action of L on T, the minimal G–invariant subtree of $T_\mathfrak{u}$, verifies*

1. *The stabilizer of any non-degenerate tripod is finite,*
2. *The stabilizer of any non-degenerate arc is finite–by–abelian,*
3. *Every subgroup of L which leaves a line in T invariant and fixes its ends is finite–by–abelian.*

These last two theorems are a generalization of the construction of Bestvina and Paulin's tree introduced in [Bes88] and [Pau88] by the use of asymptotic cones. This generalization follows the same idea as the introduction of asymptotic cones by L. Van den Dries and A.J. Wilkie, see [VdDW84], as a generalization of Gromov's methods using convergence in the Gromov-Hausdorff sense. The faithful action of a limit group L on a specific real tree allows us to use the theory of groups acting on trees to analyse the geometry of subgroups of L. This is in order to build specific elements verifying important small cancellation properties. We prove in Section 3 that free groups are equationally Noetherian. This result supported by the conclusion quoted above by Arzhantseva and Lysenok offers us an upper bound on the growth rate of $e\left(F_2, S_i\right)$. Then, using the elements possessing small cancellation properties as well as the equational Noetherianity of F_2, we show in Subsection 4.2 that there is a strong dependence between the cardinality of balls in F_2 and in the limit group L. This dependency is the last step to obtain a contradiction with the existence of a sequence of finite generating subsets of F_2 such that $e\left(F_2, S_i\right)$ is strictly decreasing. This concludes our proof and allows us to affirm that $\xi(F_2)$ is well-ordered. A diagonal argument makes it possible to generalize this result to all limit groups.

Theorem 4.2.1 is a simplification of a result obtained by Fujiwara and Sela in [FS20]. However, our approach is slightly different in the sense that it does not use the rich theory of actions on $\mathbb{R}^n$–trees and, in particular, does not use the Rips theory or the JSJ–decomposition of certain groups. These theories are used in [FS20] when Fujiwara and Sela use the fact that hyperbolic groups are equationally Noetherian. This fact, due to C. Reinfeldt and R. Weidmann, is demonstrated in [RW10] but was already known for hyperbolic groups without torsion thanks to Sela's work in [Sel09], for some relatively hyperbolic groups without torsion due to Groves in [Gro05] and for free groups proved by O. Kharlampovich and A. Myasnikov in [KM98]. This result is far from trivial and implies, in particular, that hyperbolic groups are Hopfian. In our case, we show that F_2 is equationally Noetherian exploiting the linearity of F_2, the Hilbert basis Theorem, the fact $\mathbb{C}$ is Noetherian and the use of some elementary results of model theory. Another difference between our approach and that of Fujiwara and Sela is the use of asymptotic cones of a finitely generated group instead of the Bestvina and Paulin's tree. This generalization does not use the Gromov–Hausdorff convergence, and thus opens the way to a generalization of our results to other groups with hyperbolic features. Especially, these techniques make it possible to study relatively hyperbolic groups or acylindrically hyperbolic groups.

Last but not least, we want to finish this introduction by mentioning the work of D. Groves and H. Hull in [GH19]. The latter was unknown to the author at the time of writing the first two sections of this thesis. However, the definition of limit group given in Section 3 correponds to that of this article. Thus, we hope that this document provides an introduction to the work of Groves and Hull in which they prove, using Rips machines the following two theorems.

Theorem (Groves and Hull, [GH19, Theorem D])**.** *If Γ is hyperbolic relative to equationally Noetherian groups, then Γ is equationally Noetherian.*

Moreover, in their article, they had already generalized Bestvina and Paulin's tree using asymptotic cones to any acylindrically hyperbolic groups

Theorem (Groves and Hull, [GH19, Theorem 4.4])**.** *Let G be a finitely generated group, Γ an acylindrically hyperbolic group and (φ_i) a divergent sequence from $\mathrm{Hom}(G, \Gamma)$. Then G admits a minimal action on a real tree T. Furthermore, T has no point fixed by all of G and the action of G on T induces a minimal action of the corresponding Γ–limit group $L = G/\ker_{\mathfrak{u}}(\varphi_i)$ on T.*

Finally Fujiwara uses the theory developed by Groves and Hull in [GH19], the techniques developed with Sela in [FS20] and a profound study of the geometry of groups with hyperbolic features to obtain

Theorem (Fujiwara, [Fuj21, Theorem 1.2])**.** *Let Γ be a group that is hyperbolic relative to a collection of subgroups $\{ P_1, \dots, P_n \}$. Suppose Γ is not virtually cyclic and not equal to P_k for any $k \in \{ 1, \dots, n \}$. If each P_i is finitely generated and equationnaly Noetherian, then $\xi(\Gamma)$ is well-ordered.*

Organisation

We define in a first section the construction of a limit to a family of metric spaces and introduce some model theory that motivates this construction. In particular, we review the definitions of filters, ultrafilters, ultralimits and recall the Łoś Theorem. The second section uses the limit of a family of metric spaces to define the asymptotic cones of a metric space. This allows us to obtain a limit action to a sequence of actions of one group on metric spaces. Along the way, we recall the definition of Gromov hyperbolic spaces and hyperbolic groups as well as some elements of the theory of groups acting on trees. The third section is dedicated to the study of limit groups and their actions on trees. We intend to construct and analyse a limit to a sequence of morphisms from free groups to a given group. Especially, we see that the theory of asymptotic cones allows to define, from a sequence of morphisms from free groups to a hyperbolic group, a well understood faithful action of a limit group on a real tree. Finally, the fourth and last section proves the well-ordering of the set of growth rates of limit groups by means of hyperbolic metric geometry and the study of groups acting on trees. This gives us the opportunity to introduce the growth function of a finitely generated group and to obtain a lower bound on the rate of growth for hyperbolic groups.

1 Limits of Metric Spaces and Logic

This section aims to introduce the construction of a limit to a family of metric spaces and to present the necessary background in model theory for the purpose of this thesis. We start by recalling the definitions of filters, ultrafilters and ultralimits that enable us to define, subsequently, the ultralimit of metric spaces. Finally, we introduce enough logic theory to state Łoś Theorem. In this section, we introduce all the necessary tools to give a rigorous definition of an asymptotic cone defined in Section 2 and take this opportunity to introduce objects and notations used in the rest of this document.

1.1 Filters, Ultrafilters and Ultralimits

To define a limit to a family of metric spaces, we need a notion of "largeness", to keep only the recurring features of the family. A reasonable way to proceed is by introducing an ultrafilter, that is, a set deciding which size is "big enough" to be taken into account. We follow the introduction to filters, ultrafilters and ultralimits as presented by C. Druţu and M. Kapovich in [DK18, Chapter 10].

Definition 1.1.1. A **filter** $\mathcal{F}$ on a set I is a collection of subsets of I satisfying the following conditions:

1. $\emptyset \notin \mathcal{F}$,
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

An **ultrafilter** on a set I is a filter $\mathcal{F}$ on I satisfying the additional condition: for every $A \subset I$

$$\text{either } A \in \mathcal{F} \text{ or } A^c = I \backslash A \in \mathcal{F}.$$

For any set I, one can consider the set $\text{Filter}(I) \subset 2^{2^I}$ of all filters on I. This set has a natural partial order given by the inclusion. In particular, any filter on I is an ultrafilter if and only if it is a maximal element in the ordered set $\text{Filter}(I)$. This second definition and Zorn's Lemma tell us that any filter can be extended to an ultrafilter. The following examples introduce some remarkable filters.

Example 1.1.2.

- Given a set I and an element $x \in I$, consider the collection of subsets of I containing x. This collection is an ultrafilter on I. Any ultrafilter obtained by this method is called **principal ultrafilter**.
- Suppose I is an infinite set. The set of complements of the finite subsets of I is a filter. If $I = \mathbb{N}$, this filter is called the **Fréchet filter**.

If the ultrafilter is not principal, then it is called a **non-principal ultrafilter**. On an infinite set, any ultrafilter is non-principal if and only if it contains the Fréchet filter. This is proved in [DK18, Proposition 10.16] by showing the equivalence of the negations of the two statements and using the maximality of any ultrafilter in the partially ordered set of filters. Working with ultrafilters, it is convenient to consider the following equivalent definition.

Definition 1.1.3. A map $\mu : 2^I \to [-\infty, \infty]$ on a set I is **finitely additive** if whenever A, B are disjoint sets in I, it holds $\mu(A \cup B) = \mu(A) + \mu(B)$.

Let $\mathcal{F}$ be an ultrafilter on an index set I. The characteristic map of the filter, $\mathbf{1}_{\mathcal{F}} = \mathfrak{u}$, is a finitely additive map with values in $\{0, 1\}$ verifying $\mathfrak{u}(I) = 1$. Conversely, a finitely additive map $\mathfrak{u} : 2^I \to \{0, 1\}$ such that $\mathfrak{u}(I) = 1$ defines an ultrafilter. In the sequel, we use the same name of ultrafilter for $\mathcal{F}$ and $\mathfrak{u}$. A subset $A \in 2^I$ occurs $\mathfrak{u}$–**almost surely** if $\mathfrak{u}(A) = 1$. Moreover, if a set $A \subseteq I$ has the property that $\mathfrak{u}(A) = 1$, then $\mathfrak{u}(A^c) = 0$. Thus, for any argument done $\mathfrak{u}$–almost surely, only the natural numbers in A counts. In particular, if $\mathfrak{u}$ is a non-principal ultrafilter on an infinite set I, then $\mathfrak{u}(A) = 0$ for every finite subset A of I.

Definition 1.1.4. Let I be a set, $\mathfrak{u}$ an ultrafilter on I and X a topological space. Given a function $f : I \to X$, the $\mathfrak{u}$–**limit** of f, or its **ultralimit**, is an element $x \in X$ such that, for every neighborhood U of x, the pre-image $f^{-1}(U)$ belongs to $\mathfrak{u}$. The ultralimit of the function f is denoted $\mathfrak{u}\text{–}\lim_i f(i)$ or $\mathfrak{u}\text{–}\lim f(i)$ if the index set is clear.

For $\mathfrak{u}$ a fixed non-principal ultrafilter on a set I, the $\mathfrak{u}$–limit of a function f with value in a topological space has an equivalent definition in terms of finitely additive map with values in $\{0, 1\}$ verifying $\mathfrak{u}(I) = 1$. The $\mathfrak{u}$–limit of f is an element x, such that $\mathfrak{u}\big(\{ i \in I \mid f(i) \in U \}\big) = 1$ for every neighborhood U of x. A first reason to introduce the notion of ultrafilter is to give a limit to any net on a compact topological space.

Lemma 1.1.5. *Let X be a topological space, I a set, $\mathfrak{u}$ a non-principal ultrafilter on I and $f : I \to X$ a function. Then the following two statements hold.*

1. *If X is compact, then f has an ultralimit,*
2. *If X is Hausdorff, then the ultralimit, if it exists, is unique.*

Proof. We prove each statement independently.

1. Suppose by contradiction that f has no ultralimit. Then, for every $x \in X$, there exists a neighborhood $U_x \subset X$ such that $f^{-1}(U_x) \notin \mathfrak{u}$. By compactness, we can cover X with finitely many of these neighborhoods U_{x_i} for some $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. Therefore, $I = \bigcup_{i=1}^{n} f^{-1}(U_{x_i})$ such that, using the second property in the definition of filters and the maximality of ultrafilters
$$\emptyset = \bigcap_{i=1}^{n} \left(I \backslash f^{-1}\left(U_{x_i}\right)\right) \in \mathfrak{u}.$$
This is a contradiction with the first property in the definition of a filter so that f has a limit.
2. This can be proved in a same fashion as the uniqueness of the ordinary limit in Hausdorff spaces. □

Another reason to introduce the notion of ultralimit is that, it keeps many desirable property of the usual limit. At least, the ultralimit of real-valued functions is linear.

Lemma 1.1.6. *For any ultrafilter $\mathfrak{u}$ on a set I, the $\mathfrak{u}$–limit is linear with respect to bounded real-valued functions on I.*

Proof. Let $f, g : I \to \mathbb{R}$ be two bounded functions. Their images are contained in a compact subset of $\mathbb{R}$ so that, by Lemma 1.1.5, we define x, y their respective $\mathfrak{u}$–limit. Let $\lambda, \mu \in \mathbb{R}$ and we show
$$(\lambda f + \mu g)^{-1}(U) = \{i \in I \mid \lambda f(i) + \mu g(i) \in U\} \in \mathfrak{u},$$
where U is any neighborhood of $\lambda x + \mu y$. Since the union of elements in $\mathfrak{u}$ is in the filter, it is sufficient to do it for basis neighborhood of $\lambda x + \mu y$ in $\mathbb{R}$. Moreover, using the third property of filters, it is also sufficient to show that some subset of $(\lambda f + \mu g)^{-1}(U)$ is in $\mathfrak{u}$. For $\varepsilon > 0$ and $U = (\lambda x + \mu y - \varepsilon, \lambda x + \mu y + \varepsilon)$ a neighborhood of $\lambda x + \mu y$, we have
$$A := (\lambda f)^{-1}\left(\lambda x - \frac{\varepsilon}{2}, \lambda x + \frac{\varepsilon}{2}\right) = \left\{i \in I \;\middle|\; f(i) \in \left(x - \frac{\varepsilon}{2\lambda}, x + \frac{\varepsilon}{2\lambda}\right)\right\} \in \mathfrak{u},$$
because this last set is a neighborhood of y. We can proceed similarly for $B := (\mu g)^{-1}(\mu x - \varepsilon/2, \mu x + \varepsilon/2)$ and conclude using $A \cap B \in (\lambda f + \mu g)^{-1}(U) \in \mathfrak{u}$ by definition of a filter. □

The $\mathfrak{u}$–limit on the natural numbers with an ultrafilter extending the Fréchet filter is an enrichment of the usual limit. We obtain the existence of a limit in any compact space and it avoids using diagonal arguments. Also, an ultrafilter is a device which selects accumulation points of functions with image in compact Hausdorff spaces in a coherent manner, and that generalizes the diagonal argument to uncountable sets. An ultrafilter is a specific way to define a limit of spaces and functions in a more general setting. We use this tool in the following subsection to define the ultralimit of a family of metric spaces.

1.2 Ultralimits of Metric Spaces

We now use these tools to construct a limit to a family of metric spaces; an asymptotic cone of a metric space, presented in Subsection 2.1, is a special case of this construction. We still follow the presentation of asymptotic cones as established in [DK18, Chapter 10]. To obtain the existence of an ultralimit, we first compactify $\mathbb{R}_+$ as presented in Remark 1.2.2.

Definition 1.2.1. If X is a totally ordered set, the **total order topology** on X is generated by the subbase of open rays. That is, $\{\, x \in X \mid x > a \,\}$ for some $a \in X$ or $\{\, x \in X \mid x < b \,\}$ for some $b \in X$.

Remark 1.2.2. The set $\overline{\mathbb{R}}_+ := [0, \infty]$ can be turned into a totally ordered set by considering the usual order on $\mathbb{R}_+$ and defining $0 \le a < \infty$ for every $a \in \mathbb{R}_+$. Then, $\overline{\mathbb{R}}_+$ with the total order topology is homeomorphic to $[0, 1]$ with the standard topology. In particular, $\overline{\mathbb{R}}_+$ with the total order topology is compact. Similarly, $\overline{\mathbb{R}} := [-\infty, \infty]$ with the total order topology is a compact metric space homeomorphic to $[0, 1]$.

Let X be a Hausdorff space and $\mathfrak{u}$ a principal ultrafilter on an index set I. Then, there exists $i_0 \in I$ so that any subset $A \in 2^I$ verifies $\mathfrak{u}(A) = 1$ if and only if $i_0 \in A$. Therefore, the $\mathfrak{u}$–limit of a function $f : I \to X$ is simply the element $f(i_0)$. In the sequel, all ultrafilters are assumed non-principal.

Notation 1. Given a metric space X, we denote $\operatorname{dist}_X(a, b)$ the distance between the two elements a, b in X. Also, we drop the index X in the notation when it is clear from the context.

Let $(X_i)_{i\in I}$ be a family of metric spaces parametrized by an infinite set I and $\mathfrak{u}$ a non-principal ultrafilter on I. Define a pseudo-distance on the product of these metric spaces by

$$\operatorname{dist}_{\mathfrak{u}} : \begin{array}{ccc} \prod_{i\in I} X_i \times \prod_{i\in I} X_i & \longrightarrow & [0,\infty]; \\ ((x_i)_{i\in I}, (y_i)_{i\in I}) & \longmapsto & \mathfrak{u}\text{–}\lim(i \mapsto \operatorname{dist}_{X_i}(x_i, y_i)), \end{array}$$

and consider the pair

$$(X_{\mathfrak{u}}, \operatorname{dist}_{\mathfrak{u}}) := \left(\prod_{i\in I} X_i, \operatorname{dist}_{\mathfrak{u}}\right) \Big/ \sim,$$

where we identify elements with zero $\operatorname{dist}_{\mathfrak{u}}$–distance.

Definition 1.2.3. The **ultralimit of the family of metric spaces** $(X_i)_{i\in I}$ is the pair $(X_{\mathfrak{u}}, \operatorname{dist}_{\mathfrak{u}})$ defined above that we denote

$$X_{\mathfrak{u}} := \mathfrak{u}\text{–}\lim X_i.$$

Notation 2. Given an indexed family of elements $(x_i)_{i\in I}$, where $x_i \in X_i$ for every $i \in I$, we denote its equivalence class in $X_{\mathfrak{u}}$ by $x_{\mathfrak{u}}$ or $\mathfrak{u}\text{–}\lim x_i$ or $[x_i]$. Also, we drop the indices $i \in I$ in the notation when the index set is clear.

If the spaces X_i do not have uniformly bounded diameter, the ultralimit $X_{\mathfrak{u}}$ decomposes into many components of points at mutually finite distance, where two elements in different components are at infinite distance one from the other. In order to pick one of these components, we consider **pointed metric space** (X, x) where $x \in X$ is called **base point**. For a family of pointed metric spaces $(X_i, x_i)_{i\in I}$, the sequence of base points $(x_i)_{i\in I}$ define a base point $x_{\mathfrak{u}} \in X_{\mathfrak{u}}$, and we set $X_{\mathfrak{u},x_{\mathfrak{u}}} := \{\, a_{\mathfrak{u}} \in X_{\mathfrak{u}} \mid \operatorname{dist}_{\mathfrak{u}}(a_{\mathfrak{u}}, x_{\mathfrak{u}}) < \infty \,\}$. With these notations, we introduce

Definition 1.2.4. The **based ultralimit** of the family $(X_i, x_i)_{i\in I}$ is $\mathfrak{u}\text{–}\lim(X_i, x_i) = (X_{\mathfrak{u},x_{\mathfrak{u}}}, x_{\mathfrak{u}})$.

The purpose of the based ultralimit is to select one element in $X_{\mathfrak{u}}$ and its component of elements at finite distance of it. We drop the $x_{\mathfrak{u}}$ in the notation $X_{\mathfrak{u},x_{\mathfrak{u}}}$ when the choice of the base point is clear. To get used to these definitions, we give an example of an ultralimit of metric spaces that is used in the sequel.

Notation 3. Given a point x of a metric space X and a scalar $R \geq 0$, we denote $B_{\operatorname{dist}_X}(x, R)$ the open ball in X of radius R about x for the metric dist_X. We drop the dist_X in the notation when the choice of the metric is clear.

Example 1.2.5. If $(X_{\mathfrak{u}}, \operatorname{dist}_{\mathfrak{u}})$ is an ultralimit of a constant sequence of compact metric spaces $X_i = Y$, then $X_{\mathfrak{u}}$ is isometric to Y for any ultrafilter $\mathfrak{u}$.

Proof: By looking at the equivalence class of the constant sequence in $X_{\mathfrak{u}}$, one conclude that the function $f : X_{\mathfrak{u}} \to Y$ sending $[x_i]$ to $\mathfrak{u}\text{–}\lim x_i$ is surjective. For $\varepsilon > 0$, if $[x_i], [y_i] \in X_{\mathfrak{u}}$ verify $\mathfrak{u}\text{–}\lim x_i = \mathfrak{u}\text{–}\lim y_i$, then the following sets are in $\mathfrak{u}$ by definition of a basis of neighborhood in a metric space:

$$A := \left\{ i \in I \;\middle|\; \operatorname{dist}_Y\left(x_i, \mathfrak{u}\text{–}\lim_j x_j\right) < \frac{\varepsilon}{2} \right\} \in \mathfrak{u}, \text{ and } B := \left\{ i \in I \;\middle|\; \operatorname{dist}_Y\left(y_i, \mathfrak{u}\text{–}\lim_j y_j\right) < \frac{\varepsilon}{2} \right\} \in \mathfrak{u}.$$

Then, for $g : i \mapsto \operatorname{dist}_Y(x_i, y_i)$, the set $A \cap B \in g^{-1}\big(B_{\operatorname{dist}_Y}(0, \varepsilon)\big) \in \mathfrak{u}$. Hence, the ultralimit of g is zero, such that the equivalence classes $[x_i]$ and $[y_i]$ are equal in $X_{\mathfrak{u}}$ and f is injective. Finally, we show that f preserves the distance. If g is as above and $\varepsilon > 0$, then, using a similar argument as above and the triangular inequality, the set

$$g^{-1}\big\{\, B\left(\operatorname{dist}_Y(f[x_j], f[y_j]), \varepsilon\right) \,\big\} = \big\{ i \in I \;\big|\; \operatorname{dist}_Y(f[x_j], f[y_j]) - \varepsilon \leq \operatorname{dist}_Y(x_i, y_i) \leq \operatorname{dist}_Y(f[x_j], f[y_j]) + \varepsilon \big\},$$

is in the filter $\mathfrak{u}$. Therefore, f is an isometry between $X_{\mathfrak{u}}$ and Y.

Given a sequence of metric spaces, we have defined its ultralimit. Actually, given a sequence of functions between metric spaces, we can define its ultralimit whose domain and image are the ultralimits of the metric spaces.

Lemma 1.2.6. *Let I be an index set and $(X_i, x_i)_{i\in I}$, $(X'_i, x'_i)_{i\in I}$ two families of pointed metric spaces with based ultralimits $(X_\mathfrak{u}, x_\mathfrak{u})$ and $(X'_\mathfrak{u}, x'_\mathfrak{u})$ respectively. Consider isometric embeddings $f_i : (X_i, x_i) \to (X'_i, x'_i)$ verifying*

$$\mathfrak{u}\text{–}\lim_i \operatorname{dist}_{X_i}(f_i(x_i), x'_i) < \infty. \tag{1}$$

Then the following holds.

1. *The maps f_i yield an isometric embedding between the based ultralimits $f_\mathfrak{u} : (X_\mathfrak{u}, x_\mathfrak{u}) \to (X'_\mathfrak{u}, x'_\mathfrak{u})$,*
2. *If each f_i is an isometry, then so is $f_\mathfrak{u}$,*
3. *The limit map $\phi_\mathfrak{u} : (f_i)_{i\in I} \mapsto f_\mathfrak{u}$ preserves the composition. That is, for every sequence of functions between pointed spaces $f_i : (X_i, x_i) \to (Y_i, y_i)$ and $g_i : (Y_i, y_i) \to (Z_i, z_i)$ verifying the condition* (1), we have

$$\phi_\mathfrak{u}((g_i \circ f_i)_{i\in I}) = \phi_\mathfrak{u}((g_i)_{i\in I}) \circ \phi_\mathfrak{u}((f_i)_{i\in I}).$$

Proof. Using condition (1), we define $f_\mathfrak{u}$ as $f_\mathfrak{u}(a_\mathfrak{u}) = \mathfrak{u}\text{–}\lim f_i(a_i)$ for any $a_\mathfrak{u} = \mathfrak{u}\text{–}\lim a_i \in X_\mathfrak{u}$.

1. For any pair of points $a_\mathfrak{u}, b_\mathfrak{u} \in X_\mathfrak{u}$, the construction of the distances in $X_\mathfrak{u}$, $X'_\mathfrak{u}$ leads to

$$\operatorname{dist}_\mathfrak{u}(f_\mathfrak{u}(a_\mathfrak{u}), f_\mathfrak{u}(b_\mathfrak{u})) = \mathfrak{u}\text{–}\lim_i \operatorname{dist}_{X'_i}(f_i(a_i), f_i(b_i)) = \mathfrak{u}\text{–}\lim_i \operatorname{dist}_{X_i}(a_i, b_i) = \operatorname{dist}_\mathfrak{u}(a_\mathfrak{u}, b_\mathfrak{u}),$$

where we use the isometric embedding property of all the f_i's in the second equality.

2. If each f_i is surjective, then $f_\mathfrak{u}$ is surjective as well.
3. For $a_\mathfrak{u} = \mathfrak{u}\text{–}\lim a_i \in X_\mathfrak{u}$, the following equalities hold

$$g_\mathfrak{u}(f_\mathfrak{u}(a_\mathfrak{u})) = g_\mathfrak{u}([f_i(a_i)]) = [g_i(f_i(a_i))] = [(g \circ f)_i(a_i)] = (g \circ f)_\mathfrak{u}(a_\mathfrak{u}).$$

Thus, the composition property is preserved. ☐

We aim to describe the coarse geometry of a space. To do so, we relax the notion of isometry by allowing some bounded error. Let us recall some definitions of coarse geometry.

Definition 1.2.7. Let X be a metric space, $A, B \subseteq X$ two non-empty sets and $\varepsilon > 0$. The ε–**neighborhood** of A is defined by

$$\mathcal{N}_\varepsilon(A) := \{\, x \in X \mid \operatorname{dist}_X(x, A) < \varepsilon \,\}.$$

The **Hausdorff distance** between A and B is defined by

$$\operatorname{dist}_{\text{Haus}}(A, B) := \inf\{\varepsilon > 0 \mid A \subset \mathcal{N}_\varepsilon(B) \wedge B \subset \mathcal{N}_\varepsilon(A)\}.$$

Definition 1.2.8. Let $f : X \to Y$ be a function between metric spaces X, Y and $K \geq 1$, $C \geq 0$.

- f is (K, C)–**coarse Lipschitz** if $\operatorname{dist}_Y(f(x), f(x')) \leq K \cdot \operatorname{dist}_X(x, x') + C$ for every $x, x' \in X$,
- f is a K–**bi-Lipschitz embedding** if, for every $x, x' \in X$

$$K^{-1} \cdot \operatorname{dist}_X(x, x') \leq \operatorname{dist}_Y(f(x), f(x')) \leq K \cdot \operatorname{dist}_X(x, x'),$$

- f is K–**bi-Lipschitz** if f is a K–bi-Lipschitz embedding that is surjective,
- f is a (K, C)–**quasi-isometric embedding** if, for every $x, x' \in X$

$$K^{-1} \cdot \operatorname{dist}_X(x, x') - C \leq \operatorname{dist}_Y(f(x), f(x')) \leq K \cdot \operatorname{dist}_X(x, x') + C,$$

- f is a (K, C)–**quasi-isometry** if it is a quasi-isometric embedding that is also coarsely surjective. That is

$$\operatorname{dist}_{\text{Haus}}(\operatorname{Im}(f), X') < \infty.$$

Geometrically, it is usefull to understand the metric of a space by looking at its paths. To do so, we generalize the notion of "straight line".

Definition 1.2.9. Let X be a metric space, $I \subseteq \mathbb{R}$ connected and $\gamma : I \to X$ an isometric embedding.

- The function γ is a **geodesic segment** in X if I is bounded,
- It is a **geodesic ray** if I is semi-infinite,
- This function is a **bi-infinite geodesic** if $I = \mathbb{R}$.

Moreover, we called **quasi-geodesic segment, ray** or **bi-infinite quasi-geodesic** if γ above is a quasi-isometric embedding.

Consider I an index set, $\mathfrak{u}$ an ultrafilter on I, $(X_i)_{i\in I} \subseteq \mathbb{R} \subset \overline{\mathbb{R}}$ a family of pointed intervals, $(Y_i)_{i\in I}$ a family of pointed metric spaces and $(f_i)_{i\in I}$ a family of maps between the previously defined spaces that verify the condition (1) of Lemma 1.2.6. If each f_i is a geodesic in Y_i, then the ultralimit is a geodesic into $Y_\mathfrak{u}$. In fact, the ultralimit $f_\mathfrak{u}$ is an isometric embedding from $\overline{\mathbb{R}}_\mathfrak{u}$ into $Y_\mathfrak{u}$, and we have seen in Example 1.2.5 that $\overline{\mathbb{R}}_\mathfrak{u}$ is isometric to $\overline{\mathbb{R}}$. This gives us a class of geodesics in the limit space that is totally defined by the underlying metric spaces. A geodesic f in the limit space is called **limit geodesic** in $X_\mathfrak{u}$ if there exists a sequence of geodesics in the X_i that has f for ultralimit. Some properties are preserved while passing to the limit; one of them is the geodesicity.

Lemma 1.2.10. *If $(X_i, x_i)_{i\in I}$ is a sequence of pointed geodesic metric spaces for an index set I, then the ultralimit $(X_\mathfrak{u}, x_\mathfrak{u})$ is also a geodesic metric space.*

Proof. Consider two elements in the ultralimit $a_\mathfrak{u} = [a_i],\ b_\mathfrak{u} = [b_i] \in X_\mathfrak{u}$. Since every X_i is assumed to be geodesic, there exists a sequence $\gamma_i : [0, L_i] \to X_i$ of geodesics connecting a_i to b_i for some $L_i := \mathrm{dist}_{X_i}(a_i, b_i)$ and any $i \in I$. Then

$$\mathfrak{u}\text{–}\lim L_i =: L = \mathrm{dist}_\mathfrak{u}(a_\mathfrak{u}, b_\mathfrak{u}) < \infty,$$

so that $\gamma_\mathfrak{u} : [0, L] \to X_\mathfrak{u}$ is a geodesic connecting the two points. Thus, $X_\mathfrak{u}$ is a geodesic metric space. $\square$

We attempt to describe the large scale geometry of one space in particular. Instead of taking the limit of different spaces, we consider the limit of rescaled copies of one particular space. We describe this special case in Section 2. Before that, we outline the required background in logic theory for this document. In particular, we give context to the introduction of the theory of limit groups and state Łoś Theorem which is used in Section 3.

