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Real Spectrum Compactifications of Character Varieties and Related Geometric Spaces

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To Johan, Théo, and Julie

my will to imagine a desirable future.

"Why should I be studying for a future that soon may be no more,
when no one is doing anything to save that future?"

— Greta Thunberg, TEDxStockholm, November 2018

A wishful answer:

"Those who contemplate the beauty of the earth
find reserves of strength that will endure as long as life lasts."

— Rachel Carson, *Silent Spring*

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Contents

List of Notations

Notation	Description
Groups and Group Actions	
$\pi_1(S)$	The fundamental group of S
Γ	A finitely generated group
F	A finite generating set of Γ with s elements
$\mathrm{MCG}(S)$	The mapping class group of the surface S
$\mathrm{Out}(\Gamma)$	The outer automorphism group of Γ
$\mathrm{Hom}_{\mathrm{red}}(\Gamma, G(\mathbb{R}))$	The space of reductive representations
$\mathrm{Isom}_{\mathrm{or}}(T)$	The group of orientation-preserving isometries of a tree T
$\eta(\rho)$	The norm associated with a representation ρ
$\lg(\phi)$	The displacement function of a representation ϕ
Character Varieties and Compactifications	
$T(S)$	The Teichmüller space of a surface S
$T(S)^{\mathrm{LS}}$	The length spectrum compactification of $T(S)$
$\Xi(\Gamma, G(\mathbb{R}))$	The character variety of a group Γ with values in the Lie group $G(\mathbb{R})$
$\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{RSp}}$	The real spectrum compactification
$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$	The oriented Gromov–equivariant compactification
$\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{WL}}$	The Weyl chamber length compactification
$V(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}}$	The real spectrum compactification of $V(\mathbb{K})$
π^{RSp}	The continuous extension of a map π to the real spectrum
Geometric and Symmetric Spaces	
$\mathbb{P}^{n-1}(\mathbb{R})$	The $(n-1)$-dimensional real projective space
$\mathbb{H}^2$	The hyperbolic plane
$\mathbb{H}(\mathbb{F})$	The non-Archimedean hyperbolic plane over $\mathbb{F}$
$\mathcal{P}^1(n, \mathbb{R})$	The symmetric space associated with $\mathrm{SL}_n(\mathbb{R})$
$\widehat{\mathcal{P}^1(n, \mathbb{R})}$	The algebraic cover of the symmetric space
$\overline{C^+(\mathbb{R})}$	The closed multiplicative Weyl chamber
δ	The Cartan projection
d_δ	The Cartan multiplicative distance

Notation	Description
$\mathrm{CAT}(\kappa)$	The class of metric spaces with curvature bounded above by κ
Spectrum and Field-Theoretic Notions	
$V(\mathbb{K})^{\mathrm{RSp}}$	The real spectrum of $V(\mathbb{K})$
$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$	The set of closed points of the real spectrum
$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$	The Archimedean spectrum of $V(\mathbb{K})$
$\overline{\mathbb{L}}^r$	The real closure of the ordered field $\mathbb{L}$
$\mathbb{F}, \mathbb{K}$	Real closed fields
$\hat{\mathbb{F}}$	The valuation completion of $\mathbb{F}$
$\mathcal{O}$	The valuation ring of a valued field
$\mathcal{J}$	The maximal ideal of $\mathcal{O}$
$\mathbb{F}_{\mathcal{O}}$	The quotient field $\mathcal{O}/\mathcal{J}$
$I(S)$	The ideal of polynomials vanishing on a set S
$\mathbb{F}[S]$	The coordinate ring of the $\mathbb{F}$-extension of S
$S(\mathbb{F})$	The set of $\mathbb{F}$-points of a semialgebraic set S
$\widetilde{S(\mathbb{K})}$	The constructible set associated with the semialgebraic set $S(\mathbb{K})$
E	A subspace of $(V \times W)(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ given as a union of constructible sets
$\tilde{U}(a_1, \ldots, a_p)$	A basic open set in the spectral topology
V^{an}	The Berkovich analytification of V
Asymptotic and Ultrafilter Constructions	
$\mathfrak{u}$	A non-principal ultrafilter
$X^{\mathfrak{u}}$	The ultralimit metric space
$T^{\mathfrak{u}}$	The asymptotic cone with ultrafilter $\mathfrak{u}$
$\mathrm{Cone}^{\mathfrak{u}}(X, (\lambda_k), (*_k))$	The asymptotic cone
$\mathbb{K}^{\mathfrak{u}}$	The hyper $\mathbb{K}$-field
$\mathbb{R}^{\mathfrak{u}}_{\mu}$	The Robinson field
$\phi^{\mathfrak{u}}_{\mu}$	The $(\mathfrak{u}, \mu)$-limit representation of ϕ
Trees, Orientations, and Germs	
T_ϕ	The minimal invariant subtree associated with ϕ
$\mathcal{M}$	The set of minimal vectors
$\mathcal{M}_\Gamma(\mathbb{R})$	The set of minimal vectors for the group Γ
$\partial_\infty X$	The visual boundary of a space X
$\mathcal{G}_X(P)$	The germs of oriented segments at a point P in a space X
$\mathrm{Or}(T)$	The set of orientations on a tree T
$\mathrm{Trip}(T)$	The set of germs of tripods in a tree T
$\mathcal{T}'$	The space of minimal Γ-actions on oriented $\mathbb{R}$-trees

Notation	Description
$\mathcal{T}^o$	The subspace of $\mathcal{T}'$ consisting of minimal actions with an additional condition
Push	The map from orientations to cyclic orders on the boundary

Abstract

The main part of this thesis is based on the papers [Jae24] and [Jae25], which study compactifications of character varieties $\Xi(\Gamma, G(\mathbb{R}))$ and the geometric spaces on which Γ acts via the associated representations, where Γ is a finitely generated group and $G(\mathbb{R})$ is either $\mathrm{PSL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R})$. A strong emphasis is placed on the *real spectrum compactification* $\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, from which we derive two natural Γ-actions on geometric spaces: one on oriented $\mathbb{R}$-trees, and another on the Archimedean spectrum of a space equipped with a $G(\mathbb{R})$-action.

First, we construct *universal geometric spaces* over $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ that provide geometric interpretations of boundary points in terms of Γ-actions. For an algebraic set $Y(\mathbb{R})$ on which $\mathrm{SL}_n(\mathbb{R})$ acts by algebraic automorphisms (such as $\mathbb{P}^{n-1}(\mathbb{R})$ or an algebraic cover of the symmetric space of $\mathrm{SL}_n(\mathbb{R})$), the projection map $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R})) \times Y(\mathbb{R}) \to \Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$ extends to a Γ-equivariant continuous surjection $(\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R})) \times Y(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. The fibers of this extended map, which encode the limiting behavior of Γ-actions, are homeomorphic to the Archimedean spectrum of $Y(\mathbb{F})$ for some suitably chosen real closed field $\mathbb{F}$, and form locally compact subsets of $Y(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. The Archimedean spectrum is naturally homeomorphic to the real analytification, and we use this identification to compute the image of the fibers in their Berkovich analytification. In the case $Y(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$, the image is a $\mathbb{R}$-subtree.

Second, we associate to each element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ an oriented $\mathbb{R}$-tree. This construction allows us to interpret boundary points as Γ-actions by orientation-preserving isometries on oriented $\mathbb{R}$-trees, leading to a continuous surjection from $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ to the oriented Gromov equivariant compactification of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. We justify continuity by describing how such Γ-actions arise as limits of actions on the oriented hyperbolic plane, via asymptotic cones endowed with an ultralimit orientation.

Finally the last section presents the collaborative work [ADRFJ24]. It gives an example of a homogeneous $\mathbb{R}$-tree with complete segments that is not metrically complete. This example has implications for the study of degenerations of representations, as it shows that even seemingly well-behaved trees can exhibit unexpected metric properties.

Résumé

La majeure partie de cette thèse repose sur les articles [Jae24] et [Jae25], qui étudient des compactifications des variétés de caractères $\Xi(\Gamma, G(\mathbb{R}))$ ainsi que des espaces géométriques sur lesquels Γ agit via les représentations associées, où Γ est un groupe de type fini et $G(\mathbb{R})$ désigne soit $\mathrm{PSL}_n(\mathbb{R})$, soit $\mathrm{SL}_n(\mathbb{R})$. Une attention particulière est portée à la compactification par le spectre réel $\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, à partir de laquelle on obtient deux actions naturelles de Γ sur des espaces géométriques : l'une sur des $\mathbb{R}$-arbres orientés, et l'autre sur le spectre archimédien d'un espace muni d'une action de $G(\mathbb{R})$.

Dans un premier temps, nous construisons des espaces géométriques universels au-dessus de $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, qui fournissent une interprétation géométrique des points à l'infini en termes d'actions de Γ. Pour un ensemble algébrique $Y(\mathbb{R})$ sur lequel $\mathrm{SL}_n(\mathbb{R})$ agit par automorphismes algébriques (comme $\mathbb{P}^{n-1}(\mathbb{R})$ ou un revêtement algébrique de l'espace symétrique de $\mathrm{SL}_n(\mathbb{R})$), l'application de projection $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R})) \times Y(\mathbb{R}) \to \Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$ s'étend en une surjection continue Γ-équivariante $(\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R})) \times Y(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. Les fibres de cette application, qui encodent le comportement limite des actions de Γ, sont homéomorphes au spectre archimédien de $Y(\mathbb{F})$ pour un corps réel clos bien choisi $\mathbb{F}$, et forment des sous-ensembles localement compacts de $Y(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Le spectre archimédien est naturellement homéomorphe à l'analytification réelle, identification que nous utilisons pour calculer l'image des fibres dans leur analytification au sens de Berkovich. Dans le cas $Y(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$, cette image est un $\mathbb{R}$-arbre.

Dans un second temps, nous associons à chaque élément de $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ un $\mathbb{R}$-arbre orienté. Cette construction permet d'interpréter les points à l'infini comme des actions de Γ par isométries préservant l'orientation sur des $\mathbb{R}$-arbres orientés, conduisant à une surjection continue de $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ vers la compactification de Gromov équivariante orientée de $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. La continuité de cette application est justifiée en décrivant comment de telles actions de Γ apparaissent comme limites d'actions sur le plan hyperbolique orienté, via des cônes asymptotiques munis d'une orientation ultralimite.

Enfin, la dernière section présente le travail collaboratif [ADRFJ24]. On y donne un exemple de $\mathbb{R}$-arbre homogène qui n'est pas métriquement complet. Cet

exemple a des implications pour l'étude des dégénérescences de représentations, puisqu'il montre que même des $\mathbb{R}$-arbres apparemment sans pathologie peuvent présenter des propriétés métriques inattendues.

Chapter 1

Introduction

1.1 Character varieties and their compactifications: from rank one to higher rank

This section provides historical context and background for the research carried out in this thesis. We give a non-exhaustive account of the construction of character varieties, emphasizing their connections to various areas of mathematics. Particular attention is given to the role of compactifications, both in classical and higher rank settings, as they are central to the questions addressed in this work. We conclude each subsection by stating the associated main results of the thesis, which will be developed in more detail in the subsequent sections of the introduction.

1.1.1 Character varieties: general construction and rank one compactifications

A central theme in modern geometry is the study of geometric structures on manifolds via their transformation groups. This perspective examines a manifold M equipped with local coordinate charts valued in a geometric model space X, where the transition functions lie in a group G acting on X by diffeomorphisms. This defines local geometric data—such as distances and angles—on M in a consistent way. Within this framework, the area of *Teichmüller theory* studies the case of $M = S$ being a topological surface. Building on ideas going back to Riemann [Rie04] and developed implicitly by Poincaré and Fricke–Klein [Poi84, Kle83, FK65], Teichmüller revolutionized the study of the interplay between conformal, hyperbolic, and algebraic structures on surfaces [Grö28, Tei44]. Central to this interactions is the space of marked hyperbolic structures on a surface S (called Teichmüller space $T(S)$ [BB93]), which plays a fundamental role in low-dimensional topology and geometry [JP13]. A modern viewpoint, introduced by Thurston, identifies $T(S)$

with:

$$T(S) = \{\phi \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R})) \mid \phi \text{ is faithful and discrete}\} / \mathrm{PSL}_2(\mathbb{R}),$$

where $\pi_1(S)$ is the fundamental group of S, equivalence is up to postconjugation, and $T(S)$ is equipped with the topology of pointwise convergence [Thu79]. The mapping class group $\mathrm{MCG}(S)$, the group of isotopy classes of automorphisms of S, acts naturally on $\pi_1(S)$ and consequently on $T(S)$. The quotient $T(S)/\mathrm{MCG}(S)$ can then be identified with the moduli space of all hyperbolic structures on S. This action enables a dynamical and group-theoretic study of the geometry of S by encoding the symmetries of the surface. To study degenerations of hyperbolic structures, Thurston introduced a compactification of $T(S)$ via the length spectrum

$$T(S)^{\mathrm{LS}},$$

whose boundary points correspond to projective measured laminations [Thu79, Thu88]. The MCG(S)-action extends continuously to $T(S)^{\mathrm{LS}}$, which Thurston used to classify elements of $\mathrm{MCG}(S)$, noting that topologically $T(S)^{\mathrm{LS}}$ is homeomorphic to a closed ball. This compactification is fundamental to Thurston's work, and many others such as Otal and Kapovich's proofs of Thurston's hyperbolization theorem for 3-manifolds, which gives sufficient conditions for a 3-manifold to be hyperbolic [Ota01, Kap09].

Building on these foundations, character varieties provide a natural extension of $T(S)$ and establish important connections with algebraic geometry [Hit92, KM98]. For a finitely generated group Γ and a Lie group $G(\mathbb{R}) = \mathrm{PSL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R})$, the character variety is the space

$$\Xi(\Gamma, G(\mathbb{R})) := \{\phi \in \mathrm{Hom}(\Gamma, G(\mathbb{R})) \mid \phi \text{ is reductive}\} / G(\mathbb{R}),$$

equipped with the topology of pointwise convergence, where $G(\mathbb{R})$ acts by postconjugation. Notably, in rank one ($n = 2$), the space $T(S)$ forms one connected component of $\Xi(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$. Character varieties provide a geometric point of view for studying representations of discrete groups in $G(\mathbb{R})$ and unifies several key themes:

- they support Goldman and Mirzakhani's work on the symplectic and hyperbolic geometry of moduli spaces [Gol84, Mir07];
- they are central to Thurston's and Culler–Morgan–Shalen's theory linking character varieties to the topology of manifolds and geometric group theory [Thu88, CS83, MS84];
- they are related to non-abelian Hodge theory [Hit87, Don87, Cor88, Sim92];

- they bridge to modern mathematical physics [FC99] and the Langlands program [BD91, DP12].