1.3 Elementary Theory and Łoś Theorem

One motivation to study the asymptotic cones of a metric space comes from the following Theorem 1.3.5 which gives us the opportunity to introduce some context in which limit groups have been introduced. It is a compactness theorem from logic which is used in Lemma 3.2.7. We begin by briefly introducing vocabulary of model theory and refer to [Tou03] for more details.

Definition 1.3.1. A **first-order language of predicate calculus** is composed of

- A countable set of variables $\mathcal{V} = \{ v_0, v_1, \dots \}$,
- Logical connectors $\{ \neg, \wedge, \vee, \to, \leftrightarrow \}$,
- Quantifiers $\{ \forall, \exists \}$,
- Parentheses,
- A set of constants symbols $\{ c_0, c_1, \dots \}$,
- A set of functions symbols $\left\{ f_0^{(n_0)}, f_1^{(n_1)}, \dots \right\}$ where n_i represent the arity of the i–th function,
- A finite set of relations symbols $\left\{ R_0^{(m_0)}, R_1^{(m_1)}, \dots \right\}$.

For presentation purposes, we continue the exposition with the language of group theory. That is, we consider the first-order language $\mathcal{L}_{grp} := \left\{ f^{(2)}, s^{(1)}, c, =^{(2)} \right\}$. The symbols correspond respectively to the composition, the inversion, the neutral element and the equality relation. The set of **terms** (or words) of $\mathcal{L}_{grp}$, denoted $T(\mathcal{L}_{grp})$, is the smallest set verifying

- Every variable and constant of $\mathcal{L}_{grp}$ are in $T(\mathcal{L}_{grp})$,
- For every n–ary function $f^{(n)}$ of $\mathcal{L}_{grp}$ and every n terms $t_1, \dots, t_n$, the symbol $f(t_1, \dots, t_n)$ is a term.

Then, we can define an **atomic formula**, that is, a finite sequence of $\mathcal{L}_{grp}$ composed of a relation, parentheses and terms. The set of formulas of $\mathcal{L}_{grp}$ is the smallest subset $\mathcal{F}(\mathcal{L}_{grp})$ of sequences of $\mathcal{L}_{grp}$ containing

- Every atomic formulas,
- If $\varphi, \phi \in \mathcal{F}(\mathcal{L}_{grp})$, then $\neg\varphi$ the negation of φ, $\varphi \wedge \phi$, $\varphi \vee \phi$, $\varphi \to \phi$ and $\varphi \leftrightarrow \phi$ are in $\mathcal{F}(\mathcal{L}_{grp})$,
- If x is a variable and $\varphi \in \mathcal{F}(\mathcal{L}_{grp})$, then $\forall x\varphi$ and $\exists x\varphi$ are in $\mathcal{F}(\mathcal{L}_{grp})$.

In a formula φ, the occurence of a variable x is **tied** if we might find an occurence of Qx in φ where $Q \in \{\forall, \exists\}$. The occurence is called **free** if it is not tied and a formula without free variable is **closed**. Also, a formula is **universal** if it can be written

$$\forall x_1 \cdots \forall x_n \; \varphi(x_1, \ldots, x_n),$$

for some quantifier free formula $\varphi(x_1, \ldots, x_n)$. Similarly, a formula is **existential** if the above quatifiers are replaced by existential ones. We say that a term t is **substitutable** to the variable x in the formula φ if, for every variable y that appears in t, there is no subformula of φ of the form $Qy\phi$ in which x has a free occurence and where $Q \in \{\forall, \exists\}$. A formula with no free variable is called a **sentence**.

Remark 1.3.2. The statement $\forall x_1 \exists k \in \mathbb{N} \; \left(x_1^k = 1\right)$ is not a first-order formula. Indeed, it quantifies over an integer, and not a group element. In fact, the quantifiers do not mention to which group the variables belong. It is the interpretation which specifies the group.

The **elementary theory** of a group G is the set of sentences which are satisfied by G. Two groups are **elementarily equivalent** if they have the same elementary theory. Likewise, the **universal (existential) theory** of a group G is the set of universal (existential) sentences which are satisfied by G. Sela introduced in [Sel01] the concept of limit groups as part of his research on the Tarski problem which asked whether the free groups on two or more generators have the same elementary theory. A partial result to this problem is presented in Corollary 3.2.3. While studying elementary theory of a group, it is convenient to introduce its ultrapowers.

Definition 1.3.3. The **ultraproduct** with respect to the ultrafilter $\mathfrak{u}$ of a sequence of groups (G_i) is the group

$$\left(\prod_{i\in\mathbb{N}} G_i\right) \Big/ \sim_{\mathfrak{u}},$$

where two elements $(g_i), (h_i)$ are identified if and only if $(g_i = h_i)$ $\mathfrak{u}$–almost surely. The ultraproduct of a constant sequence of groups $G_i = G$ is called an **ultrapower** of G and is denoted ${}^*G$.

In our context, the main interest of ultraproducts and ultrapowers comes from Łoś Theorem which states that ultrapowers of a group G have the same elementary theory as G.

Definition 1.3.4. A $\mathcal{L}_{grp}$–**structure** is a sequence $\mathcal{M} := \left\langle M, \left(f^{(2)}\right)^{\mathcal{M}}, \left(s^{(1)}\right)^{\mathcal{M}}, (c)^{\mathcal{M}} \right\rangle$ where:

- $M = |\mathcal{M}|$ is a non-empty set called **basis domain**,
- $(c)^{\mathcal{M}}$ is an element of M,
- $\left(f^{(2)}\right)^{\mathcal{M}}$ and $\left(s^{(1)}\right)^{\mathcal{M}}$ are functions $\left(f^{(2)}\right)^{\mathcal{M}} : M \times M \to M$ and $\left(s^{(1)}\right)^{\mathcal{M}} : M \to M$ respectively.

Let $\mathcal{M}$ be a $\mathcal{L}_{grp}$–structure, φ a first-order formula with free variables in $x_1, \ldots, x_n$ and $t_1, \ldots, t_n \in |\mathcal{M}|$. We denote $\mathcal{M} \models_{(t_1/x_1, \ldots, t_n/x_n)} \varphi$ if, φ is satisfied in $\mathcal{M}$ when the variables $x_1, \ldots, x_n$ are substituted by $t_1, \ldots, t_n$. Also, given a set of closed formulas S, a $\mathcal{L}_{grp}$–structure $\mathcal{M}$ is a **model** of S, denoted $\mathcal{M} \models S$, if any formula of S is satisfied in $\mathcal{M}$. We can finally state Łoś Theorem in the special case of group theory, see [BS06].

Theorem 1.3.5 (Łoś)**.** *If G is a group and ${}^*G$ an ultrapower of G, then G and ${}^*G$ have the same elementary theory. In particular, for every first-order formula $\varphi(x_1, \ldots, x_n)$, the following two assertions are equivalent.*

- ${}^*G \models_{(x_1 = (g_{1,k})_{k\in\mathbb{N}}, \ldots, x_n = (g_{n,k})_{k\in\mathbb{N}})} \varphi(x_1, \ldots, x_n)$,
- $G \models_{(x_1 = g_{1,k}, \ldots, x_n = g_{n,k})} \varphi(x_1, \ldots, x_n)$ *for almost every* $k \in \mathbb{N}$.

In particular, any finite system of equations stated in a proper way is satisfied in a group if and only if it is satisfied in an ultrapower of this group. We use this theory in Section 3 while studying limit groups, but first we use the theory developed so far to study actions of groups on real trees.

2 Hyperbolic Groups and Groups Acting on Trees

We gather here the objects introduced in the previous section to define, under certain conditions, a limit action to a sequence of actions of one group on hyperbolic spaces. At first, we construct the asymptotic cones of a hyperbolic space and study their geometry. Then, in a second time, we define a nice action of a group on one of its asymptotic cones generalizing work carried out independently by Bestvina in [Bes88] and Paulin in [Pau88]. The purpose of this section is to collect enough theory about hyperbolic groups and groups actions on trees to define a metric space equivalent to Bestvina and Paulin's tree studied in Section 3.

2.1 Hyperbolic Groups and their Asymptotic Cones

In this subsection, we recall in a first paragraph, some definitions of hyperbolic geometry. In a second one, we construct the asymptotic cones of one metric space and, finally, we bring together both concepts to study the asymptotic cones of hyperbolic metric spaces in a third paragraph. We follow the introduction to Gromov-hyperbolic geometry as exposed in [DK18, Chapter 11]. This first paragraph gives us the opportunity to recall definitions from hyperbolic geometry and, in particular, the definition of real tree. For more details, we refer to [BH13, Part III. H and Γ] and [DK18, Chapter 11].

For a metric space X and a triple of points $x, y, w \in X$, the **Gromov product** between x and y with respect to w is

$$(x, y)_w := \frac{1}{2}\left(\mathrm{dist}_X(x, w) + \mathrm{dist}_X(y, w) - \mathrm{dist}_X(x, y)\right).$$

This quantity represents the failure of the triangular inequality to be an equality. In a geodesic hyperbolic metric space, it is up to an additive constant, the distance between w and a geodesic from x to y.

Definition 2.1.1. A metric space X is called δ–**Gromov-hyperbolic** for some $\delta \geq 0$, if for some point $w \in X$, and for every $x, y, z \in X$ we have

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta.$$

A metric space is **Gromov-hyperbolic** if it is δ–Gromov-hyperbolic for some **hyperbolicity constant** $\delta \geq 0$.

Up to considering a different δ in the definition of a hyperbolic metric space (in the worst case, 2δ), we can show that this definition does not depend on the choice of the base point w. In a geodesic metric space, the previous notion of hyperbolicity is equivalent (up to a rescaling of the hyperbolicity constant) to the property that every triangle in X is **thin**. That is, there exists $\delta \geq 0$ such that every geodesic triangle with sides x-y, y-z and x-z in X satisfies $x\text{-}y \subseteq \mathcal{N}_\delta(x\text{-}z \cup y\text{-}z)$, where x-y denotes a geodesic segment between x and y and $\mathcal{N}_\delta(x\text{-}z \cup y\text{-}z)$ is a δ–neighborhood around the two geodesics. A geodesic space with δ–thin triangles is called δ–**hyperbolic** and it is called **hyperbolic** if it is δ–hyperbolic for some $\delta \geq 0$. One can show as in [BH13, Chapter III.H, Theorem 1.9] that hyperbolicity is a quasi-isometry invariant. This permit us to define hyperbolic groups without referring to a particular finite generating set.

Definition 2.1.2. A finitely generated group is **hyperbolic** if it has one hyperbolic Cayley graph.

Hyperbolicity has many equivalent definitions. A list of many of them and the proof of their equivalence can be found in [ABC+91] or [BH13, Chapter III.H] where many results are proved. The geometry of triangles in a geodesic metric space characterizes the hyperbolicity. It turns out that triangles in hyperbolic geodesic metric spaces are thin. Consequently, shrinking all distances, we should not be able to distinguish edges of the considered triangle; it should look like a tripod. This is the idea we express using asymptotic cones and real trees.

Definition 2.1.3. A **real tree** T is a geodesic metric space such that

1. If two segments of T intersect in a single point, which is an endpoint of both, then their union is a segment,

2. The intersection of two segments with a common endpoints is also a segment.

Equivalently, real trees are exactly the geodesic 0–hyperbolic spaces. In the next subsection, we use the following property of segments in a real tree.

Remark 2.1.4. If x-y, y-z are two geodesic segments of a real tree, then the geodesic segment from x to z is the concatenation of x-o and o-z where o is the crux of x, y, z.

The asymptotic cones of a metric space are a particular case of the construction studied in Subsection 1.2. The family of spaces is rescaled copies of one particular space. These transformations are made to get rid of the bounded features; to keep only the large scale geometry and give a picture of a metric space as seen "from infinitely far away". Consider a metric space X, a non-principal ultrafilter $\mathfrak{u}$ on an index set I, $(x_i)_{i\in I}$ a family of base points in X and $(\lambda_i)_{i\in I} \subseteq \mathbb{R}_+$ a collection of positive scalars verifying $\mathfrak{u}\text{–}\lim \lambda_i = 0$. We define

$$\mathcal{C} := \left\{ (y_i)_{i\in I} \subseteq X \;\middle|\; \mathfrak{u}\text{–}\lim_i \lambda_i \cdot \operatorname{dist}_X (y_i, x_i) \text{ is finite} \right\},$$

and the equivalence relation on this set

$$(y_i)_{i\in I} \sim (z_i)_{i\in I} \Leftrightarrow \mathfrak{u}\text{–}\lim_i \lambda_i \cdot \operatorname{dist}_X (y_i, z_i) = 0.$$

Definition 2.1.5. The **asymptotic cone** of a metric space X with respect to the **sequence of scalars** $(\lambda_i)_{i\in I}$, the sequence of **observation centers** $(x_i)_{i\in I}$ and the non-principal ultrafilter $\mathfrak{u}$, is the quotient space

$$\operatorname{Cone}_{\mathfrak{u}}(X, (x_i), (\lambda_i)) := \mathcal{C}/\sim\, = \mathfrak{u}\text{–}\lim_i (\lambda_i \cdot X, x_i).$$

As in Section 1.2, $\operatorname{Cone}_{\mathfrak{u}}(X, (x_i), (\lambda_i))$ is a metric space with $\operatorname{dist}_{\mathfrak{u}} ([y_i], [z_i]) := \mathfrak{u}\text{–}\lim \lambda_i \cdot \operatorname{dist}_X (y_i, z_i)$ for every $[y_i], [z_i]$ in $\operatorname{Cone}_{\mathfrak{u}}(X, (x_i), (\lambda_i))$. This metric space is often denoted $X_{\mathfrak{u}} ((x_i), (\lambda_i))$ and notice that the image of a set at unbounded $\mathfrak{u}$–distance from $[x_i]$ is empty. In particular, for any family of points $(y_i)_{i\in I} \subseteq X$, the corresponding image in the asymptotic cone is either a one-point set, or the empty set if $\mathfrak{u}\text{–}\lim \lambda_i \cdot \operatorname{dist}_X (y_i, x_i) = \infty$. An asymptotic cone of a finitely generated group is the asymptotic cone of this group seen as a metric space, where we use the word metric defined by a given finite generating set. For a group, the bi-Lipschitz homeomorphism class of asymptotic cones is independent of the generating set and the choice of base point, but does depend on the ultrafilter and the scaling family. We state the bi-Lipschitz equivalence of the asymptotic cones of quasi-isometric spaces and refer to [TV00] for the dependence described above.

Lemma 2.1.6. *Let X, X' be pointed metric spaces, I an index set and $(x_i), (x_i')$ sequences of elements in X and X' respectively. Let (λ_i) be a sequence of scalars going to zero, $\mathfrak{u}$ a non-principal ultrafilter on I and consider a family of (K, C)–coarse Lipschitz maps $f_i : X \to X'$ satisfying $\mathfrak{u}\text{–}\lim \lambda_i \cdot \operatorname{dist}_{X'} (f_i(x_i), x_i') < \infty$. For*

$$f_{\mathfrak{u}} : \begin{array}{ccc} X_{\mathfrak{u}} ((x_i), (\lambda_i)) & \longrightarrow & X'_{\mathfrak{u}} ((x_i'), (\lambda_i)) \,; \\ [x_i] & \longmapsto & [f_i(x_i)], \end{array}$$

the ultralimit of this family, the following holds.

1. *$f_{\mathfrak{u}}$ is K–Lipschitz,*
2. *If the f_i are (K, C)–quasi-isometric embedding, then $f_{\mathfrak{u}}$ is a K–bi-Lipschitz embedding,*
3. *The limit map $\phi_{\mathfrak{u}} : (f_i) \mapsto f_{\mathfrak{u}}$ preserves the composition,*
4. *If $X = X'$ and the f_i verify $\operatorname{dist}_X (f_i(x), x) \le C$ for every $x \in X$ and $\mathfrak{u}$–almost every i, then $f_{\mathfrak{u}} = \operatorname{Id}_{X_{\mathfrak{u}}}$,*
5. *If each f_i is a (K, C)–quasi-isometry, then $f_{\mathfrak{u}}$ is a K–bi-Lipschitz homeomorphism.*

Proof. The map $f_{\mathfrak{u}}$ is the image of the family (f_i) by the limit map defined in the third item of Lemma 1.2.6.

1. Let $[y_i], [z_i] \in X_{\mathfrak{u}} ((x_i), (\lambda_i))$ and consider the following inequalities

$$\begin{aligned} \operatorname{dist}_{X'_{\mathfrak{u}}} (f_{\mathfrak{u}}([y_i]), f_{\mathfrak{u}}([z_i])) &= \mathfrak{u}\text{–}\lim_i \lambda_i \cdot \operatorname{dist}_{X'} (f_i(y_i), f_i(z_i)) \\ &\le \mathfrak{u}\text{–}\lim_i (\lambda_i K \cdot \operatorname{dist}_X (y_i, z_i) + \lambda_i C) \\ &= \mathfrak{u}\text{–}\lim_i \lambda_i K \cdot \operatorname{dist}_X (y_i, z_i)\,, \end{aligned}$$

   where we used that the f_i are (K, C)–coarse Lipschitz in the first inequality and the linearity of the $\mathfrak{u}$–limit in the third equality. So, $\operatorname{dist}_{X'_{\mathfrak{u}}} (f_{\mathfrak{u}}([y_i]), f_{\mathfrak{u}}([z_i])) = K \cdot \operatorname{dist}_{\mathfrak{u}} ([y_i], [z_i])$ and $f_{\mathfrak{u}}$ is K–Lipschitz.

2. This is the same calculation as before with the reverse inequality.
3. This is done as in Lemma 1.2.6.
4. For $[y_i] \in X_{\mathfrak{u}}$, since the family (λ_i) goes to zero, we get

$$\operatorname{dist}_{\mathfrak{u}} (f_{\mathfrak{u}}([y_i]), [y_i]) = \mathfrak{u}\text{–}\lim_i \lambda_i \cdot \operatorname{dist}_X (f_i(y_i), y_i) \le \mathfrak{u}\text{–}\lim_i \lambda_i \cdot C = 0.$$

   Thus, $f_{\mathfrak{u}} = \operatorname{Id}_{X_{\mathfrak{u}}}$.

5. By 2. above, $f_\mathfrak{u}$ is a K–bi-Lipschitz embedding. Then, using the third and fourth properties above, together with the existence of the quasi-inverse of a quasi-isometry, we get the bijective property of a bi-Lipschitz homeomorphism. ☐

In particular, an asymptotic cone of a finitely generated group does not depend on the generating set up to bi-Lipschitz homeomorphism. Therefore, one can use topological results on the asymptotic cones to study quasi-isometric properties of the underlying group. This motivates the characterization of hyperbolic geodesic metric spaces in terms of the geometry of their asymptotic cones. The next theorem is a first step in this direction and is a direct consequence of Lemma 1.2.10.

Lemma 2.1.7. *The asymptotic cones of a geodesic metric space are geodesic metric spaces.*

The geometry of a metric space is well understood by the study of sequences within it. Thus, we consider ultrafilters on the set of natural numbers; in the sequel, if it is not precised, all utrafilters are on the index set of the natural numbers. Moreover, sequences $(x_i)_{i\in\mathbb{N}}$ of points in a metric space are denoted (x_i). We are now able to show, as done in [DK18, Chapter 11], that all asymptotic cones of hyperbolic geodesic metric spaces are real trees.

Theorem 2.1.8. *If a geodesic metric space X is hyperbolic, then every asymptotic cone of it is a real tree.*

Proof. Let δ be a constant such that X is δ–hyperbolic. Consider $[a_i], [b_i], [c_i], [y_i] \in X_\mathfrak{u}\left((x_i), (\lambda_i)\right)$ four elements in the asymptotic cone of X with respect to a sequence of observation centers (x_i), a sequence of scalars (λ_i) and a non-principal ultrafilter $\mathfrak{u}$. Since X is δ–hyperbolic, the following holds

$$\begin{aligned}
([a_i], [c_i])_{[y_i]} &= \mathfrak{u}\text{–}\lim_i \frac{1}{2}\left(\lambda_i \cdot \operatorname{dist}_X(a_i, y_i) + \lambda_i \cdot \operatorname{dist}_X(c_i, y_i) - \lambda_i \cdot \operatorname{dist}_X(a_i, c_i)\right) \\
&= \mathfrak{u}\text{–}\lim_i \lambda_i (a_i, c_i)_{y_i} \\
&\geq \mathfrak{u}\text{–}\lim_i \lambda_i \min\left\{(a_i, b_i)_{y_i}, (c_i, b_i)_{y_i}\right\} - \lambda_i \delta \\
&= \min\left\{([a_i], [b_i])_{[y_i]}, ([c_i], [b_i])_{[y_i]}\right\},
\end{aligned}$$

so that by definition $X_\mathfrak{u}\left((x_i), (\lambda_i)\right)$ is 0–Gromov-hyperbolic. Moreover, by Lemma 2.1.7, the asymptotic cone is geodesic. So, by the equivalence between Gromov-hyperbolicity and hyperbolicity in geodesic metric spaces, the asymptotic cone is a real tree. ☐

For a geodesic metric space X, the converse also holds. That is, if there exists $\mathfrak{u}$ such that for every (x_i) and (λ_i) the asymptotic cone $\operatorname{Cone}_\mathfrak{u}\left(X, (x_i), (\lambda_i)\right)$ is a real tree, then X is hyperbolic, see [Gro91, Section 2.A.]. Note that this equivalent characterization give a short proof of the quasi-isometry invariance of hyperbolicity for geodesic metric space. We can intuitively understand asymptotic cones as presented in [TV00]. Imagine an observer moving into a space X from a base point to another base point. Then, $(\lambda_i \cdot X, x_i)$ is the environment they can look at if they pause at x_i; the points of X that would have appeared at distance λ_i^{-1} from him, now appear at distance one. As the observer continues to move in the space, any finite configuration become indistinguishable from a single point. However, they may observe finite configurations which may look like earlier configurations. The asymptotic cones of X are the spaces which encode all these recurring finite configurations. In the following subsection, we use the theory of asymptotic cones to define, under certain conditions, a limit action to a sequence of actions on a hyperbolic metric space. This limit action is of first interest to prove the main theorem in Subsection 4.2.

2.2 Groups Acting on Trees and applications to Hyperbolic Groups

We first use the theory of asymptotic cones developed previously to construct a limit to a sequence of actions on hyperbolic spaces. Thereafter, we show that this limiting process generalizes a construction of limit space introduced by Bestvina and Paulin in [Bes88] and [Pau88] respectively. This limit action is studied further in Subsection 3.3 and is of primary interest for the proof of the main theorem in Subsection 4.2. To construct this particular limit action and because asymptotic cones of hyperbolic metric spaces are real trees, we start by introducing vocabulary of groups acting on trees as exposed by J.-S. Serre in [Ser77] and M. Culler and J.W. Morgan in [CM87].

Notation 4. Given an isometry γ of a metric space X, we denote $\lg(\gamma) := \inf_{x\in X} \operatorname{dist}_X(x, \gamma x)$.

Definition 2.2.1. Let γ be an isometry of a metric space X.

- If γ fixes an element of X, then γ is called **elliptic**,
- If $\lg(\gamma) > 0$ and is attained (there exists $x \in X$ so that $\lg(\gamma) = \operatorname{dist}_X(x, \gamma x)$ then γ is called **hyperbolic**,

- The isometry γ is called **parabolic** otherwise.

The following results investigate the geometric properties of isometries acting on real trees. In particular, the next lemma states that, in fact, only two types of isometries of a real tree exist.

Lemma 2.2.2. *If γ is an isometry of a real tree T without fixed point, then there exists a unique embedded line, the characteristic axis of γ denoted $\mathrm{Axis}(\gamma) \subseteq T$, on which γ acts by translation. In particular, γ is hyperbolic.*

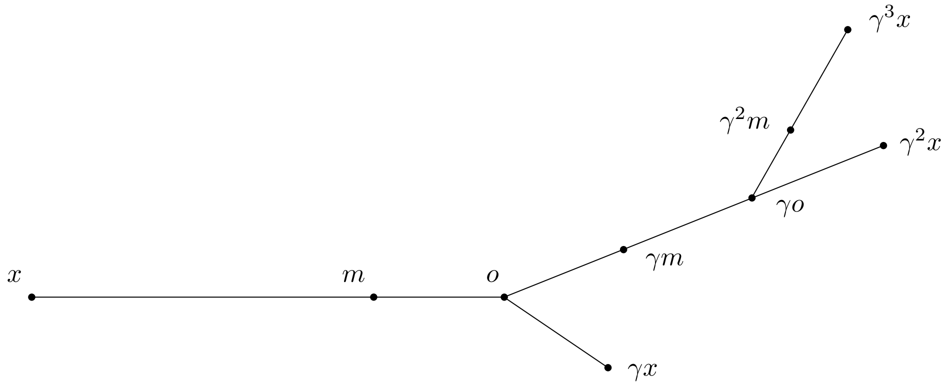

Figure 1: The axis for a non-elliptic isometry γ and notations for Lemma 2.2.2.

Proof. To find $\mathrm{Axis}(\gamma)$, we look for an element $m \in T$ that verifies $\mathrm{dist}_T\left(m, \gamma^2 m\right) = 2\,\mathrm{dist}_T\left(m, \gamma m\right)$. Indeed, using Remark 2.1.4 and the second item of Definition 2.1.3, this equality holds if and only if m-γm and γm-$\gamma^2 m$ intersect at a single point γm. Therefore, the γ–translates of the geodesic segment m-γm form a γ–invariant line on which γ acts by translation. Consider a tripod with vertices $x, \gamma x, \gamma^2 x$ in T, o its crux and m the mid-point of the segment x-γx. If $\mathrm{dist}_T\left(x, m\right) \geq \mathrm{dist}_T\left(x, o\right)$, then m is on the path γx-$\gamma^2 x$. So, using $\mathrm{dist}_T\left(m, x\right) = \mathrm{dist}_T\left(m, \gamma x\right)$, we obtain

$$\mathrm{dist}_T\left(m, \gamma m\right) = \mathrm{dist}_T\left(\gamma x, \gamma m\right) - \mathrm{dist}_T\left(m, \gamma x\right) = \mathrm{dist}_T\left(x, m\right) - \mathrm{dist}_T\left(m, \gamma x\right) = 0,$$

so that m is a fixed point of γ. This is a contradiction with γ has no fixed point, therefore, $\mathrm{dist}_T\left(x, m\right) < \mathrm{dist}_T\left(x, o\right)$. Thus, we are in the situation presented in Figure 1 and, to conclude, we show that $\mathrm{dist}_T\left(m, \gamma^2 m\right) = 2\,\mathrm{dist}_T\left(m, \gamma m\right)$. Since $o \in m$-γm and $\gamma o \in \gamma m$-$\gamma^2 m$, it is sufficient to prove $\mathrm{dist}_T\left(o, \gamma o\right) = 2\,\mathrm{dist}_T\left(o, \gamma m\right)$. Indeed, if this equality holds, then

$$\begin{aligned}
\mathrm{dist}_T\left(m, \gamma^2 m\right) &= \mathrm{dist}_T\left(m, o\right) + \mathrm{dist}_T\left(o, \gamma o\right) + \mathrm{dist}_T\left(\gamma o, \gamma^2 m\right) \\
&= \mathrm{dist}_T\left(m, \gamma m\right) + \underbrace{\mathrm{dist}_T\left(\gamma m, \gamma o\right)}_{2^{-1}\mathrm{dist}_T(o,\gamma o)} + \frac{1}{2}\,\mathrm{dist}_T\left(o, \gamma o\right) \\
&= \underbrace{\mathrm{dist}_T\left(o, \gamma o\right)}_{\mathrm{dist}_T(o,\gamma m)+\mathrm{dist}_T(o,m)} + \mathrm{dist}_T\left(m, \gamma m\right),
\end{aligned}$$

that is equal to $2\,\mathrm{dist}_T\left(m, \gamma m\right)$ since $o \in m$-γm. Finally, $\mathrm{dist}_T\left(o, \gamma o\right) = 2\,\mathrm{dist}_T\left(o, \gamma m\right)$ by the following equalities

$$\begin{aligned}
\mathrm{dist}_T\left(o, \gamma o\right) &= \mathrm{dist}_T\left(\gamma x, \gamma^2 x\right) - 2\,\mathrm{dist}_T\left(o, \gamma x\right) \\
&= \mathrm{dist}_T\left(x, \gamma x\right) - 2\left(\frac{1}{2}\,\mathrm{dist}_T\left(x, \gamma x\right) - \mathrm{dist}_T\left(o, \gamma m\right)\right) \\
&= 2\,\mathrm{dist}_T\left(o, \gamma m\right).
\end{aligned}$$

Hence γ is a hyperbolic isometry with axis $\bigcup_{i\in\mathbb{Z}}\left(\gamma^i m\text{-}\gamma^{i+1} m\right)$. $\square$

We aim to find, under certain conditions, hyperbolic elements in the group of isometries of a real tree. In the next lemma, we construct hyperbolic isometries by studying the composition of two isometries of the same type.

Proposition 2.2.3. *Let η, γ be two isometries of a real tree. Then the following holds.*

1. *If η, γ are elliptic and their fixed-point sets are disjoint, then their composition $\eta\gamma$ is hyperbolic.*
2. *If η, γ are hyperbolic and their axes are disjoint, then their composition $\eta\gamma$ is hyperbolic with* $\mathrm{Axis}(\eta\gamma)$ *intersects* $\mathrm{Axis}(\gamma)$ *and* $\mathrm{Axis}(\eta)$.

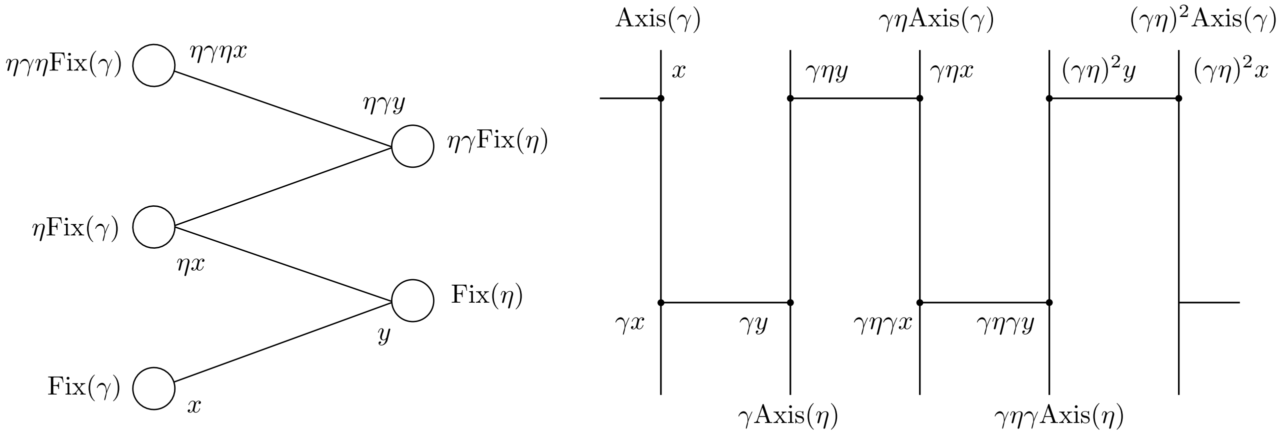

Figure 2: The axes of the compositions of: two elliptic isometries with disjoint fixed-point sets (left), two hyperbolic elements with disjoint axes (right).

Proof. We construct an axis for $\eta\gamma$, see Figure 2, on which the isometry acts by translation.

1. Suppose that γ, η are elliptic with disjoint fixed-point sets. Since γ, η are isometries, the sets $\mathrm{Fix}(\gamma)$ and $\mathrm{Fix}(\eta)$ are closed subtrees of T. Let $x \in \mathrm{Fix}(\gamma)$ be the unique point closest to $\mathrm{Fix}(\eta)$, and $y \in \mathrm{Fix}(\gamma)$ the unique point closest to $\mathrm{Fix}(\eta)$. Because no point on the segment x-y is fixed by η, the unique geodesic from x to $\eta\gamma x$ is $x\text{-}y \cup \eta(y\text{-}x)$ so that $\mathrm{dist}_T(x, \eta\gamma x) = \mathrm{dist}_T(x, \eta x) = 2\,\mathrm{dist}_T(x, y)$. Likewise, the geodesic from x to $(\eta\gamma)^2 x$ is the concatenation
$$x\text{-}y \cup \eta(y\text{-}x) \cup \eta\gamma(x\text{-}y) \cup \eta\gamma\eta(y\text{-}x),$$
so that $\mathrm{dist}_T\left(x, (\eta\gamma)^2 x\right) = 4\,\mathrm{dist}_T(x, y) = 2\,\mathrm{dist}_T(x, \eta\gamma x)$. Thus, $\eta\gamma$ is hyperbolic as wanted.

2. Suppose that γ, η are hyperbolic with disjoint axes. Let x be the unique point of $\mathrm{Axis}(\gamma)$ closest to $\mathrm{Axis}(\eta)$, and y the unique point of $\mathrm{Axis}(\eta)$ closest to $\mathrm{Axis}(\gamma)$. As in the first item, the unique geodesic from x to $\gamma\eta x$ is the concatenation
$$x\text{-}\gamma x \cup \gamma(x\text{-}y) \cup \gamma(y\text{-}\eta y) \cup \gamma\eta(y\text{-}x),$$
so that $\mathrm{dist}_T(x, \gamma\eta x) = \lg(\gamma) + \lg(\eta) + 2\,\mathrm{dist}_T(x, y)$. Likewise, the geodesic from x to $(\gamma\eta)^2 x$ is
$$x\text{-}\gamma x \cup \gamma(x\text{-}y) \cup \gamma(y\text{-}\eta y) \cup \gamma\eta(y\text{-}x) \cup \gamma\eta(x\text{-}\gamma x) \cup \gamma\eta\gamma(x\text{-}y) \cup \gamma\eta\gamma(y\text{-}\eta y) \cup \gamma\eta\gamma\eta(y\text{-}x).$$
Hence, $\mathrm{dist}_T\left(x, (\gamma\eta)^2 x\right) = 2\lg(\gamma) + 2\lg(\eta) + 4\,\mathrm{dist}_T(x, y) = 2\,\mathrm{dist}_T(x, \gamma\eta x)$ so that $\gamma\eta$ is hyperbolic with axis intersecting $\mathrm{Axis}(\gamma)$ and $\mathrm{Axis}(\eta)$ at x and y respectively. ☐

This proposition and the following lemma are the main steps to prove Lemma 2.2.7. Since this is not a detour, we prove the next lemma for convex spaces.

Definition 2.2.4. A subset C of a geodesic metric space is **convex** if, given any two elements in C, there is a unique geodesic segment contained within C that joins those two elements.

Lemma 2.2.5. *Suppose $C_1, \dots, C_n$ are convex subsets of a geodesic metric space X verifying $C_i \cap C_j \neq \emptyset$ for every $i, j \in \{1, \dots, n\}$. Then, the intersection $\cap_{i=1}^n C_i$ is non-empty.*

Proof. The proof is by induction on the number of convex subsets n. The initial case $n = 2$ is verified by assumption. By induction, if $n > 2$ there exist elements
$$x_k \in \bigcap_{i \neq k} C_i \text{ for every } 1 \leq k \leq n.$$
So, $x_1 \in C_2$, $x_2 \in C_1$ and x_1-$x_2 \subseteq C_k$ for each $k > 2$. Therefore, x_1-x_2 must pass through the intersection $C_1 \cap C_2$ which is a convex subset X. Hence, there exists an element of the geodesic metric space on the segment x_1-x_2 that lies in every convex subsets. ☐

Since the fixed-point set of an elliptic isometry of a real tree is a convex set, we obtain:

Corollary 2.2.6. *Suppose $\gamma_1, \dots, \gamma_n$ are elliptic isometries of a real tree T verifying $\mathrm{Fix}(\gamma_i) \cap \mathrm{Fix}(\gamma_j) \neq \emptyset$ for every $i, j \in \{1, \dots, n\}$. Then*
$$\bigcap_{i=1}^n \mathrm{Fix}(\gamma_i) \neq \emptyset.$$

We now turn to the study of finitely generated groups acting on metric spaces by isometry. An **action** of a group G on a metric space X is a homomorphism from G to the isometry group of X. Then, a G–**space** is a tuple (X, x_0, ρ) consisting of a metric space X, a base point $x_0 \in X$ and an action ρ of G on X. If a group G acts on a real tree T, then T is called a G–**tree**. In this case, the G–action on T is **hooked** if there exists a point of T fixed by all of G.

Lemma 2.2.7. *If G is a finitely generated group acting unhooked on a G–tree T, then G contains a hyperbolic element. That is, the image of this element in the isometry group of T is a hyperbolic isometry.*

Proof. Let $g_1, \dots, g_\ell$ be a finite generating set for G and suppose that every g_i is elliptic, as otherwise, we conclude using Lemma 2.2.2. Suppose by contradiction that every two generators have fixed-point sets intersecting non-trivially. Then, by Corollary 2.2.6, T has an element fixed by all generators of G, which contradicts the assumption that T is unhooked. Thus, there exist two generators g_i, g_j that have disjoint fixed-point sets. Hence, by the first item of Proposition 2.2.3, the element $g_i g_j$ is hyperbolic. □

In Subsection 4.2, we study combinatorial properties of a group from its action on a real tree. In this sense, it seems natural to analyse the "simplest" subtree which is invariant under the group action.

Definition 2.2.8. Let G be a group and T a G–tree. If T does not contain any proper, unhooked, G–invariant subtree, then T is called **minimal**.

Theorem 2.2.9. *If G is a finitely generated group and T is an unhooked G–tree, then T contains a unique minimal G–invariant subtree which is the countable union of the axes of hyperbolic elements in G.*

Proof. Consider the subspace $T' \subseteq T$ defined as the union of all axes of every hyperbolic element of G. By Proposition 2.2.7, this subspace is non-empty and, by the second item of Lemma 2.2.3, T' is actually a tree. For every hyperbolic element $g \in G$, any g–invariant subtree contains $\mathrm{Axis}(g)$. Thus, any G–invariant subtree contains T' and, to conclude, we prove that T' is itself G–invariant. However, $h\,\mathrm{Axis}(g) = \mathrm{Axis}\left(hgh^{-1}\right)$ for any $g, h \in G$ where g is a hyperbolic element. Hence, the union of all axes is G–invariant as wanted. □

The asymptotic cones of a hyperbolic group G are real trees. Moreover, given a fixed asymptotic cone of G, there exists an action of G on this tree that makes it a G–tree. The purpose of the following is to clarify this process, extend this idea to G–spaces and apply the above theory of groups acting on trees to this limit action.

Definition 2.2.10. Let G be a finitely generated group with finite generating set S_G and (X, x_0, ρ) a based G–space. The **displacement of the action** ρ with respect to the base point x_0, denoted $|\rho|_{x_0}$, is

$$|\rho|_{x_0} := \max_{s \in S_G} \mathrm{dist}_X\left(x_0, \rho(s)x_0\right).$$

We may simply write $|\rho|$ if the base point x_0 is clear from the context.

Theorem 2.2.11. *Consider a finitely generated group G with finite generating set S_G and (X, x_i, ρ_i) a sequence of G–spaces where X is hyperbolic. If $(|\rho_i|_{x_i})$ diverges towards infinity, then G acts by isometry on the asymptotic cone $T_{\mathfrak{u}} := \mathrm{Cone}_{\mathfrak{u}}\left(X, (x_i), \left(|\rho_i|_{x_i}^{-1}\right)\right)$ for any non-principal ultrafilter $\mathfrak{u}$. Moreover, if for every $i \in \mathbb{N}$ it holds*

$$|\rho_i|_y \geq |\rho_i|_{x_i} \text{ for every } y \in X, \tag{2}$$

then $T_{\mathfrak{u}}$ is unhooked and admits a minimal G–invariant subtree.

Proof. Fix $\mathfrak{u}$ a non-principal ultrafilter. By Theorem 2.1.8, the asymptotic cone $T_{\mathfrak{u}}$ is a real tree and, using the third item of Lemma 1.2.6, we define the limit homomorphism $\rho : G \to \mathrm{End}(T_{\mathfrak{u}})$ via

$$\rho(g)[z_i] := [\rho_i(g)z_i], \text{ for every } [z_i] \in T_{\mathfrak{u}},\ g \in G.$$

By the second item of Lemma 1.2.6, ρ is an action of G on $T_{\mathfrak{u}}$; this conclude the first part of the theorem. To prove the second part, we show that $T_{\mathfrak{u}}$ has no element fixed by all of G. Let $[z_i]$ be an element of $T_{\mathfrak{u}}$. Since S_G is finite and $\mathfrak{u}$ is finitely additive as an ultrafilter, there exists $s \in S_G$ verifying

$$\max_{s' \in S_G} \mathrm{dist}_X\left(z_i, \rho_i\left(s'\right) z_i\right) = \mathrm{dist}_X\left(z_i, \rho_i(s)z_i\right) \text{ for } \mathfrak{u}\text{–almost every } i \in \mathbb{N}.$$

Thus, by condition (2) above, $\mathrm{dist}_X\left(z_i, \rho_i(s)z_i\right) \geq |\rho_i|_{x_i}$ for $\mathfrak{u}$–almost every $i \in \mathbb{N}$ so that

$$\mathrm{dist}_{\mathfrak{u}}\left([z_i], \rho(s)[z_i]\right) = \mathfrak{u}\text{–}\lim |\rho_i|_{x_i}^{-1} \mathrm{dist}_X\left(z_i, \rho_i(s)z_i\right) \geq \mathfrak{u}\text{–}\lim 1 = 1.$$

Hence, $T_{\mathfrak{u}}$ has no element fixed by all of G and, by Theorem 2.2.9, it contains a minimal G–invariant subtree. □

Based on [RW10, Theorem 1.12], from which we take the idea of projection, we build a model for this subtree.

Notation 5. Given a G–space (X, x_0, ρ), we denote gx the action of $g \in G$ on $x \in X$ if the choice of ρ is clear.

Proposition 2.2.12. *Let G be a finitely generated group with finite generating set S_G and ρ an unhooked action of G on a real tree T. If x is an element of T realizing the minimum of the displacement function, then the subtree spanned by $G \cdot x$*

$$T_x := \bigcup_{g \in G} x\text{-}gx \subseteq T,$$

is the unique G–invariant minimal subtree of T containing x.

Proof. By construction, every element of T_x is connected to x. So, this subspace is a tree. Let y be an element of T_x, $g \in G$ such that $y \in x\text{-}gx$ and $h \in G$. Since G acts on T by isometry, $hy \in h(x\text{-}gx) = hx\text{-}hgx$ and, using that T is a tree, $hy \in T_x$. Therefore, T_x is a G–invariant unhooked tree and it remains to show that T_x is minimal. Suppose $T' \subset T_x$ is a proper, unhooked, G–invariant subtree. By definition of T_x and properness of T', the element x is not in T'. Indeed, if x is in T', then $G \cdot x$ is in T' by G–invariance. However, T' is a tree so that any segment between x and gx is in T' for every $g \in G$. This is a contradiction with T' is a proper subspace of T_x. Hence, $x \in T_x$ is not contained in T' and we consider its projection p_x on T'. Since S_G is finite and x has minimal displacement for the ρ action, there exists $s, s' \in S_G$ with

$$\operatorname{dist}_T(x, \rho(s)x) =: |\rho|_x \leq |\rho|_{p_x} := \operatorname{dist}_T(p_x, \rho(s')\, p_x).$$

Moreover, for every $g \in G$, either p_x is fixed by g or $x\text{-}gx = x\text{-}p_x \cup p_x\text{-}gp_x \cup gp_x\text{-}gx$. Thus,

$$\operatorname{dist}_T(p_x, \rho(s')\, p_x) \leq \operatorname{dist}_T(x, \rho(s')x) \leq \operatorname{dist}_T(x, \rho(s)x) \leq \operatorname{dist}_T(p_x, \rho(s')\, p_x).$$

Therefore, p_x is fixed by s' so that x is fixed by s. That is a contradiction with T is unhooked. Hence, the action of G on T_x is minimal and this conclude the proof. ☐

Since ultrafilters are finitely additive, the following corollary is a direct consequence of Proposition 2.2.12.

Corollary 2.2.13. *With the notations and hypothesis of Theorem 2.2.11, the subtree spanned by $G \cdot x_{\mathfrak{u}}$*

$$T := \bigcup_{g \in G} x_{\mathfrak{u}}\text{-}gx_{\mathfrak{u}} \subseteq T_{\mathfrak{u}},$$

is the unique G–invariant minimal subtree of $T_{\mathfrak{u}}$ containing $x_{\mathfrak{u}}$, where $x_{\mathfrak{u}}$ is the ultralimit of the sequence $(x_i) \subset X$.

Bestvina in [Bes88] and Paulin in [Pau88] obtained similar results while studying the space of actions of one group on hyperbolic spaces $\mathbb{H}^n$. The remainder of this subsection aims to construct a G–equivariant isometry between the minimal tree obtained in Theorem 2.2.11 and the tree provided by Bestvina and Paulin's method. In the following, we present Bestvina's tree which is slightly different from Paulin's tree but equivalent up to G–equivariant isometry.

Consider a natural number n and a sequence of faithful representations from a finitely generated non-virtually abelian group G to the group of orientation preserving isometry of the hyperbolic n–space $\rho_i : G \to \operatorname{Isom}_+(\mathbb{H}^n)$. For S_G a finite symmetric generating set of G, Bestvina proved the following proposition.

Proposition 2.2.14 (Bestvina, [Bes88, Proposition 2.1])**.** *If G is not virtually abelian, then for every discrete and faithful representation $\rho : G \to \operatorname{Isom}_+(\mathbb{H}^n)$, there exists $x_0 \in \mathbb{H}^n$ such that $|\rho|_{x_0} \leq |\rho|_y$ for every $y \in \mathbb{H}^n$.*

Thus, there exist base points x_i which move minimally for the G–actions via the ρ_i representations. Moreover, if $(|\rho_i|_{x_i})$ diverges towards infinity, there exists a subsequence of (ρ_i) converging to a discrete and faithful representation $\rho : G \to \operatorname{Isom}(\mathcal{T})$ where $\mathcal{T}$ is a real tree. This result relies heavily on Gromov-convergence that is introduced by M. Gromov in [Gro91]. A sequence (X_i) of closed subsets of a compact metric space Q converges in the **Hausdorff sense** to a closed subset X of Q if, for every $\varepsilon > 0$ there exists $m(\varepsilon)$ such that, for every $m > m(\varepsilon)$ the set X is contained in the ε–neighborhood of X_m and X_m is contained in the ε–neighborhood of X.

Definition 2.2.15. A sequence (X_i) of compact metric spaces converges in the **Gromov sense** to a compact metric space X if, there exists a compact metric space Q and isometric embeddings $X \hookrightarrow Q$ and $X_i \hookrightarrow Q$, so that the sequence (X_i) converges to X in the Hausdorff sense as subsets of Q.

Let X,Y be proper metric spaces. We make a remark on Gromov-convergence taken from [KL95]. If the sequence of rescaled metric spaces with distinguished points $\left(X, x_i, \lambda_i^{-1} \cdot \mathrm{dist}_X\right)$ converges to $(Y, y_0, \mathrm{dist}_Y)$ in the Gromov sense, then for all ultrafilters $\mathfrak{u}$, there exists an isometry between $\mathrm{Cone}_{\mathfrak{u}}\left(X, (x_i), \left(\lambda_i^{-1}\right)\right)$ and (Y, y_0) such that the image of $[x_i]$ is y_0. However, the converse is not true in general and relies on the asymptotic cone's property of being a proper metric space. Thus, instead of considering the entire rescaled space $\left(X, x_i, |\rho_i|^{-1} \cdot \mathrm{dist}_X\right)$, Bestvina considers, in [Bes88], a sequence of specific subspaces. We proceed similarly and define the following subspaces

$$\begin{aligned} W^k &:= \{ g \in G \mid g \text{ can be represented as a word of length smaller than } k \text{ in the alphabet } S_G \}, \\ F_i^k &:= \text{ convex hull in } \mathbb{H}^n \text{ of } \left\{\rho_i(g)x_i \;\middle|\; g \in W^k \right\}, \\ X_i^k &:= F_i^k \text{ setwise, with the rescaled metric } \mathrm{dist}_{X_i^k}(a,b) := |\rho_i|_{x_i}^{-1} \mathrm{dist}_{\mathbb{H}^n}(a,b) \text{ for every } a, b \in X_i^k. \end{aligned}$$

Proposition 2.2.16 (Bestvina, [Bes88, Theorem 3.4])**.** *There exists a subsequence of (ρ_i) so that $\left(X_i^k\right)_{i\in\mathbb{N}}$ converges in the Gromov sense to finite trees $\mathcal{T}^k$ for every $k \in \mathbb{N}$. These trees verify the inclusions*

$$\mathcal{T}^1 \subset \mathcal{T}^2 \subset \cdots \subset \mathcal{T}^k \subset \cdots .$$

Moreover, if $g \in W^k$, then the sequence $(\rho_i(g)x_i)_{i\in\mathbb{N}}$ converges to a point $p(g) \in \mathcal{T}^k$ in the metric space containing both X_i^k and $\mathcal{T}^k$ provided by the Hausdorff convergence.

We endow the real tree $\mathcal{T}_{BP} := \cup_{k\in\mathbb{N}} \mathcal{T}^k$ with the limit metric. That is, for every $a, b \in \mathcal{T}_{BP}$

$$\mathrm{dist}_{\mathcal{T}_{BP}}(a,b) := \lim_{i\to\infty} \mathrm{dist}_{X_i^k}(a_i, b_i) = \lim_{i\to\infty} |\rho_i|^{-1} \mathrm{dist}_{\mathbb{H}^n}(a_i, b_i),$$

where k is sufficiently large so that $a, b \in \mathcal{T}^k$ and a_i, b_i are points of X_i^k for every i that verify $a = \lim_i a_i$, $b = \lim_i b_i$. Finally, this tree is equipped with a G–space structure via $gp(h) := p(gh)$ for every $g, h \in G$. This construction leads to the following result.

Theorem 2.2.17 (Bestvina, [Bes88, Theorem 4.3])**.** *With the above notations, the following holds.*

1. *The group G contains an element that acts without fixed points on $\mathcal{T}_{BP}$,*
2. *The action of G on $\mathcal{T}_{BP}$ is minimal.*

We give a model to Bestvina and Paulin's tree. Consider the subspace of $\mathcal{T}_{BP}$ spanned by $G \cdot p(e)$

$$\mathcal{T} := \bigcup_{g\in G} p(e)\text{-}gp(e) \subseteq \mathcal{T}_{BP}.$$

Because every element of $\mathcal{T}$ is connected to $p(e)$, this subspace is in fact a tree. Then, as in Corollary 2.2.13, the G–action is by isometry so that $\mathcal{T}$ is G–invariant. Moreover, by the first item of Theorem 2.2.17, this subtree is unhooked. Finally, notice that the limit metric on $\mathcal{T}_{BP}$ is, for every $a = \lim a_i, b = \lim b_i \in \mathcal{T}_{BP}$, the limit

$$\mathrm{dist}_{\mathcal{T}_{BP}}(a,b) = \mathfrak{u}\text{–}\lim |\rho_i|^{-1} \mathrm{dist}_{\mathbb{H}^n}(a_i, b_i),$$

where $\mathfrak{u}$ is the Fréchet filter. So, using the assumption $|\rho_i|_{x_i} \leq |\rho_i|_y$ for every $y \in \mathbb{H}^n$, we obtain by a similar method as in Corollary 2.2.13, using Proposition 2.2.12, the conclusion that $\mathcal{T}$ is a unhooked, minimal G–invariant subtree of $\mathcal{T}_{BP}$. By uniqueness of the minimal G–invariant subtree of $\mathcal{T}_{BP}$ and the second item of Theorem 2.2.17, we obtain $\mathcal{T} = \mathcal{T}_{BP}$.

We are now able to construct, in the special case of a group G acting on the hyperbolic n–space, a G–equivariant isometry between both trees constructed in this subsection.

Theorem 2.2.18. *Let G be a finitely generated non-virtually abelian group, $\mathfrak{u}$ a non-principal ultrafilter and (ρ_i) a sequence of faithful representations $\rho_i : G \to \mathrm{Isom}_+(\mathbb{H}^n)$. Fix S_G a finite symmetric generating set for G and $x_i \in \mathbb{H}^n$ elements verifying $|\rho_i|_{x_i} \leq |\rho_i|_y$ for every $y \in \mathbb{H}^n$. If $(|\rho_i|_{x_i})$ diverges towards infinity, then there exists a G–equivariant isometry between T, the minimal G–invariant subtree of $T_{\mathfrak{u}} := \mathrm{Cone}_{\mathfrak{u}}\left(\mathbb{H}^n, (x_i), \left(|\rho_i|_{x_i}^{-1}\right)\right)$, and the tree obtained by Bestvina and Paulin's method $\mathcal{T}_{BP}$.*

Proof. By definition of the metrics on both trees, it holds for every $g, h \in G$

$$\mathrm{dist}_{\mathfrak{u}}(gx_{\mathfrak{u}}, hx_{\mathfrak{u}}) = \mathfrak{u}\text{–}\lim |\rho_i|^{-1} \mathrm{dist}_{\mathbb{H}^n}(\rho_i(g)x_i, \rho_i(h)x_i) = \lim |\rho_i|^{-1} \mathrm{dist}_{\mathbb{H}^n}(\rho_i(g)x_i, \rho_i(h)x_i) = \mathrm{dist}_{\mathcal{T}_{BP}}(p(g), p(h)).$$

So, the map sending $x_{\mathfrak{u}} \mapsto p(e)$ can be extended by G–equivariance to an isometry ϕ from $G \cdot x_{\mathfrak{u}}$ to $G \cdot p(e)$. By the above equalities, for $g \in G$, the segments $x_{\mathfrak{u}}$-$gx_{\mathfrak{u}}$ and $p(e)$-$p(g)$ are isometric embedding of the interval

$[0, \mathrm{dist}_{\mathrm{u}}(x_{\mathrm{u}}, gx_{\mathrm{u}})] \subset \mathbb{R}$ in T and $\mathcal{T}$ respectively. Consequently, there exist isometries $\phi_g : x_{\mathrm{u}}\text{-}gx_{\mathrm{u}} \to p(e)\text{-}p(g)$ sending x_{u} to $p(e)$ and gx_{u} to $p(g)$ for every $g \in G$. Since any two elements of T, respectively $\mathcal{T}$, are joined by a unique geodesic

$$\varphi := \bigcup_{g \in G} \phi_g : T \to \mathcal{T},$$

is a well-defined isometry that extend ϕ by construction. Since ϕ is G–equivariant and the action of G on both spaces is by isometry, the extension of ϕ to φ is also G–equivariant. This conclude the proof of the theorem. $\square$

The theory of asymptotic cones as been studied first by L. Van den Dries and J.A. Wilkie in [VdDW84] to generalize results obtained studying limits in the Gromov sense. Using their results, Paulin had already carried out a similar construction to the one above in [Pau97] and Groves had already delivered a model for this tree in [Gro09]. The limit action of a group on a real tree as obtained above is studied in Section 3 and used to prove the main Theorem 4.2.1. To deepen our understanding of this action, we introduce the concept of limit group which is the cornerstone of this text.

3 Preliminaries in the Theory of Limit Groups

This section is intended to construct, under certain conditions, a limit to a sequence of morphisms from a free group to a given group. We start by studying the limit of a sequence of morphisms from free groups to arbitrary groups. Subsequently, we analyse more precisely this limit in the specific case of morphisms from free groups to free groups. Finally, we use the theory introduced in this and the previous sections to study the action of limit groups on limit spaces. In this section, we introduce all the necessary results to present Fujiwara and Sela's proof of Theorem 4.2.1 as done in [FS20] and take this occasion to give an introduction to the theory of limit groups.

3.1 Space of Marked Groups and Limit Groups over an arbitrary Group

At first, we define the concept of G–limit groups where G is an arbitrary finitely generated group. Then, we study some properties of these groups and give several equivalent definitions. This framework gives us the opportunity to define the category of marked groups as well as the Grigorchuk space introduced in [Gri84]. We synthesize approches to limit groups written by C. Champetier and V. Guiradel [CG05] and P. Paulin [Pau03], by giving four equivalent definitions of marked groups. Each definition gives new insight on the Grigorchuk space which is the basis to our topological definition of limit groups.

We define the category of marked groups as follows. A **marking** of a group G is a finite ordered generating family of G. In this definition, repetitions of elements are allowed in the generating family. The objects in the category are marked groups.

Definition 3.1.1. A **marked group** is a pair (G, S) where G is a group and S is a marking of G.

A morphism of marked groups is a homomorphism of groups commuting with the markings. In formula, $\varphi : (G, (s_1, \dots, s_\ell)) \to (G', (s'_1, \dots, s'_\ell))$ is a morphism of marked groups if and only if φ is a morphism of groups between G and G' that sends s_i on s'_i for every $i \in \{1, \dots, \ell\}$. Notice that in the category of marked groups, there exists at most one morphism between two objects and that any morphism is an epimorphism.

Notation 6. For $\ell \in \mathbb{N}$, we denote $\mathcal{G}_\ell$ the set of groups marked by ℓ elements up to isomorphism of marked groups.

Before equipping this set with a topology, we give three other equivalent definitions of $\mathcal{G}_\ell$, as in [CG05], that illustrate different aspects of this space. If not precised, a morphism between marked (labeled) objects is always a morphism respecting the marking (labeling).

<u>Marked groups as Cayley graphs:</u> Each marked group (G, S) has a natural labeled Cayley graph, denoted $\mathrm{Cay}(G, S)$, whose edges are labeled by integers in $\{1, \dots, \ell\}$ referring to the generating elements $s_1, \dots, s_\ell$ of S. With these definitions, two marked groups are isomorphic if and only if their labelled Cayley graphs are isomorphic in a label preserving way. As a result, $\mathcal{G}_\ell$ may be viewed as the set of labeled Cayley graphs on ℓ generators up to isomorphism.

<u>Marked groups as epimorphisms from free groups:</u> Let ℓ be a natural number and $F_\ell := \langle s_1, \dots, s_\ell \rangle$ denote the free group generated by $s_1, \dots, s_\ell$ that we mark by the free basis $(s_1, \dots, s_\ell)$. For a group G generated by at most ℓ elements, the epimorphisms from F_ℓ onto G are in one-to-one correspondence with the markings of this group. Then, two epimorphisms $h_1 : F_\ell \to G_1$ and $h_2 : F_\ell \to G_2$ correspond to isomorphic marked groups if and only if h_1 and h_2 are equivalent. In formula, h_1 and h_2 are equivalent if there exists a group isomorphism $f : G_1 \to G_2$ such that the following diagram commutes

$$\begin{tikzcd}
(F_\ell, S) \arrow[rr, two heads, "h_2"] \arrow[dr, two heads, "h_1"'] & & (G_2, h_2(S)) \\
& (G_1, h_1(S)). \arrow[ur, hook, two heads, "f"'] &
\end{tikzcd}$$

Therefore, $\mathcal{G}_\ell$ is the set of epimorphisms with source the free group on ℓ elements up to equivalence.

<u>Marked groups as subgroups of free groups:</u> Two epimorphisms $h_1 : F_\ell \to G_1$ and $h_2 : F_\ell \to G_2$ describe the same element in $\mathcal{G}_\ell$ if and only if their kernels are equal. Thus, $\mathcal{G}_\ell$ can be represented by the set of normal subgroups of F_ℓ. Equivalently, it can be viewed as the set of quotients of F_ℓ where each element is marked by the image of $(s_1, \ldots, s_\ell)$ under the quotient map. As a convention, we use a presentation $\langle s_1, \ldots, s_\ell \mid r_1, r_2, \ldots \rangle$ to represent the marked group $(\langle s_1, \ldots, s_\ell \mid r_1, r_2, \ldots \rangle, (s_1, \ldots, s_\ell))$ in $\mathcal{G}_\ell$.