As in the case of $T(S)$, character varieties are in general not compact. Thus there exist sequences of representations that do not have a limit. To investigate their asymptotic behavior, we examine compactifications of $\Xi(\Gamma, G(\mathbb{R}))$. In the rest of this subsection, we focus on character varieties of rank one, that is when $n = 2$. Culler, Morgan, and Shalen [CS83, MS84] extended the length spectrum compactification to the algebraic set

$$\Xi(\Gamma, \mathrm{SL}_2(\mathbb{C})).$$

Using the theory of valuations, they interpret the boundary points of this compactification as Γ-actions by isometries on $\mathbb{R}$-trees. This provides tools for constructing new group actions on $\mathbb{R}$-trees and, in the case $\Gamma = \pi_1(M)$ for a 3-manifold M, for detecting essential subsurfaces. Independently, Bestvina and Paulin developed geometric constructions for $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}}$ [Bes88, Pau89]. They study the space of Γ-actions on hyperbolic n-spaces and $\mathbb{R}$-trees endowed with the Gromov equivariant topology, based on the Hausdorff distance between topological spaces (following a definition in [Gro81a]). The mapping class group action on $T(S)$ extends continuously to an

$$\mathrm{Out}(\Gamma)\text{-action on } \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}}.$$

Here $\mathrm{Out}(\Gamma)$ is the outer automorphism group of Γ, that is, automorphisms of Γ modulo those that come from conjugation by elements of Γ itself. Wolff proves in [Wol11] that $\Xi(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}}$ is a connected topological space, whereas $\Xi(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ has $4g - 3$ connected components, for a topological surface of genus $g \geq 2$ [Wol11, Corollary 1.2]. This mismatch complicates the study of the $\mathrm{MCG}(S)$-action on individual components. To avoid this degeneracy, Wolff introduced the oriented Gromov equivariant compactification

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}},$$

that retains the information of the orientation on $\mathbb{H}^2$ and the crucial fact that $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by orientation preserving isometries. This compactification is a refinement, equipped with a natural forgetful map [Wol11, Proposition 3.22]

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}},$$

which discards the orientation data. It allows the Euler class to extend continuously to the boundary. In particular, it preserves the connected components, as the Euler class distinguishes them (see Chapter 5).

Remark. *While the convergence of representations to boundary elements in this compactification are well understood, topological properties of* $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$*, such as the homology groups, are unknown.*

An additional perspective arises from real algebraic geometry. Brumfiel recognized that the algebraic approach developed by Morgan and Shalen to study $\Xi(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ could be extended to $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, using that $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is a semialgebraic set [Bru88a]. This leads to the natural compactification

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$$

of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, constructed using the real spectrum of a semialgebraic set, see Chapter 2 and 3. It preserves the connected components, the $\mathrm{Out}(\Gamma)$-action extends continuously to the boundary, and it satisfies a semialgebraic version of Brouwer's fixed point theorem, see Chapter 3 and [Bru88b]. Moreover, using the strength of the Transfer Principle (Theorem 2.2.2), Brumfiel shows:

Theorem ([Bru88a, Proposition 7.2])**.** *There exists a continuous surjection*

$$T(S)^{\mathrm{RSp}}_{\mathrm{cl}} \to T(S)^{\mathrm{LS}},$$

which is $\mathrm{Out}(\pi_1(S))$*-equivariant.*

In this thesis, we build on Brumfiel's construction to refine the comparison between compactifications. Using asymptotic cones and ultralimits of orientations, we establish the following result:

Theorem (Theorem 5.3.4)**.** *There exists a continuous surjection*

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}},$$

which is $\mathrm{Out}(\Gamma)$*-equivariant.*

In particular, the map $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}}$ from above factors through the surjection $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ constructed in our theorem. Figure 1.1 summarizes the relationships between the compactifications discussed, where Γ is finitely generated and the inclusion $T(S)^{\mathrm{LS}} \hookrightarrow \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{LS}}$ holds only when $\Gamma = \pi_1(S)$.

Remark. *One possible application that we intend to pursue in the future is to use the continuous surjection* $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$*, and the knowledge about the homology groups of* $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ *([BCR13, Section 11]), to compute the homology groups of* $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$*.*

Several other compactifications of character varieties have been proposed—for example, the Gardiner–Masur compactification [GM91] and the wonderful compactification [BLR19]. In this thesis, we focus on the oriented Gromov equivariant and the real spectrum compactifications.

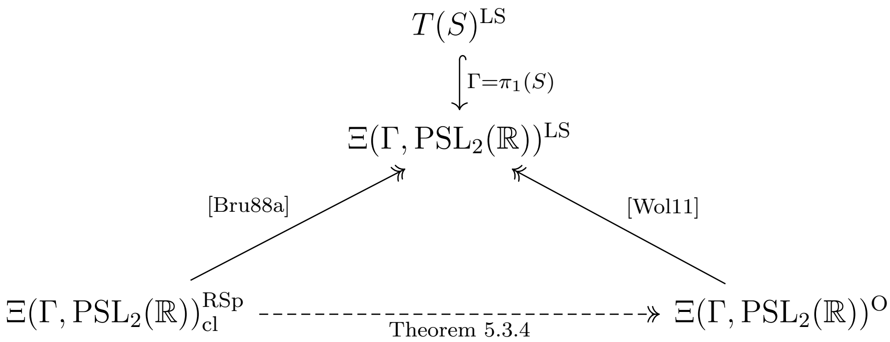


Figure 1.1: Relations between compactifications of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ and $T(S)$.

1.1.2 Geometric interpretation of higher rank compactifications

When $G(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{PSL}_n(\mathbb{R})$ for $n > 2$, the associated character variety $\Xi(\Gamma, G(\mathbb{R}))$ is said to be of higher rank. This space generalizes the rank one theory of Teichmüller space and lies at the heart of several active areas of research, including:

- Higher Teichmüller spaces, a term introduced by Fock and Goncharov [FG06] to describe certain connected components of character varieties, include examples such as Hitchin components [Hit92, Lab04] and maximal representations [BIW10, FG06]. We refer to [Wie18] for a survey of the theory.

- Anosov representations, initiated by Labourie [Lab04] and further studied by Guichard and Wienhard [GW12]. These representations form open subsets of character varieties, are stable under deformation, and offer powerful dynamical tools to study representations, see [Kas18];

- Convex cocompact representations in the sense of Danciger–Guéritaud–Kassel [DGK23] extend the classical notion of convex cocompactness from rank one to higher rank, offering new tools for studying geometric structures on manifolds.

In general, the study of $\Xi(\Gamma, G(\mathbb{R}))$ plays a central role in modern geometry, raising questions, for example, about the dynamics of mapping class group actions (see [Gol06] and [Pap14]) or about their connections to geometric structures on manifolds (see [CG93, CTT19, GW08]).

In this thesis we study a complementary question by analyzing the asymptotic behavior of representations. This naturally leads to studying compactifications of higher rank character varieties, generalizing the rank 1 case discussed in the previous section. A natural generalization of the length spectrum compactification is the Weyl chamber length compactification

$$\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{WL}}$$

of $\Xi(\Gamma, G(\mathbb{R}))$, introduced by Parreau [Par12]. This construction analyzes sequences of representations via their induced actions on the symmetric space associated to $G(\mathbb{R})$ and on affine buildings, which arise as rescaled limits of symmetric spaces. Like the length spectrum compactification, this approach interprets degenerations of representations geometrically. However, it does not preserve the connected components of $\Xi(\Gamma, G(\mathbb{R}))$, complicating the analysis of the mapping class group action on individual components.

Remark. *Although beyond the scope of this thesis, a natural question is whether the oriented Gromov equivariant compactification extends to higher rank, by endowing affine buildings with an orientation. We plan to investigate whether such a compactification preserves the connected components of* $\Xi(\Gamma, G(\mathbb{R}))$.

To overcome these limitations, Burger, Iozzi, Parreau, and Pozzetti study the real spectrum compactification, extending Brumfiel's ideas to higher rank [BIPP23]. This approach provides a refinement of the Weyl chamber length compactification by offering a more algebraic perspective:

Theorem ([BIPP23, Theorem 8.2])**.** *There exists a continuous surjection*

$$\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, G(\mathbb{R}))^{\mathrm{WL}},$$

which is $\mathrm{Out}(\Gamma)$*-equivariant.*

A boundary element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ is represented by an equivalence class $[\phi, \mathbb{F}]$, where $\phi\colon \Gamma \to G(\mathbb{F})$ is a reductive representation and $\mathbb{F}$ is a suitably chosen non-Archimedean real closed field satisfying some minimality conditions [Bru88a, BIPP23], see Section 5.2. In this way, the compactification admits a purely representation-theoretic description of its boundary points. While such an algebraic characterization exists, a central challenge raised by Wienhard at the ICM [Wie18] is to develop a geometric interpretation of the boundary points of this compactification. This thesis aims to progress toward that goal.

In [BIPP23], the authors associate to each $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, G(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ a Γ-action by isometries without fixed points on an affine building (see also [KT02, App24]) giving a first geometric interpretation of boundary elements. A key feature of this compactification is that each boundary element (with some property) defines a geodesic current—a higher rank analogue of measured laminations—on the quotient of the symmetric space of $G(\mathbb{R})$ by the Γ-action. Yet, the modular interpretation of this compactification remains incomplete (that is, we lack a full understanding of what geometric structures the boundary points represent). This thesis takes a step towards this goal by analyzing an algebraic subset of minimal vectors

$$\mathcal{M}_\Gamma(\mathbb{R}) \subset \mathrm{Hom}_{\mathrm{red}}(\Gamma, \mathrm{SL}_n(\mathbb{R})),$$

which is a cover of $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$. More precisely $\mathcal{M}_\Gamma(\mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ is semialgebraically isomorphic to $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$. We examine the geometric actions induced by $\mathcal{M}_\Gamma(\mathbb{R})$ on $\mathbb{P}^{n-1}(\mathbb{R})$, the $(n-1)$-dimensional projective space (or on an algebraic cover $\widehat{\mathcal{P}^1(n, \mathbb{R})}$ of the symmetric space $\mathcal{P}^1(n, \mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$, see Subsection 4.1.2 and Subsection 4.1.3). The algebraic group $\mathrm{SL}_n(\mathbb{R})$ acts on $\mathbb{P}^{n-1}(\mathbb{R})$ by algebraic automorphisms so that any element of $\mathcal{M}_\Gamma(\mathbb{R})$ induces a Γ-action on $\mathbb{P}^{n-1}(\mathbb{R})$. We call

$$\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$$

the *universal projective space* over $\mathcal{M}_\Gamma(\mathbb{R})$. It is an algebraic set that parametrizes all Γ-actions induced by elements of $\mathcal{M}_\Gamma(\mathbb{R})$ on $\mathbb{P}^{n-1}(\mathbb{R})$, as illustrated in Figure 1.2 (see Subsection 4.1.1 for details).

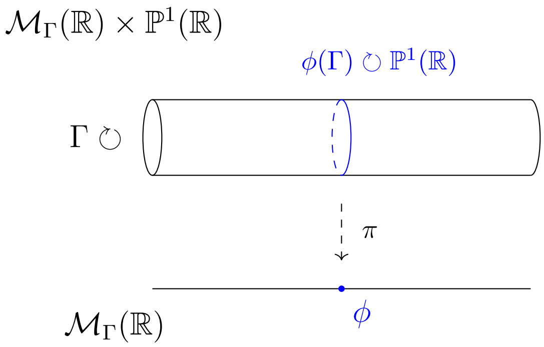


Figure 1.2: Schematic picture of the universal projective line ($n = 2$) over $\mathcal{M}_\Gamma(\mathbb{R})$, with the fiber over ϕ in blue.

A natural question is whether such a geometric space exists over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ and whether it contains information about the Γ-actions induced by elements in the boundary of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. We provide this universal geometric space by considering

$$(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$$

and analyzing its geometric and dynamical properties. The *Archimedean spectrum* $\mathbb{P}^{n-1}(\mathbb{F})^{\mathrm{RSp}}_{\mathrm{Arch}}$ of $\mathbb{P}^{n-1}(\mathbb{F})$ is a dense and locally compact open subset of $\mathbb{P}^{n-1}(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ that contains $\mathbb{P}^{n-1}(\mathbb{F})$ with desirable geometric properties. A detailed study of this space is given in Section 3. Our main result, illustrated in Figure 1.3, can be summarized as follows:

Theorem (Theorem 4.1.8). *The projection map* $\pi\colon \mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}) \to \mathcal{M}_\Gamma(\mathbb{R})$ *extends to a continuous map*

$$\pi_{\text{cl}}^{\text{RSp}} : (\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))_{\text{cl}}^{\text{RSp}} \to \mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}},$$

which is surjective, Γ*-equivariant, with the fiber over* $(\rho, \mathbb{F}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}}$ *homeomorphic to* $\mathbb{P}^{n-1}(\mathbb{F})_{\text{Arch}}^{\text{RSp}}$.

Using this theorem, we analyze $\mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}}$ through the study of its fibers in $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))_{\text{cl}}^{\text{RSp}}$. We understand the Γ-actions on $\mathbb{P}^{n-1}(\mathbb{F})_{\text{Arch}}^{\text{RSp}}$ induced by representations in $\mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}}$ by comparing them to Γ-actions on $\mathbb{P}^{n-1}(\mathbb{R})$ coming from representations in $\mathcal{M}_\Gamma(\mathbb{R})$. In the following, an element of the fiber $(\pi_{\text{cl}}^{\text{RSp}})^{-1}(\rho, \mathbb{K})$ is written as a triple $(\rho, A, \mathbb{F})$ where $A \in \mathbb{P}^{n-1}(\mathbb{F})_{\text{cl}}^{\text{RSp}}$ encodes the decomposition into factors, see Remark 4.1.1.

Corollary (Corollary 4.1.10). *Let* $(\rho_n, \mathbb{K}_n) \subset \mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}}$ *be a sequence converging to* $(\rho, \mathbb{K}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\text{cl}}^{\text{RSp}}$. *For every element of the fiber* $(\rho, A, \mathbb{F}) \in (\pi_{\text{cl}}^{\text{RSp}})^{-1}(\rho, \mathbb{K})$, *there exists a sequence* $(\rho_n, A_n, \mathbb{F}_n) \in (\pi_{\text{cl}}^{\text{RSp}})^{-1}(\rho_n, \mathbb{K}_n)$ *such that* $(\rho_n, A_n, \mathbb{K}_n)$ *converges to* $(\rho, A, \mathbb{F})$ *in the spectral topology.*

Both results hold if $\mathbb{P}^{n-1}(\mathbb{R})$ is replaced by the algebraic cover $\widehat{\mathcal{P}^1(n, \mathbb{R})}$ of the symmetric space $\mathcal{P}^1(n, \mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$; see Subsection 4.1.2 and 4.1.3 for the precise statements.