Notation 7. Given a group G with finite generating set S_G, we denote $|\cdot|_{S_G} : G \to \mathbb{R}_+$, the length function on G with respect to S_G. That is, the function defined by $|g|_{S_G}$ is the length of the shortest word expressing an element $g \in G$ in the alphabet S_G.

We endow $\mathcal{G}_\ell$ with a topology by studying it as a subspace of a metric space (topology of Gromov-Hausdorff and Chabauty). Let 2^{F_ℓ} be the set of all subsets of the free group on ℓ elements and S a finite symmetric generating family for F_ℓ. For any subsets $A, A' \in 2^{F_\ell}$, consider the maximal radius of the balls on which A and A' coincide:

$$v(A, A') := \max\{ R \in \mathbb{N} \cup \{\infty\} \mid A \cap B_S(F_\ell, R) = A' \cap B_S(F_\ell, R) \},$$

where $B_S(F_\ell, R)$ denote the open ball of radius R in F_ℓ about the identity for the S–word length. This induces an ultrametric on 2^{F_ℓ} defined by

$$\mathrm{dist}_{2^{F_\ell}}(A, A') := e^{-v(A, A')}.$$

Then, a base for the metric topology is given by $\left\{ B_{\mathrm{dist}_{2^{F_\ell}}}(X, R) \;\middle|\; R \geq 0, X \in 2^{F_\ell} \right\}$. Each ball $B_{\mathrm{dist}_{2^{F_\ell}}}(X, R)$ is a product of sets V_g where

$$V_g = \begin{cases} \{1\} & \text{if } g \in X \ \wedge\ |g|_S \leq -\log(r) \\ \{0\} & \text{if } g \notin X \ \wedge\ |g|_S \leq -\log(r) \\ \{0, 1\} & \text{otherwise.} \end{cases}$$

Thus, every ball in this base is made of the whole 2^{F_ℓ} except finitely many elements. Similarly, a base for the product topology is composed of sets which have the form 2^{F_ℓ} minus finitely many points. Hence, the metric topology on 2^{F_ℓ} coincides with the product topology so that 2^{F_ℓ} is compact by Tychonoff's Theorem.

Proposition 3.1.2. *The set $\mathcal{G}_\ell$ of normal subgroups of F_ℓ can be endowed with a structure of compact metric space.*

Proof. We show that $\mathcal{G}_\ell$ is a closed subspace of the compact metric space 2^{F_ℓ}. Consider a sequence (H_i) of normal subgroups of F_ℓ that converges towards a subset $H \subseteq F_\ell$. Let g, h be elements in H, $R := \max\{ |g|_S, |h|_S \}$ and N sufficiently large so that $H_i \in B_{\mathrm{dist}_{2^{F_\ell}}}\left(H, e^{-2R}\right)$ for every $i \geq N$. By choice of R and N, the elements g, h are in H_N and since the latter is a group, the element gh^{-1} is also in H_N. Then, by symmetry of S, it holds $\left|gh^{-1}\right|_S \leq 2R$ so that gh^{-1} is contained in H. Hence, H is a subgroup of F_ℓ. By a similar method, H is a normal subgroup of F_ℓ so that H is a closed subspace of the compact metric space 2^{F_ℓ}. Hence, $\mathcal{G}_\ell$ can be equipped with the structure of a compact metric space. ☐

The space of marked groups was first introduced by R.I. Grigorchuk in [Gri84, Section 6] and is sometimes called **Grigorchuk space**. In his article, he proves among other results, the compactness of the space of marked groups without using Tychonoff's Theorem. We now translate the metric topology defined for normal subgroups of F_ℓ to the other definitions of $\mathcal{G}_\ell$. If we view $\mathcal{G}_\ell$ as the set of epimorphisms from the free group on the alphabet $s_1, \ldots, s_\ell$, then the metric topology can be described as follows. Two epimorphisms $h_1 : F_\ell \to G_1$, $h_2 : F_\ell \to G_2$ are at distance at most e^{-R} if, for every word $w(s_1, \ldots, s_\ell)$ in F_ℓ of length at most R, then

$$w(h_1(s_1), \ldots, h_1(s_\ell)) = 1 \text{ in } G_1 \text{ if and only if } w(h_2(s_1), \ldots, h_2(s_\ell)) = 1 \text{ in } G_2.$$

In this setting, another base system for the product topology on $\mathcal{G}_\ell$ is $\{ V_n \mid n \in \mathbb{N} \}$ where V_n is the set of couples of epimorphisms $h_1 : F_\ell \to G_1$, $h_2 : F_\ell \to G_2$ such that there exists an isometry between $B_{h_1(S)}(G, n)$ and $B_{h_2(S')}(G', n)$ making the following diagram commutative:

$$\begin{tikzcd}
B_S(F_\ell, n) \arrow[rr, hook, two heads, "h_2"'] \arrow[dr, hook, two heads, "h_1"'] & & B_{h_2(S)}(G', n) \\
& B_{h_1(S)}(G, n). \arrow[ur, hook, two heads, dashed, "\exists f"'] &
\end{tikzcd}$$

To translate the metric topology, in the case $\mathcal{G}_\ell$ is defined by groups and their markings, we analyse the relations in each group for the alphabet given by the marking. A **relation** in a marked group (G, S) is an S–word representing the identity in G. Then, two marked groups $(G, S), (G', S')$ are at distance at most e^{-R} if they have exactly the same relations of length at most R. In this description, marked groups are always considered up to isomorphism. Thus, we identify an S–word in G with the corresponding S'–word in G' under the canonical bijection from S to S' as ordered family of cardinality ℓ. Finally, in a marked group (G, S), the set of relations of length at most $2R + 1$ contains the same information as the ball of radius R of its Cayley graph. Thus, the metric on $\mathcal{G}_\ell$ viewed as space of Cayley graphs can be expressed as follows. Two marked groups $(G, S), (G', S')$ are at distance at most e^{-R} if their labeled Cayley graphs have isomorphic labeled balls of radius R. Hence, we have a description of the metric topology on $\mathcal{G}_\ell$ for each equivalent definition of the space of marked groups. The limit of a sequence in $\mathcal{G}_\ell$ depends on the sequence of ambient groups as well as on their markings.

Example 3.1.3. *The free abelian group $\mathbb{Z}^\ell$ is a limit of markings of $\mathbb{Z}$:* Consider $(G_i, S_i) := \big(\mathbb{Z}, \big(1, i, \dots, i^{\ell-1}\big)\big)$ a sequence of markings of $\mathbb{Z}$ in $\mathcal{G}_\ell$. For any $R > 0$, if $i \geq 100R$ and j, k are distinct elements of $\{0, \dots, \ell - 1\}$, then the only relations between i^j and i^k in the ball of radius R in (G_i, S_i) are relations of commutation, see Figure 3. Thus, the ball of radius R of (G_i, S_i) is the same as the ball of radius R in $\big(\mathbb{Z}^\ell, (e_1, \dots, e_\ell)\big)$, where e_n denotes the n–th vector of the canonical base of $\mathbb{Z}^\ell$. Consequently, the sequence of marked groups $\big(\mathbb{Z}, \big(1, i, \dots, i^{\ell-1}\big)\big)$ converges to the marked group $\big(\mathbb{Z}^\ell, (e_1, \dots, e_\ell)\big)$. So, being generated by k elements for $k < \ell$ is not a closed condition in $\mathcal{G}_\ell$.

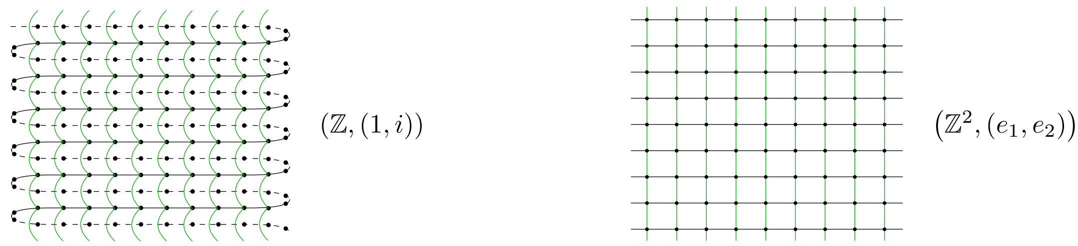

Figure 3: Convergence of the sequence $(\mathbb{Z}, (1, i))$ towards the marked group $\big(\mathbb{Z}^2, (e_1, e_2)\big)$.

The free group $\mathbb{Z}$ is a limit of finite cyclic groups: Consider the sequence $(\mathbb{Z}/i\mathbb{Z}, (1)) \subset \mathcal{G}_\ell$. For any natural number i, the ball of radius $i/3$ about the identity in the marked group $(\mathbb{Z}/i\mathbb{Z}, (1))$ is the same as the ball of radius $i/3$ about the identity in $(\mathbb{Z}, (1))$, see Figure 4. Thus, the sequence of marked groups $(\mathbb{Z}/i\mathbb{Z}, (1))$ converges to the marked group $(\mathbb{Z}, (1))$ when i diverges towards infinity. Hence having torsion is not a closed condition in $\mathcal{G}_\ell$.

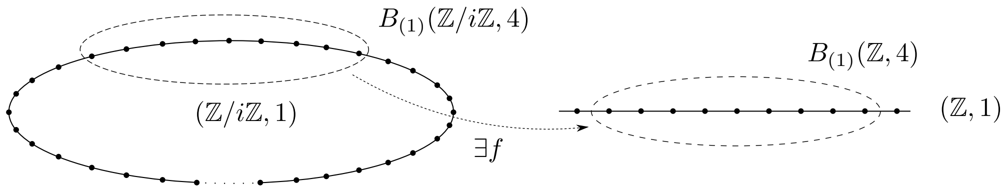

Figure 4: Convergence of the sequence $(\mathbb{Z}/i\mathbb{Z}, (1))$ towards the marked group $(\mathbb{Z}, (1))$.

The subsequent results examine the algebraic properties of a group that can be deduced by studying its neighborhood in $\mathcal{G}_\ell$. All these results come from [CG05] and are of major importance in the study of limit groups.

Proposition 3.1.4. *For $k \leq \ell$, being generated by at most k elements and having torsion are open but not closed properties in $\mathcal{G}_\ell$.*

Proof. Being generated by at most k elements: Let $g_1, \dots, g_k$ be k generators of a marked group (G, S) where $S := (s_1, \dots, s_\ell)$ and define $S_G := (g_1, \dots, g_k)$. Each element $s_i \in S$ is represented by a word $w_i(g_1, \dots, g_k)$ for every $i \in \{1, \dots, \ell\}$. Then, since S is finite, we can define R_1 the maximal S–length of an element g_i; R_2 the maximal S_G–length of a word w_i. Consider a marked group (G', S') at distance less than $e^{-R_1 R_2}$ from (G, S) so that $B_S(G, R_1 R_2)$ and $B_{S'}(G', R_1 R_2)$ are isometric. By choice of R_1, R_2, the elements $g_1, \dots, g_k$ are in correspondance with k elements of G', denoted $g'_1, \dots, g'_k$, and one can read the relations $s'_i = w_i(g'_1, \dots, g'_k)$ in (G', S') for every $i \in \{1, \dots, \ell\}$. Hence, $g'_1, \dots, g'_k$ generate G'. This is true for any marked group at distance less than $e^{-R_1 R_2}$ from (G, S) so that being generated by less than k elements is an open property. However, as seen in the first example above, this property is not closed in $\mathcal{G}_\ell$.

Having torsion: Let $g \in G$ be a torsion element such that $g^i = 1$ for some natural number i. If R is i times the S–length of g, then any marked group (G', S') at distance less than e^{-R} from (G, S) contains an element $g' \in G'$

corresponding to g. This element is non-trivial and verifies the relation $(g')^i = 1$. Hence, any marked group in a neighborhood of (G, S) has torsion. However, this property is not closed since the sequence $(\mathbb{Z}/i\mathbb{Z}, (1))$ converges towards the torsion-free marked group $(\mathbb{Z}, (1))$. $\square$

Proposition 3.1.5. *Being abelian is an open and closed property in $\mathcal{G}_\ell$.*

Proof. A marked group (G, S) is abelian if and only if the commutators in the generators define the identity element in G. Thus, two marked groups at distance less than e^{-4} in $\mathcal{G}_\ell$ are either both abelian or both non-abelian. Hence, being abelian is an open and closed property. $\square$

The following proposition and its corollary are a great source of open or closed properties in $\mathcal{G}_\ell$. Two formulas φ, ϕ of $\mathcal{L}_{grp}$ are **elementarily equivalent** if, any group G that is a model of φ, denoted $G \models \varphi$, is a model of ϕ and vice versa.

Proposition 3.1.6 (Champetier and Guirardel)**.** *If φ is an existential sentence, then $G \models \varphi$ is an open property in $\mathcal{G}_\ell$.*

Proof. Consider an existential sentence φ and a marked group $(G, S) \in \mathcal{G}_\ell$ verifying $G \models \varphi$. There exists a quantifier free sentence $\phi(x_1, \ldots, x_n)$ so that φ is elementarily equivalent to the formula $\exists\, x_1 \cdots \exists\, x_n\ \phi(x_1, \ldots, x_n)$. Using distributivity of $\wedge$ with respect to $\vee$, the formula $\phi(x_1, \ldots, x_n)$ is elementarily equivalent to a formula

$$\Sigma_1(x_1, \ldots, x_n) \vee \cdots \vee \Sigma_p(x_1, \ldots, x_n),$$

where each Σ_i is a system of equations or inequations. That is, a set of equations or inequations of the form $(w(x_1, \ldots, x_n) = 1)$ or $(w(x_1, \ldots, x_n) \neq 1)$ separated by the symbol $\wedge$ where $w(x_1, \ldots, x_n)$ is a word in the variables $x_1, \ldots, x_n$ and their inverses. Since G is a model of φ, there exists $a_1, \ldots, a_n \in G$ and $i \in \{1, \ldots, p\}$ such that $\Sigma_i(a_1, \ldots, a_n)$ holds. Consider R large enough so that the ball of radius R in (G, S) contains $\{a_1, \ldots, a_n\}$ and so that for each word w occuring in Σ_i, the corresponding word on $\{a_1, \ldots, a_n\}$ can be read in this ball (one can take R to be the maximal length of the words in Σ_i times the maximal S–length of the a_i's). If a marked group (G', S') is at distance less than e^{-R} from (G, S), then its ball $B_{S'}(G', R)$ is isometric to $B_S(G, R)$. Consequently the elements $a_1, \ldots, a_n$ correspond to elements $a'_1, \ldots, a'_n$ which satisfy Σ_i. Hence, G' is a model of φ. $\square$

In addition, the negation of an existential formula is a universal formula. Thus, Proposition 3.1.6 proves

Corollary 3.1.7. *If φ is a universal sentence, then $G \models \varphi$ is a closed property in $\mathcal{G}_\ell$.*

Z. Sela introduced in [Sel01] the concept of limit group as Gromov's limit of a sequence of homomorphisms. Then, Bestvina and Feighn generalized, in [BF09], Sela's methods to the notion of G–limit groups for G a finitely generated group. We analyse two equivalent definitions of G–limit groups before giving that of Bestvina and Feighn.

Definition 3.1.8 (Groves and Hull, [GH19])**.** Let $\mathbb{G}$ be a family of groups. A marked group (H, S) is a $\mathbb{G}$–**limit group** if there exists $\ell \in \mathbb{N}$ such that (H, S) is the limit of marked groups (H_i, S_i) in $\mathcal{G}_\ell$ where each H_i is a finitely generated subgroup of an element of $\mathbb{G}$. If $\mathbb{G}$ reduces to a single element G, then we talk about G–limit groups.

This class of groups first appeared in [GH19] where it is studied in depth. For presentation purposes, we examine a family of groups reduced to a single element, although most of the results generalize to the broader concept.

Proposition 3.1.9 (Champetier and Guirardel)**.** *If $(G_i, S_i) \subset \mathcal{G}_\ell$ is a sequence of marked groups which converges to $(G, S) \in \mathcal{G}_\ell$, then (G, S) embeds in any ultraproduct $\prod_{i\in\mathbb{N}} G_i/\sim_{\mathfrak{u}}$ where $\mathfrak{u}$ is a non-principal ultrafilter.*

Proof. Write $S_i = (s_{1,i}, \ldots, s_{\ell,i})$ and consider a non-principal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$. Define the subgroup $\overline{G}$ of the ultrapoduct $\prod_{i\in\mathbb{N}} G_i/\sim_{\mathfrak{u}}$ generated by $\overline{S} := ([s_{1,i}], \ldots, [s_{\ell,i}])$. We prove that $(\overline{G}, \overline{S})$ is isomorphic to (G, S) as a marked group. Let w be a word on ℓ variables and their inverses. If w is trivial in (G, S), then it is trivial in (G_i, S_i) for all but finitely many i. Thus, w is trivial for $\mathfrak{u}$–almost every $i \in \mathbb{N}$ so that w is trivial in $(\overline{G}, \overline{S})$. Since w has finite length, the same result occurs when w is non-trivial and proves that (G, S) is isomorphic to $(\overline{G}, \overline{S})$. $\square$

Proposition 3.1.10 (Champetier and Guirardel)**.** *Let $\mathfrak{u}$ be a non-principal ultrafilter on the natural numbers, G a finitely generated subgroup of an ultraproduct $\prod_{i\in\mathbb{N}} G_i/\sim_{\mathfrak{u}}$ and S a marking of G. Then there exists a sequence of finitely generated subgroups $H_{i_k} < G_{i_k}$ and markings S_{i_k} of H_{i_k} so that (H_{i_k}, S_{i_k}) converges towards (G, S).*

Proof. Let $S = (s_1, \ldots, s_\ell)$ and write each generator s_n as a sequence $(s_{n,i})_{i\in\mathbb{N}}$ for any $n \in \{1, \ldots, \ell\}$. Define $S_i := (s_{1,i}, \ldots, s_{\ell,i})$ and H_i the subgroup of G_i generated by S_i. We prove that, up to passing to a subsequence, (H_i, S_i) converges towards (G, S). Consider a ball of radius R in (G, S) and a word w in it. By definition of the ultraproduct, the word w is trivial in (G, S) if and only if it is trivial in (H_i, S_i) for $\mathfrak{u}$–almost every i. Thus, as an intersection of finitely many full $\mathfrak{u}$–measure subsets, the set of indices i such that the ball of radius R of (H_i, S_i) coincides with the ball of radius R of (G, S) has full $\mathfrak{u}$–measure. Since $\mathfrak{u}$ is non-principal, this set of indices is infinite. Hence (H_i, S_i) accumulates on (G, S) so that, passing to a subsequence, we obtain (H_{i_k}, S_{i_k}) that converges to (G, S). □

The last two propositions prove the following corollaries in which G is an arbitrary group. These corollaries motivate the algebraic definition of limit groups.

Corollary 3.1.11. *A group is a G–limit group if and only if it is a finitely generated subgroup of an ultrapower of G. Moreover, any ultrapower of G contains every G–limit group.*

Corollary 3.1.12. *A finitely generated subgroup of a G–limit group is a G–limit group.*

Definition 3.1.13 (Groves and Hull, [GH19])**.** Let G be an arbitrary group, H a finitely generated group and $\mathfrak{u}$ a non-principal ultrafilter. The $\mathfrak{u}$**–kernel** of a sequence of homomorphisms $(\varphi_i) \subset \mathrm{Hom}(H, G)$ is defined as

$$\ker_{\mathfrak{u}}(\varphi_i) := \{\, h \in H \mid \varphi_i(h) = 1 \text{ for } \mathfrak{u}\text{–almost every } i \,\}.$$

Then, the quotient $H/\ker_{\mathfrak{u}}(\varphi_i)$ is called G**–limit group** associated to (φ_i) and we denote η the quotient map.

Consequently, the next proposition, taken from [GH19], is a consequence of Corollary 3.1.11.

Proposition 3.1.14. *Let G be an arbitrary group and $\mathfrak{u}$ a non-principal ultrafilter. The G–limit groups as limit of marked groups are in one-to-one correspondence with G–limit groups as quotients of finitely generated groups by $\mathfrak{u}$–kernels.*

The remainder of this subsection intends to prove that being a G–limit group does not depend on the marking, nor on the space $\mathcal{G}_\ell$ in which the marking is chosen. To this end, we introduce an equivalence relation on $\mathcal{G}_\ell$ that gives us new insight on the structure of subgroups of a G–limit group. Fix ℓ a natural number, the space $\mathcal{G}_\ell$ is endowed with the isomorphism equivalence relation: two marked groups are equivalent if their underlying group (forgetting about the marking) are isomorphic as groups. We denote by $[G]_{\mathcal{G}_\ell}$ the equivalence class of G in $\mathcal{G}_\ell$. From the definition of $\mathcal{G}_\ell$ as epimorphisms from a free group, the elements in a class $[G]_{\mathcal{G}_\ell}$ are in bijection with $\mathrm{Epi}(F_\ell \twoheadrightarrow G)/\mathrm{Aut}(G)$ for the action of $\mathrm{Aut}(G)$ on $\mathrm{Epi}(F_\ell \twoheadrightarrow G)$ by post composition.

Definition 3.1.15. A subset $A \subseteq \mathcal{G}_\ell$ is **saturated** if it is a union of equivalence classes for the isomorphism relation. The **saturation** of a subset $A \subseteq \mathcal{G}_\ell$ is the union of equivalence classes meeting A. That is, the smallest saturated set containing A.

Lemma 3.1.16. *The saturation of an open set in $\mathcal{G}_\ell$ is open.*

Proof. Consider an open subset $U \subseteq \mathcal{G}_\ell$, V its saturation and two elements $(G, S) \in U$, $(G, S') \in V$. Since U is open, there exists $R > 0$ such that any marked group having the same ball of radius R as (G, S) lies in U. We prove that there exists $R' > 0$ such that any marked group (H, T') in a ball of radius $e^{-R'}$ about (G, S') in $\mathcal{G}_\ell$ is in the saturation of U. That is, we show that H has a marking T so that (H, T) has the same ball of radius R as (G, S). To do so, express the elements of $S = (s_1, \ldots, s_\ell)$ as S'–words $s_i = w_i(s'_1, \ldots, s'_\ell)$ and define L the maximum length of the words w_i as well as $R' := RL$. Let t_i be the elements of H corresponding to the word $w_i(t'_1, \ldots, t'_\ell)$ where $T' = (t'_1, \ldots, t'_\ell)$. Given a word $r(x_1, \ldots, x_\ell)$ of length at most $2R$ in the variables $\{\, x_1, \ldots, x_\ell \,\}$, the following properties are equivalent:

- the word $r(t_1, \ldots, t_\ell)$ defines a relation in (H, T),
- the word $r(w_1(t'_1, \ldots, t'_\ell), \ldots, w_\ell(t'_1, \ldots, t'_\ell))$ of length at most $2R'$ defines a relation in (H, T'),
- the word $r(w_1(s'_1, \ldots, s'_\ell), \ldots, w_\ell(s'_1, \ldots, s'_\ell))$ of length at most $2R'$ defines a relation in (G, S'),
- the word $r(s_1, \ldots, s_\ell)$ defines a relation in (G, S).

The second and third points are equivalent by choice of (H, T') with the same balls of radius R' as (G, S'). Thus (H, T) has the same balls of radius R as (G, S) so that (H, T') is in V. □

Corollary 3.1.17. *The closure in $\mathcal{G}_\ell$ of a saturated set is saturated.*

Proof. Consider a saturated set $A \subseteq \mathcal{G}_\ell$. Then, the interior U of its complement is the largest open set which does not intersect A. Since A is saturated, the saturation V of U does not meet A. By Lemma 3.1.16, V is an open set so that $V = U$ and $\overline{A} = \mathcal{G}_\ell \backslash V$ is saturated. □

Corollary 3.1.18. *Being a G–limit group does not depend on the choice of marking.*

Proof. By Lemma 3.1.16, the set of G–limit groups is saturated. That is, if a marking of a group $H \in \mathcal{G}_\ell$ is a limit of markings of G, then any other marking of H is also a limit of markings of G in $\mathcal{G}_\ell$. $\square$

Lemma 3.1.19. *Let $h : F_{\ell+1} \to F_\ell$ be an epimorphism and $h_* : \mathcal{G}_\ell \to \mathcal{G}_{\ell+1}$ the induced map defined in terms of epimorphisms by $h_* : f \mapsto f \circ h$. Then, h_* is an open homeomorphism onto its image.*

Proof. The injectivity of h_* comes from the surjectivity of h. First, we show that h_* is a homeomorphim onto its image. Consider a sequence of epimorphisms $f_i : F_\ell \to G_i$ converging towards an epimorphism $f : F_\ell \to G$ in $\mathcal{G}_\ell$. If $g \in \ker(f \circ h)$, then, by convergence of the sequence (f_i) towards f, there exists $N \in \mathbb{N}$ so that for every $i > N$ it holds $h(g) \in \ker(f_i)$. Therefore, for every $i > N$, we obtain $g \in \ker(f_i \circ h)$ so that $h_* \circ f_i$ converges towards $h_* \circ f$. Hence, h_* is an injective continuous map from a compact space to a metric space so that h_* is a homeomorphism onto its image. Second, we show that the image of h_* in $\mathcal{G}_{\ell+1}$ is open. Consider $f := f_1 \circ h : F_{\ell+1} \to G_1$ an element in the image of h_*. Since F_ℓ is finitely presented, $\ker(h)$ is the normal closure of a finite set. So, R the maximum length of any word of $\ker(h)$ in the canonical alphabet of $F_{\ell+1}$ is well-defined. Let $k : F_{\ell+1} \to G_2$ be an element in a ball of radius smaller than e^{-R} about f, we define a homomorphism $k_1 : F_\ell \to G_2$ verifying $k = k_1 \circ h$. Define k_1 on basis elements of F_ℓ by $k_1(a) := k(a')$ for any $a' \in h^{-1}(a)$, where a is a basis element of F_ℓ, and extend k_1 to a homomorphism. Since $bc^{-1} \in \ker(h)$ for any $b, c \in h^{-1}(a)$, we obtain

$$k\left(bc^{-1}\right) = f\left(bc^{-1}\right) = f_1 \circ h\left(bc^{-1}\right) = \mathrm{Id}_{F_{\ell+1}}.$$

Thus, the homomorphism k_1 is well-defined and verifies $k = k_1 \circ h$ so that h' is an open map. $\square$

Therefore, by Lemma 3.1.19, there exist embeddings from $\mathcal{G}_n$ to $\mathcal{G}_m$ for all $n \leq m$. Indeed, consider the epimorphism $h : F_m = \langle e_1, \dots, e_m \rangle \to F_n = \langle f_1, \dots, f_n \rangle$ which sends e_i to f_i for any $i \in \{1, \dots, n\}$ and to 1 for any $i \in \{n+1, \dots, m\}$. Then, by Lemma 3.1.19, the map h_* embeds $\mathcal{G}_n$ into $\mathcal{G}_m$. In formula, a marked group $(G, (g_1, \dots, g_n))$ of $\mathcal{G}_n$ correspond to the marked group

$$h_*(G, (g_1, \dots, g_n)) = (G, (g_1, \dots, g_n, 1, \dots, 1)) \in \mathcal{G}_m.$$

Corollary 3.1.20. *Being a G–limit group does not depend on the set $\mathcal{G}_\ell$ in which the marking is chosen.*

In comparison to Bestvina and Paulin's work which give a compactification of the space of actions of one group on hyperbolic spaces, Sela's work on equations in free groups leads to a compactification of the space of epimorphisms from an arbitrary group onto free groups. This compactification consists of epimorphisms from an arbitrary group onto potentially non free groups, called limit groups. In the upcoming subsection, we focus our attention on the framework studied by Sela in [Sel01], that is, limits of markings of free groups.

3.2 Limit Groups over Free Groups and Equationally Noetherian Groups

We first study F_2–limit groups, also called limit groups, with an approach similar to that used by Champetier and Guirardel in [CG05]. More specifically, we characterize their subgroups generated by two elements and show that limit groups are equationally Noetherian. Along the way, we present two equivalent definitions of limit groups that are interesting for historical reasons. Subsequently, to give a framework in which Theorem 4.2.1 applies, we present examples of limit groups as well as hyperbolic groups that are not limit groups. In the following, ℓ is a natural number large enough for the groups considered to be elements of $\mathcal{G}_\ell$.

Lemma 3.2.1. *For $k \in \{1, \dots, \ell\}$, the closure of all the markings of the free abelian group $\mathbb{Z}^k$ in $\mathcal{G}_\ell$ is the set of all markings of the groups $\mathbb{Z}^k, \mathbb{Z}^{k+1}, \dots, \mathbb{Z}^\ell$. That is*

$$\overline{[\mathbb{Z}^k]}_{\mathcal{G}_\ell} = \left[\mathbb{Z}^k\right]_{\mathcal{G}_\ell} \cup \left[\mathbb{Z}^{k+1}\right]_{\mathcal{G}_\ell} \cup \dots \cup \left[\mathbb{Z}^\ell\right]_{\mathcal{G}_\ell}.$$

Proof. Let p be an element of $\{k, \dots, \ell\}$. As in Example 3.1.3, there exists a sequence of markings of $\mathbb{Z}^k$ converging to a particular marking of $\mathbb{Z}^p$. Since the closure of all markings of $\mathbb{Z}^k$ is saturated by Lemma 3.1.16, every marking of $\mathbb{Z}^p$ is a limit of marking of $\mathbb{Z}^k$. Conversely, if a marked group (G, S) is a limit of markings of $\mathbb{Z}^k$, then it is abelian, torsion-free and its rank is at least k by Proposition 3.1.4 and Proposition 3.1.5. Hence, (G, S) is contained in $[\mathbb{Z}^k]_{\mathcal{G}_\ell} \cup [\mathbb{Z}^{k+1}]_{\mathcal{G}_\ell} \cup \dots \cup [\mathbb{Z}^\ell]_{\mathcal{G}_\ell}$. $\square$

Theorem 3.2.2. *Two elements of a limit group generate a free abelian group $\left(\{1\}, \mathbb{Z} \text{ or } \mathbb{Z}^2\right)$ or a non-abelian free group of rank 2.*

Proof. Using Corollary 3.1.12, it is sufficient to prove that any 2–generated limit group is isomorphic to F_2, $\mathbb{Z}^2, \mathbb{Z}$ or $\{1\}$. So, consider a marked group $(G, \{ a, b \})$ in $\mathcal{G}_2$ which is a limit of marked free groups $(G_i, \{ a_i, b_i \})$ and assume that a and b satisfy a non-trivial relation. Then, for i large enough, a_i and b_i satisfy the same relation. Since G_i is a free group, a_i and b_i generate a (maybe trivial) cyclic group. By Lemma 3.2.1

$$\overline{[\mathbb{Z}, \{ 1 \}]_{\mathcal{G}_2}} = [\{ 1 \}]_{\mathcal{G}_2} \cup [\mathbb{Z}]_{\mathcal{G}_2} \cup \left[\mathbb{Z}^2\right]_{\mathcal{G}_2},$$

so that a, b generate a free abelian group. □

The next corollary comes from [Pau03] where Univ(G) denotes the universal theory of a group G. Historically, limit groups were a tool for studying Tarski's problem on the elementary theory of free groups. The following corollary, first proved by V.N. Remeslennikov in [Rem89], allows without further development, to show that all free groups have the same universal theory.

Corollary 3.2.3 (Remeslennikov)**.** *A finitely generated group has the same universal theory as a non-abelian free group (infinite cyclic group) if and only if it is a non-abelian limit group (abelian limit group).*

Proof. By Theorem 3.2.2, a non-abelian limit group G contains F_2 so that $\mathrm{Univ}(G) \subseteq \mathrm{Univ}(F_2)$. Moreover, by Corollary 3.1.7, being a model of a universal formula is a closed property in the space of marked groups. Hence, $\mathrm{Univ}(F_2) \subseteq \mathrm{Univ}(G)$ so that the equality holds. We now prove that if G has the same existential theory as F_2, then it is a limit group. Let $S = (s_1, \dots, s_\ell)$ be a marking of G, N a natural number and $w_1, \dots, w_n$ the reduced words of length at most N in the variables $x_1, \dots, x_\ell$ and their inverses. Consider the finite system Σ_N of equations and inequations $\wedge_{i=1}^{n} \varphi_i$ in the variables $x_1, \dots, x_\ell$ where each φ_i is the formula

$$w_i(x_1, \dots, x_\ell) = 1 \text{ if } w_i(s_1, \dots, s_\ell) = 1 \text{ in } G \text{ or } w_i(x_1, \dots, x_\ell) \neq 1 \text{ if } w_i(s_1, \dots, s_\ell) \neq 1 \text{ in } G.$$

Then, the existential formula $\exists\, x_1 \cdots \exists\, x_\ell\, \Sigma_N$ is verified in G. So, by assumption, there exists $S_N = (s_{1,N}, \dots, s_{\ell,N})$ a solution in F_2 to the system Σ_N. If G_N is the subgroup of F_2 generated by $\langle S_N \rangle$, then the sequence (G_N, S_N) converges towards (G, S) by construction. Since G is not abelian, the subgroups G_N are not abelian for N sufficiently large and therefore G is a limit group. □

Unlike Fujiwara and Sela which prove Theorem 4.2.1 for arbitrary hyperbolic groups in [FS20], we restrict ourselves to the study of hyperbolic limit groups. This allows to describe certain properties on elements' neighborhood in $\mathcal{G}_\ell$ without refering to Rips theory. This description is the purpose of the following development that begins by showing the equivalent characterization of limit groups as fully residually free groups. Also, this class of groups is extensively studied by B. Baumslag in [Bau67] where examples and results may be found.

Definition 3.2.4. A group G is **fully residually free** if for any finite set of distinct elements $g_1, \dots, g_k \in G$, there exists a group homomorphism h from G to a free group such that $h(g_1), \dots, h(g_k)$ are distinct.

Lemma 3.2.5. *Any fully residually free group is a limit group.*

Proof. Let G be a fully residually free group and S a marking of G. For every $i \in \mathbb{N}$, there exists a free quotient G_i of G in which the ball of radius i of (G, S) embeds. Let S_i be the image of S in G_i by the quotient map and consider the sequence of marked groups (G_i, S_i). For any $i \in \mathbb{N}$, by construction, (G_i, S_i) has the same balls of radius i as (G, S). Hence, (G, S) is the limit of the sequence of marked free groups (G_i, S_i). □

We now prove that the converse also holds. That is, any limit group is fully residually free. We proceed as Champetier and Guirardel in [CG05] using the Noetherianity of $\mathbb{Z}$, $\mathbb{C}$ and the linearity of F_2.

Definition 3.2.6. A commutative ring is **Noetherian** if it satisfies the ascending chain condition. That is, every ascending chain of ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ is constant from a certain point.

Lemma 3.2.7. *Any finitely generated subring R of an ultrapower ${}^*\mathbb{Z}$ is, as a ring, fully residually $\mathbb{Z}$. That is, for any $a_1, \dots, a_k \in R \backslash \{0\}$, there exists a ring morphism $\gamma : R \to \mathbb{Z}$ such that $\gamma(a_i) \neq 0$ for every $i \in \{ 1, \dots, k \}$.*

Proof. By assumption, there exists $t_1, \dots, t_\ell \in {}^*\mathbb{Z}$ such that $R = \mathbb{Z}[t_1, \dots, t_\ell]$. Consider the related exact sequence

$$J \hookrightarrow \mathbb{Z}[T_1, \dots, T_\ell] \twoheadrightarrow R,$$

where $\mathbb{Z}[T_1, \dots, T_\ell]$ is the ring of polynomials with ℓ commuting indeterminates. Since $\mathbb{Z}[T_1, \dots, T_\ell]$ is Noetherian by Hilbert Basis Theorem, see [Rot10, Theorem 6.42], the ideal J is generated by finitely many polynomials $f_1, \dots, f_q$. Consider $a_1, \dots, a_k \in R \backslash \{0\}$ and we define a ring morphism γ from R to $\mathbb{Z}$ so that all $\gamma(a_i)$ are distinct. Let

$g_1, \ldots, g_k$ be preimages of $a_1, \ldots, a_k$ in $\mathbb{Z}[T_1, \ldots, T_\ell] \backslash J$ and note that $(t_1, \ldots, t_\ell)$ is a solution to the system of equations and inequations

$$\begin{cases} f_i(x_1, \ldots, x_\ell) = 0 & \text{for every } i \in \{1, \ldots, q\}, \\ g_j(x_1, \ldots, x_\ell) \neq 0 & \text{for every } j \in \{1, \ldots, k\}. \end{cases}$$

This system is finite so that Łós Theorem applies and gives a solution $(s_1, \ldots, s_\ell)$ in $\mathbb{Z}$ to this system. Consequently, the morphism $\mathbb{Z}[T_1, \ldots, T_\ell] \to \mathbb{Z}$ sending T_i to s_i induces the desired morphism $\gamma : R \to \mathbb{Z}$. ☐

In view of the next lemma, let p be an odd prime and ${}^*\varphi : {}^*\mathbb{Z} \to {}^*\mathbb{Z}/p\mathbb{Z}$ the morphism induced by the quotient map $\varphi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. As seen in Lemma 3.2.7, for any finitely generated subring $R < {}^*\mathbb{Z}$, there exists $t_1, \ldots, t_n \in {}^*\mathbb{Z}$ so that $R = \mathbb{Z}[t_1, \ldots, t_n]$. Then, the image ${}^*\varphi\,(R)$ is also a finitely generated subring of ${}^*\mathbb{Z}/p\mathbb{Z}$ which is isomorphic, by naturality of ${}^*\varphi$, to $\mathbb{Z}/p\mathbb{Z}\,[{}^*\varphi(t_1), \ldots, {}^*\varphi(t_n)]$. If $\phi : R \to \mathbb{Z}$ is a ring morphism such that the images $\phi(t_i) = s_i \in \mathbb{Z}$ are distinct, then, since all morphisms are natural, the following diagram is commutative

$$\begin{tikzcd} {}^*\mathbb{Z} \arrow[d, "{}^*\varphi"] & R \arrow[l, hook'] \arrow[r, "\phi:\ t_i \mapsto s_i"] \arrow[d, "{}^*\varphi"] \arrow[l, phantom, "\circlearrowleft", shift right=8] & \mathbb{Z} \arrow[d, "\varphi"] \\ {}^*(\mathbb{Z}/p\mathbb{Z}) & {}^*\varphi(R) \arrow[l, hook'] \arrow[r, "{}^*\varphi(t_i) \mapsto \varphi(s_i)"'] & \mathbb{Z}/p\mathbb{Z}. \end{tikzcd}$$

These observations are the main ideas to prove the following lemma.

Theorem 3.2.8 (Remeslennikov)**.** *A finitely generated subgroup of an ultrapower ${}^*F_2$ is fully residually free.*

Proof. Fix an ultrafilter $\mathfrak{u}$, the corresponding ultrapower ${}^*F_2$ of F_2 and G a finitely generated subgroup of ${}^*F_2$. For any odd prime p, the kernel of $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is a non-abelian free group. So, F_2 embeds in $\ker(\varphi)$ so that ${}^*F_2$ embeds in the kernel of the natural morphism ${}^*\varphi : \mathrm{SL}_2({}^*\mathbb{Z}) \to \mathrm{SL}_2({}^*(\mathbb{Z}/p\mathbb{Z}))$ where ${}^*\mathbb{Z}$ is the ring obtained by taking the $\mathfrak{u}$–ultrapower of the ring $\mathbb{Z}$. Since G is a finitely generated subgroup of ${}^*F_2$, it embeds in a finitely generated subgroup $\mathrm{SL}_2(R)$ of $\mathrm{SL}_2({}^*\mathbb{Z})$ where R is a finitely generated subring of ${}^*\mathbb{Z}$. To show that G is fully residually free, consider finitely many elements $g_1, \ldots, g_k \in G \backslash \{1\} \subseteq \mathrm{SL}_2(R)$. Then, we construct a morphism $\phi : \mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{Z})$ such that $\phi(g_i) \neq \mathrm{Id}_{\mathrm{SL}_2(\mathbb{Z})}$ for every $i \in \{1, \ldots, k\}$ and verifying $\phi(G)$ is contained in a free group. Let $a_1, \ldots, a_q$ be the set of non-zero coefficients of the matrices $g_j - \mathrm{Id}_{\mathrm{SL}_2(R)}$ for $j \in \{1, \ldots, k\}$. By Lemma 3.2.7, there exists a ring morphism $\gamma : R \to \mathbb{Z}$ so that $\gamma(a_i) \neq 0$ for every $i \in \{1, \ldots, q\}$. Then, the induced group morphism $\phi : \mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{Z})$ maps the elements g_i to non-trivial elements. By naturality of the defined morphisms, the following diagram is commutative

$$\begin{tikzcd} G \arrow[r, hook] & \mathrm{SL}_2(R) \arrow[r, "{}^*\varphi"] \arrow[d, "\phi"'] & \mathrm{SL}_2(R') \arrow[d, "\phi'"] \\ & \mathrm{SL}_2(\mathbb{Z}) \arrow[r, "\varphi"'] & \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}). \end{tikzcd}$$

Thus, with $G \subseteq \ker({}^*\varphi)$ and the commutativity of the above diagram, we obtain $\phi(G) \subseteq \ker(\varphi)$ which is free. Consequently, $\phi(G)$ is free and G is fully residually free. ☐

Corollary 3.2.9 (Kharlampovich, Myasnikov and Sela)**.** *A finitely generated group is fully residually free if and only if it is a limit group.*

Before proving the next crucial theorem, we introduce some notions of algebra and more precisely that of affine algebraic variety. This treatment is due to J.J. Rotman and comes from [Rot10]. In the sequel, k denotes a field.

Definition 3.2.10. If F is a subset of $k[x_1, \ldots, x_\ell]$, then the **variety** defined by F is

$$\mathrm{Var}(F) := \left\{ a \in k^\ell \;\middle|\; f(a) = 0 \text{ for every } f(x_1, \ldots, x_\ell) \in F \right\}.$$

Thus, $\mathrm{Var}\,(F)$ consists of every $a \in k^\ell$ which are zeros for every $f(x_1, \ldots, x_\ell) \in F$.

Definition 3.2.11. A **representation** of a group G on a k–vector space V is a group morphism from G to $\mathrm{GL}(V)$.

Let G be a finitely generated group and $S = (s_1, \ldots, s_\ell)$ a finite generating set of G. The subset $\mathrm{Hom}\,(G, \mathrm{SL}_2(\mathbb{C}))$ of the set of representations of G on $\mathbb{C}^2$ defines a variety. Indeed, if W denotes the set of relations of G in the alphabet S, then for every $\varphi \in \mathrm{Hom}(G, \mathrm{SL}_2(\mathbb{C}))$ and every $w \in W$, it holds

$$\varphi(w(s_1, \ldots, s_\ell)) = w(\varphi(s_1), \ldots, \varphi(s_\ell)) = \mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})}.$$

Also, any map $\varphi : S \to \mathrm{SL}_2(\mathbb{C})$ that satisfies $w(\varphi(s_1), \ldots, \varphi(s_\ell)) = \mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})}$ for every $w \in W$ extends to a morphism from G to $\mathrm{SL}_2(\mathbb{C})$. Since words in W are equations in ℓ variables, this defines a **variety of representations**

$$V := \left\{ (\varphi(s_1), \ldots, \varphi(s_\ell)) \in \mathrm{SL}_2(\mathbb{C})^\ell \;\middle|\; \varphi \in \mathrm{Hom}(G, \mathrm{SL}_2(\mathbb{C})),\ w(\varphi(s_1), \ldots, \varphi(s_\ell)) = \mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})} \text{ for every } w \in W \right\}.$$

Definition 3.2.12. If $A \subseteq \mathbb{C}^\ell$, we denote

$$\mathrm{I}(A) := \{ f(x_1, \ldots, x_\ell) \in \mathbb{C}[x_1, \ldots, x_\ell] \mid f(a) = 0 \text{ for every } a \in A \}.$$

From these definitions, we obtain the following useful proposition.

Proposition 3.2.13.

1. *If $A \subset B$ are subset of $\mathbb{C}^\ell$, then $\mathrm{I}(B) \subseteq \mathrm{I}(A)$,*
2. *If $F \subset G$ are subset of $\mathbb{C}[x_1, \ldots, x_\ell]$, then $\mathrm{Var}(G) \subseteq \mathrm{Var}(F)$.*

It is a classical result, see [Rot10, Theorem 6.42, Hilbert Basis Theorem], that $\mathbb{C}$ and any finite extension of it are Noetherian. As a result, the forthcoming proposition is a direct consequence of the Nullstellensatz Theorem, see [Rot10, Theorem 6.102, Nullstellensatz].

Proposition 3.2.14. *If V_1 and V_2 are varieties and $\mathrm{I}(V_1) = \mathrm{I}(V_2)$, then $V_1 = V_2$.*

We now have enough results from the theory of varieties to state the first finitness lemma on limit groups.

Theorem 3.2.15. *Consider a sequence of quotients of finitely generated groups*

$$G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow \cdots .$$

If every group G_i is residually free, then all but finitely many epimorphisms are isomorphisms.

Proof. Let $S_1 := (s_1, \ldots, s_\ell)$ be a finite generating set of G_1 and define $S_i := (s_{1,i}, \ldots, s_{\ell,i})$ its image in G_i under the quotient map. For every $i \in \mathbb{N}$, consider also $V_i \subseteq \mathrm{SL}_2(\mathbb{C})^\ell$ the variety of representations of (G_i, S_i) in $\mathrm{SL}_2(\mathbb{C})$:

$$V_i := \left\{ (M_1, \ldots, M_\ell) \in \mathrm{SL}_2(\mathbb{C})^\ell \;\middle|\; \text{for every relation } w \text{ of } (G_i, S_i),\ w\,(M_1, \ldots, M_\ell) = \mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})} \right\}.$$

These sets V_i are varieties in $\mathbb{C}^{4\ell}$ which verify the inclusions $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k \cdots$. Using Proposition 3.2.13, we obtain an ascending chain of ideals $\mathrm{I}(V_1) \subseteq \mathrm{I}(V_2) \subseteq \cdots$. Then, by Noetherianity of $\mathbb{C}[x_1, \ldots, x_{4\ell}]$, for every but finitely many indices i it holds $\mathrm{I}(V_i) = \mathrm{I}(V_{i+1})$. Hence, by Propositon 3.2.14, $V_i = V_{i+1}$ for all but finitely many indices. There remains to check that if G_{i+1} is a strict quotient of G_i, then V_{i+1} is strictly contained in V_i. Consider a word w on ℓ variables which is trivial in G_{i+1} but not in G_i when evaluate with the respective alphabet S_{i+1} and S_i. Since G_i is residually free, there exists a morphism $\varphi : G_i \to F_2$ such that $\varphi\,(w\,(s_{1,i}, \ldots, s_{\ell,i})) \neq 1$ in F_2. Since F_2 embeds in $\mathrm{SL}_2(\mathbb{C})$, there exists a representation $\gamma : G_i \to \mathrm{SL}_2(\mathbb{C})$ such that $\gamma\,(w\,(s_{1,i}, \ldots, s_{\ell,i})) \neq 1$. This representation provides an element in $V_i \backslash V_{i+1}$. ☐

Definition 3.2.16. A group G is **Hopfian** if every epimorphism $G \twoheadrightarrow G$ is an isomorphism.

Corollary 3.2.17. *Limit groups are Hopfian groups.*

The following remark comes from [Pau03] and describes the neighborhood of marked finitely presented groups.

Remark 3.2.18. For a marked group (G, S), by a theorem of Neumann [Bau12, Theorem 12, page 65], asking that G has finite presentation is equivalent to ask that G has finite presentation on the generating family S. Subsequently, every marked group (G, S) sufficiently closed to a marked group (G', S') of finite presentation is a quotient of the latter. Indeed, if $\{ w_i(s_1, \ldots, s_\ell) \}_{i \in I}$ is a finite family that generates normaly the kernel of the marking $F_\ell \to G$, and if (G, S) is sufficiently close to (G', S'), then the relations $w_i(s_1, \ldots, s_\ell) = 1$ are verified by S. Hence, the unique ordered application sending S' to S induces a morphism $(G', S') \to (G, S)$.

We now use topological properties of $\mathcal{G}_\ell$ to obtain information on groups in a neighborhood of a marked group. There exists a natural partial order on the set $\mathcal{G}_\ell$: a marked group (G_1, S_1) is smaller than (G_2, S_2), denoted $(G_1, S_1) \leq (G_2, S_2)$, if and only if the marked epimorphism $F_\ell \twoheadrightarrow G_1$ factorizes through the marked epimorphism $F_\ell \twoheadrightarrow G_2$. That is, G_1 is a marked quotient of G_2 and the following diagram is commutative

$$\begin{tikzcd}
(F_\ell, S) \arrow[rr, two heads] \arrow[dr, two heads] & & (G_1, S_1) \\
& (G_2, S_2)\,. \arrow[ur, dashed, two heads] &
\end{tikzcd}$$

Therefore, any group G has a largest residually free quotient $\mathrm{RF}(G)$ for the order on $\mathcal{G}_\ell$: the quotient of G by the intersection of all kernels of morphisms from G to free groups. The next lemma proves that any residually free group has a presentation with finitely many relations plus the relations necessary to make it residually free.

Theorem 3.2.19 (Chatzidakis). *Let G be a residually free group with finite generating set S. Then, there exist finitely many S–words $w_1, \dots, w_n$ such that $G = \mathrm{RF}(H)$ where H is the group presented by $\langle\, S \mid w_1, \dots, w_n \,\rangle$.*

Proof. Enumerate the relations w_i of G in the alphabet S and define

$$G_k := \mathrm{RF}\left(\langle\, S \mid w_1, \dots, w_k \,\rangle\right).$$

Then, the sequence (G_k, S) converges towards (G, S) and Theorem 3.2.15 applied to the sequence $G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots$ implies that for k large enough $(G, S) = (G_k, S)$. □

Corollary 3.2.20. *Given a residually free marked group (G, S), there exists a neighborhood $V_{(G,S)}$ of (G, S) in $\mathcal{G}_\ell$ so that every residually free group in $V_{(G,S)}$ is a quotient of (G, S).*

Proof. Let $w_1, \dots, w_n$ be the relations of G in the alphabet S given by Theorem 3.2.19. This set of relations is finite and $G = \mathrm{RF}\left(\langle\, S \mid w_1, \dots, w_n \,\rangle\right)$. Therefore, there exists a neighborhood $V_{(G,S)}$ of (G, S) small enough so that any marked group $(G', S') \in V_{(G,S)}$ verifies the relations $w_1, \dots, w_n$. Hence, by Remark 3.2.18, (G', S') is a marked quotient of (G, S). □

This development is part of a larger framework studied by Groves and Hull in [GH19]. We state some of their results on equationally Noetherian groups that generalize the above results to larger class of groups. In the remainder of this subsection, an element $w \in F_\ell$ is a word in the variables $x_1, \dots, x_\ell$. Then, given a group G and $(s_1, \dots, s_\ell) \in G^\ell$, the element $w(s_1, \dots, s_\ell)$ in G is obtained by evaluation of w in the elements s_i and their inverses. For $W \subseteq F_\ell$, we denote

$$V_G(W) := \left\{(s_1, \dots, s_\ell) \in G^\ell \;\middle|\; w(s_1, \dots, s_\ell) = 1 \text{ for every } w \in W\right\}.$$

Definition 3.2.21. A group G is **equationally Noetherian** if for any natural number ℓ and any set of words $W \subseteq F_\ell$, there exists a finite set $W_0 \subseteq W$ such that $V_G(W_0) = V_G(W)$.

The proof of Theorem 4.2.1 relies heavily on Corollary 3.2.20 which implies that residually free groups are equationally Noetherian. In fact, more can be stated as done in [GH19, Theorem 3.5].

Theorem 3.2.22 (Groves and Hull, [GH19, Theorem 3.5]). *For $\mathfrak{u}$ a non-principal ultrafilter and G a finitely generated group, the following are equivalent*

1. *G is equationally Noetherian.*
2. *For any sequence of homomorphisms $\varphi_i : F_\ell \to G$, the homomorphisms φ_i $\mathfrak{u}$–almost surely factors through the quotient map $\eta : F_\ell \to F_\ell/\ker_{\mathfrak{u}}(\varphi_i)$.*
3. *For any sequence of homomorphisms $\varphi_i : F_\ell \to G$, some φ_i factors through the map $\eta : F_\ell \to F_\ell/\ker_{\mathfrak{u}}(\varphi_i)$.*

Proof. We prove the implications in order and keep the notations defined above. Note that these equivalences remain valid if we replace F_ℓ by an arbitrary finitely generated group.

1. Suppose that G is equationally Noetherian; we prove that G verifies the statement of the second item. Let $W := \ker_{\mathfrak{u}}(\varphi_i)$ and consider a finite subset $W_0 \subseteq W$ such that $V_{F_\ell}(W_0) = V_{F_\ell}(W)$ which exists by assumption. By definition of the $\mathfrak{u}$–kernel, for $\mathfrak{u}$–almost every $i \in \mathbb{N}$ and any $w \in W$ it holds $\varphi_i(w) = 1$. Since W_0 is finite and $\mathfrak{u}$ is finitely additive, it holds for $\mathfrak{u}$–almost every i

$$W_0 \subseteq \ker(\varphi_i) \text{ and } V_{F_\ell}\left(\ker\left(\varphi_i\right)\right) \subseteq V_{F_\ell}(W_0) = V_{F_\ell}(W).$$

Hence, for $\mathfrak{u}$–almost every i, if $(x_1, \dots, x_\ell)$ is an element of $V_{F_\ell}(\ker(\varphi_i)) \subseteq V_{F_\ell}(W)$ and $w \in W$, then $\varphi_i(w(x_1, \dots, x_\ell)) = w(\varphi_i(x_1), \dots, \varphi_i(x_\ell)) = 1$ in G. Thus, $\ker(\eta) = \ker_{\mathfrak{u}}(\varphi_i) \subseteq \ker(\varphi_i)$ for $\mathfrak{u}$–almost every i and for these indexes, the morphisms φ_i factor through η.

2. If the statement holds $\mathfrak{u}$–almost surely, then it holds for some elements of the sequence.

3. Suppose by contradiction that G is not equationally Noetherian but verifies the property stated in the third item. Then, there exists $W := \{\, w_1, w_2, \dots \,\} \subseteq F_\ell$ so that for any finite subset $W_i := \{\, w_1, \dots, w_i \,\} \subset W$, it holds $V_G(W_i) \neq V_G(W)$. So, there exists $\varphi_i : F_\ell \to G$ such that $W_i \subseteq \ker(\varphi_i)$ but $W \not\subseteq \ker(\varphi_i)$. By construction, $w_j \in \ker(\varphi_k)$ for every $k \geq j$ and $W \subseteq \ker_{\mathfrak{u}}(\varphi_i)$. By assumption, there exists an index i such that φ_i factors through η the quotient map $\eta : F_\ell \to F_\ell/\ker_{\mathfrak{u}}(\varphi_i)$. Hence, $W \subseteq \ker(\eta) \subseteq \ker(\varphi_i)$ and we obtain a contradiction. Therefore, G is equationally Noetherian. □

Studying actions of groups on $\mathbb{R}^n$–trees and using the theory of Rips machinery, Groves and Hull proved in [GH19] that many relatively hyperbolic groups are equationally Noetherian. Before giving examples of limit groups, we state two consequences of Theorem 3.2.22 which motivate the study of Noetherian groups.

Proposition 3.2.23 (Groves and Hull, [GH19, Theorem 3.13.])**.** *Suppose G is an equationally Noetherian group and $(h_i : L_i \to L_{i+1})$ is a sequence of epimorphisms between G–limit group. Then h_i is an isomorphism for all but finitely many i.*

Corollary 3.2.24. *If G is an equationally Noetherian group, then every G–limit group is Hopfian.*

In [FS20], Fujiwara and Sela prove the well-ordering of the set of growth rates of an arbitrary hyperbolic group. In contrast, we prove in Subsection 4.2, the well-ordering of the same set but only for hyperbolic limit groups. Therefore, the remainder of this subsection is intended to provide examples of hyperbolic limit groups as well as hyperbolic groups that are not limit groups.

Many surface groups are hyperbolic limit groups. A **surface** is a real connected compact 2–dimensional manifold with boundaries (potentially empty) and a **closed surface** is a surface without boundaries. A **surface group** is a group isomorphic to the fundamental group of a surface. Then, the next proposition is copied from [Pau03].

Proposition 3.2.25 (Paulin, [Pau03, Proposition 3.3.])**.** *Every surface group, except that of the projective plane, the Klein bottle and the connected sum of three projective planes, are limit groups.*

Proof. If a surface has non-empty boundaries, then its fundamental group is free, hence, a limit group. Thus, we treat the case of closed surfaces. To show that a given sequence of marked groups converges to a marked group, we refer to Example 3.1.3, since the methods are in every detail identical.

- If Σ_2^+ is the connected sum of two torus, then $\langle\, a, b, c, d \mid [a, b] = [c, d]\,\rangle$ is a presentation of its fundamental group. Define $c_i := [a, b]^i a[a, b]^{-i}$ and $d_i := [a, b]^i b[a, b]^{-i}$ two elements of the free group F_2 on a, b and $S_i := (a, b, c_i, d_i)$. Then, the sequence of marked groups (F_2, S_i) converges towards $\left(\pi_1\left(\Sigma_2^+\right), (a, b, c, d)\right)$,
- If Σ_4^- is the connected sum of four projective planes, then $\left\langle\, a, b, c, d \mid a^2b^2 = c^2d^2 \,\right\rangle$ is a presentation of its fundamental group. Define $c_i := \left(a^2b^2\right)^i a \left(a^2b^2\right)^{-i}$ and $d_i := \left(a^2b^2\right)^i b \left(a^2b^2\right)^{-i}$ two elements of $F_2 = \langle\, a, b\,\rangle$ and $S_i := (a, b, c_i, d_i)$. So, the sequence of marked groups (F_2, S_i) converges towards $\left(\pi_1\left(\Sigma_4^-\right), (a, b, c, d)\right)$,
- It is a topological result that any closed surface of Euler characteristic at most -2 is a covering of Σ_2^+ or Σ_4^-. Thus, their fundamental groups are limit groups as subgroups of limit groups,
- Any closed surface of Euler characteristic at least -1 is a sphere, a torus or a connected sum of at most three projective planes. Since the fundamental groups of the sphere and the torus are limit groups, we conclude. $\square$

The above proposition gives examples of hyperbolic groups that are also limit groups. Also, the three exceptions not considered in this proposition are not limit groups. Indeed, the fundamental group of the projective plane has torsion, thus it is not a limit group by Proposition 3.1.4. For the other two cases, we introduce another property of limit groups.

Definition 3.2.26. A group G is **commutative transitive** if commutativity is a transitive relation on $G\backslash\{1\}$:

$$\text{for every } a, b, c \in G\backslash\{1\},\ \text{ if } [a, b] = [b, c] = 1 \text{ then } [a, c] = 1.$$

This formula is a universal sentence so that, by Corollary 3.1.7, being commutative transitive is a closed property of $\mathcal{G}_\ell$. Since free groups are commutative transitive, limit groups are too. We prove that the fundamental group of the Klein bottle Σ_2^- and the connected sum of three projective planes Σ_3^- are not limit groups.

- The fundamental group of the Klein bottle is not commutative transitive. Indeed, a presentation of $\pi_1\left(\Sigma_2^-\right)$ is given by $\left\langle\, a, b \mid a^2b^{-2} = 1 \,\right\rangle$. So, a and b commute with $a^2 = b^2$. However, $\pi_1\left(\Sigma_2^-\right)$ is not abelian, hence, it is not a limit group,
- The fundamental group of Σ_3^- is not commutative transitive. Indeed, a presentation of $\pi_1\left(\Sigma_3^-\right)$ is given by $\left\langle\, a, b, c \mid a^2b^2c^2 = 1 \,\right\rangle$. It is shown in [Lyn59] that the equation $a^2b^2c^2 = 1$ in a free group implies the commutativity of a, b and c. Thus, the fundamental group of the connected sum of three projective planes is not a limit group since it is not abelian.