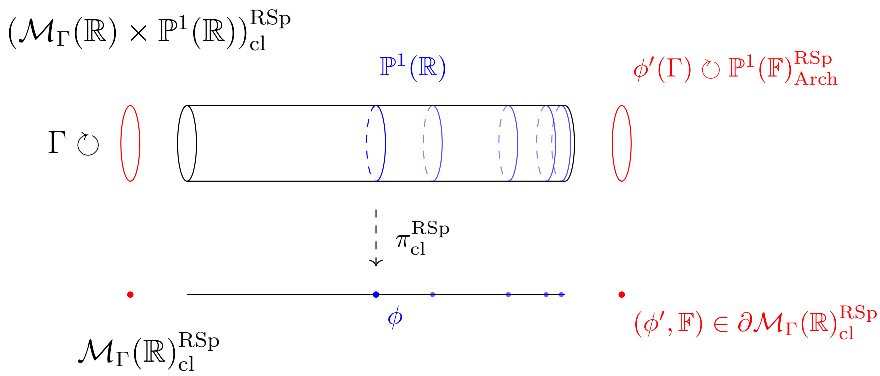


Figure 1.3: Schematic extension of the universal projective line over $\mathcal{M}_\Gamma(\mathbb{R})$ to the real spectrum, illustrating the convergence of interior fibers (in blue) to the fiber over the boundary point (in red).

These results establish a connection between the study of degenerations of representations in character varieties and techniques from analytic geometry. Building on the natural homeomorphism between $\mathbb{P}^{n-1}(\mathbb{F})_{\text{Arch}}^{\text{RSp}}$ and the real analytification of $\mathbb{P}^{n-1}(\mathbb{F})$ [JSY22], we construct a proper continuous map

$$\psi\colon \mathbb{P}^{n-1}(\mathbb{F})_{\text{Arch}}^{\text{RSp}} \to \mathbb{P}^{n-1}(\mathbb{F})^{\text{an}},$$

where $\mathbb{P}^{n-1}(\mathbb{F})^{\mathrm{an}}$ is the Berkovich analytification of $\mathbb{P}^{n-1}(\mathbb{F})$—a connected, locally compact space containing $\mathbb{P}^{n-1}(\mathbb{F})$ and endowed with an analytic structure. Theorem 4.2.14 provides a description of the image of ψ; in the special case of $\mathbb{P}^1(\mathbb{F})$, this yields the following result:

Corollary (Corollary 4.2.32)**.** *Let $\mathbb{F}$ be a non-Archimedean real closed field endowed with an absolute value. The image under ψ of $\mathbb{P}^1(\mathbb{F})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{F})$ inside $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{F})$ is a closed, $\mathrm{PSL}_2(\mathbb{F})$-invariant $\mathbb{R}$-subtree.*

The natural Γ-action on $\mathbb{P}^{n-1}(\mathbb{F})^{\mathrm{RSp}}_{\mathrm{Arch}}$, induced by an element of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, provides an analytic perspective to the study of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ with the aim of identifying dynamical properties of boundary representations. The following subsections outline these constructions and the main methods used in this article.

1.2 Archimedean spectrum and real spectrum compactification

This section first motivates and gives a more detailed overview of the real spectrum compactification. It then presents our main results on the Archimedean spectrum, which yields a locally compact topological space to study questions from non-Archimedean geometry.

1.2.1 Real spectrum compactification

Introduced by Coste and Roy in [CC80, CR82], the real spectrum compactification applies to semialgebraic sets and preserves the structure determined by the equalities and inequalities that define them. In other words, the semialgebraic properties that characterize interior points extend naturally to points at infinity. To simplify technical details, we present the definitions for algebraic sets and refer to Section 3.1 for further details. A field $\mathbb{F}$ is a *real field* if it is endowed with an order compatible with its field operations. Moreover, $\mathbb{F}$ is *real closed* if it has no proper algebraic ordered field extension, see Section 2.2. As an example, one may consider $\mathbb{F} = \mathbb{R}$ in what follows. Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields and $V \subset \mathbb{L}^n$ an algebraic set. Denote by $\mathbb{K}[V]$ the *coordinate ring* of the $\mathbb{K}$-extension $V(\mathbb{K})$ of V, and, by abuse of notation, we use the same symbol for the $\mathbb{K}$-extension of V and the set of $\mathbb{K}$-points $V(\mathbb{K})$. We endow $V(\mathbb{K})$ with its *Euclidean topology*, see Section 2.2.

Definition ([BCR13, Proposition 7.1.2])**.** Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields. The *real spectrum* of the $\mathbb{K}$-points of an algebraic set $V \subset \mathbb{L}^n$ is

$$V(\mathbb{K})^{\mathrm{RSp}} := \left\{ (\rho, \mathbb{F}) \,\middle|\, \begin{array}{c} \mathbb{F} \text{ a real closed field,} \\ \rho \colon \mathbb{K}[V] \to \mathbb{F} \ \text{ a ring morphism} \end{array} \right\} / \sim,$$

where $\sim$ is the smallest equivalence relation such that $(\rho_1, \mathbb{F}_1) \sim (\rho_2, \mathbb{F}_2)$ if there is an ordered field homomorphism $\varphi\colon \mathbb{F}_1 \to \mathbb{F}_2$ for which $\rho_2 = \varphi \circ \rho_1$.

The real spectrum is endowed with a natural topology called the *spectral topology*, see Subsection 3.1. It offers a real counterpart to the spectrum in algebraic geometry and serves as a central object of study in real algebraic geometry [BCR13]. The real spectrum defines a functor from the category of real algebraic sets to the category of compact topological spaces, which associates to every algebraic map $\pi\colon V(\mathbb{K}) \to W(\mathbb{K})$ between algebraic sets a continuous map

$$\pi^{\mathrm{RSp}}\colon V(\mathbb{K})^{\mathrm{RSp}} \to W(\mathbb{K})^{\mathrm{RSp}}.$$

With its spectral topology $V(\mathbb{K})^{\mathrm{RSp}}$ is not Hausdorff in general. However, the subspace of closed points $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} \subset V(\mathbb{K})^{\mathrm{RSp}}$ is a compact Hausdorff space, which we refer to as the *real spectrum compactification* of $V(\mathbb{K})$. Note that given an algebraic map $\pi\colon V(\mathbb{K}) \to W(\mathbb{K})$ between algebraic sets $V(\mathbb{K})$ and $W(\mathbb{K})$, the induced continuous map π^{RSp} does not necessarily map closed points to closed points.

Theorem (Theorem 3.2.4)**.** *Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be algebraic sets. If $\pi\colon V(\mathbb{R}) \to W(\mathbb{R})$ is a proper algebraic map, then the image of $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$, the restriction to the closed points of the induced map π^{RSp}, is $W(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. That is, we have a continuous surjective map*

$$\pi^{\mathrm{RSp}}_{\mathrm{cl}}\colon V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \twoheadrightarrow W(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Brumfiel deduces from [CS83] that $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is a semialgebraic set, and Burger, Iozzi, Parreau, and Pozzetti extend this result to $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$ via Richardson–Slowdowy theory [RS90], as described in [Bru88a, BIPP23]. These papers examine real spectrum compactifications with a focus on their topology and their related objects at infinity, with the aim to better understand the degenerations of elements in the representation space. The work [BIPP23] initiates a program dedicated at studying $\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ and $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, with the goal of analyzing representations of finitely generated groups in higher rank reductive Lie groups. In particular, they characterize elements of $\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ as equivalence classes of representations

$$\phi\colon \Gamma \to \mathrm{SL}_n(\mathbb{F}),$$

for suitable real closed fields $\mathbb{F}$ [BIPP23, Proposition 6.3]. Such a representation induces a Γ-action by isometries on an affine building (a geometric space with a rich combinatorial structure [App24, Theorem 8.1]) that has no fixed point [BIPP23, Proposition 6.4]. This construction has implications for the study of Anosov, maximal, Hitchin, and Θ-positive representations [BIPP23, Examples 6.19]. This method of studying boundary points of $\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ naturally leads to the examination of non-Archimedean geometry. For a non-Archimedean real closed field $\mathbb{F}$, the projective space $\mathbb{P}^{n-1}(\mathbb{F})$ is totally disconnected, not locally compact, and not open in $\mathbb{P}^{n-1}(\mathbb{F})^{\mathrm{RSp}}_{\mathrm{cl}}$. To develop a deeper understanding of non-Archimedean geometry, we use the Archimedean spectrum.

1.2.2 Non-Archimedean geometry and Archimedean spectrum

Let $\mathbb{L}$ be an ordered field. An ordered algebraic extension $\mathbb{F}$ is the *real closure* of $\mathbb{L}$ if $\mathbb{F}$ is a real closed field and the ordering on $\mathbb{L}$ extends to the ordering of $\mathbb{F}$. The real closure, denoted by $\overline{\mathbb{L}}^r$, always exists and is unique up to order-preserving isomorphism over $\mathbb{L}$, see Subsection 2.2. In addition, if $R_1 \subset R_2$ are two subrings of an ordered field, then the subring R_2 is *Archimedean* over R_1 if every element of R_2 is bounded above by some element of R_1.

Definition. Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields and $V \subset \mathbb{L}^n$ an algebraic set. The *Archimedean spectrum* of $V(\mathbb{K})$ is the following subspace of $V(\mathbb{K})^{\mathrm{RSp}}$

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} := \left\{ (\rho, \mathbb{F}) \in V(\mathbb{K})^{\mathrm{RSp}} \,\middle|\, \begin{array}{c} \mathbb{F} = \overline{\mathrm{Frac}(\rho(\mathbb{K}[V]))}^r, \\ \mathbb{F} \text{ is Archimedean over } \mathbb{K} \end{array} \right\},$$

where $\mathrm{Frac}(\rho(\mathbb{K}[V]))$ is the fraction field of $\rho(\mathbb{K}[V])$.

The Archimedean spectrum defines a functor from the category of algebraic sets to the category of topological spaces, see Subsection 3.2. An element b in a real closed field $\mathbb{K}$ is a *big element* if for every $a \in \mathbb{K}$ there exists $n \in \mathbb{N}$ with $a \leq b^n$.

Theorem (Theorem 3.4.3)**.** *Let $\mathbb{L}$ be a real closed field, $\mathbb{L} \subset \mathbb{K}$ a real closed field with a big element b, and $V \subset \mathbb{L}^n$ an algebraic set. The Archimedean spectrum of $V(\mathbb{K})$ is an open subset of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ which is a countable union of compact subsets of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. In particular, the space $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is σ-compact and locally compact.*

The Archimedean spectrum thus provides a well-behaved topological space for studying group actions on non-Archimedean algebraic sets. These favorable properties are further supported by its identification with real analytifications, as discussed in [JSY22].

1.3 Universal geometric spaces and Berkovich analytification

This section introduces the main content of [Jae25]. We give more details about the construction of universal geometric spaces over the real spectrum compactification and examine their fibers under natural projection maps. We then explore the connection between the Archimedean spectrum and the Berkovich analytification.

1.3.1 Universal geometric space over the real spectrum compactification

To better understand the advantages of universal geometric spaces over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, consider the two following constructions (where $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ is an algebraic cover of the symmetric space $\mathcal{P}^1(n,\mathbb{R})$ associated to $\mathrm{SL}_n(\mathbb{R})$). First, we define a topological space that contains $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$:

$$E_1 \subset \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}\right)^{\mathrm{RSp}}_{\mathrm{cl}}$$

which is a countable union of specifically chosen open subsets, see Subsection 4.1.2. These open sets are constructed to yield a σ-compact space equipped with a natural Γ-action by homeomorphisms, induced by the $\mathrm{SL}_n(\mathbb{R})$ action on $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ by algebraic isomorphisms. Second, define

$$E_2 := \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}}.$$

The algebraic group $\mathrm{SL}_n(\mathbb{R})$ acts on $\mathbb{P}^{n-1}(\mathbb{R})$ by algebraic isomorphisms. So, Γ acts on the universal projective space $\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ over $\mathcal{M}_\Gamma(\mathbb{R})$ via

$$\gamma.\,(\phi, x) = (\phi, \phi(\gamma)x) \quad \forall \gamma \in \Gamma.$$

Both spaces share important structural properties. For either construction, we consider the natural projection maps

$$\pi_1 \colon \mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \to \mathcal{M}_\Gamma(\mathbb{R}),$$
$$\pi_2 \colon \mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}) \to \mathcal{M}_\Gamma(\mathbb{R}),$$

which are algebraic, surjective, Γ-equivariant for the trivial action on $\mathcal{M}_\Gamma(\mathbb{R})$, and open. For each element $\phi \in \mathcal{M}_\Gamma(\mathbb{R})$, Γ acts naturally via ϕ on the fiber $\pi_i^{-1}(\phi)$ which is homeomorphic to either $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ for $i = 1$ or $\mathbb{P}^{n-1}(\mathbb{R})$ for $i = 2$, see Subsection 4.1.1. To capture the degeneration of Γ-actions on these geometric spaces induced by representations in $\mathcal{M}_\Gamma(\mathbb{R})$, we extend these universal geometric spaces over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Our candidate for such an extension is E_i. However

$$(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \ncong \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \times \mathbb{P}^{n-1}(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$$

in general (and similarly when replacing $\mathbb{P}^{n-1}(\mathbb{R})$ by $\widehat{\mathcal{P}^1(n,\mathbb{R})}$), making the study of the fibers of π_i^{RSp} less straightforward. We refine the study of the real spectrum compactification of the product of algebraic sets. We use the functorial properties of the real spectrum and topological properties of E_i (for example, the compactness of $\mathbb{P}^{n-1}(\mathbb{R})$) to deduce that $\pi_i^{\mathrm{RSp}}|_{E_i}$ takes values in $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$. It leads to the central result of this thesis: a characterization of $\partial\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ as Γ-actions on the Archimedean spectrum, which in turn are interpreted as actions on real analytifications.

Theorem (Theorem 4.1.8 and Theorem 4.1.19). *For $i = 1, 2$, the projection map π_i induces a continuous map*

$$\pi_i^{\mathrm{RSp}}|_{E_i} : E_i \to \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}},$$

which is surjective, Γ-equivariant, with the fiber over $(\rho, \mathbb{F}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ homeomorphic to $\widehat{\mathcal{P}^1(n,\mathbb{F})}_{\mathrm{Arch}}^{\mathrm{RSp}}$ for $i = 1$ and $\mathbb{P}^{n-1}(\mathbb{F})_{\mathrm{Arch}}^{\mathrm{RSp}}$ for $i = 2$.