The connected sum of three projective planes has negative Euler characteristic. So, by a uniformisation theorem of Poincaré, Klein and Koebe, its fundamental group is hyperbolic although it is not a limit group. Since $\mathbb{Z}^2$ is a limit group, there are also examples of limit groups that are not hyperbolic. Finally, we present some other interesting examples of hyperbolic groups that are not limit groups. Let $\mathcal{U}(\mathcal{H})$ denote the set of unitary operators of a complex Hilbert space $\mathcal{H}$. We call **unitary representation** of a group G, a morphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that $\xi \mapsto \pi(g)\xi$ is continuous for every $g \in G$.

Definition 3.2.27. Let G be a locally compact topological group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G on a complex Hilbert space $\mathcal{H}$ with a norm denoted by $\|\cdot\|$. If $\varepsilon > 0$ and K is a finite subset of G, then a unit vector ξ in $\mathcal{H}$ is called (ε, K)–**invariant** if

$$\text{for every } g \in K \;\; \|\pi(g)\xi - \xi\| < \varepsilon.$$

Definition 3.2.28. A locally compact topological group G has **Kazhdan's property (T)**, also refered to as **property (T)**, if every unitary representation of G that has an (ε, K)–invariant unit vector for any $\varepsilon > 0$ and any finite subset K, has a non-zero invariant vector.

It is a classical result that any group with Kazhdan's property (T) is finitely generated and that all its quotients also have property (T). Moreover, Y. Shalom proved in [Sha00] that every finitely generated group with Kazhdan's property (T) is a quotient of a finitely presented group with property (T). Using these results, the following proposition based on [CG05, Proposition 2.15] shows that limit groups do not have Kazhdan's property (T).

Proposition 3.2.29. *Having Kazhdan's property (T) is an open property in $\mathcal{G}_\ell$.*

Proof. Let G be a group with Kazhdan's property (T), $S_G = (s_1, \ldots, s_\ell)$ a marking of G and H a finitely presented group with property (T) such that G is a quotient of H. Let $\langle\, S_H \mid v_1(h_1, \ldots, h_n), \ldots, v_m(h_1, \ldots, h_n)\,\rangle$ be a finite presentation of H and S'_G the image of S_H by the quotient map from H to G. By writing each element $s'_k \in S'_G$ as a word $u_k(s_1, \ldots, s_\ell)$ for every $k \in \{\, 1, \ldots, n\,\}$, we obtain for every $i \in \{\, 1, \ldots, m\,\}$

$$w_i(s_1, \ldots, s_\ell) := v_i(u_1(s_1, \ldots, s_\ell), \ldots, u_n(s_1, \ldots, s_\ell)),$$

which are relations in G. By construction, any marked group at distance at most e^{-R} from (G, S_G), where R is the maximum length of a word w_i in the alphabet S_G, is a quotient of H. Hence, there exists an open set around (G, S) so that any marked group (G', S') in it verifies G' has Kazhdan's property (T). $\square$

Since the free group on two elements does not have Kazhdan's property (T), limit groups does not have neither. Hence, any hyperbolic group that has Kazhdan's property (T) is not a limit group. As stated in [Pau03], this is the case of the uniform lattices in $Sp(n, 1)$: the group of linear transformations of the right $\mathbb{H}$–vector space $\mathbb{H}^{n+1}$ preserving the sesquilinear form

$$(u, v) := \sum_{i=1}^{n} \overline{u}_i v_i - \overline{u}_{n+1} v_{n+1} \text{ for every } u = (u_i), v = (v_i) \in \mathbb{H}^{n+1}.$$

More complicated examples of hyperbolic groups that are not limit groups are non-coherent hyperbolic groups. In the succeding subsection, we gather the theory of G–limit groups and the theory of groups acting on trees seen in Subsection 2.2. This permits us to define from a sequence of morphisms from free groups to an hyperbolic group G, under certain conditions, a faithful action of a G–limit group on a real tree. The construction of this faithful action is central to the proof of the main theorem presented in Subsection 4.2.

3.3 Limit Groups of Hyperbolic Groups

Given a sequence of homomorphisms between a group G and a hyperbolic group Γ, we defined in Section 2, under certain conditions, a limit action of G on an asymptotic cone of Γ. In this subsection, we show that this process defines a limit action of a Γ–limit group on the asymptotic cone and analyse the kernel of this action. To do this, we follow Reinfeldt and Weidmann's presentation as done in [RW10] and recall first some results of hyperbolic groups theory. The following treatment comes from [RW10, Subsection 1.3] that we detail when necessary.

Proposition 3.3.1 ([RW10, Proposition 1.17])**.** *If Γ is a finitely generated hyperbolic group, then:*

1. *There exists a constant N depending only on the hyperbolic constant of Γ such that every torsion subgroup of Γ has at most N elements,*

2. *There exists a constant M depending only on the hyperbolic constant of Γ such that for every subgroup H of Γ, one of the following holds.*
   (a) *H is a finite group,*
   (b) *For every finite generating set S of H there exists a hyperbolic element $\gamma \in H$ such that $|\gamma|_S \leq M$, where $|\cdot|_S$ denotes the word length on H relative to S.*

In the upcoming, we describe the geometry of a hyperbolic space as seen from "infinitly far away". To proceed, it is convenient to introduce the boundary of a metric space as done in [GdlH90]. If X is a metric space, then a sequence $(x_i) \subset X$ **converges towards infinity** if $\lim_{i,j\to\infty}(x_i \cdot x_j)_w = \infty$ for every w in X. It is a classical result that this definition does not depend on the choice of base point w. Define an equivalence relation on the set of sequences in X converging towards infinity by

$$(x_i) \sim (y_j) \Leftrightarrow \lim_{i,j\to\infty}(x_i \cdot y_j)_w = \infty.$$

Definition 3.3.2. The **ideal boundary** of a metric space X is the quotient set

$$\partial_\infty X := \{\, (x_i) \mid (x_i) \text{ converges towards infinity} \,\}/\sim,$$

where the equivalence relation is the one defined in the last paragraph.

Throughout this subsection, let G be a finitely generated group and Γ an infinite hyperbolic group without torsion. We endow these two groups with word metrics relative to fixed generating sets S_G and S_Γ respectively and denoted $|\cdot|_{S_G}$, $|\cdot|_{S_\Gamma}$. Consider $X := \mathrm{Cay}(\Gamma, S_\Gamma)$ the Cayley graph of Γ with respect to S_Γ. One can show that any hyperbolic element $\gamma \in \Gamma$ fixes points p_γ^+, p_γ^- in $\partial_\infty X$, see [Ghy90, Subsection 8.2, Theorem 16.]. Then, as done in [RW10], define the **axis of** γ, denoted A_γ, as the union of all geodesics connecting p_γ^+ to p_γ^-.

Definition 3.3.3. Let $\mathcal{P}, \mathcal{Q}$ be two properties of groups. A group G is $\mathcal{P}$–**by**–$\mathcal{Q}$ if there exists a normal subgroup N of G such that N has property $\mathcal{P}$ and G/N has property $\mathcal{Q}$.

The next proposition was independently proven by Bestvina and Paulin in [Bes88] and [Pau88] respectively.

Proposition 3.3.4 (Bestvina and Paulin, [Bes88] and [Pau88])**.** *If Γ is a hyperbolic group with finite generating set S_Γ and $X := \mathrm{Cay}(\Gamma, S_\Gamma)$, then there exist a constant K and strictly positive real numbers C, ε such that*

1. *For any hyperbolic element $\gamma \in \Gamma$ and $x \in X$ it holds $\mathrm{dist}_X(x, \gamma x) \geq 2\,\mathrm{dist}_X(x, A_\gamma) - K$,*
2. *If $Y \subset \Gamma$ verifies: there exists $x_1, x_2 \in X$ with $\mathrm{dist}_X(x_1, x_2) > C$ and $\mathrm{dist}_X(x_i, hx_i) < \varepsilon\,\mathrm{dist}_X(x_1, x_2)$ for all $h \in Y$ and $i \in \{1, 2\}$, then the group generated by Y is either finite or finite–by–$\mathbb{Z}$.*

We generalize these results on hyperbolic groups to Γ–limit groups where Γ is a finitely generated hyperbolic group. The following lemma is essential to prove the main theorem of this subsection.

Lemma 3.3.5 (Reinfeldt and Weidmann)**.** *If Γ is a finitely generated hyperbolic group, L is a Γ–limit group and N is the constant such that every torsion subgroup of Γ has at most N elements, then the following holds.*

1. *Every torsion subgroup of L has at most N elements.*
2. *A subgroup $A \leq L$ is finite–by–abelian if and only if all finitely generated subgroups of A are finite–by–abelian.*

Proof. Let $\mathfrak{u}$ be a non-principal ultrafilter, $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ a sequence of homomorphisms with induced Γ–limit group $L := G/\ker_{\mathfrak{u}}(\varphi_i)$ where G is a finitely generated group and denote $\eta : G \to L$ the quotient map. We first prove the first item by contradiction and then prove *2*.

1. Suppose there exists a torsion subgroup $H \leq L$ that contains $N+1$ pairwise distinct elements $g_0, \ldots, g_N$. For each $n \in \{0, \ldots, N\}$ choose $g'_n \in G$ such that $\eta(g'_n) = g_n$. Then $\varphi_i(g'_n) \neq \varphi_i(g'_m)$ for $\mathfrak{u}$–almost every i and every $0 \leq n \neq m \leq N$ so that, by the first item of Proposition 3.3.1, the subgroup

$$\langle\, \varphi_i(g'_0), \ldots, \varphi_i(g'_N) \,\rangle \leq \Gamma \text{ is infinite for } \mathfrak{u}\text{–almost every } i.$$

Then, by the second item of Proposition 3.3.1, there exists a word w_i in the alphabet $g'_0, \ldots, g'_N$ of length at most M for $\mathfrak{u}$–almost every i such that $\varphi_i(w_i)$ is of infinite order. Since there are only finitely many such words, there exists a word w verifying $w = w_i$ for $\mathfrak{u}$–almost every i. As H is a torsion subgroup, there exists k so that $\eta(w)^k = 1$. That is, $w^k \in \ker_{\mathfrak{u}}(\varphi_i)$ and $w^k \in \ker(\varphi_i)$ for $\mathfrak{u}$–almost every i. That is a contradiction with $\varphi_i(w)$ is of infinite order and so the first item is proven.

2. In one hand, if A is finite–by–abelian, so are all its finitely generated subgroups. In the other hand, we use the following characterization of finite–by–abelian groups: a group B is finite–by–abelian if and only if its commutator subgroup, denoted $[B, B]$, is finite. Suppose the commutator subgroup of A is infinite. Then, by the first item of Proposition 3.3.1, it contains $N+1$ distinct elements $g_0, \dots, g_N$ and we show that the same holds for some finitely generated subgroup of A. However, any element of $[A, A]$ is the product of finitely many commutators and, therefore, lies in the commutator subgroup of a finitely generated subgroup. □

In Subsection 4.2, we study a finitely generated hyperbolic group Γ and a sequence of morphisms from a group G to Γ. In this setting, the following definitions describe the necessary conditions to apply Theorem 2.2.11 with the induced action of G on a Cayley graph of Γ. This allows us to get a nice action of a Γ–limit group on a real tree.

Notation 8. Given a group G with finite generating set S_G, we denote $\operatorname{dist}_{S_G}(a, b)$ the distance between the two elements a, b in G with respect to S_G. That is, the S_G–word length of $a^{-1}b$. In addition, $B_{S_G}(G, R)$ denotes the open ball in G of radius R about the identity element for the metric induced by $\operatorname{dist}_{S_G}$.

Definition 3.3.6. Let G and Γ be groups with finite generating sets S_G and S_Γ respectively. A homomorphism $\varphi \in \operatorname{Hom}(G, \Gamma)$ is **conjugacy short** if

$$\max_{s \in S_G} \operatorname{dist}_{S_\Gamma}(e, \varphi(s)) =: |\varphi|_e \leq |\varphi|_g = \left|g\varphi g^{-1}\right|_e \text{ for every } g \in \Gamma,$$

where $g\varphi g^{-1}$ is the homomorphism that sends any $h \in G$ to $g\varphi(h)g^{-1}$ in Γ.

Definition 3.3.7. Let $\mathfrak{u}$ be a non-principal ultrafilter. A sequence $(\varphi_i) \subset \operatorname{Hom}(G, \Gamma)$ is called $\mathfrak{u}$–**strict** if:

1. The morphism φ_i is conjugacy short for $\mathfrak{u}$–almost every i,
2. The sequence (φ_i) does not have a $\mathfrak{u}$–constant subsequence. That is, there is no homomorphism φ' such that $\varphi_i = \varphi'$ for $\mathfrak{u}$–almost every i.

Consider $\mathfrak{u}$ a non-principal ultrafilter, G and Γ two groups with finite generating sets S_G and S_Γ respectively and an arbitrary sequence $(\varphi_i) \subset \operatorname{Hom}(G, \Gamma)$ with associated Γ–limit map $\eta : G \to L$. Up to conjugation, we may assume that every φ_i is conjugacy short since this does not change its kernel. Then, the second item of Definition 3.3.7 implies that $\mathfrak{u}$–almost all φ_i are pairwise distinct. Also, for any real number R, there are only finitely many homomorphisms of norm at most R. So, in particular, $\mathfrak{u}\text{–}\lim |\varphi_i|_e = \infty$. Thus, by Theorem 2.2.11, there exists an isometric action ρ of G on the asymptotic cone

$$T_{\mathfrak{u}} := \operatorname{Cone}_{\mathfrak{u}}\left(\operatorname{Cay}(\Gamma, S_\Gamma), (e), \left(|\varphi_i|^{-1}\right)\right).$$

Moreover, this action ρ factors through the quotient map η to define an action of L on the tree T, still denoted ρ.

Theorem 3.3.8 (Reinfeldt and Weidmann)**.** *With the above notation, the action of L on T, the minimal G–invariant subtree of $T_{\mathfrak{u}}$, verifies*

1. *The stabilizer of any non-degenerate tripod is finite,*
2. *The stabilizer of any non-degenerate arc is finite–by–abelian,*
3. *Every subgroup of L which leaves a line in T invariant and fixes its ends is finite–by–abelian.*

Proof. In the following, X is the Cayley graph of Γ with respect to the finite generating set S_Γ.

1. Let D be a non-degenerate tripod in T spanned by $[x_i] = x_{\mathfrak{u}}, [y_i] = y_{\mathfrak{u}}, [z_i] = z_{\mathfrak{u}} \in T$. By the first item of Lemma 3.3.5, it suffices to show that $H := \operatorname{stab}_L(D)$ is a torsion subgroup. Consider h an element in H and choose $h' \in G$ such that $\eta(h') = h$. We show that $\varphi_i(h')$ is of finite order for $\mathfrak{u}$–almost every i so that $h'^{N!} \in \ker_{\mathfrak{u}}(\varphi_i)$ which implies that h is a torsion element, where N is the constant introduced in Lemma 3.3.5 so that any torsion subgroup of L has at most N elements. Suppose by contradiction that $\varphi_i(h')$ is of infinite order for $\mathfrak{u}$–almost every i. Since any element of infinite order in a hyperbolic group is hyperbolic, see [Ghy90, Subsection 8.3.], the element $\varphi_i(h')$ is hyperbolic for $\mathfrak{u}$–almost every i. By Lemma 3.3.4, there exists a constant K such that

$$\operatorname{dist}_X\left(x_i, A_{\varphi_i(h')}\right) \leq \frac{1}{2}\left(\operatorname{dist}_X\left(x_i, \varphi_i\left(h'\right)x_i\right) + K\right),$$

for $\mathfrak{u}$–almost every i. Therefore, the following inequality holds

$$\begin{aligned}\mathfrak{u}\text{–}\lim \frac{1}{|\varphi_i|}\mathrm{dist}_X\left(x_i, A_{\varphi_i(h')}\right) &\leq \mathfrak{u}\text{–}\lim \frac{1}{2|\varphi_i|}\left(\mathrm{dist}_X\left(x_i, \varphi_i\left(h'\right)x_i\right)+K\right)\\ &= \frac{1}{2}\mathrm{dist}_T\left(x_{\mathfrak{u}}, \varphi(h')x_{\mathfrak{u}}\right)=0.\end{aligned}$$

Consequently, $\mathrm{dist}_T\left(x_{\mathfrak{u}}, A_h\right)=\mathfrak{u}\text{–}\lim|\varphi_i|^{-1}\mathrm{dist}_X\left(x_i, A_{\varphi_i(h')}\right)=0$ so that $x_{\mathfrak{u}}$ lies on A_h. A similar argument shows that $y_{\mathfrak{u}}$ and $z_{\mathfrak{u}}$ lie on A_h. Since A_h is a line, that is a contradiction with the assumption that $x_{\mathfrak{u}}, y_{\mathfrak{u}}$ and $z_{\mathfrak{u}}$ span a non-degenerate tripod.

2. Let H be a subgroup of G that stabilises a non-degenerate arc $x_{\mathfrak{u}}$-$y_{\mathfrak{u}} \subset T$ where $x_{\mathfrak{u}}=[x_i], y_{\mathfrak{u}}=[y_i] \in T$. Since $x_{\mathfrak{u}}$-$y_{\mathfrak{u}}$ is non-degenerate, $x_{\mathfrak{u}}$ and $y_{\mathfrak{u}}$ are distinct so that $\mathfrak{u}\text{–}\lim \mathrm{dist}_X\left(x_i, y_i\right)=\infty$ and for any $h \in H$

$$\mathfrak{u}\text{–}\lim \frac{\mathrm{dist}_X\left(x_i, \varphi_i(h)x_i\right)}{\mathrm{dist}_X\left(x_i, y_i\right)}=0=\mathfrak{u}\text{–}\lim \frac{\mathrm{dist}_X\left(y_i, \varphi_i(h)y_i\right)}{\mathrm{dist}_X\left(x_i, y_i\right)}. \tag{3}$$

Let A be a subgroup of H with finite generating set S_A. By equation (3), the hypothesis for the second item of Proposition 3.3.4 are satisfied by $S_i:=\varphi_i(S_A)$ for $\mathfrak{u}$–almost every i. Therefore, $\langle\, \varphi_i(A)\,\rangle=\langle\, S_i\,\rangle$ is finite–by–abelian. That is, $\varphi_i\left([A, A]\right)=[\varphi_i(A), \varphi_i(A)]$ is finite of order at most N for $\mathfrak{u}$–almost every i so that the commutator $[\eta(A), \eta(A)]$ is a torsion group. Therefore, this subgroup is finite and $\eta(A)$ is finite–by–abelian so that H is finite–by–abelian by the second item of Lemma 3.3.5.

3. We proceed as in the second item. Let H be a subgroup of G that leaves a line $Y \subset T$ invariant and fixes its ends $a_{\mathfrak{u}}$ and $b_{\mathfrak{u}}$. Consider sequences $(x_{\mathfrak{u},k})_{k\in\mathbb{N}}$ and $(y_{\mathfrak{u},k})_{k\in\mathbb{N}}$ of elements in T converging toward the ends $a_{\mathfrak{u}}$ and $b_{\mathfrak{u}}$ respectively. By definition of ends, it holds $\mathfrak{u}\text{–}\lim \mathrm{dist}_T\left(x_{\mathfrak{u},k}, y_{\mathfrak{u},k}\right)=\infty$ so that for all $h \in H$

$$\mathfrak{u}\text{–}\lim \frac{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, \eta(h)x_{\mathfrak{u},k}\right)}{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, y_{\mathfrak{u},k}\right)}=0=\mathfrak{u}\text{–}\lim \frac{\mathrm{dist}_T\left(y_{\mathfrak{u},k}, \eta(h)y_{\mathfrak{u},k}\right)}{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, y_{\mathfrak{u},k}\right)}.$$

Indeed, $\mathrm{dist}_T\left(x_{\mathfrak{u},k}, \eta(h)x_{\mathfrak{u},k}\right)=\mathrm{dist}_T\left(y_{\mathfrak{u},k}, \eta(h)y_{\mathfrak{u},k}\right)$ is the translation length of $\eta(h)$ and so is independent of k. For each k, consider $(x_{i,k})_{i\in\mathbb{N}}$ and $(y_{i,k})_{i\in\mathbb{N}}$ defining sequences in X of $x_{\mathfrak{u},k}$ and $y_{\mathfrak{u},k}$ respectively. By the choice of sequence of scalars $\left|\varphi_i^{-1}\right|$, for fixed $h \in H$ and $k \in \mathbb{N}$, it holds

$$\begin{aligned}\mathfrak{u}\text{–}\lim_i \frac{\mathrm{dist}_X\left(x_{i,k}, \varphi_i(h)x_{i,k}\right)}{\mathrm{dist}_X\left(x_{i,k}, y_{i,k}\right)} &= \frac{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, \eta(h)x_{\mathfrak{u},k}\right)}{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, y_{\mathfrak{u},k}\right)},\\ \mathfrak{u}\text{–}\lim_i \frac{\mathrm{dist}_X\left(y_{i,k}, \varphi_i(h)y_{i,k}\right)}{\mathrm{dist}_X\left(x_{i,k}, y_{i,k}\right)} &= \frac{\mathrm{dist}_T\left(y_{\mathfrak{u},k}, \eta(h)y_{\mathfrak{u},k}\right)}{\mathrm{dist}_T\left(x_{\mathfrak{u},k}, y_{\mathfrak{u},k}\right)}.\end{aligned}$$

Since the right-hand sides of these equations tend to 0 as k diverges towards infinity, there exists (φ_{m_i}) a subsequence of (φ_i) so that

$$\mathfrak{u}\text{–}\lim_i \frac{\mathrm{dist}_X\left(x_{m_i,i}, \varphi_{m_i}(h)x_{m_i,i}\right)}{\mathrm{dist}_X\left(x_{m_i,i}, y_{m_i,i}\right)}=0=\mathfrak{u}\text{–}\lim_i \frac{\mathrm{dist}_X\left(y_{m_i,i}, \varphi_{m_i}(h)y_{m_i,i}\right)}{\mathrm{dist}_X\left(x_{m_i,i}, y_{m_i,i}\right)}.$$

Since H is finitely generated, it is countable and a diagonal argument shows that these equalities hold for all h in H after passing to a subsequence. Thus, a similar argument as in the second item conclude. □

Corollary 3.3.9. *With the above notations, the subtree T is not a line if and only if the torsion-free finitely generated hyperbolic group Γ is non-elementary.*

We studied several aspects of the theory of limit groups. This allows us to understand the subgroups generated by two elements of a limit group but especially to describe the asymptotic behavior of specific sequences of morphisms. In particular, certain sequences of morphisms generate well understood actions of limit groups on real trees. In the next section, we use the whole theory seen so far to study the geometry of limit groups through their actions on real trees. More specifically, we analyse their growth rates.

4 Rate of Growth for Limit Groups

This section is the culmination of this thesis and uses all techniques developed previously to analyse the geometry of limit groups. We begin by introducing the growth function of a group with respect to a finite generating set and some devices which, from this function, permit us to examine this group. Then, we rewrite the argument developed by G.N. Arzhantseva and I.G. Lysenok in [AL06] to give a lower bound on the rate of growth for non-elementary hyperbolic groups. Finally, we study a theorem presented by Fujiwara and Sela in [FS20] to prove the well-ordering of the set of growth rates of any limit group and comment on its generalization to other classes of groups.

4.1 A lower bound on the Rate of Growth for Hyperbolic Groups

In this subsection, we give a lower bound on the rate of growth of a non-elementary hyperbolic group. This gives us the opportunity to introduce the growth function of a finitely generated group and some notations used in the rest of this document. The first part of this subsection follows the treatment on growth functions as outlined by P. De la Harpe in [dlH00, Chapter VI, VII].

Notation 9. Given a set A, we denote $\#A$ the cardinality of A.

Definition 4.1.1. A **growth function** is a non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$. The **growth function of a group** G with respect to a finite generating set S_G is the growth function defined by $t \mapsto \#B_{S_G}(G,t)$.

This positive function may vary depending on the choice of finite generating set. To define an invariant of groups, we introduce an equivalence relation on the set of growth functions. A growth function α_1 **dominates** a growth function α_2, denoted $\alpha_2 \preceq \alpha_1$, if there exist constants $\lambda \geq 1$ and $C \geq 0$ such that

$$\alpha_2(t) \leq \lambda\alpha_1(\lambda t + C) + C.$$

Then, two growth functions α_1, α_2 are equivalent, denoted $\alpha_1 \sim \alpha_2$, if each dominates the other. It is shown in [dlH00, Subsection VI.B] that two growth functions of one group with respect to two generating sets are equivalent. So, this defines an invariant of groups: the **growth type** of a finitely generated group is the equivalence class of its growth functions. In addition, one can show that two quasi-isometric groups have the same growth type. So, the growth type of a group is an invariant by quasi-isometry.

Example 4.1.2 (De la Harpe, [dlH00, Examples 28, page 166])**.** For $a, b \in \mathbb{R}_{>0}$, the function $t \mapsto t^b$ dominates $t \mapsto t^a$ if and only if $a \leq b$. Also, for any $a, b \in \mathbb{R}_{>0}$, the maps $t \mapsto e^{at}$ and $t \mapsto e^{bt}$ are equivalent.

We analyse the different growth types that a group may have. Given a group G with finite symmetric generating set S_G, the length function $|\cdot|_{S_G}$ is symmetric and subadditive:

$$\left|g^{-1}\right|_{S_G} = |g|_{S_G} \text{ and } |gh|_{S_G} \leq |g|_{S_G} + |h|_{S_G},$$

for every $g, h \in G$. Thus, the growth function of a group is submultiplicative. That is

$$\#B_{S_G}(G, t+s) \leq \#B_{S_G}(G,t)\,\#B_{S_G}(G,s),$$

for every $s, t \in \mathbb{N}$ and, consequently, $\lim_{t\to\infty}\left(\#B_{S_G}(G,t)\right)^{1/t}$ is finite.

Definition 4.1.3. The **exponential growth rate** of G with respect to S_G is

$$e\left(G, S_G\right) := \lim_{t\to\infty}\left(\#B_{S_G}(G,t)\right)^{\frac{1}{t}}.$$

Then, a group G with finite generating set S_G is necessarily of one of the following three growth types:

- If $e(G, S_G) > 1$, then G has **exponential growth**,
- If one growth function of G is dominated by a polynomial function, then G has **polynomial growth**,
- If $e(G, S_G) = 1$ and G is not of polynomial growth, then G has **intermediate growth**.

The next proposition gives an efficient way to prove that a group has exponential growth.

Proposition 4.1.4 (De la Harpe, [dlH00, page 187])**.** *A finitely generated group which contains a free subsemigroup on two generators is of exponential growth.*

In particular, it is a classical result that non-elementary hyperbolic groups contain a free subgroup on two generators. Thus, they have exponential growth. In fact, M. Koubi strengthened this result by proving that non-elementary hyperbolic groups are of uniform exponential growth. A group G has **uniform exponential growth** if there exists $c > 1$, such that $e(G, S_G) > c > 1$ for every finite generating set S_G of G.

Theorem 4.1.5 (Koubi, [Kou98, Theorem 1.1])**.** *A non-elementary hyperbolic group has uniform exponential growth.*

In Subsection 4.2, we are interested in the countable set of exponential growth rates with respect to all possible finite generating sets of a given group. Note that, as stated by A. Sambusetti in [Sam99], the uniform exponential growth of non-elementary hyperbolic groups does not imply that their set of growth rates admits a minimum.

Notation 10. Given a finitely generated group G, we denote $\xi(G)$ the set of growth rates of G. That is

$$\xi(G) := \{\, e\,(G, S_G) \;\mid S_G \text{ is a finite generating set of } G\,\}.$$

In the remainder of this subsection, we retrieve the idea of Proposition 4.1.4 to construct, for a given non-elementary hyperbolic group Γ, a free subgroup with certain properties. We realize this construction in order to obtain a lower bound on the growth rate of Γ. This method comes from [AL06] which gives proofs of all the lemmas that follow. Nevertheless, we modify slightly the main result of this article in Theorem 4.1.17 of which we detail the computations.

Definition 4.1.6. Let X be a metric space with length function $|\cdot|_X$ and r a positive real number. Two subsets $A, B \subseteq X$ are r**–close** if A and B lie in the Hausdorff r–neighborhood of each other. That is, for every $x \in A$ there exists $y \in B$ so that $|x-y|_X \leq r$ and vice versa. Also, let $p : I \to X$ and $q : J \to X$ be two paths in a metric space X where I and J are connected subsets of $\mathbb{R}$. The paths p and q are **strictly** r**–close** if $\mathrm{Im}(p)$ and $\mathrm{Im}(q)$ are r–close and the closeness is monotone. That is, there exists a binary relation $R \in I \times J$ such that

1. If xRy, then $|p(x) - q(y)|_X \leq r$,
2. For every $x \in I$, there exists at least one $y \in J$ with xRy and vice versa,
3. If xRx', yRy' and $x \leq x'$, then $y \leq y'$.

The next two lemmas are consequences of the geometry of hyperbolic metric spaces.

Lemma 4.1.7 (Arzhantseva and Lysenok, [AL06, Lemma 1])**.** *Consider a natural number $k \geq 3$ and points $x_1, \ldots, x_k$ in a δ–hyperbolic geodesic metric space X. If $(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_i} < |x_i - x_{i+1}|_X - 3\delta$ for every $i \in \{\,2, \ldots, k-2\,\}$, then the following hold*

1. $|x_1 - x_k|_X \geq \sum_{i=1}^{k-1} |x_i - x_{i+1}|_X - 2\sum_{i=2}^{k-1} (x_{i-1}, x_{i+1})_{x_i} - 2(k-3)\delta$,
2. *The segment x_1-x_k and the broken path $\bigcup_{i=1}^{k-1} x_i$-x_{i+1} are strictly r–close for $r := \max_i (x_{i-1}, x_{i+1})_{x_i} + 14\delta$.*

Lemma 4.1.8 (Arzhantseva and Lysenok, [AL06, Lemma 2])**.** *Let x, x', y, y' be four elements in a δ–hyperbolic geodesic metric space X and $r > 0$. If*

$$|x-y|_X + |x'-y'|_X \geq |x-x'|_X + |y-y'|_X + 2r,$$

then the geodesic segments x-y and x'-y' have strictly 8δ–close subsegments u-v and u'-v' with $|u-v|_X, |u'-v'|_X \geq r$.