To better understand the arrangement of the fibers of $\pi_i^{\mathrm{RSp}}|_{E_i}$ over the space $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$, a finer analysis is required. By [CR82], π_i^{RSp} is an open map, and fibers over $(\rho, \mathbb{F}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}$ are homeomorphic to $\widehat{\mathcal{P}^1(n,\mathbb{F})}^{\mathrm{RSp}}$ for $i = 1$ and $\mathbb{P}^{n-1}(\mathbb{F})^{\mathrm{RSp}}$ for $i = 2$. In our setting, we refine this description.

Theorem (Theorem 4.1.5). *If π_i is the projection map defined above, then*

$$\pi_i^{\mathrm{RSp}}|_{E_i} : E_i \to \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$$

is open.

We deduce structural behavior of the fibers in E_i, which strengthens the accessibility results presented in [BIPP23, Subsection 3.2] for our specific context. In the following, an element of the fiber $(\pi_{\mathrm{cl}}^{\mathrm{RSp}})^{-1}(\rho, \mathbb{K})$ is written as a triple $(\rho, A, \mathbb{F})$ where $A \in \mathbb{P}^{n-1}(\mathbb{F})_{\mathrm{cl}}^{\mathrm{RSp}}$ encodes the decomposition into factors, see Remark 4.1.1.

Corollary (Corollary 4.1.10 and Corollary 4.1.21). *Let $(\rho_n, \mathbb{K}_n) \subset \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ be a sequence converging to $(\rho, \mathbb{K}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$. For any $(\rho, A, \mathbb{F}) \in (\pi_{\mathrm{cl}}^{\mathrm{RSp}})^{-1}(\rho, \mathbb{K})$, there exists a sequence $(\rho_n, A_n, \mathbb{F}_n) \in (\pi_{\mathrm{cl}}^{\mathrm{RSp}})^{-1}(\rho_n, \mathbb{K}_n)$ such that $(\rho_n, A_n, \mathbb{K}_n)$ converges to $(\rho, A, \mathbb{F})$ in the spectral topology.*

It is therefore useful to gain a better understanding of the Γ-actions on the Archimedean spectrum. To this end, we use an identification of the Archimedean spectrum with analytic spaces.

1.3.2 Archimedean spectrum and Berkovich analytification

Let $\mathbb{K}$ be a non-Archimedean field $\mathbb{K}$ with an absolute value $|\cdot|_{\mathbb{K}}$. The $\mathbb{K}$-points of an algebraic set V are then totally disconnected, making the study of its analytical properties less straightforward. To address this, Berkovich introduces the *Berkovich analytification* $V(\mathbb{K})^{\mathrm{an}}$ in [Ber90], which contains $V(\mathbb{K})$ and has a richer topological structure.

A *multiplicative seminorm* on $V(\mathbb{K})$ is a map $\eta\colon \mathbb{K}[V] \to \mathbb{R}_{\geq 0}$ such that

1. $\eta(f) = |f|_{\mathbb{K}}$ for every $f \in \mathbb{K}$,
2. $\eta(fg) = \eta(f)\eta(g)$ for every $f, g \in \mathbb{K}[V]$,
3. $\eta(f+g) \leq \max\{\,\eta(f), \eta(g)\,\}$ for every $f, g \in \mathbb{K}[V]$.

Definition ([Ber90, Definition 1.5.1])**.** The *Berkovich analytification* of $V(\mathbb{K})$ is the set

$$V(\mathbb{K})^{\mathrm{an}} := \{\, \eta\colon \mathbb{K}[V] \to \mathbb{R}_{\geq 0} \,|\, \eta \text{ is a multiplicative seminorm} \,\},$$

with the coarsest topology that makes the evaluation maps on elements of $\mathbb{K}[V]$ continuous.

Remark 1.3.1. Some authors define the Berkovich analytification using bounded multiplicative seminorms on a Banach ring A, see [Ber90, Subsection 1.5]. For a $\mathbb{K}$-algebra of finite type, this boundedness condition is equivalent to requiring that η extend the absolute value on $\mathbb{K}$, so both definitions lead to the same analytification. In this thesis we adopt the latter point of view, emphasizing that the key requirement is that η extends $|\cdot|_{\mathbb{K}}$.

This space has good topological properties. For example, $V(\mathbb{K})^{\mathrm{an}}$ is locally compact, locally contractible, and contains $V(\mathbb{K})$ as an open and dense subset, see Subsection 4.2.1. Moreover, $V(\mathbb{K})$ is connected in the Zariski topology if and only if $V(\mathbb{K})^{\mathrm{an}}$ is connected. This makes the Berkovich analytification a powerful tool for studying non-Archimedean geometry using analytic techniques [BR10]. In this work, we investigate the relationship between the Archimedean spectrum and the Berkovich analytification. Specifically, using the good topological properties of the Berkovich analytification and its canonical measure [BR10, Chapter 10], we aim to study group actions on the Archimedean spectrum using tools from dynamical systems.

Theorem ([JSY22, Theorem 3.17 and Lemma 3.9])**.** *Let $\mathbb{K}$ be a real closed field with a non-trivial absolute value and $V(\mathbb{K})$ an algebraic set. There exists a canonical map ψ from $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ to $V(\mathbb{K})^{\mathrm{an}}$ which is continuous and proper.*

To establish this result, the authors of [JSY22] introduce the real analytification, a real counterpart of the Berkovich analytification, and show that it is homeomorphic to the Archimedean spectrum. We give an explicit description of the image of the Archimedean spectrum inside the Berkovich analytification:

Theorem (Theorem 4.2.14). *Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields with non-trivial absolute values and $V \subset \mathbb{L}^n$ an algebraic set. The image of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ in $V(\mathbb{K})^{\mathrm{an}}$ is*

$$\left\{ \eta \in V(\mathbb{K})^{\mathrm{an}} \,\middle|\, \eta\left(f_1^2 + \cdots + f_q^2\right) = \max_i \eta\left(f_i^2\right) \quad \forall f_1, \ldots, f_q \in \mathbb{K}[V] \right\}.$$

In the case $V(\mathbb{K}) = \mathbb{P}^1(\mathbb{K})$, the space $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ is uniquely path connected [BR10].

Theorem (Corollary 4.2.31 and Corollary 4.2.32). *Let $\mathbb{K}$ be a non-Archimedean real closed field with a non-trivial absolute value. The image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ in $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ is uniquely path connected and the image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{K})$ in $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{K})$ is a closed $\mathbb{R}$-subtree of $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ on which $\mathrm{PSL}_2(\mathbb{K})$ acts by isometries.*

Remark. *A possible direction for future research would be to investigate whether an analogous statement holds in higher rank settings. Let $\mathbb{K}$ be a complete non-Archimedean real closed field. There is a canonical $\mathrm{PGL}_n(\mathbb{K})$-equivariant map*

$$\tau : \mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{an}} \longrightarrow \overline{B},$$

where $\overline{B}$ is the seminorm compactification of the Bruhat–Tits building B of $\mathrm{PGL}_n(\mathbb{K})$ [RTW15, Subsection 2.2], see also [Wer04]. Let $\psi : \mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \to \mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{an}}$ be the canonical map from the Archimedean spectrum to the Berkovich analytification. It would be interesting to study the image of $\mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ in $\overline{B}$ via ψ and τ. In particular, one may ask whether this image lies in B, and if so, whether it is convex in the Bruhat–Tits metric.

1.4 A continuous surjection from the real spectrum to the oriented Gromov equivariant compactifications

This section introduces the main content of [Jae24]. We construct a continuous surjection from the real spectrum compactification to the oriented Gromov equivariant compactification of character varieties.

1.4.1 The oriented Gromov equivariant compactification

The study of the oriented Gromov equivariant compactification builds on Bestvina and Paulin's geometric approach to the length spectrum compactification of character varieties [Bes88, Pau89]. The asymptotic cone

$$\mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lambda_k), (*_k)\right)$$

is an $\mathbb{R}$-tree obtained from a non-principal ultrafilter $\mathfrak{u}$, a sequence of scalars (λ_k) such that $\lambda_k \to \infty$, and basepoints $(*_k)$. Let Γ be a finitely generated group. Denote by $\lg(\phi)$ the *displacement function* of a class of representations $[\phi, \mathbb{H}^2] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ and by $\phi^{\mathfrak{u}} := \lim_{\mathfrak{u}} \phi_k$ the *ultralimit* of a sequence of representations $\phi_k \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ (see Section 2 or [DK18, Section 10]).

Proposition ([Pau09, Page 434])**.** *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $[\phi_k, \mathbb{H}^2]$ a sequence in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ such that $\lg(\phi_k) \to \infty$, and $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$. If $*_k \in \mathbb{H}^2$ is an element in $\mathbb{H}^2$ achieving the infimum of the displacement function of ϕ_k for every $k \in \mathbb{N}$ and $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}(\mathbb{H}^2, (\lg(\phi_k)), (*_k))$, then*

$$\lim_{\mathfrak{u}} [\phi_k, \mathbb{H}^2] = [\phi^{\mathfrak{u}}, T^{\mathfrak{u}}]$$

$\mathfrak{u}$*-almost surely in the Gromov equivariant topology.*

Wolff proves in [Wol11] that the extension of the length spectrum compactification to $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ leads to a wild space and defines a refinement of the Gromov equivariant topology that preserves the orientation on $\mathbb{H}^2$. This requires a notion of orientation on $\mathbb{R}$-trees using cyclic orders. A *cyclic order* on a set Ω is a map

$$o \colon \Omega^3 \to \{-1, 0, 1\}$$

satisfying a cocycle property (see Definition 5.1.1), and encoding a circular arrangement of points. In Subsection 5.1.2, we describe the standard orientation on $\mathbb{H}^2$ via a cyclic order on its visual boundary $\partial_\infty \mathbb{H}^2 \cong \mathbb{S}^1$ by a semialgebraic equation:

Lemma (Lemma 5.1.13)**.** *If* sgn *denotes the sign function on $\mathbb{R}$, then*

$$o_{\mathbb{R}} : \begin{array}{ccc} (\mathbb{S}^1)^3 & \longrightarrow & \{-1, 0, 1\}; \\ (z_1, z_2, z_3) & \longmapsto & \mathrm{sgn}\left(\det(z_2 - z_1, z_3 - z_2)\right), \end{array}$$

defines a cyclic order on $\mathbb{S}^1$, which encodes the orientation on $\mathbb{H}^2$.

An $\mathbb{R}$-tree T is *oriented* if for every $P \in T$, there is a cyclic order $or(P)$ defined on $\mathcal{G}_T(P)$, the set of *germs of oriented segments* at P (we define similarly $\mathcal{G}_X(P)$ when X is a geodesic Gromov hyperbolic space, see Subsection 5.1.1). An isometry is orientation preserving if it preserves the cyclic orders at every $P \in T$, see Definition 5.1.5. Denote by $\mathrm{Isom}_{\mathrm{or}}(T)$ the group of isometries preserving the orientation of T. Wolff defines the *oriented Gromov equivariant* topology (Definition 5.1.22) on a subspace of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}'$ where

$$\mathcal{T}' := \left\{ (\phi, T, or) \;\middle|\; \begin{array}{l} T \text{ an } \mathbb{R}\text{-tree not reduced to a point,} \\ \phi\colon \Gamma \to \mathrm{Isom}_{\mathrm{or}}(T) \text{ a minimal action, and} \\ or \text{ an orientation on } T \end{array} \right\} / \sim,$$

and the equivalence is by orientation preserving equivariant isometries [Wol11, Page 1273]. The closure of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}'$ is the oriented Gromov equivariant compactification $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$, which is a first countable compact Hausdorff space. Based on the work of Paulin [Pau89], we describe the boundary elements of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ as Γ-actions on asymptotic cones, which we endow with an orientation. The description of the standard orientation on $\mathbb{H}^2$ allows us to construct an *ultralimit orientation* on the asymptotic cones of $\mathbb{H}^2$.

Theorem (Theorem 5.1.30)**.** *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $(\lambda_k) \subset \mathbb{R}$ a sequence such that $\lambda_k \to \infty$, and $(*_k) \subset \mathbb{H}^2$ a sequence of basepoints. Let*

$$T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lambda_k), (*_k)\right)$$

be the corresponding asymptotic cone, and $[P_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$. Using an identification between germs of oriented segments at $[P_k]^{\mathfrak{u}}$ and elements $[x_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$ representing directions, define the map $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})\colon (\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}}))^3 \to \{-1, 0, 1\}$ by

$$([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}) \mapsto \begin{cases} 0 & \text{if } \mathrm{Card}\{ [x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}} \} \leq 2, \\ \lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) & \text{otherwise,} \end{cases}$$

where $or(P_k)$ is the cyclic order on $\mathcal{G}_{\mathbb{H}^2}(P_k)$ given by the standard orientation on $\mathbb{H}^2$. Then $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ defines a cyclic order on $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$.

This in turn allows us to give a characterization of the limit of sequences in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ in terms of asymptotic cones.

Theorem (Theorem 5.1.31)**.** *Let $[\phi_k, \mathbb{H}^2, or]$ be a sequence in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ such that $\lg(\phi_k) \to \infty$ and $\mathfrak{u}$ a non-principal ultrafilter on $\mathbb{N}$. For each $k \in \mathbb{N}$, let $*_k \in \mathbb{H}^2$ be a point that realizes the infimum of the displacement function of ϕ_k and set $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$. If $T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}$ is the $\phi^{\mathfrak{u}}$-invariant minimal subtree inside the asymptotic cone $\mathrm{Cone}^{\mathfrak{u}}(\mathbb{H}^2, (\lg(\phi_k)), (*_k))$, then*

$$\lim[\phi_k, \mathbb{H}^2, or] = [\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, or^{\mathfrak{u}}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$$

$\mathfrak{u}$-almost surely, where $or^{\mathfrak{u}}$ is the restriction to $T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}$ of the ultralimit orientation defined in Theorem 5.1.30.

This result shows that the oriented Gromov equivariant compactification is described in terms of asymptotic cones. To compare $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ with the real spectrum compactification, we associate to every element of the real spectrum compactification an oriented $\mathbb{R}$-tree and show that this oriented $\mathbb{R}$-tree is also well described using asymptotic cones.