Notation 11. Given a group G with finite generating set S_G, we denote $\langle\, g, h \,\rangle$ the Gromov product between g and h in G with respect to the identity element. We also denote $\|g\|_{S_G}$ the length of a shortest element in the conjugacy class of $g \in G$.

With these notations, an element $g \in G$ is **cyclically minimal** if $\|g\|_{S_G} = |g|_{S_G}$. Then the following two results are non trivial consequences of Lemma 4.1.7. In the remainder of this subsection, Γ is a δ– hyperbolic group with finite generating set S_Γ.

Lemma 4.1.9 (Arzhantseva and Lysenok, [AL06, Corollary 1])**.** *If g is an element in Γ that verifies $|g|_{S_\Gamma} - 2\,\langle\, g, g^{-1} \,\rangle > 3\delta$, then for every $n \geq 2$*

$$|g^n|_{S_\Gamma} \geq n|g|_{S_\Gamma} - 2(n-1)\,\langle\, g, g^{-1} \,\rangle - 2(n-2)\delta.$$

Moreover, the segment e-g^n and the broken geodesic $\bigcup_{i=0}^{n-1} g^i$-g^{i+1} are strictly r–close for $r := \langle\, g, g^{-1} \,\rangle + 14\delta$.

Lemma 4.1.10 (Arzhantseva and Lysenok, [AL06, Lemma 3])**.** *If g is an element in Γ, then the following hold.*

1. *If $|g|_{S_\Gamma} - 2\,\langle\, g, g^{-1} \,\rangle > 3\delta$, then $\|g\|_{S_\Gamma} \geq |g|_{S_\Gamma} - 2\,\langle\, g, g^{-1} \,\rangle - 2\delta$,*
2. *If $\|g\|_{S_\Gamma} > 4\delta$, then $\|g\|_{S_\Gamma} \leq |g|_{S_\Gamma} - 2\,\langle\, g, g^{-1} \,\rangle + 4\delta$. In particular, if $|g|_{S_\Gamma} - 2\,\langle\, g, g^{-1} \,\rangle > 3\delta$ and g is cyclically minimal, then $\langle\, g, g^{-1} \,\rangle \leq 2\delta$,*
3. *If $\|g\|_{S_\Gamma} > 7\delta$, then $h^{-1}gh$ is cyclically minimal for some $h \in \Gamma$ with $|h|_{S_\Gamma} \leq 1/2\,(|g|_{S_\Gamma} - \|g\|_{S_\Gamma}) + 5\delta$.*

The following group $E(g)$, for g an element of infinite order in Γ, has a natural definition in terms of stabilizer of boundary elements in $\partial_\infty\Gamma$. However, we give a convenient definition of $E(g)$ in terms of relations in Γ.

Definition 4.1.11. Let $g \in \Gamma$ be an element of infinite order. We denote

$$\begin{aligned} E(g) &:= \{h \in \Gamma \mid h^{-1}g^t h = g^{\varepsilon t} \text{ for some integer } t \neq 0 \text{ and } \varepsilon = \pm 1\}, \\ E^+(g) &:= \{h \in \Gamma \mid h^{-1}g^t h = g^t \text{ for some integer } t \neq 0\}, \\ E^-(g) &:= E(g) \backslash E^+(g), \\ E^*(g) &:= \{h \in E^+(g) \mid h \text{ has finite order}\}. \end{aligned}$$

The following three lemmas give geometric and algebraic properties of the subgroup $E(g)$ of Γ.

Lemma 4.1.12 (Arzhantseva and Lysenok, [AL06, Lemma 4]). *For any $g \in \Gamma$ of infinite order, the following hold.*

1. *$E(g)$ is the maximal elementary subgroup of Γ containing g,*
2. *$E^*(g)$ is a finite normal subgroup of $E(g)$,*
3. *For every $h \in E^-(g)$, the element h^2 is in $E^*(g)$.*

Lemma 4.1.13 (Arzhantseva and Lysenok, [AL06, Lemma 5]). *Let g be a cyclically minimal element of Γ. If $|g|_{S_\Gamma} > 7\delta$, then $|h|_{S_\Gamma} \leq 140\delta$ for any $h \in E^*(g)$.*

Lemma 4.1.14 (Arzhantseva and Lysenok, [AL06, Lemma 6]). *Let g be a cyclically minimal element in Γ verifying $|g|_{S_\Gamma} > 7\delta$ and $N := \#B_{S_\Gamma}(\Gamma, 200\delta)+10$. Consider $x, y \in \Gamma$ such that the geodesics x-xg^n and y-$yg^{\varepsilon m}$ have strictly 8δ–close subsegments S and T for some constants $m, n > 0$ and $\varepsilon = \pm 1$. If $|S|_{S_\Gamma} \geq N|g|_{S_\Gamma}$, then $x^{-1}y \in E^+(g)$ if $\varepsilon = 1$ and $x^{-1}y \in E^-(g)$ if $\varepsilon = -1$.*

The upcoming two lemmas rely on the geometry of hyperbolic groups and are, together with Lemma 4.1.14, the main preliminary results to find a lower bound for the growth rates of a non-elementary hyperbolic group.

Lemma 4.1.15 (Arzhantseva and Lysenok, [AL06, Lemma 7]). *Let S'_Γ be a finite generating set of Γ and suppose that*

$$\min_{g \in \Gamma} \max_{x \in S'_\Gamma} \left|g^{-1}xg\right|_{S_\Gamma} \geq 40\delta.$$

Then, after an appropriate conjugation of S'_Γ by some element of Γ, there exists a cyclically minimal element $b \in \Gamma$ verifying $|b|_{S'_\Gamma} \leq 2$ and $L - 24\delta \leq |b|_{S_\Gamma} \leq 2L$. Moreover, $\max_{x \in S'_\Gamma} |x|_{S_\Gamma} \leq L + 26\delta$ after the conjugation.

Lemma 4.1.16 (Arzhantseva and Lysenok, [AL06, Lemma 8]). *Let S'_Γ be a finite generating set of Γ and suppose that*

$$L := \min_{g \in G} \max_{x \in S'_\Gamma} \left|g^{-1}xg\right|_{S_\Gamma} \geq 40\delta. \tag{4}$$

If $M > 0$ and $N := \#B_{S_\Gamma}(\Gamma, 200\delta) + 10$, then, after conjugation of S'_Γ by an appropriate element of Γ, there exists a cyclically minimal element $d \in \Gamma$ such that

$$\begin{aligned} |d|_{S'_\Gamma} &\leq 5M + 200N + 300, \\ ML &\leq |d|_{S_\Gamma}, \\ \langle d, d^{-1} \rangle &\leq N|b|_{S_\Gamma} + 2|v|_{S_\Gamma}, \end{aligned}$$

where b is the element of Γ defined in Lemma 4.1.15 and v is an element in S'_Γ that is not in $E(b)$. Moreover, any $h \in E^+(d)$ may be represented as $h = d^t u$ for some $t \in \mathbb{Z}$ and $u \in E^*(d)$ verifying $|u|_{S_\Gamma} < (4N+12)L$. In addition, $\max_{x \in S'_\Gamma} |x|_{S_\Gamma} < (4N + 10)L$ after the conjugation.

We finally give a lower bound on the exponential growth rate of a non-elementary hyperbolic group that depends only on the hyperbolic constant of the group and the cardinality of the studied finite generating set. The upcoming theorem and its proof come from [AL06] which we reproduce, with the exception of the calculation of the lower bound once the free generating subset Y is constructed. Our short argument does not allow to obtain the linear lower bound obtained by Arzhantseva and Lysenok but is sufficient for our use in Subsection 4.2.

Theorem 4.1.17 (Arzhantseva and Lysenok). *Let Γ be a δ–hyperbolic group with a finite generating set S_Γ. Then, there exist constants α, β that depend only on $\#S_\Gamma$ and δ so that, for any finite generating set S'_Γ of Γ*

$$e\left(\Gamma, S'_\Gamma\right) \geq \left(2\alpha \, \#S'_\Gamma - 1\right)^\beta.$$

Proof. The central idea of this proof is to construct a finite set $Y \subset \Gamma$ that verifies the following properties

- Every $y \in Y$ is a product of at most D generators in $(S'_\Gamma)^{\pm 1}$ where D depends only on δ,
- There exists $K > 0$ depending only on δ such that $\# Y \geq K^{-1} \# S'_\Gamma$,
- The set Y freely generates a free subgroup of Γ.

Since the growth function of $\langle Y \rangle$ with respect to the basis Y is given by $t \mapsto \# B_Y(\langle Y \rangle, t) = 2\#Y(2\#Y-1)^{t-1}$, we deduce a lower bound on the growth rate of Γ with respect to S'_Γ

$$\# B_{S'_\Gamma}(\Gamma, t) \geq \# B_{S'_\Gamma}(\Gamma, Dt)^{\frac{1}{D}} \geq \# B_Y(\langle Y \rangle, t)^{\frac{1}{D}} \geq \left(2\frac{1}{K}\#S'_\Gamma\right)^{\frac{1}{D}}\left(2\frac{1}{K}\#S'_\Gamma - 1\right)^{\frac{t-1}{D}}.$$

Then, we obtain a lower bound on the exponential growth rate of Γ with respect to S'_Γ where K and D depend only on $\#S_\Gamma$ and δ.

$$e(\Gamma, S'_\Gamma) \geq \lim_{t\to\infty}\left(2\frac{1}{K}\#S'_\Gamma\right)^{\frac{1}{Dt}}\left(2\frac{1}{K}\#S'_\Gamma - 1\right)^{\frac{t-1}{Dt}} = \left(2\frac{1}{K}\#S'_\Gamma - 1\right)^{\frac{1}{D}}. \tag{5}$$

The remainder of this proof is intended to construct a subset $Y \subset \Gamma$ with the above properties. Keep the notations defined above and, quite to consider $\alpha \leq \# B_{S_\Gamma}(\Gamma, 40\delta)^{-1}$, assume that

$$\# S'_\Gamma \geq \# B_{S_\Gamma}(\Gamma, 40\delta).$$

This assumption guarantees that the constant $L := \min_{g\in G}\max_{x\in S'_\Gamma}\left|g^{-1}xg\right|_{S_\Gamma}$ is greater than 40δ. Thus, the inequality (4) in Lemma 4.1.16 is verified and with the notations of this lemma, define the constants

$$R := 2(4N+10),\ N := \# B_{S_\Gamma}(\Gamma, 200\delta) + 10 \text{ and } M := 2R + 4N + 12.$$

After an appropriate conjugation of S'_Γ, there exists $d \in \Gamma$ cyclically minimal such that

$$|d|_{S'_\Gamma} \leq 5M + 200N + 300,\ ML \leq |d|_{S_\Gamma} \text{ and } \left\langle d, d^{-1} \right\rangle \leq N|b|_{S_\Gamma} + 2|v|_{S_\Gamma}.$$

where b and v are the elements of Γ and S_Γ respectively defined in Lemma 4.1.15.
<u>Claim:</u> The intersection $B_{S_\Gamma}(\Gamma, RL) \cap E(d)$ is contained in a finite subgroup F of Γ.
Proof: As stated in Lemma 4.1.16, for any $h \in E^+(d)$, there exists $t \in \mathbb{Z}$ and $u \in E^*(d)$ verifying $|u|_{S_\Gamma} < (4N+12)L$ such that $h = d^t u$. Moreover, $\left\langle d, d^{-1}\right\rangle \leq N|b|_{S_\Gamma} + 2|v|_{S_\Gamma}$ so that

$$\left|d^2\right|_{S_\Gamma} - |d|_{S_\Gamma} \geq ML - 4|v|_{S_\Gamma} - 2N|b|_{S_\Gamma}.$$

Thus, enlarging M if necessary by a constant that depends only on L, we may assume $\left|d^2\right|_{S_\Gamma} - |d|_{S_\Gamma} \geq 2\delta$. So, by Lemma 4.1.9, $|d^t| > |d|$ for any $t \geq 2$ and, consequently, for any $h \in E^+(d) \backslash E^*(d)$, it holds

$$|h| > |d| - (4N+12)L \geq ML - (4N+12)L > RL.$$

Hence, $B_{S_\Gamma}(\Gamma, RL) \cap E^+(d) \subseteq E^*(d)$. If $B_{S_\Gamma}(\Gamma, RL) \cap E^-(d)$ is the empty set, then consider $F = E^*(d)$ and conclude. Otherwise, consider $v \in B_{S_\Gamma}(\Gamma, RL) \cap E^-(d)$. By Lemma 4.1.12, $E^*(d)$ is a finite group, v^2 is an element of $E^*(d)$ and $v^{-1}E^*(d)v \subseteq E^*(d)$ so that the subgroup $F = \langle E^*(d), v \rangle$ of Γ is finite. As $E^+(d)$ is a subgroup of index two of $E(d)$, every $g \in E^-(d)$ is represented as $g = d^t uv$ for some $t \in \mathbb{Z}$ and $u \in E^*(d)$ that verifies $|u|_{S_\Gamma} < (4N+10)L$. If $|g|_{S_\Gamma} \leq RL$, it holds

$$\left|d^t\right|_{S_\Gamma} \leq |g|_{S_\Gamma} + |u|_{S_\Gamma} + |v|_{S_\Gamma} < 2RL + (4N+10)L = ML \leq |d|_{S_\Gamma}.$$

That is, $t = 0$ so that g is an element in F as wanted. ∎

According to the first item of Proposition 3.3.1, there exists a constant K depending only on δ so that F has at most K elements. Choose a largest subset $W := \{ w_1, \ldots, w_n \} \subseteq S'_\Gamma$ representing distinct right cosets of Γ modulo F. For $m := 3N$, construct the set

$$Y := \left\{ w_i d^m w_i^{-1} \right\}_{i=1}^n.$$

<u>Claim:</u> The set Y verifies the three properties stated at the beginning of this proof.

- Every $y \in Y$ is a product of at most D generators in $(S'_\Gamma)^{\pm 1}$ where D depends only on δ,
- There exists $K > 0$ depending only on δ such that $\#Y \geq K^{-1}\#S'_\Gamma$,
- The set Y freely generates a free subgroup of Γ.

Proof: Every $y \in Y$ is a product of at most $\left|w_i d^m w_i^{-1}\right|_{S'_\Gamma} \leq 2|w_i|_{S'_\Gamma} + |d^m|_{S'_\Gamma} < 2 + m|d|_{S'_\Gamma} =: D$ elements in S'_Γ which only depends on the hyperbolicity constant of Γ. By construction $\#Y = \#W \geq K^{-1}\#S'_\Gamma$. So, it remains to show that Y freely generates a free subgroup of Γ. To this purpose, we show that any non-cancellable product $p_1 p_2 \cdots p_k$ is not trivial for $p_i := v_i d^{mt_i} v_i^{-1}$, $v_i \in W$ and $t_i = \pm 1$ for any $i \in \{1, \ldots, k\}$. Consider two neighboring factors $vd^{mt}v^{-1}wd^{ms}w^{-1}$ where $s, t = \pm 1$. Suppose by contradiction that

$$\left|d^{mt}v^{-1}wd^{ms}\right|_{S_\Gamma} \leq \left|d^{mt}\right|_{S_\Gamma} + \left|d^{ms}\right|_{S_\Gamma} - \left|vw^{-1}\right|_{S_\Gamma} - 2N\,|d|_{S_\Gamma}.$$

Then, with $r := N|d|_{S_\Gamma} > 0$ and the elements $x := v^{-1}$, $x' := w^{-1}$, $y := d^{-mt}v^{-1}$, $y' := d^{-ms}w^{-1}$, it holds

$$|x - y|_{S_\Gamma} + |x' - y'|_{S_\Gamma} \geq |x - x'|_{S_\Gamma} + |y - y'|_{S_\Gamma} + 2r.$$

So, by Lemma 4.1.8, the segments v^{-1}-$d^{-mt}v^{-1}$ and w^{-1}-$d^{-ms}w^{-1}$ have strictly 8δ–close subsegment which have S_Γ–length greater than $2N|d|_{S_\Gamma}$. The element d is cyclically minimal so that, by Lemma 4.1.14, the element vw^{-1} is in $E^+(d)$ and, by Lemma 4.1.15

$$\left|vw^{-1}\right|_{S_\Gamma} \leq 2L + 52\delta \leq 4L \leq 2(4N + 10)L \text{ and } 40\delta \leq L.$$

Therefore $vw^{-1} \in B_{S_\Gamma}(\Gamma, RL) \cap E(d) \subseteq F$ so that $v \in Fw$. That is a contradiction with the hypothesis that $p_i p_{i+1}$ are non-cancelling factors and v, w are distinct representatives of F. Therefore

$$\left|d^{mt}v^{-1}wd^{ms}\right|_{S_\Gamma} > \left|d^{mt}\right|_{S_\Gamma} + \left|d^{ms}\right|_{S_\Gamma} - \left|vw^{-1}\right|_{S_\Gamma} - 2N\,|d|_{S_\Gamma}, \tag{6}$$

which implies that

$$\begin{aligned}
\left|vd^{mt}v^{-1}wd^{ms}w^{-1}\right|_{S_\Gamma} &\geq \left|vd^{mt}v^{-1}wd^{ms}w^{-1}\right|_{S_\Gamma} + \left|v^{-1}w\right|_{S_\Gamma} - |v|_{S_\Gamma} - |w|_{S_\Gamma} \\
&\geq \left|d^{mt}v^{-1}wd^{ms}\right|_{S_\Gamma} + \left|v^{-1}w\right|_{S_\Gamma} - 2\,|v|_{S_\Gamma} + 2\,|w|_{S_\Gamma} \\
&> 2\,|v|_{S_\Gamma} + 2\,|w|_{S_\Gamma} + \left|d^{mt}\right|_{S_\Gamma} + \left|d^{ms}\right|_{S_\Gamma} - 4\left(|v|_{S_\Gamma} + |w|_{S_\Gamma}\right) - 2N\,|d|_{S_\Gamma} \\
&\geq \left|vd^{mt}v^{-1}\right|_{S_\Gamma} + \left|wd^{ms}w^{-1}\right|_{S_\Gamma} - 4\left(|v|_{S_\Gamma} + |w|_{S_\Gamma}\right) - 2N\,|d|_{S_\Gamma},
\end{aligned}$$

where we used the inequality (6) in the third inequality. Using Lemma 4.1.9, we obtain

$$\left|vd^{mt}v^{-1}\right|_{S_\Gamma} \geq \left|d^{mt}\right|_{S_\Gamma} - 2\,|v|_{S_\Gamma} \geq m\,|t|\,|d|_{S_\Gamma} - 2\,(m\,|t| - 1)\left\langle d, d^{-1}\right\rangle - 2\,(m\,|t| - 2)\,\delta - 2\,|v|_{S_\Gamma}.$$

Since d is cyclically minimal and verifies $|d|_{S_\Gamma} > 4\delta$, it holds by the second item of Lemma 4.1.10

$$|d|_{S_\Gamma} \leq |d|_{S_\Gamma} - 2\left\langle d, d^{-1}\right\rangle + 4\delta.$$

Hence, $2\delta \geq \left\langle d, d^{-1}\right\rangle$ so that $\left|vd^{mt}v^{-1}\right|_{S_\Gamma} \geq m|t||d|_{S_\Gamma} - 6m|t|\delta + 2\delta \geq m|t|\left(|d|_{S_\Gamma} - 6\delta\right)$. Consequently, using the bound given by Lemma 4.1.16 for the length of any element in S'_Γ, it holds

$$\begin{aligned}
|p_i|_{S_\Gamma} &\geq m(|d|_{S_\Gamma} - 6\delta) - 2(4N + 10)L \\
N|d|_{S_\Gamma} + 4(4N + 10)L &\geq \left\langle p_i^{-1}, p_{i+1}\right\rangle.
\end{aligned}$$

Thus, with $m = 3N$ we get $\left\langle p_{i-1}^{-1}, p_i\right\rangle + \left\langle p_i^{-1}, p_{i+1}\right\rangle < |p_i|_{S_\Gamma} - 3\delta$ if and only if $40NL + 100L < N|d|_{S_\Gamma} - 18N\delta - 3\delta$. This last inequality is verified since $N|d|_{S_\Gamma} \geq NML$ and $N \geq 10$. Then, by the first item of Lemma 4.1.7 with $x_1 := p_k^{-1} \cdots p_1^{-1}, \ldots, x_k := p_1^{-1}$, we see that $|p_1 \cdots p_k|_{S'_\Gamma} > 0$ so that the product is not trivial. ■

Finally, define the constants $\alpha := K^{-1}, \beta := D^{-1}$ that depend only on δ and $\#S_\Gamma$. Then, the inequality (5) gives as wanted

$$e\left(\Gamma, S'_\Gamma\right) \geq \left(2\alpha\#S'_\Gamma - 1\right)^\beta.$$

□

To prove this theorem, we brought to light the existence of a free subgroup of the studied non-elementary hyperbolic group. So, in particular, these groups have exponential growth. Moreover, this result gives a first approach to strengthen Koubi's Theorem on growth of hyperbolic groups and find a finite generating set, if it exists, that achieves the minimum growth rate. In the upcoming subsection, we prove that the set of growth rates of any limit group admits a minimum and further prove that this set is well-ordered.

4.2 Well-Ordering of the set of Growth Rates

We bring together the theory developed so far to study the set of growth rates of limit groups. First, we prove that this set is well-ordered for hyperbolic limit groups, and then extend this result for all limit groups. This analysis and these results come from [FS20] in which Fujiwara and Sela also prove the well-ordering of the set of growth rates for arbitrary hyperbolic groups.

Theorem 4.2.1 (Fujiwara and Sela). *If Γ is a hyperbolic limit group, then the set $\xi(\Gamma)$ is well-ordered.*

If Γ is elementary, then $\xi(\Gamma) = \{ 1 \}$ so that the set of growth rates of Γ is well-ordered. Thus, suppose Γ is a non-elementary hyperbolic limit group. Let $\mathfrak{u}$ be a non-principal ultrafilter, S_Γ a finite generating set of Γ and suppose by contradiction that $\xi(\Gamma)$ is not well-ordered. Then, there exists a sequence of finite generating sets (S_i) of Γ such that $(e(\Gamma, S_i)) \subset \xi(\Gamma)$ is strictly decreasing with $\mathfrak{u}\text{–}\lim e(\Gamma, S_i) \geq 1$. In one hand, we prove that the sequence $(e(\Gamma, S_i))$ is bounded from above by the exponential growth rate of a limit group with respect to a specific generating set. In the other hand, we prove that $(e(\Gamma, S_i))$ converges towards this upper bound. This leads to a contradiction since the sequence $(e(\Gamma, S_i))$ is strictly decreasing so that it can not converge to an upper bound.
Claim: Quit to consider a subsequence of (S_i), we may assume that all generating sets have the same cardinality.
Proof: By Theorem 4.1.17, if $(\#S_i)$ has a subsequence that diverges towards infinity, then $(e(\Gamma, S_i))$ has also a subsequence that grows to infinity. Therefore, since $(e(\Gamma, S_i))$ is decreasing, the sequence $(\#S_i)$ is bounded and we may consider a subsequence verifying $\#S_i = \ell$ for every $i \in \mathbb{N}$. ∎

Explicit the elements of $S_i := \{ x_{1,i}, \dots, x_{\ell,i} \}$ and consider the free group F_ℓ on ℓ elements $S := \{ s_1, \dots, s_\ell \}$. Define the epimorphisms described on basis elements

$$\varphi_i : \begin{array}{ccc} F_\ell & \longrightarrow & \Gamma; \\ s_n & \longmapsto & x_{n,i}, \end{array}$$

that form a sequence of elements in the compact space $\mathcal{G}_\ell$ and define the limit group $L := F_\ell/\ker_{\mathfrak{u}}(\varphi_i)$ with associated quotient map $\eta : F_\ell \to L$. The set of limit groups is closed in $\mathcal{G}_\ell$ so that L is a limit group and, in particular, is residually free by Theorem 3.2.8. Consequently, by the second item of Theorem 3.2.22, the homomorphisms φ_i $\mathfrak{u}$–almost surely factor through the quotient map $\eta : F_\ell \to F_\ell/\ker_{\mathfrak{u}}(\varphi_i)$. Thus, there exists a subsequence of (φ_i), still denoted (φ_i), so that for every natural number i, it holds $\varphi_i = \phi_i \circ \eta$ for some epimorphisms $\phi_i : L \to \Gamma$. That is, the following diagram is commutative

$$\begin{tikzcd} (F_\ell, S) \arrow[rr, two heads, "\varphi_i"] \arrow[dr, two heads, "\eta"'] & & (\Gamma, \varphi_i(S)) \\ & (L, \eta(S)). \arrow[ur, two heads, "\phi_i"'] & \end{tikzcd}$$

Hence, $e(\Gamma, \varphi_i(S)) \leq e(L, \eta(S))$ so that $(e(\Gamma, S_i))$ is bounded from above by the exponential growth rate of a limit group with respect to a specific generating set.

Lemma 4.2.2 (Fujiwara and Sela). *With the notations defined above, it holds* $\mathfrak{u}\text{–}\lim e(\Gamma, \varphi_i(S)) = e(L, \eta(S))$.

Proof. We present a proof of this result in three steps. First, using Theorem 2.2.11 and Theorem 3.3.8, we construct a faithful action of the limit group L on a real tree in order to construct elements of L verifying a small cancellation property. Second, we use these elements to build feasible elements that have interesting geometric properties. Finally, we use these feasible elements to show a strong link between $e(\Gamma, \varphi_i(S))$ and $e(L, \eta(S))$.

The following construction is similar to that produced in Subsection 3.3. Fix a Cayley graph X of Γ with respect to S_Γ and denote $|\cdot|_{S_\Gamma}$ the word length in Γ with repespect to S_Γ. In the remainder of this document, we use the same notation for elements in Γ and their representatives in the Cayley graph X. The group Γ acts isometrically by translations on X so that each epimorphism φ_i induces a transitive action of F_ℓ on the hyperbolic metric space X. Consider a sequence (h_i) of elements in Γ such that

$$\left|h_i \varphi_i h_i^{-1}\right|_e := \max_{1 \leq n \leq \ell} \left|h_i \varphi_i(s_n) h_i^{-1}\right|_{S_\Gamma} = \min_{h \in \Gamma} \max_{1 \leq n \leq \ell} \left|h \varphi_i(s_n) h^{-1}\right|_{S_\Gamma}.$$

Replace the epimorphisms φ_i by $h_i \varphi_i h_i^{-1}$ and keep the notation φ_i for the conjugated epimorphisms. Notice that the point in X representing the identity element $e \in \Gamma$ has minimal displacement among all elements in X for the defined actions of F_ℓ on this Cayley graph. So, each φ_i is conjugacy short and since the growth rate is invariant by conjugation, $(e(\Gamma, \varphi_i(S)))$ is still strictly decreasing.
Claim: The sequence of scalars $(|\varphi_i|_e)$ is not bounded.
Proof: Suppose by contradiction that the sequence $(|\varphi_i|_e)$ is bounded. Then, the image $\varphi_i(S)$ is contained in a fixed bounded set of Γ. Since any morphism is completely determined by its values on a generating set, there

is only finitely many distinct φ_i. This is a contradiction with $(e(\Gamma, \varphi_i(S)))$ is strictly decreasing. Hence, the sequence $(|\varphi_i|_e)$ is not bounded. ∎

Consider a subsequence of (φ_i), still denoted (φ_i), so that $(|\varphi_i|_e)$ diverges towards infinity. Using Theorem 2.2.11, the free group F_ℓ acts on the asymptotic cone $T_{\mathfrak{u}} := \mathrm{Cone}_{\mathfrak{u}}\left(X, (e), \left(|\varphi_i|_e^{-1}\right)\right)$ which is an unhooked F_ℓ–tree that admits a minimal F_ℓ–invariant subtree T. In addition, the $\mathfrak{u}$–kernel of the sequence (φ_i) acts trivially on T. So, this defines an unhooked action of the limit group L on the real tree T that is F_ℓ–minimal.

Claim: The action ρ of L on T defined above is faithful.

Proof: Suppose by contradiction that L acts on T non-faithfully. By Corollary 3.3.9, the subtree T is not a line. So, the kernel of ρ stabilises a tripod in T and by Theorem 3.3.8, this kernel is finite. Therefore, kerρ is a finite subgroup of the limit group L that is torsion-free. That is a contradiction so that ρ is faithful. ∎

By definition of Γ–limit group, for any natural number m, there exists $N \in \mathbb{N}$ such that for every $i \geq N$, the epimorphisms ϕ_i induce isometries

$$\psi_i := \phi_i|_{B_{\eta(S)}(L,m)} : B_{\eta(S)}(L, m) \to B_{\varphi_i(S)}(\Gamma, m).$$

Then, using that each ϕ_i is a morphism, we dispose of maps

$$\begin{array}{ccc} B_{\eta(S)}(L, m)^q & \longrightarrow & B_{\varphi_i(S)}(\Gamma, qm); \\ (\omega_1, \ldots, \omega_q) & \longmapsto & \phi_i(\omega_1 \cdots \omega_q), \end{array}$$

that are not injective in general. In the following, we define elements $u_j \in L$ of bounded $\eta(S)$–length, called **separators**, to introduce another map $\nu_i : (\omega_1, \ldots, \omega_q) \mapsto \phi_i(\omega_1 u_1 \cdots \omega_q u_q)$ which we show is injective for all natural number q on specific elements of L using the Γ–action on X.

Definition 4.2.3. Let x be a point in a real tree T. The **space of directions** at x, denoted Σ_x, is the quotient of

$$\mathcal{R}_x := \{\, p : [0, a) \to T \mid a > 0,\ p \text{ is an isometry, } p(0) = x \,\},$$

via the equivalence relation

$$p_1 \sim p_2 \Leftrightarrow \text{there exists } \varepsilon > 0 \text{ such that } p_1|_{[0,\varepsilon)} \equiv p_2|_{[0,\varepsilon)}.$$

Let x, y be two elements in T and $\gamma : [0, a] \to T$ a geodesic between x and y where $a := \mathrm{dist}_T(x, y)$. The geodesic γ **starts** with a direction $p_1 : [0, b) \to T$ at x if there exists $\varepsilon > 0$ such that $p_1|_{[0,\varepsilon)} \equiv \gamma|_{[0,\varepsilon)}$. Also, γ **ends** with a direction $p_2 : [0, c) \to T$ at y if there exists $\varepsilon > 0$ such that $\overline{p_2}|_{[0,\varepsilon)} \equiv \gamma|_{(1-\varepsilon,1]}$, where $\overline{p_2}(t) := p_2(b - t)$ for $t \in [0, c)$.

Let $[x_i]$ and $[y_i]$ be elements in T. A direction $p_1 : [0, a) \to T$ at $[x_i]$ induces a direction $p_2 : [0, a) \to T$ at $[y_i]$ defined by $p_2(t) := \left[y_i x_i^{-1} p_1(t)_i\right]$ for any $t \in [0, a)$ where $p_1(t) = [p_1(t)_i]$ in T. With the constructions described so far, we now construct elements of L verifying a small cancellation property.

Lemma 4.2.4 (Fujiwara and Sela). *Let p_1, p_2 be two distinct directions at $[e] \in T$. There exist elements $u_{i,j} \in L$ for $i, j \in \{1, 2\}$, called separators, with the following properties.*

1. *For every $i, j \in \{1, 2\}$, the segment $[e]$-$u_{i,j}[e]$ starts with p_i at $[e]$ and ends with the direction p_j at $u_{i,j}[e]$,*
2. *For every $i, j \in \{1, 2\}$, it holds* $\mathrm{dist}_T([e], u_{i,j}[e]) > 10$,
3. *For every ω in L and every two pairs $(i_1, j_1), (i_2, j_2)$ with $i_1, j_1, i_2, j_2 \in \{1, 2\}$, if the segment $[e]$-$u_{i_1,j_1}[e]$ intersects the segment $\omega[e]$-$\omega u_{i_2,j_2}[e]$ non-trivially, then the length of the intersection is bounded from above by*

$$\frac{1}{10}\mathrm{dist}_T([e], u_{i_1,j_1}[e]).$$

If the two pairs $(i_1, j_1), (i_2, j_2)$ are equal, we assume in addition that ω is not the identity element.

*This last property is called **small cancellation** property of separators.*

Notation 12. Given an isometry γ of a metric space X with base point $x \in X$, we denote $\mu(\gamma) := \mathrm{dist}_X(x, \gamma x)$.

Proof. Since $\eta(S)$ is finite and $\mathfrak{u}$ is finitely additive as an ultrafilter, there exists $t_1 \in \eta(S)$ such that

$$\max_{t \in \eta(S)} \mathrm{dist}_T([e], t[e]) = \mathrm{dist}_T([e], t_1[e]) = 1.$$

Because $[e]$ is a point of T that has minimal displacement for the ρ action of L on T, the isometry t_1 is hyperbolic and, without loss of generality, we assume $t_1 = \eta(s_1)$. In addition, $[e]$ is on the axis of $\eta(s_1)$ so that there exist at least two distinct directions p_1, p_2 at $[e]$ in T. Consider two generators of S, without loss of generality s_1 and s_2, for which $[e]$-$\eta(s_i)[e]$ starts with p_i for $i \in \{1, 2\}$.

Claim: There exists $y, z \in L$ such that the subgroups $\langle y, z \rangle, \langle y, s_i \rangle, \langle z, s_i \rangle$ of L are free for any $i \in \{1, 2\}$.

Proof: We proceed in several cases, looking at the characteristic axis of different hyperbolic elements in L.

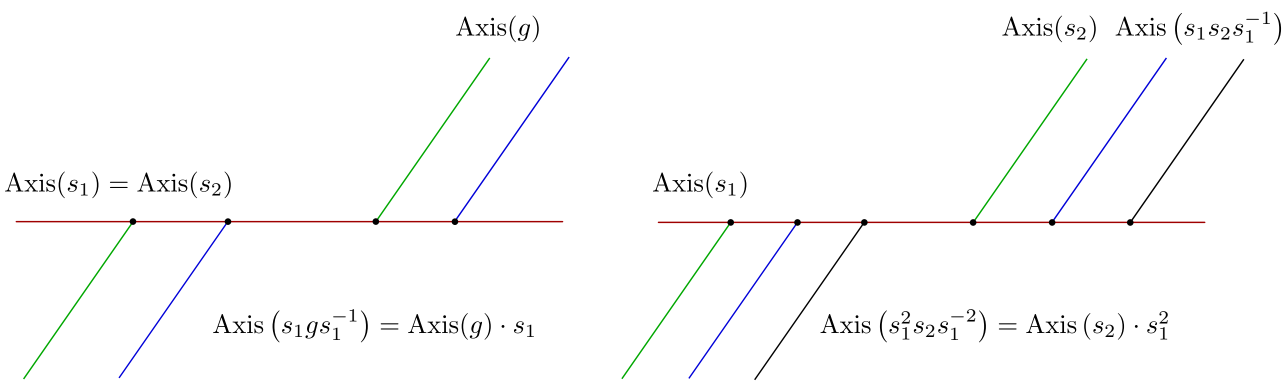

Figure 5: The axes of s_1, s_2 and the construction of axes for y and z as translations of Axis(s_1).

- If Axis(s_1) = Axis(s_2), there exists another hyperbolic element $g \in L$ such that Axis(g) $\neq$ Axis(s_1). Indeed, by Theorem 2.2.9, T is the union of all axes of hyperbolic elements in L and, by Corollary 3.3.9, it is not a line. So, as seen in Figure 5, the three hyperbolic elements $s_1, g, s_1 g s_1^{-1}$ of L have distinct characteristic axes. If one of the subgroups of L
$$\left\langle\, g, s_1 g s_1^{-1} \,\right\rangle, \left\langle\, g, s_i \,\right\rangle, \left\langle\, s_1 g s_1^{-1}, s_i \,\right\rangle$$
satisfies a non-trivial relation for $i \in \{\,1,2\,\}$, then, as a 2–generated subgroup of a limit group, it is abelian by Lemma 3.2.2. This produces a cycle in T which is a contradiction. Hence, these five groups are free.

- Otherwise, Axis(s_1) $\neq$ Axis(s_2) and, as seen in Figure 5, the hyperbolic elements $s_1, s_2, s_1 s_2 s_1^{-1}$ and $s_1^2 s_2 s_1^{-2}$ in L have distinct characteristic axes. As above, if one of the subgroups of L
$$\left\langle\, s_1, s_1 s_2 s_1^{-1} \,\right\rangle, \left\langle\, s_2, s_1 s_2 s_1^{-1} \,\right\rangle, \left\langle\, s_1, s_1^2 s_2 s_1^{-2} \,\right\rangle, \left\langle\, s_1, s_1^2 s_2 s_1^{-2} \,\right\rangle \text{ or } \left\langle\, s_1 s_2 s_1^{-1}, s_1^2 s_2 s_1^{-2} \,\right\rangle$$
satisfies a non-trivial relation, then, as a 2–generated subgroup of a limit group, it is abelian by Lemma 3.2.2. This produces a cycle in T which is a contradiction. Hence, these subgroups are free of rank 2 as wanted. ■

Define the elements $u_{i,j} \in L$ for $i, j \in \{\,1,2\,\}$ as
$$u_{i,j} := s_i^{\beta_i} y^{\,\alpha_1+i+3j} z y^{\,\alpha_2+i+3j} z \cdots y^{\,\alpha_{29}+i+3j} z y^{\,\alpha_{30}+i+3j} s_j^{-\beta_j},$$
where the parameters β_i for $i \in \{\,1,2\,\}$ and α_k for $k \in \{\,1,\ldots,30\,\}$ satisfy

- $\beta_i \lg(s_i) > 5\,\mu(s_i)$ and $\beta_i \lg(s_i) > 5\,\mu(y)$ for any $i \in \{\,1,2\,\}$,
- $\alpha_1 \lg(y) \geq \max(200\mu(y),\, 20\left(\beta_1\mu(s_1) + \beta_2\mu(s_2)\right),\, 20\mu(z),\, 1)$,
- The elements α_k are defined recursively by $\alpha_k := \alpha_1 + 6k$ for $k \in \{\,2,\ldots,30\,\}$.

The conditions on the parameters β_i and α_k for $i \in \{\,1,2\,\}$ and $k \in \{\,1,\ldots,30\,\}$ guarantee that the cancellations between consecutive geodesics in the sequence
$$[e]\text{-}s_i^{\beta_i}[e],\ [e]\text{-}y^{\,\alpha_1}[e],\ [e]\text{-}zy^{\,\alpha_k}[e] \text{ and } [e]\text{-}s_j^{-\beta_j}[e],$$
for $i, j \in \{\,1,2\,\}$ and $k \in \{\,1,\ldots,30\,\}$ are limited to a small proportion of the lengths of these segments. Hence, the segment $[e]$-$u_{i,j}[e]$ starts with the same direction as $[e]$-$s_i[e]$ at $[e]$ and ends with the same direction as $[e]$-$s_j^{-1}[e]$ at $u_{i,j}$ (if $[e]$-$s_j^{-1}[e]$ ends with direction p_3 at s_j^{-1}, then $[e]$-$u_{i,j}[e]$ ends with the direction induced by p_3 at $u_{i,j}$). Consequently, the element $u_{i,j}$ satisfy the conditions of the first item of this lemma.

The second item of Lemma 4.2.4 follows from the bound on the cancellations between consecutive geodesics and the definition of α_1.

The third item follows from the structure of the elements $u_{i,j}$ as products of high powers of an element y, separated by an element that does not commute with it, and the bound on the cancellations between consecutive segments that correspond to these high powers. □

This result permits us to define feasible elements in L that have interesting geometric properties in order to highlight a link between $e\,(\Gamma, \varphi_i(S))$ and $e\,(L, \eta(S))$. Let b be the maximum $\eta(S)$–length of the words $u_{i,j}$ for $i, j \in \{\,1,2\,\}$ constructed in Lemma 4.2.4 and m a natural number. Because the sequence $(\Gamma, \varphi_i(S))$ converges to

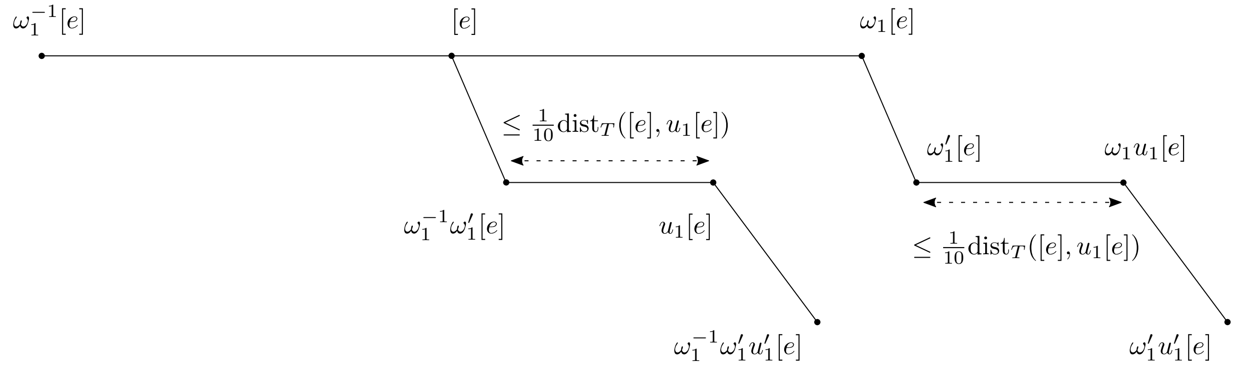

Figure 6: Notations for two subpaths of $[e]$-$\omega_1 u_1 \cdots \omega_q u_q[e]$ and their study.

the limit group $(L, \eta(S))$ in $\mathcal{G}_\ell$, there exists $N \in \mathbb{N}$ such that for every $i \geq N$, the ball $B_{\eta(S)}(L, m+b)$ is isometric to $B_{\varphi_i(S)}(\Gamma, m+b)$ and thus, for every $i \geq N$, the following diagram is commutative

$$\begin{tikzcd}
B_S(F_\ell, m+b) \arrow[rr, two heads, "\varphi_i"] \arrow[dr, two heads, "\eta"'] & & B_{\varphi_i(S)}(\Gamma, m+b) \\
& B_{\eta(S)}(L, m+b). \arrow[ur, hook, two heads, "\psi_i := \phi_i|_{B_{\eta(S)}(L,m+b)}"'] &
\end{tikzcd}$$

Definition 4.2.5. Let ω be an element in L and denote p_1 the direction in which the segment $[e]$-$\omega[e]$ ends at $\omega[e]$. By the first item of Lemma 4.2.4, there exists a separator u in L such that $\omega[e]$-$\omega u[e]$ starts with a different direction at $\omega[e]$ than p_1. Any separator u verifying this property is called **admissible** for ω. Then, given ω, ω' elements in L, there exists similarly, a separator u which is admissible for ω and such that u^{-1} is admissible for ω'^{-1}. Any separator with this property is **admissible** for ω, ω'.

In other words, if ω, ω' are elements in L such that $[e]$-$\omega[e]$ ends with a direction p_1 at $\omega[e]$ and the geodesic $[e]$-$\omega'[e]$ starts with a direction p_2 at $[e]$, then a separator u is admissible for ω, ω' if $[e]$-$u[e]$ starts with a direction at $[e]$ different than p_1 and ends with a direction at $u[e]$ different than p_2.

For each pair of non-trivial elements $\omega_1, \omega_2 \in B_{\eta(S)}(L, m+b)$, choose a separator u_1 that is admissible for ω_1, ω_2. Then inductively, for q an arbitrary natural number, $\omega_1, \ldots, \omega_q$ a collection of non-trivial elements in $B_{\eta(S)}(L, m)$ and $t \in \{1, \ldots, q-1\}$, choose a separator u_t that is admissible for ω_t, ω_{t+1}. So far, we know that all the elements considered are mapped to non-trivial elements by the isometries ψ_i. But the maps ψ_i are not defined on all of L and the epimorphisms ϕ_i are not isometries on all of L. However, we show that the map

$$\nu_i : \begin{array}{ccc} B_{\eta(S)}(L, m)^q & \longrightarrow & B_{\varphi_i(S)}(\Gamma, q(m+b)); \\ (\omega_1, \ldots, \omega_q) & \longmapsto & \phi_i(\omega_1 u_1 \cdots \omega_q u_q), \end{array}$$

are injective on feasible elements of L.

Definition 4.2.6. A non-trivial element $\omega_1 \in B_{m+b}(L, \eta(S))$ is **forbidden** if there exists an element ω_2 in the same ball and a separator u constructed in Lemma 4.2.4 such that:

1. the separator u is admissible for ω_1,
2. $\mathrm{dist}_T(\omega_2[e], \omega_1 u[e]) \leq 5^{-1}\mathrm{dist}_T([e], u[e])$.

Then, an element $\omega_1 u_1 \cdots \omega_q u_q$ as constructed inductively above is **feasible** of type q if all the elements ω_t for $t \in \{1, \ldots, q\}$ are not forbidden.

Lemma 4.2.7 (Fujiwara and Sela)**.** *Let m and q be natural numbers. If i is a natural number large enough so that $B_{\eta(S)}(L, m+b)$ and $B_{\varphi_i(S)}(\Gamma, m+b)$ are isometric, then ϕ_i maps the collection of feasible elements of type q in L to distinct elements in $B_{\varphi_i(S)}(\Gamma, m+b)$.*

Proof. We argue by induction on the type q of the feasible elements. If $q = 1$, then $\omega_1 u_1 \in B_{\eta(S)}(L, m+b)$ and the restriction of ϕ_i to this set is an isometry. So, the morphism ϕ_i maps feasible elements of type 1 in L to distinct elements in Γ. Suppose the conclusion holds for feasible elements of type less than q and suppose by contradiction that ϕ_i maps two distinct feasible elements of type q, denoted $\omega_1 u_1 \cdots \omega_q u_q$ and $\omega_1' u_1' \cdots \omega_q' u_q'$, to the same element of Γ. In one hand, assume $u_1 \neq u_1'$ and refer to Figure 6 for the notations. Then, using the choice of separators in the construction of feasible elements and the fact that T is a real tree, either the segment $[e]$-$u_1[e]$

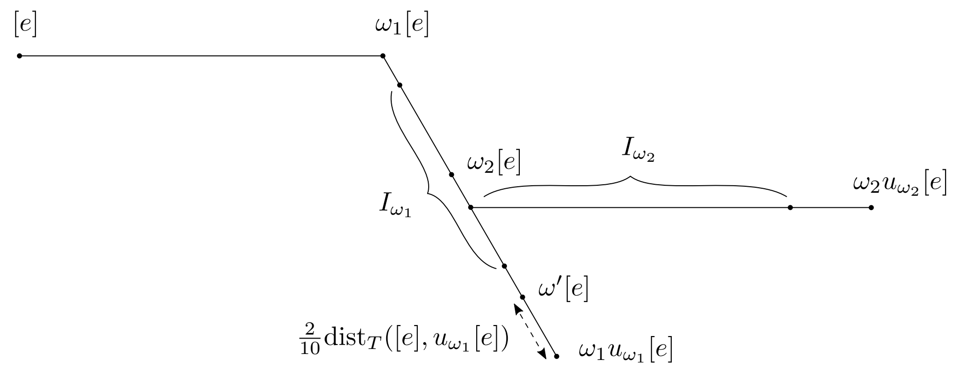

Figure 7: A representation of the subsegments I_ω and notations for Lemma 4.2.8.

intersects the geodesic $[e]$-$\omega_1^{-1}\omega_1' u_1'[e]$ or the segment $[e]$-$u_1'[e]$ intersects the geodesic $[e]$-${\omega_1'}^{-1}\omega_1 u_1[e]$. By choice of u_1, respectively u_1', the segment $[e]$-$u_1[e]$ intersects the geodesic $\omega_1^{-1}\omega_1'[e]$-$\omega_1^{-1}\omega_1' u_1'[e]$, or respectively $[e]$-$u_1'[e]$ intersects the geodesic ${\omega_1'}^{-1}\omega_1[e]$-${\omega_1'}^{-1}\omega_1 u_1[e]$. So, by the third item of Lemma 4.2.4, it holds

$$\mathrm{dist}_T\left(\omega_1'[e], \omega_1 u_1[e]\right) \leq \frac{1}{10}\mathrm{dist}_T\left([e], u_1[e]\right) \text{ or } \mathrm{dist}_T\left(\omega_1[e], \omega_1' u_1'[e]\right) \leq \frac{1}{10}\mathrm{dist}_T\left([e], u_1'[e]\right).$$

Therefore, one of the element ω_1' or ω_1 is forbidden. That is a contradiction so that $u_1 = u_1'$. In the other hand, suppose $u_1 = u_1'$ and $\omega_1 \neq \omega_1'$. Then, $\omega_1^{-1}\omega_1' \neq 1$ and a similar reasoning as above, using the small cancellation property of the separators, shows that ω_1' or ω_1 is forbidden. That is a contradiction so that $\omega_1 = \omega_1'$.

Since ϕ_i is a morphism, it maps the two feasible elements $\omega_2 u_2 \cdots \omega_q u_q$ and $\omega_2' u_2' \cdots \omega_q' u_q'$ of type $q-1$ to the same element of Γ. Consequently, by induction hypothesis, these two elements are equal so that $\omega_1 u_1 \cdots \omega_q u_q$ and $\omega_1' u_1' \cdots \omega_q' u_q'$ are also equal. Hence, ϕ_i maps the set of feasible elements of type q in L to distinct elements in Γ. $\square$

For i large enough, the morphism ϕ_i maps the feasible elements of L injectively into Γ. So, we can estimate from below the cardinality of balls with a radius greater than b in Γ by counting the number of feasible elements in L. That is the purpose of the upcoming lemma.

Lemma 4.2.8 (Fujiwara and Sela)**.** *If m is a natural number, then the following are lower bounds on the number of non-forbidden and feasible elements in $B_{\eta(S)}(L, m)$.*

- *The number of non-forbidden elements in $B_{\eta(S)}(L, m)$ is at least $5/6\,\#B_{\eta(S)}(L, m)$,*
- *For every natural number q, the number of feasible elements of type q in L is at least $\left(5/6\,\#B_{\eta(S)}(L, m)\right)^q$.*

Proof. The second item follows from the first one. Indeed, given natural numbers m and q, feasible elements of type q in L are built from all the possible q concatenations of non-forbidden elements in $B_{\eta(S)}(L, m)$ with separators between the non-forbidden elements.

To prove the first item, define for every natural number m the finite subtree of T spanned by $B_{\eta(S)}(L, m) \cdot [e]$

$$T_m := \bigcup_{g \in B_{\eta(S)}(L,m)} [e]\text{-}g[e] \subset T.$$

By construction, $T_1 \subset T_2 \subset \cdots \subset T_m$ and since $|\eta|_{[e]} = 1$, every element in $B_{\eta(S)}(L, m)$ adds at most one to the total length of the edges in T_m. Therefore, the sum of the lengths of the edges in the finite tree T_m is bounded by $\#B_{\eta(S)}(L, m)$. We use this fact to count the number of forbidden elements in $B_{\eta(S)}(L, m)$. By Lemma 4.2.4 and the definition of forbidden element, for every forbidden element $\omega \in B_{\eta(S)}(L, m)$, there exists $\omega' \in B_{\eta(S)}(L, m)$ and an element u_ω constructed in Lemma 4.2.4 so that

$$\mathrm{dist}_T\left(\omega'[e], \omega\, u_\omega[e]\right) < 5^{-1}\mathrm{dist}_T\left([e], u_\omega[e]\right). \tag{7}$$

With these notations, define for every forbidden element $\omega \in B_{\eta(S)}(L, m)$, a subsegment I_ω of the segment $\omega[e]$-$\omega u_\omega[e]$ which starts after the first $1/10$ of the segment $\omega[e]$-$\omega u_\omega[e]$ and ends at $7/10$ of the segment $\omega[e]$-$\omega u_\omega[e]$. Then, by construction, the total length of the edges in I_ω is $6/10\,\mathrm{dist}_T\left([e], u_\omega[e]\right)$ and the following holds.
<u>Claim:</u> Let ω be a forbidden element and ω_1, ω_2 two distinct forbidden elements.

1. The segment I_ω is a subset of T_m,
2. The intersection $I_{\omega_1} \cap I_{\omega_2}$ is either empty or a point.

Proof: The first item follows from equation (7). Indeed, by uniqueness of the geodesics between two elements in T, the element $\omega'[e]$ is on the geodesic $[e]$-$\omega u_\omega[e]$ and the segment $[e]$-$\omega'[e]$ is a subset of T_m. By equation (7), the first 8/10 of the segment $\omega[e]$-$\omega u_\omega[e]$ is in T_m so that $I_\omega \subset T_m$, see Figure 7. We prove the second item of the claim using the small cancellation property of the separators. Given two distinct forbidden elements ω_1, ω_2, the overlap between the geodesics $\omega_1[e]$-$\omega_1 u_{\omega_1}[e]$ and $\omega_2[e]$-$\omega_2 u_{\omega_2}[e]$ is bounded by

$$10^{-1}\mathrm{dist}_T\left([e], u_{\omega_k}[e]\right) \text{ for every } k \in \{1, 2\}.$$

Then, by uniqueness of the geodesics between two elements in T and the construction of the segments I_ω as subsegments of $\omega[e]$-$\omega u_\omega[e]$, the intersection $I_{\omega_1} \cap I_{\omega_2}$ is either empty or a point for every distinct forbidden elements $\omega_1, \omega_2 \in B_{\eta(S)}(L, m)$. ■

By the second item of Lemma 4.2.4, the length of a geodesic $[e]$-$u_{i,j}[e]$ is at least 10 for any element $u_{i,j}$ constructed in Lemma 4.2.4. Thus, for every forbidden element ω, the length of I_ω is at least 6. Hence, the collection of subintervals I_ω covers the edges of T_m for a total length of 6 times the number of forbidden elements in $B_{\eta(S)}(L, m)$. Since the total length of the edges in T_m is bounded by $\#B_{\eta(S)}(L, m)$, the number of forbidden elements in $B_{\eta(S)}(L, m)$ is bounded by $6^{-1}\#B_{\eta(S)}(L, m)$. That is, the lower bound on the number of non-forbidden elements stated in the first item of this lemma. □

We finally conclude the proofs of Lemma 4.2.2 and Theorem 4.2.1. Fix m a natural number and i large enough so that the conclusion of Lemma 4.2.7 holds. By Lemma 4.2.7, for any $q \in \mathbb{N}$, the ball $B_{\varphi_i(S)}(\Gamma, q(m+b))$ contains all the distinct elements $\phi_i(\omega_1 u_1 \cdots \omega_q u_q)$ where $\omega_1 u_1 \cdots \omega_q u_q$ are feasible elements of type q. So, using Lemma 4.2.8 (for large enough i, where large enough does not depend on q), the cardinality of the ball $B_{\varphi_i(S)}(\Gamma, q(m+b))$ is bounded from below by

$$\#B_{\varphi_i(S)}(\Gamma, q(m+b)) \geq \left(\frac{5}{6}\#B_{\eta(S)}(L, m)\right)^q.$$

Therefore, the following inequalities hold.

$$\begin{aligned}
\log\left(e\left(L, \eta(S)\right)\right) &\geq \lim_{i\to\infty} \log\left(e\left(\Gamma, \varphi_i(S)\right)\right) \\
&= \lim_{i\to\infty}\lim_{q\to\infty} \frac{\log\left(\#B_{\varphi_i(S)}(\Gamma, q(m+b))\right)}{q(m+b)} \\
&\geq \lim_{m\to\infty}\lim_{q\to\infty} \frac{q\log\left(\#B_{\eta(S)}(L, m)\right) + q\log\left(\frac{5}{6}\right)}{q(m+b)} \\
&= \log(e\left(L, \eta(S)\right)),
\end{aligned}$$

where the second inequality holds because i does not depend on q. Hence, this concludes the proof of Lemma 4.2.2 and so the proof of Theorem 4.2.1. □

Using a diagonal argument, we extend the result of Theorem 4.2.1 to all non-abelian limit groups.

Theorem 4.2.9 (Fujiwara and Sela)**.** *If L is a limit group, then the set $\xi(L)$ is well-ordered.*

Proof. If L is abelian, then $\xi(L) = \{1\}$ is well-ordered. So, assume L is a non-abelian limit group and suppose by contradiction that $\xi(L)$ is not well-ordered. Then, there exists a sequence (S_i) of finite generating sets of the limit group L such that the sequence $(e(L, S_i))$ is strictly decreasing. Since L is a limit group and F_2 is equationally Noetherian, there exists a sequence of epimorphisms $(\varphi_i : L \to F_2)$ that converges towards the identity. By Lemma 4.2.2 and because F_2 is a hyperbolic limit group, for every index n

$$\lim_{i\to\infty} e\left(F_2, \varphi_i(S_n)\right) = e\left(L, S_n\right).$$

Since the sequence $(e(L, S_i))$ is strictly decreasing, for every $n \in \mathbb{N}$, there exists an index i_n such that

$$e\left(L, S_{n+1}\right) < e\left(F_2, \varphi_{i_n}(S_n)\right) \leq e\left(L, S_n\right).$$

Therefore, the sequence $(e(F_2, \varphi_{i_n}(S_n)))_{n\in\mathbb{N}}$ is strictly decreasing. That is a contradiction with Theorem 4.2.1. Hence, $\xi(L)$ is well-ordered as wanted. □

Using the theory developed in the previous sections, we showed that the set of growth rates of any limit group is well-ordered. In particular, we prove that it admits a minimum and therefore associate to the set of growth rates of any limit group an ordinal. Actually, the techniques exposed in this thesis permit us to study a wider class of groups. Fujiwara and Sela prove in [FS20], using similar methods, that the set of growth rates of any hyperbolic group is well-ordered. In addition, using that any hyperbolic group relative to equationally Noetherian groups is equationally Noetherian, see [GH19, Theorem D], Fujiwara generalize in [Fuj21] the results obtained in this section.

Theorem 4.2.10 (Fujiwara, [Fuj21, Theorem 1.3]). *Let Γ be a group that is hyperbolic relative to a collection of subgroups $\{ P_1, \ldots, P_n \}$. Suppose Γ is not virtually cyclic and not equal to P_k for any $k \in \{ 1, \ldots, n \}$. If each P_i is finitely generated and equationnaly Noetherian, then $\xi(\Gamma)$ is well-ordered.*

In particular, this theorem extend the conclusion of Theorem 4.2.1 to many other groups.

Corollary 4.2.11 (Fujiwara, [Fuj21, Theorem 1.2]). *Let Γ be one of the following groups*

- *A lattice in a simple Lie group of rank one,*
- *The fundamental group of a complete Riemannian manifold M of finite volume such that there exists $a, b > 0$ with $-b^2 \leq K \leq -a^2$, where K denotes the sectional curvature.*

Then, the set of growth rates of Γ is well-ordered.

One question that remains open is the well-ordering of the set of growth rates of the mapping class group, denoted $MCG(\Sigma_{g,p})$, of a compact oriented surface $\Sigma_{g,p}$ with genus g, punctures p and complexity $c(\Sigma_{g,p}) := 3g+p$. By studying its action on its curve graph, Fujiwara proves this set to be well-ordered assuming $MCG(\Sigma_{g,p})$ is equationally Noetherian, see [GH19, page 7] for advances on whether $MCG(\Sigma_{g,p})$ is equationally Noetherian.

Theorem 4.2.12 (Fujiwara, [Fuj21, Theorem 6.6]). *Let $\Sigma_{g,p}$ be a compact oriented surface of complexity greater than 4. If $MCG(\Sigma_{g,p})$ is equationally Noetherian, then $\xi(MCG(\Sigma_{g,p}))$ is well-ordered.*

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