1.4.2 Oriented trees associated to the real spectrum compactification

The real spectrum compactification $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ is a compact Hausdorff metrizable space that contains $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ as an open and dense subset. We are interested in the following characterization of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$:

Definition (Definition 5.2.1)**.** Given a representation $\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F})$, the real closed field $\mathbb{F}$ is *ϕ-minimal* if ϕ can not be $\mathrm{PSL}_2(\mathbb{F})$-conjugated into a representation $\phi'\colon \Gamma \to \mathrm{PSL}_2(\mathbb{L})$, where $\mathbb{L} \subset \mathbb{F}$ is a proper real closed subfield.

If $\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F})$ is reductive and $\mathbb{F}$ is real closed, a minimal real closed field always exists and is unique [BIPP23, Corollary 7.9]. If $\mathbb{F}_1$ and $\mathbb{F}_2$ are two real closed fields, we say that two representations $(\phi_1, \mathbb{F}_1)$ and $(\phi_2, \mathbb{F}_2)$ are equivalent if there exists a real closed field morphism $\psi\colon \mathbb{F}_1 \to \mathbb{F}_2$ such that

$$\psi \circ \phi_1 \quad \text{is } \mathrm{PSL}_2(\mathbb{F}_2)\text{-conjugated to} \quad \phi_2.$$

Theorem ([BIPP23, Theorem 1.1 and Corollary 7.9])**.** *Elements in the boundary of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ are in bijective correspondence with equivalence classes of pairs $[\phi, \mathbb{F}]$, where ϕ is a reductive representation*

$$\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F}),$$

and $\mathbb{F}$ is real closed, non-Archimedean and ϕ-minimal.

For any representative $(\phi, \mathbb{F})$ of a point in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, we study the induced Γ-action on the non-Archimedean hyperbolic plane $\mathbb{H}^2(\mathbb{F})$, which one equips with a well described pseudo-metric, see Subsection 5.2.1. The quotient

$$T\mathbb{F} := \mathbb{H}^2(\mathbb{F}) / \left\{ \mathrm{dist}_{\mathbb{H}^2(\mathbb{F})} = 0 \right\}$$

is a Λ-tree [Bru88c, Theorem 28] which by completing the segments produces an $\mathbb{R}$-tree $T\mathbb{F}^{\mathrm{sc}}$ on which $\phi(\Gamma)$ acts by isometries and without global fixed points, see Subsection 5.2.1. By [Pau89, Proposition 2.4], there exists, up to isometry, a unique ϕ-minimal invariant subtree

$$T_\phi \subset T\mathbb{F}^{\mathrm{sc}},$$

which we endow with an orientation. Brumfiel gives in [Bru88c, Proposition 41] an explicit bijective correspondence between the set $\mathcal{G}_{\mathbb{H}^2(\mathbb{F})}(P)$ of germs of oriented segments at $P \in \mathbb{H}^2(\mathbb{F})$ and $\mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$. Here, $\mathbb{F}_{\mathcal{O}}$ denotes the quotient field of the valuation ring of $\mathbb{F}$, which is itself a real closed field (see Remark 5.2.12). By the transfer principle and Lemma 5.1.13, the $\mathbb{F}_{\mathcal{O}}$-extension of $o_{\mathbb{R}}$ gives a cyclic order on $\mathcal{G}_{\mathbb{H}^2(\mathbb{F})}(P)$ for every $P \in \mathbb{H}^2(\mathbb{F})$.

Corollary (Corollary 5.2.11). *The cyclic order $or_{\mathbb{F}_{\mathcal{O}}}$ from Lemma 5.1.13 defines a cyclic order on $\mathcal{G}_{T\mathbb{F}^{\mathrm{sc}}}(P) \cong \mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$ for every $P \in T\mathbb{F}^{\mathrm{sc}}$.*

Consider T_ϕ with the orientation or_ϕ induced by the restriction to T_ϕ of the orientation of $T\mathbb{F}^{\mathrm{sc}}$. We show in Theorem 5.2.14 that $[\phi, T_\phi, or_\phi]$ does not depend on the choice of representative of the class $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$.

Lemma (Lemma 5.2.15). *Each class $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ defines a canonical Γ-action by orientation preserving isometries on a ϕ-minimal oriented $\mathbb{R}$-tree $[\phi, T_\phi, or_\phi]$, up to Γ-equivariant orientation preserving isometries.*

An essential result to prove Theorem 5.3.4, see Figure 1.1, is the following accessibility result [BIPP23] that allows us to represent any element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ by a representation of Γ in $\mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$, where $\mathbb{R}^{\mathfrak{u}}_\mu$ is a Robinson field, see Example 2.2.3. We refer to Subsection 5.2.2 for the notations. Let F be a finite generating set of Γ. In the following theorem, given $\mathfrak{u}$ a non-principal ultrafilter, the sequence of scales (μ_k) is *well adapted* to a sequence of representations $\phi_k \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ if there exists $c_1, c_2 \in \mathbb{R}_{>0}$ such that for $\mathfrak{u}$-almost every $k \in \mathbb{N}$

$$c_1(\mu_k) \leq \sum_{\gamma \in F} \left(\mathrm{tr}\left(\phi_k(\gamma)\phi_k(\gamma)^T\right)\right) \leq c_2(\mu_k).$$

If only the second inequality holds, then the sequence (μ_k) is *adapted*. In this setting, if $\mu = (\mu_k)$, we denote by $\phi^{\mathfrak{u}}_\mu \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$ the $(\mathfrak{u}, \mu)$-limit representation

$$\phi^{\mathfrak{u}}_\mu(\gamma) = \begin{pmatrix} (\phi_k(\gamma)^{1,1})_k & (\phi_k(\gamma)^{1,2})_k \\ (\phi_k(\gamma)^{2,1})_k & (\phi_k(\gamma)^{2,2})_k \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu).$$

Theorem ([BIPP23, Theorem 7.16]). *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $(\phi_k, \mathbb{R})_k \in \mathcal{M}_\Gamma(\mathbb{R})$ and $(\phi^{\mathfrak{u}}_\mu, \mathbb{R}^{\mathfrak{u}}_\mu)$ its $(\mathfrak{u}, \mu)$-limit representation for an adapted sequence of scales $\mu := (\mu_k)$. Then:*

- *$\phi^{\mathfrak{u}}_\mu$ is reductive, and*
- *if μ is well adapted, infinite, and $\mathbb{F}_{\phi^{\mathfrak{u}}_\mu}$ denotes the $\phi^{\mathfrak{u}}_\mu$-minimal field, then $(\phi^{\mathfrak{u}}_\mu, \mathbb{R}^{\mathfrak{u}}_\mu)$ is $\mathrm{SO}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$-conjugate to a representation $(\phi, \mathbb{F}_{\phi^{\mathfrak{u}}_\mu})$ that represents an element in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$.*

Conversely, any element in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ *arises in this way.*

As in the length spectrum compactification, this permits to describe the trees associated to elements of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ using asymptotic cones. The following is a consequence of [BIPP23, Theorem 5.10 and Lemma 5.12], where $(\mathbb{H}^2)^{\mathfrak{u}}_{\lambda}$ is the ultralimit of the rescaled copies of $\mathbb{H}^2$ by a sequence of scalars $\lambda := \lambda_k$ and $\mathfrak{u}$ a non-principal ultrafilter.

Corollary 1.4.1 (Corollary 5.2.21)**.** *The map*

$$\begin{array}{rccc} \Psi\colon & (\mathbb{H}^2)^{\mathfrak{u}}_{\lambda} & \longrightarrow & \mathbb{H}^2\left(\mathbb{R}^{\mathfrak{u}}\right); \\ & (x+iy)_k & \longmapsto & (x_k)+i(y_k) \end{array}$$

induces an isometry between the asymptotic cone $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lambda_k), (0)\right)$ *and* $\mathbb{H}^2(\mathbb{R}^{\mathfrak{u}}_{\mu})/\{\, \mathrm{dist}_{\mathbb{H}^2(\mathbb{R}^{\mathfrak{u}}_{\mu})} = 0 \,\} =: T\mathbb{R}^{\mathfrak{u}}_{\mu}$*. Moreover, with the above notations, the isometry is* Γ*-equivariant for the induced* Γ*-actions by* $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$ *on* $T^{\mathfrak{u}}$ *and by* $\phi^{\mathfrak{u}}_{\mu}$ *on* $T\mathbb{R}^{\mathfrak{u}}_{\mu}$*.*

As in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$, we can enhance this description via asymptotic cones to take account of the orientation. We describe the oriented $\mathbb{R}$-tree associated to any element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ using asymptotic cones and ultralimit orientations.

Theorem (Subsection 5.2.2 and Theorem 5.2.22)**.** *Let* $\mathfrak{u}$ *be a non-principal ultrafilter on* $\mathbb{N}$*,* $(\phi_k, \mathbb{R})_k$ *a sequence in* $\mathrm{Hom}_{\mathrm{red}}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ *such that* $\lg(\phi_k) \to \infty$*, and* $*_k$ *a point of* $\mathbb{H}^2$ *that realizes the infimum of the displacement function of* ϕ_k *for each* $k \in \mathbb{N}$*. Consider*

$$T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lg\phi_k), (*_k)\right)$$

endowed with the limit orientation as defined in Theorem 5.1.30. Let also $\mu := (e^{\lg(\phi_k)})$*,* $T\mathbb{R}^{\mathfrak{u}}_{\mu}$ *the* $\mathbb{R}$*-tree associated to* $\mathbb{R}^{\mathfrak{u}}_{\mu}$*, which we endow with its orientation* $or_{\mathbb{R}^{\mathfrak{u}}_{\mu}}$*, and* $(\phi^{\mathfrak{u}}_{\mu}, \mathbb{R}^{\mathfrak{u}}_{\mu})$ *its* $(\mathfrak{u}, \mu)$*-limit representation. Then*

$$\Psi\colon T^{\mathfrak{u}} \to T\mathbb{R}^{\mathfrak{u}}_{\mu}$$

is an orientation preserving isometry which is Γ*-equivariant for the actions induced by* $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$ *and* $\phi^{\mathfrak{u}}_{\mu}$*.*

This theorem allows us to compare the real spectrum compactification with the oriented Gromov equivariant compactification of the character variety defined by Wolff in [Wol11].

1.4.3 Relationship between the compactifications

By associating an oriented $\mathbb{R}$-tree to every element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, one obtains a map between the two compactifications.

Definition (Definition 5.3.1)**.**

$$\begin{aligned} \beth\colon\ \Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}} &\longrightarrow \qquad \Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right) \cup \mathcal{T}'; \\ [\phi, \mathbb{F}] &\longmapsto [\phi, X, or_\phi] = \begin{cases} [\phi, \mathbb{H}^2, or] & \text{if } \mathbb{F} = \mathbb{R} \\ [\phi, T_\phi, or_\phi] & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\mathcal{T}'$ is a space of Γ-actions on oriented ℝ-trees, see Subsection 5.1.1.

Using the accessibility result (Theorem 5.2.17) and the uniqueness of the minimal invariant ℝ-tree up to isometry, we obtain the continuity of $\beth$.

Lemma (Lemma 5.3.3)**.** *The map $\beth$ from Definition 5.3.1 is continuous.*

As a consequence, a topological argument based on the density of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ in both compactifications allows us to upgrade the function to a continuous surjection.

Theorem (Theorem 5.3.4)**.** *The map* $\beth\colon \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ *from Definition 5.3.1 is a continuous surjection.*

Moreover, the oriented Gromov equivariant compactification of the character variety surjects continuously on its length spectrum compactification [Wol11]. Thus, Theorem 5.3.4 provides a new construction of the continuous surjection between the real spectrum and the length spectrum compactifications of $T(S)$ [Bru88a].

1.5 Outline

We begin in Chapter 2 with general background material needed throughout the thesis; more specific background is provided at the beginning of each chapter as necessary. Chapter 3 reviews useful results from the theory of the real spectrum, establishes a functoriality property for the closed points of the real spectrum, and develops the theory of the Archimedean spectrum of algebraic sets. Chapter 4 constructs universal geometric spaces over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Chapter 5 constructs a continuous surjection from $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ to the oriented Gromov equivariant compactification of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. Detailed outlines are provided at the beginning of each of these chapters. Finally, Chapter 6 presents an example of an homogeneous ℝ-tree which is incomplete written is a work in collaboration.

Chapter 2

Preliminaries

This chapter introduces the necessary background and notation. We begin with the theory of asymptotic cones, highlighting that those of the hyperbolic plane are $\mathbb{R}$-trees. Next, we review key concepts from real algebraic geometry, including semialgebraic sets and their Euclidean topology. The last two sections present classical examples of algebraic and semialgebraic sets. We recall the definition of minimal vectors $\mathcal{M}_\Gamma(\mathbb{R})$ for representations of a finitely generated group Γ in $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{PSL}_n(\mathbb{R})$. These sets are algebraic in the $\mathrm{SL}_n(\mathbb{R})$ case and semialgebraic in the $\mathrm{PSL}_n(\mathbb{R})$ case. Their quotients, the character varieties $\Xi(\Gamma, \mathrm{SL}_n(\mathbb{R}))$ and $\Xi(\Gamma, \mathrm{PSL}_n(\mathbb{R}))$, are semialgebraic sets. Finally, we construct a cover $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ of the symmetric space associated to $\mathrm{SL}_n(\mathbb{R})$, which can be endowed with a semialgebraic multiplicative norm. These algebraic and semialgebraic models allow for their study in the following chapters using tools from real algebraic geometry.

2.1 Asymptotic cones

We recall briefly the classical theory of asymptotic cones necessary for this text, as presented for example in [DK18, Chapter 10] and [Dru02]. Roughly speaking, the asymptotic cone of a metric space gives a picture of the metric space as "seen from infinitely far away". It was introduced by Gromov in [Gro81b], and formally defined in [vdDW84] to construct a limit to a family of rescaled metric spaces.

Definition 2.1.1. A *filter* $\mathfrak{u}$ on a set I is a collection of subsets of I satisfying:

1. $\varnothing \notin \mathfrak{u}$,

2. If $I_1, I_2 \in \mathfrak{u}$ then $I_1 \cap I_2 \in \mathfrak{u}$,

3. If $I_1 \in \mathfrak{u}$ and $I_1 \subset I_2 \subset I$, then $I_2 \in \mathfrak{u}$.

An *ultrafilter* on I is a filter $\mathfrak{u}$ such that for every $J \subset I$ either $J \in \mathfrak{u}$ or $I \backslash J \in \mathfrak{u}$.

For simplicity, in the rest of this text, all ultrafilters are on $\mathbb{N}$ and are *non-principal*. That is, they contain the set of complements of finite subsets [DK18, Proposition 10.16]. Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$. A subset $J \in 2^{\mathbb{N}}$ occurs $\mathfrak{u}$-*almost surely* if $J \in \mathfrak{u}$.

Definition 2.1.2. Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, X a topological space, and $f\colon \mathbb{N} \to X$ a map. The *ultralimit* of f is an element $x \in X$ such that

$$\forall x \in U \text{ open, then } f^{-1}(U) \in \mathfrak{u}.$$

The ultralimit of f is denoted by $\lim_{\mathfrak{u}} f$.

A first reason to introduce ultrafilters is to give a limit to any sequence on a compact topological space.

Proposition 2.1.3 ([DK18, Lemma 10.25])**.** *Let X be a topological space, $\mathfrak{u}$ a non-principal ultrafilter on $\mathbb{N}$ and $f : \mathbb{N} \to X$ a map.*

1. *If X is compact, then f has an ultralimit,*
2. *If X is Hausdorff, then the ultralimit, if it exists, is unique.*

Remark 2.1.4. The standard order on $\mathbb{R}_{>0}$ extend to an order on $[0, \infty]$ by setting $a < \infty$ for all $a \in \mathbb{R}_{>0}$. Then, $[0, \infty]$ with the total order topology is compact.

Let (X_k) be a sequence of metric spaces and $\mathfrak{u}$ a non-principal ultrafilter on $\mathbb{N}$. Define a pseudo-distance on the product by

$$\begin{array}{rccc} \mathrm{dist}_{\mathfrak{u}}\colon & \prod_{k \in \mathbb{N}} X_k \times \prod_{k \in \mathbb{N}} X_k & \longrightarrow & [0, \infty]; \\ & ((x_k), (y_k)) & \longmapsto & \lim_{\mathfrak{u}} (k \mapsto \mathrm{dist}_{X_k}(x_k, y_k)). \end{array}$$

Then, the *ultralimit metric space* is

$$(X^{\mathfrak{u}}, \mathrm{dist}_{\mathfrak{u}}) := \left(\prod_{k \in \mathbb{N}} X_k, \mathrm{dist}_{\mathfrak{u}} \right) \Big/ \sim,$$

where $(x_k) \sim (y_k)$ if and only if $\mathrm{dist}_{\mathfrak{u}}((x_k), (y_k)) = 0$.

Notation 2.1.5. Given a sequence of elements (x_k), where $x_k \in X_k$ for every $k \in \mathbb{N}$, denote its equivalence class in $X^{\mathfrak{u}}$ by $[x_k]^{\mathfrak{u}}$.

If the spaces X_k do not have uniformly bounded diameter, the ultralimit $X^{\mathfrak{u}}$ decomposes into many components of points at mutually finite distance, where two elements in different components are at infinite distance one from the other. In order to pick one of these components, we consider *pointed metric spaces* $(X, *)$ where $* \in X$ is a *base point*. For a family of pointed metric spaces $(X_k, *_k)$, the sequence of base points $(*_k)$ defines a base point $[*_k]^{\mathfrak{u}} \in X^{\mathfrak{u}}$, and we set

$$X^{\mathfrak{u}}_{[*_k]^{\mathfrak{u}}} := \{ [y_k]^{\mathfrak{u}} \in X^{\mathfrak{u}} \,|\, \mathrm{dist}_{\mathfrak{u}} ([y_k]^{\mathfrak{u}}, [*_k]^{\mathfrak{u}}) < \infty \}.$$

Definition 2.1.6. The *pointed ultralimit* of the sequence $(X_k, *_k)$ is

$$\lim_{\mathfrak{u}} (X_k, *_k) = \left(X^{\mathfrak{u}}_{[*_k]^{\mathfrak{u}}}, [*_k]^{\mathfrak{u}} \right) =: (X^{\mathfrak{u}}, [*_k]).$$

Denote also elements $[x_k]^{\mathfrak{u}}$ by $[x_k]$ when the notation is clear.

For two sequences of pointed metric spaces $(X_k, *_k), (X'_k, *'_k)$ and a sequence of maps $f_k : (X_k, *_k) \to (X'_k, *'_k)$ such that $\lim_{\mathfrak{u}} \mathrm{dist}_{X_k} (f_k(*_k), *'_k) < \infty$, the *ultralimit* of (f_k) is

$$\begin{array}{rccc} f^{\mathfrak{u}}\colon & (X^{\mathfrak{u}}, [*_k]) & \longrightarrow & (X'^{\mathfrak{u}}, [*'_k])\,; \\ & [y_k] & \longmapsto & [f_k(y_k)]. \end{array}$$

If in addition, $\mathfrak{u}$-almost every f_k is an isometric embedding, then $f^{\mathfrak{u}}$ is an isometric embedding [DK18, Lemma 10.48]. We now study more in depth the limit of rescaled copies of metric spaces as in [DK18, Chapter 10].

Definition 2.1.7. Let $(\lambda_k) \subset \mathbb{R}$ be a sequence such that $\lambda_k \to \infty$. The *asymptotic cone* of a metric space X with respect to the *sequence of scalars* (λ_k), the sequence of *observation centers* $(*_k)$ and the non-principal ultrafilter $\mathfrak{u}$ is

$$\mathrm{Cone}^{\mathfrak{u}} (X, (\lambda_k), (*_k)) := \lim_{\mathfrak{u}} (X_{\lambda_k}, *_k),$$

where X_{λ_k} is the metric space X endowed with the rescaled distance $(\lambda_k)^{-1} \cdot \mathrm{dist}_X$.

A $\mathbb{R}$-*tree* T is a geodesic metric space which is 0-hyperbolic in the sense of Gromov [DK18, Lemma 11.30].

Theorem 2.1.8 ([GdlH90, Chapter 2, Section 1, Proposition 11])**.** *If a geodesic metric space X is hyperbolic, then every asymptotic cone of it is a $\mathbb{R}$-tree.*

In Chapter 5, we study the limits of actions by isometries on the hyperbolic plane. The theory of asymptotic cones allows us to characterize these limits as actions by isometries on $\mathbb{R}$-trees, which is central in the theory of compactification of character varieties.

2.2 Preliminaries in real algebraic geometry

Our approach to the theory of character varieties is based on real algebraic geometry. In particular, our main objects of study are semialgebraic sets defined over real closed fields. To set up this framework, we first introduce key definitions and notation, following [BCR13, Section 1 and 2] and [BIPP23, Section 2].

Definition 2.2.1. A field $\mathbb{K}$ is *ordered* if there exists a total order $\leq$ compatible with its field operations. Formally, $\leq$ satisfies: for every $a, b, c \in \mathbb{K}$

$$\text{if } a \leq b, \text{ then } a + c \leq b + c \quad \text{and} \quad \text{if } 0 \leq a, b, \text{ then } 0 \leq ab.$$

A *real field* is a field that can be ordered. A stronger notion is that of *real closed field* $\mathbb{K}$, which is a real field that has no proper algebraic ordered field extension. Equivalently, this means that every positive element has a square root, and every polynomial of odd degree has a root in $\mathbb{K}$ [BCR13, Theorem 1.2.2]. The following theorem sheds a little more light on the nature of real closed fields. It gives an equivalent definition, which is highlighted by the Transfer principle —a result of the Tarski–Seidenberg principle.

Theorem 2.2.2 (Transfer principle [BCR13, Proposition 5.2.3])**.** *Let $\mathbb{K}$ be a real closed field, Ψ a formula in the first-order language of ordered rings with parameters in $\mathbb{K}$ without a free variable, and $\mathbb{F}$ a real closed extension of $\mathbb{K}$. Then Ψ holds true in $\mathbb{K}$ if and only if it holds true in $\mathbb{F}$.*

Real closed fields possess the same elementary theory as the reals. They give a natural framework to extend results from $\mathbb{R}$ to more general fields which are real closed.

Example 2.2.3. We are particularly interested in the following examples of real closed fields.

(i) The field $\overline{\mathbb{Q}}^r$ of real algebraic numbers and the field $\mathbb{R}$ of real numbers are real closed fields [BCR13, Example 1.3.6].

(ii) Another example of a real field is given by the *real Puiseux series*

$$\left\{ \sum_{k=-\infty}^{k_0} c_k x^{\frac{k}{m}} \;\middle|\; k_0, m \in \mathbb{Z},\ m > 0,\ c_k \in \mathbb{R},\ c_{k_0} \neq 0 \right\},$$

which, if endowed with the order such that $\sum_{k=-\infty}^{k_0} c_k x^{\frac{k}{m}} > 0$ if $c_{k_0} > 0$, is real closed.

(iii) Consider a field $\mathbb{K}$ and a non-principal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$. Define the *hyper $\mathbb{K}$-field* as $\mathbb{K}^{\mathfrak{u}} := \mathbb{K}^{\mathbb{N}}/\sim$, where the equivalence relation is given by $(x_k) \sim (y_k)$ if and only if the two sequences coincide $\mathfrak{u}$-almost surely. If $\mathbb{K}$ is an ordered field, then $\mathbb{K}^{\mathfrak{u}}$ is also an ordered field. Indeed, the order is determined by

$$[x_k]^{\mathfrak{u}} > 0 \text{ if and only if } x_k > 0 \text{ } \mathfrak{u}\text{-almost surely.}$$

Moreover, it is a real closed field if $\mathbb{K}$ itself is real closed. It also has *positive infinite elements*, that is elements larger than any integer. Fields with positive infinite elements are called *non-Archimedean*. For such an infinite element μ, define

$$\mathcal{O}_\mu := \left\{ x \in \mathbb{K}^{\mathfrak{u}} \,\middle|\, x < \mu^k \text{ for some } k \in \mathbb{Z} \right\},$$
$$\mathcal{J}_\mu := \left\{ x \in \mathbb{K}^{\mathfrak{u}} \,\middle|\, x < \mu^k \text{ for every } k \in \mathbb{Z} \right\},$$

where $\mathcal{J}_\mu$ is a maximal ideal inside the subring $\mathcal{O}_\mu$ of the hyper $\mathbb{K}$-field [LR75]. Now, the *Robinson field* associated to the non-principal ultrafilter $\mathfrak{u}$ and the infinite element μ is the quotient

$$\mathbb{K}^{\mathfrak{u}}_\mu := \mathcal{O}_\mu / \mathcal{J}_\mu.$$

This is necessarily a non-Archimedean field, as all rational numbers are smaller than μ.

Let $\mathbb{L}$ be an ordered field. An ordered algebraic extension $\mathbb{F}$ is the *real closure* of $\mathbb{L}$ if $\mathbb{F}$ is a real closed field and the ordering on $\mathbb{L}$ extends to the ordering of $\mathbb{F}$. The real closure of $\mathbb{L}$ always exists and is unique up to a unique order preserving isomorphism over $\mathbb{L}$. That is, if $\mathbb{F}_1$ and $\mathbb{F}_2$ are real closures of $\mathbb{L}$, there exists a unique order preserving isomorphism $\mathbb{F}_1 \to \mathbb{F}_2$ that is the identity on $\mathbb{L}$. Denote the real closure of $\mathbb{L}$ by:

$$\overline{\mathbb{L}}^r.$$

In real algebraic geometry, the fundamental objects of study are real algebraic and semialgebraic sets. Intuitively, algebraic sets consist of points satisfying polynomial equations, while semialgebraic sets allow polynomial inequalities as well.

Definition 2.2.4. Let $\mathbb{K}$ be a real closed field. A set $V \subset \mathbb{K}^n$ is an *algebraic set* defined over $\mathbb{K}$ if there exists $B \subset \mathbb{K}[x_1, \ldots, x_n]$ such that

$$V := \{v \in \mathbb{K}^n \mid f(v) = 0 \quad \forall f \in B\}.$$

Then, $S \subset \mathbb{K}^n$ is a *semialgebraic set* defined over $\mathbb{K}$ if there exists polynomials $f_i, g_j \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$S := \bigcup_{finite} \bigcap_{finite} \{s \in \mathbb{K}^n \mid f_i(s) = 0\} \cap \{s \in \mathbb{K}^n \mid g_j(s) > 0\}.$$

Denote by $I(S) := \{ f \in \mathbb{K}[x_1, \dots, x_n] \mid f(s) = 0 \ \forall s \in S \}$ the ideal of polynomials vanishing on S.

For the remainder of this section, let $\mathbb{K}$ be a real closed field and $\mathbb{F}$ a real closed extension of $\mathbb{K}$.

Definition 2.2.5. Let $S \subset \mathbb{K}^n$ be a semialgebraic set. The *coordinate ring* of the $\mathbb{F}$-extension of S is

$$\mathbb{F}[S] := \mathbb{F}[x_1, \dots, x_n]/I(S).$$

Consider a semialgebraic set S defined over $\mathbb{K}$. The $\mathbb{F}$*-points* of S are the solutions in $\mathbb{F}^n$ of the polynomials defining S

$$S(\mathbb{F}) := \{s \in \mathbb{F}^n \mid f(v) = 0 \quad \forall s \in B\},$$

where $B \subset \mathbb{K}[x_1, \dots, x_n]$ is a set of polynomials defining S. It can be shown that the set $S(\mathbb{F})$ is independent of the choice of B [BCR13, Proposition 5.1.1]. By abuse of notation, we use the same symbol for the $\mathbb{F}$-points and the $\mathbb{F}$-extension of S. To define a topology on $S(\mathbb{F})$, introduce the norm $N : \mathbb{F}^n \to \mathbb{F}_{\geq 0}$, given by

$$N(s) := \sqrt{\sum_{i=1}^{n} s_i^2} \quad \forall s = (s_1, \dots, s_n) \in \mathbb{F}^n.$$

The *open ball centered at* $s \in \mathbb{F}^n$ *with radius* $r \in \mathbb{F}_{\geq 0}$ is then defined as

$$B(s, r) := \{ y \in \mathbb{F}^n \mid N(s - y) < r \}.$$

These open balls form a basis for the *Euclidean topology* on $\mathbb{F}^n$. Finally, to study the links between algebraic sets we will use maps between semialgebraic sets with good algebraic properties.

Definition 2.2.6. A map $\pi \colon S_1 \to S_2$ between semialgebraic sets $S_1 \subset \mathbb{K}^n$, $S_2 \subset \mathbb{K}^m$ is *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{K}^n \times \mathbb{K}^m$.

Proposition 2.2.7 ([BCR13, Proposition 5.3.1])**.** *Let* $S_1 \subset \mathbb{K}^n$*,* $S_2 \subset \mathbb{K}^m$ *be semialgebraic sets and* $\pi \colon S_1 \to S_2$ *a semialgebraic map with graph* X*. The* $\mathbb{F}$*-extension* $X(\mathbb{F})$ *is the graph of a semialgebraic map* $\pi_{\mathbb{F}}$ *called the* $\mathbb{F}$-extension *of* π*.*

We defined our primary objects of study: semialgebraic sets and semialgebraic maps over real closed fields. In Section 4.2.3, we employ semialgebraic maps to construct universal geometric spaces over character varieties, thereby providing a geometric description of the real spectrum compactification of character varieties. In the next subsection, we construct some algebraic and semialgebraic examples that are of particular interest to us.

2.3 Character varieties and minimal vectors

This section recalls the classical theory of character varieties of a finitely generated group Γ in $G(\mathbb{R}) = \mathrm{PSL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R})$, based on reductive representations and their quotient by postconjugation by $G(\mathbb{R})$. Using the theory of minimal vectors, we construct a semialgebraic structure of the character variety, which serves as the foundation for defining its real spectrum compactification using Chapter 3. This introduction to character varieties is inspired by [Fla25] and based on [RS90] (on ideas of Kempf–Ness for the complex reductive case). See also [BL17] and [BIPP23, Section 6, Section 7] for more general treatments. We begin with the following proposition, which realizes $\mathrm{PSL}_n(\mathbb{R})$ as a semialgebraic subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$. This allows us to use the language of linear representations throughout the remainder of the section.

Proposition 2.3.1. *The group* $\mathrm{PGL}_n(\mathbb{R})$ *is an algebraic subset of* $\mathbb{R}^{n^4}$ *and* $\mathrm{PSL}_n(\mathbb{R})$ *is a semialgebraic subset of* $\mathrm{PGL}_n(\mathbb{R})$.

Proof. The algebra $M_{n\times n}(\mathbb{R})$ of n by n matrices with real coefficients is central and simple. By the Skolem–Noether Theorem [GS17, Theorem 2.7.2] (originally in [Sko27]), every $\mathbb{R}$-algebra automorphism of $M_n(\mathbb{R})$ is inner. Hence the adjoint representation

$$\begin{array}{rccc} \mathrm{Ad}\colon & \mathrm{PGL}_n(\mathbb{R}) & \longrightarrow & \mathrm{Aut}(M_{n\times n}(\mathbb{R})) \subset \mathrm{GL}_{n^2}(\mathbb{R}); \\ & A & \longmapsto & B \mapsto ABA^{-1} \end{array}$$

is an isomorphism. The condition of being an automorphism of $M_{n\times n}(\mathbb{R})$ is given by finitely many algebraic equations. Thus $\mathrm{PGL}_n(\mathbb{R})$ is a real algebraic subset of $\mathrm{GL}_{n^2}(\mathbb{R})$. Moreover $\mathrm{PSL}_n(\mathbb{R})$ is a connected component of the algebraic set $\mathrm{PGL}_n(\mathbb{R})$ and thus is a semialgebraic set by [BCR13, Theorem 2.1.11]. □

As a semialgebraic subgroup of $\mathrm{GL}_{n^2}(\mathbb{R})$, the group $\mathrm{PSL}_n(\mathbb{R})$ is linear. This gives a convenient framework to define reductive representations with target group $\mathrm{PSL}_n(\mathbb{R})$.

Notation 2.3.2. Let Γ be a finitely generated group with finite generating set F with s elements and $G(\mathbb{R})$ be either $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{PSL}_n(\mathbb{R})$.

Definition 2.3.3. A representation $\phi\colon \Gamma \to G(\mathbb{R})$ is *reductive* if, seen as a linear representation on $\mathbb{R}^{n^2}$, it is completely reducible. That is, a direct sum of irreducible representations.

Denote by $\mathrm{Hom}_{\mathrm{red}}(\Gamma, G(\mathbb{R})) \subset \mathrm{Hom}(\Gamma, G(\mathbb{R}))$ the subspace of reductive representations which is invariant under postconjugation by the target group.

Definition 2.3.4. The *character variety* of the finitely generated group Γ and the semialgebraic group $G(\mathbb{R})$ is the quotient of $\mathrm{Hom}_{\mathrm{red}}(\Gamma, G(\mathbb{R}))$ via postconjugation by $G(\mathbb{R})$:

$$\Xi(\Gamma, G(\mathbb{R})) := \mathrm{Hom}_{\mathrm{red}}(\Gamma, G(\mathbb{R}))/G(\mathbb{R}).$$

Theorem 2.3.5 ([Bou12, §20, page 376, Corollaire a])**.** *Let ϕ, ϕ' be two reductive linear representations of Γ in $G(\mathbb{R})$ that verify*

$$\mathrm{tr} \circ \mathrm{Ad}(\phi(\gamma)) = \mathrm{tr} \circ \mathrm{Ad}(\phi'(\gamma)) \quad \forall \gamma \in \Gamma.$$

Then $\mathrm{Ad}(\phi)$ and $\mathrm{Ad}(\phi')$ are conjugate by an element in $\mathrm{GL}_{n^2}(\mathbb{R})$.

In other words, reductive representations are, up to postconjugation, determined by their trace functions—hence the term character variety. The character variety corresponds to the maximal Hausdorff quotient of $\mathrm{Hom}(\Gamma, G(\mathbb{R}))$ by postconjugation by the target group; see [Par11, Théorème 23], building on ideas from [Wol07].

Using the theory of minimal vectors introduced by Richardson–Slowdowy [RS90], we recall that $\Xi(\Gamma, G(\mathbb{R}))$ is a semialgebraic set. Throughout the remainder of this section, we look at $G(\mathbb{R})$ as a semialgebraic subset of $M_{m\times m}(\mathbb{R})$ using the adjoint representation (see Proposition 2.3.1). The semialgebraic group $G(\mathbb{R})$ acts by conjugation on the real vector space $M_{m\times m}(\mathbb{R})^F$, which is endowed with the $\mathrm{SO}_n(\mathbb{R})$-invariant scalar product

$$\langle\, (A_1, \ldots, A_s)\ ,\ (B_1, \ldots, B_s)\, \rangle := \sum_{i=1}^{s} \mathrm{tr}\left(A_i^T B_i\right).$$

Denote by $\|\cdot\|$ its associated norm. The set of *minimal vectors* of $M_{m\times m}(\mathbb{R})^F$ for the $G(\mathbb{R})$-action by conjugation is

$$\mathcal{M} := \left\{ v \in M_{m\times m}(\mathbb{R})^F \;\middle|\; \|g.v\| \geq \|v\| \text{ for every } g \in G(\mathbb{R}) \right\}.$$

This defines a closed subset of $M_{m\times m}(\mathbb{R})^F$, and we show that it defines an algebraic set. Consider the involution $\sigma : g \mapsto (g^T)^{-1}$ on $G(\mathbb{R})$ which sends g to the inverse of its transpose. Then, $\mathrm{SO}_n(\mathbb{R})$ is the subgroup of fixed points of σ and

$$sym_n^0(\mathbb{R}) := \left\{ A \in M_{n\times n}(\mathbb{R}) \;\middle|\; \mathrm{tr}(A) = 0, A = A^T \right\}$$

is the -1 eigenspace of the Cartan involution $d_{\mathrm{Id}}\sigma$ on the Lie algebra of $G(\mathbb{R})$. This eigenspace also acts on $M_{m\times m}(\mathbb{R})^F$ by

$$\begin{aligned} Z.(A_1, \ldots, A_s) &= \frac{d}{dt}_{|t=0} \exp(tZ)(A_1, \ldots, A_s) \exp(-tZ) \\ &= ([Z, A_1], \ldots, [Z, A_s]) \quad \forall Z \in sym_n^0(\mathbb{R}). \end{aligned}$$

In this context, the following result from the Richardson–Slowdowy theory of minimal vectors holds.

Theorem 2.3.6 ([RS90, Theorem 4.3]). *An element $v \in M_{m\times m}(\mathbb{R})^F$ is in $\mathcal{M}$ if and only if $\langle\, Z.v\ ,\ v\,\rangle = 0$ for every $Z \in sym_n^0(\mathbb{R})$.*

Using the description of the action of $sym_n^0(\mathbb{R})$ on $M_{m\times m}(\mathbb{R})^F$ and, in the last equality, that $[A, A^T] \in sym_m^0(\mathbb{R})$ for all $A \in M_{m\times m}(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{M} &= \left\{ (A_1,\ldots,A_s) \in M_{m\times m}(\mathbb{R})^F \,\middle|\, \operatorname{tr}\left(\sum_{i=1}^{s} [Z, A_i]^T A_i\right) = 0\ \forall Z \in sym_n^0(\mathbb{R}) \right\} \\
&= \left\{ (A_1,\ldots,A_s) \in M_{m\times m}(\mathbb{R})^F \,\middle|\, \operatorname{tr}\left(\sum_{i=1}^{s} \left[A_i, A_i^T\right] Z\right) = 0\ \forall Z \in sym_n^0(\mathbb{R}) \right\} \\
&= \left\{ (A_1,\ldots,A_s) \in M_{m\times m}(\mathbb{R})^F \,\middle|\, \sum_{i=1}^{s} \left[A_i, A_i^T\right] = 0 \right\}.
\end{aligned}$$

In particular, $\mathcal{M}$ is an algebraic set defined over $\overline{\mathbb{Q}}^r$.

We now turn to the connection between the space of minimal vectors as described above and the set of representations of Γ in $G(\mathbb{R})$. Using Proposition 2.3.1 and the adjoint representation, we study the representation space within the real vector space $M_{m\times m}(\mathbb{R})^F$ using the evaluation of morphisms on the generating set F:

$$\begin{array}{rccc}
ev\colon & \operatorname{Hom}(\Gamma, G(\mathbb{R})) & \longrightarrow & G(\mathbb{R})^F \subset M_{m\times m}(\mathbb{R})^F; \\
& \rho & \longmapsto & (\rho(\gamma))_{\gamma\in F}.
\end{array}$$

The evaluation map is injective and its image $R^F(\Gamma, G(\mathbb{R}))$ is a closed real algebraic subset of $M_{m\times m}(\mathbb{R})^F$. The group $G(\mathbb{R})$ acts by conjugation on both $\operatorname{Hom}(\Gamma, G(\mathbb{R}))$ and $M_{m\times m}(\mathbb{R})^F$. Moreover, the evaluation map is $G(\mathbb{R})$-equivariant with respect to these actions and so its image is $G(\mathbb{R})$-invariant. From [Sik14, Theorem 30] (following an argument in [JM87]), the restriction to reductive homomorphisms has image

$$R^F_{red}(\Gamma, G(\mathbb{R})) = \left\{ v \in R^F(\Gamma, G(\mathbb{R})) \,\middle|\, G(\mathbb{R}).v \text{ is closed} \right\}. \tag{2.1}$$

The link with minimal vectors comes from the following result.

Theorem 2.3.7 ([RS90, Theorem 4.4]). *If $v \in M_{m\times m}(\mathbb{R})^F$, then the intersection $G(\mathbb{R}).v \cap \mathcal{M}$ is not empty if and only if $G(\mathbb{R}).v$ is closed.*

Hence, the set

$$\mathcal{M}_\Gamma := \mathcal{M} \cap R^F_{red}(\Gamma, G(\mathbb{R})) = \mathcal{M} \cap R^F(\Gamma, G(\mathbb{R}))$$

is a closed algebraic subset of $M_{m\times m}(\mathbb{R})^F$. Now, using results in the study of quotients by compact Lie groups [RS90, Subsection 7.1] (using [Sch75]), the quotient $\mathcal{M}_\Gamma/\mathrm{SO}_n(\mathbb{R})$ is homeomorphic to a closed semialgebraic set.

Theorem 2.3.8 ([RS90, Theorem 7.7])**.** *The inclusion* $\mathcal{M}_\Gamma \subseteq R^F_{red}(\Gamma, G(\mathbb{R}))$ *induces a homeomorphism between* $\mathcal{M}_\Gamma/\mathrm{SO}_n(\mathbb{R})$ *and the topological quotient*

$$R^F_{red}(\Gamma, G(\mathbb{R}))/G(\mathbb{R}).$$

This theorem and Equation (2.1) proves the wanted result.

Theorem 2.3.9 ([RS90, Section 7.1])**.** *The character variety of a finitely generated group* Γ *in* $G(\mathbb{R})$ *is a semialgebraic set which is homeomorphic to* $\mathcal{M}_\Gamma(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$*.*

Remark 2.3.10 ([Fla25, Lemma 3.7])**.** A semialgebraic model for the character variety is unique up to semialgebraic isomorphism. That is, if

$$\phi\colon \Xi(\Gamma, G(\mathbb{R})) \to \mathbb{R}^p \quad \text{and} \quad \phi'\colon \Xi(\Gamma, G(\mathbb{R})) \to \mathbb{R}^q$$

are two semialgebraic models, then there exists a unique semialgebraic isomorphism $f\colon \mathrm{Im}(\phi) \to \mathrm{Im}(\phi')$ with $f \circ \phi = \phi'$.

It is worth noting that the character variety is a semialgebraic set in the more general case when $G(\mathbb{R})$ is replaced by a connected semisimple algebraic group defined over the reals, see [Fla25, Section 3]. One aspect of our study is to define the real spectrum compactification of $\mathcal{M}_\Gamma(\mathbb{R})$ and $\Xi(\Gamma, G(\mathbb{R}))$, which requires them to be semialgebraic sets, see Chapter 4 and 5.

2.4 The symmetric space associated to $\mathrm{SL}_n(\mathbb{R})$

This section recalls the basic theory of the symmetric space associated to $\mathrm{SL}_n(\mathbb{R})$ and introduces a cover of it, which is an algebraic set. We then review the construction of a continuous semialgebraic multiplicative norm on both spaces using the Cartan projection defined on $\mathrm{SL}_n(\mathbb{R})$. This allows us to further study the space of minimal vectors and their dynamics in Subsection 4.2.3. This introduction is inspired by [BIPP23, Section 5].

The symmetric space associated to $\mathrm{SL}_n(\mathbb{R})$ is

$$\mathcal{P}^1(n, \mathbb{R}) := \{\, A \in M_{n\times n}(\mathbb{R}) \,|\, \det(A) = 1, A \text{ is symmetric and positive definite} \,\}.$$

From this definition, $\mathcal{P}^1(n)$ is a semialgebraic set and $\mathrm{SL}_n(\mathbb{R})$ acts transitively on $\mathcal{P}^1(n, \mathbb{R})$ via

$$gA := gAg^T$$

for all $g \in \mathrm{SL}_n(\mathbb{R})$ and all $A \in \mathcal{P}^1(n, \mathbb{R})$. Moreover, by Sylvester's criterion (see for example [HJ85, Theorem 7.2.5]), a symmetric matrix is positive definite if and only if all its principal minors are positive. So we define a cover of $\mathcal{P}^1(n, \mathbb{R})$ as

$$\widehat{\mathcal{P}^1(n, \mathbb{R})} := \left\{ (A, t) \in M_{n\times n}(\mathbb{R}) \times \mathbb{R}^{n-1} \,\middle|\, \begin{array}{c} A \text{ is symmetric, } \det(A) = 1, \\ \det\left(A\,[j]\right) t_j^2 = 1 \text{ for } 1 \le j \le n-1 \end{array} \right\}.$$

where $A[j]$ denotes the j-th leading principal minor of the matrix A. The group $\mathrm{SL}_n(\mathbb{R})$ acts on $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ via

$$g.(A, t_1, \ldots, t_{n-1}) := \left(gAg^T, t'_1, \ldots, t'_{n-1}\right),$$

for every $g \in \mathrm{SL}_n(\mathbb{R})$ and $(A, t_1, \ldots, t_{n-1}) \in \widehat{\mathcal{P}^1(n,\mathbb{R})}$ where

$$t'_j := \left(\frac{\det(A[j])}{\det(gAg^T[j])}\right)^{1/2} t_j$$

for every $1 \leq j \leq n-1$.

Remark 2.4.1. Note that $\mathcal{P}^1(n,\mathbb{R})$ is the quotient of $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ by the action of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ defined by $(z_1, \ldots, z_{n-1})(A, t_1, \ldots, t_{n-1}) = (A, \varepsilon_1 t_1, \ldots, \varepsilon_{n-1} t_{n-1})$, where $\varepsilon_i = 1$ if $z_i = 0$ or $\varepsilon_i = -1$ if $z_i = 1$. Moreover, the quotient map is $\mathrm{SL}_n(\mathbb{R})$-equivariant for the actions described above.

We now use the Cartan decomposition of $\mathrm{SL}_n(\mathbb{R})$ to construct a multiplicative distance on $\mathcal{P}^1(n,\mathbb{R})$, which we promote to $\widehat{\mathcal{P}^1(n,\mathbb{R})}$. As in the previous section, consider $\mathrm{SO}_n(\mathbb{R})$ the maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ associated with the involution $\sigma : g \mapsto (g^T)^{-1}$. Consider also $S(\mathbb{R})$ the maximal split torus of diagonal matrices in $\mathrm{SL}_n(\mathbb{R})$ and its *closed multiplicative Weyl chamber*

$$\overline{C^+(\mathbb{R})} := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{R}) \,\middle|\, \lambda_1 \geq \cdots \geq \lambda_n > 0 \right\}.$$

Then $\mathrm{SL}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R})\overline{C^+(\mathbb{R})}\mathrm{SO}_n(\mathbb{R})$, that is, each $g \in \mathrm{SL}_n(\mathbb{R})$ can be written as

$$g = k_1 c(g) k_2, \text{ where } k_1, k_2 \in \mathrm{SO}_n(\mathbb{R}), c(g) \in \overline{C^+(\mathbb{R})}.$$

Moreover, the element $c(g)$ in the decomposition is unique (see [Hel78, Chapter IX, Theorem 1.1] or [Kna02, Theorem 7.39]). This decomposition is called the *Cartan decomposition* of $\mathrm{SL}_n(\mathbb{R})$. In addition, the map $\mathrm{SL}_n(\mathbb{R}) \to \overline{C^+(\mathbb{R})}$ that sends g to $c(g)$ is semialgebraically continuous [BIPP23, Proposition 4.4]. A consequence of this decomposition is the existence of a *Cartan projection*, which is semialgebraically continuous by [BIPP23, Corollary 5.1]:

Lemma 2.4.2. *For every $A, B \in \mathcal{P}^1(n,\mathbb{R})$ the $\mathrm{SL}_n(\mathbb{R})$-orbit of (A, B) intersects $\{\mathrm{Id}\} \times \overline{C^+(\mathbb{R})}.\{\mathrm{Id}\}$ in exactly one point $(\mathrm{Id}, \delta(A,B).\mathrm{Id})$. Moreover, the Cartan projection*

$$\begin{array}{rccc} \delta\colon & \mathcal{P}^1(n,\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R}) & \longrightarrow & \overline{C^+(\mathbb{R})}; \\ & (A,B) & \longmapsto & \delta(A,B) \end{array}$$

is well-defined, invariant under the $\mathrm{SL}_n(\mathbb{R})$-action, and semialgebraically continuous.

From this map, we define the Cartan multiplicative distance d_δ and the genuine distance dist_δ on the symmetric space $\mathcal{P}^1(n,\mathbb{R})$. To show that it is indeed a multiplicative distance, we first need the forthcoming result of Planche [Pla95, Théorème 1]. Let $\mathfrak{a}$ be the Lie algebra of the maximal split torus $S(\mathbb{R})$ defined above, $\mathrm{Exp}\colon \mathfrak{a} \to S(\mathbb{R})$ the Riemann exponential map between $\mathfrak{a}$ and $S(\mathbb{R})$, and $L\colon S(\mathbb{R}) \to \mathfrak{a}$ its inverse.

Theorem 2.4.3 ([Pla95, Théorème 1])**.** *Given a Weyl group invariant norm $\|\cdot\|$ on $\mathfrak{a}$, the function*

$$\begin{array}{ccc} \mathcal{P}^1(n,\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R}) & \longrightarrow & \mathbb{R}_{\geq 0}; \\ (A,B) & \longmapsto & \|L(\delta(A,B))\|, \end{array}$$

is an $\mathrm{SL}_n(\mathbb{R})$-invariant distance function on $\mathcal{P}^1(n,\mathbb{R})$.

To apply results from real algebraic geometry, we are interested in a semialgebraically continuous multiplicative distance. To do this, consider the semialgebraically continuous map $N\colon S(\mathbb{R}) \to \mathbb{R}_{>0}$ that sends $\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ to $\max_{i\neq j} \lambda_i\lambda_j^{-1}$. It is a *multiplicative norm* $N\colon S(\mathbb{R}) \to \mathbb{R}_{>0}$. That is, a map which is invariant for the Weyl group action, and verifies:

- $N(gh) \leq N(g)N(h)$ for every $g,h \in S(\mathbb{R})$,
- $N(g) \geq 1$ for every $g \in S(\mathbb{R})$ with equality if and only if $g = \mathrm{Id}$,
- $N(g^n) = N(g)^{|n|}$ for every $g \in S(\mathbb{R})$ and $n \in \mathbb{Z}$.

We now introduce the Cartan's multiplicative distance which allows us, in Section 4.2.3, to define a universal symmetric space over the real spectrum compactification of $\mathcal{M}_\Gamma(\mathbb{R})$.

Proposition 2.4.4 ([BIPP23, Proposition 5.5])**.** *The Cartan's multiplicative distance is defined as*

$$\begin{array}{cccccc} d_\delta: & \mathcal{P}^1(n,\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R}) & \longrightarrow & \overline{C^+(\mathbb{R})} & \longrightarrow & \mathbb{R}_{\geq 1}; \\ & (A,B) & \longmapsto & \delta(A,B) & \longmapsto & N(\delta(A,B)) = \frac{\lambda_1}{\lambda_n}, \end{array}$$

where $\delta(A,B) = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. It is $\mathrm{SL}_n(\mathbb{R})$-invariant, semialgebraically continuous and verifies

1. *$d_\delta(x,z) \leq d_\delta(x,y)d_\delta(y,z)$ for every $x,y,z \in \mathcal{P}^1(n,\mathbb{R})$,*
2. *$d_\delta(x,y) = 1$ if and only if $x = y$.*

Proof. The action of the Weyl group, associated with $\mathrm{SL}_n(\mathbb{R})$, on $S(\mathbb{R})$ consists of permutations of the diagonal entries. Since $S(\mathbb{R})$ consists of diagonal matrices

$$\begin{array}{rrcl} \|\cdot\|: & \mathfrak{a} & \longrightarrow & \mathbb{R}_{\geq 0}; \\ & a & \longmapsto & \ln(N(\mathrm{Exp}(a))) \end{array}$$

is a Weyl group invariant norm on the Lie algebra $\mathfrak{a}$. Hence, by Theorem 2.4.3

$$\begin{array}{rlcl} \mathrm{dist}_\delta\colon & \mathcal{P}^1(n,\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R}) & \longrightarrow & \mathbb{R}_{\geq 0}; \\ & (A,B) & \longmapsto & \ln\left(N\left(\delta\left(A,B\right)\right)\right) \end{array}$$

is a $\mathrm{SL}_n(\mathbb{R})$-invariant distance so that d_δ is a $\mathrm{SL}_n(\mathbb{R})$-invariant multiplicative distance. Finally, from Lemma 2.4.2, δ is semialgebraically continuous, so d_δ is semialgebraically continuous as a composition of semialgebraically continuous maps. □

The multiplicative distance defined on $\mathcal{P}^1(n,\mathbb{R})$ extends to a multiplicative pseudo-distance on the cover $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ via

$$\begin{array}{rccc} \widehat{d_\delta}\colon & \widehat{\mathcal{P}^1(n,\mathbb{R})} \times \widehat{\mathcal{P}^1(n,\mathbb{R})} & \longrightarrow & \mathbb{R}_{\geq 1}; \\ & ((A,t_1,\ldots,t_{n-1}),(B,t'_1,\ldots,t'_{n-1})) & \longmapsto & d_\delta(A,B). \end{array}$$

We use the objects presented above in the following sections to introduce a subspace of the real spectrum compactification of $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$, invariant under the action of Γ. This subspace encodes the behavior of minimal representations and their induced Γ-actions on their associated symmetric space in the real spectrum. To this end, in the next section we introduce the real spectrum and the Archimedean spectrum of an algebraic set.

Chapter 3

Real spectrum compactification and Archimedean spectrum

We want to understand the degeneration of minimal vectors of representations. To do this, it is interesting to understand the manner in which representations go to infinity. To understand this behavior, we study the real spectrum compactification and the Archimedean spectrum of minimal vectors. The former has the advantage of providing a natural compactification for real points of a semialgebraic set with good topological properties. The second provides a natural framework for studying the $\mathbb{F}$-extension of the semialgebraic set when $\mathbb{F}$ is a non-Archimedean real closed field, and provides a better understanding of the geometry and dynamics of its $\mathbb{F}$-points.

3.1 The real and Archimedean spectra of rings

The real spectrum applies to commutative rings with unity and provides a natural functor from commutative rings to compact spaces. In this subsection, we first present this compactification and the accompanying notions essential for its study in our text. Our presentation follows [BCR13, Chapter 7], [Bru88a, Bru88c], and adapts [BIPP23, Section 2] to our context. Second, we define the concept of Archimedicity between rings, which allows us to characterize the closed points of the real spectrum and to introduce the Archimedean spectrum of a $\mathbb{K}$-algebra—both of which form notable subspaces of the real spectrum with interesting topological properties. Throughout this subsection, A denotes a commutative ring with unity.

Definition 3.1.1. The *real spectrum* A^{RSp} of a commutative ring A with unity is the set of *prime cones* of A. That is, the subsets $\alpha \subset A$ such that

- $-1 \notin \alpha$,

- $\alpha + \alpha \subset \alpha$ and $\alpha \cdot \alpha \subset \alpha$,
- $\alpha \cup (-\alpha) = A$,
- $\alpha \cap (-\alpha)$ is a prime ideal in A.

Another characterization of the points of the real spectrum uses real algebraic geometry. In particular ring morphisms to real closed fields.

Proposition 3.1.2 ([BCR13, Proposition 7.1.2]). *The following data are equivalent:*

1. *a prime cone $\alpha \subset A$,*
2. *a pair $(p, \leq_p)$ consisting of a prime ideal p and an ordering $\leq_p$ on the field of fractions of A/p,*
3. *an equivalence class of pairs $(\rho, \mathbb{F}_\rho)$ where $\rho\colon A \to \mathbb{F}_\rho$ is a ring homomorphism to a real closed field $\mathbb{F}_\rho$ which is the real closure of the field of fractions of $\rho(A)$ and $(\rho_1, \mathbb{F}_{\rho_1})$, $(\rho_2, \mathbb{F}_{\rho_2})$ are equivalent if there exists an ordered field isomorphism $\varphi\colon \mathbb{F}_{\rho_1} \to \mathbb{F}_{\rho_2}$ such that $\rho_2 = \varphi \circ \rho_1$.*
4. *an equivalence class of pairs $(\rho, \mathbb{F})$ where $\rho\colon A \to \mathbb{F}$ is a ring homomorphism to a real closed field $\mathbb{F}$, for the smallest equivalence relation such that $(\rho_1, \mathbb{F}_1)$ and $(\rho_2, \mathbb{F}_2)$ are equivalent if there is an ordered field homomorphism $\varphi\colon \mathbb{F}_1 \to \mathbb{F}_2$ such that $\rho_2 = \varphi \circ \rho_1$.*

One goes from 1 to 2 by considering the prime ideal $p = \alpha \cap (-\alpha)$ and the unique order $\leq_\alpha$ on $\mathrm{Frac}(A/p)$ whose set of positive elements is given by

$$\left\{ \frac{\overline{a}}{\overline{b}} \,\middle|\, ab \in \alpha, b \notin p \right\},$$

where $\overline{\cdot}\colon A \to A/p$ denotes the reduction modulo p and $\mathrm{Frac}(A/p)$ the fraction field of A/p. One goes from 2 to 3 and 4 by composing the reduction modulo p with the inclusion of $\mathrm{Frac}(A/p)$ into its real closure $\mathbb{F}$ with respect to the ordering $\leq_p$. Finally, one goes from 4 to 1 by considering the prime cone

$$\alpha := \{ a \in A \,|\, \rho(a) \geq 0 \},$$

where $(\rho, \mathbb{F})$ is the given ring homomorphism to a real closed field $\mathbb{F}$.

Employing the notation from the fourth item of Proposition 3.1.2, the *spectral topology* on the real spectrum is defined using a basis of open sets

$$\tilde{U}(a_1, \dots, a_p) := \{ (\rho, \mathbb{F}) \in A^{\mathrm{RSp}} \,|\, \rho(a_k) > 0 \quad \forall k \in \{ 1, \dots, p \} \},$$

where a_k are elements in A for every $k \in \{ 1, \dots, p \}$. With this topology, the real spectrum of a ring and its closed points have good properties.

Theorem 3.1.3 ([BCR13, Proposition 7.1.25 (ii)]). *The real spectrum of A is compact and its subset of closed points $A_{\mathrm{cl}}^{\mathrm{RSp}}$ is Hausdorff and compact.*

So, to each commutative ring with a unit, we have associated a compact topological space. Furthermore, if $\pi\colon A \to B$ is a ring homomorphism between the commutative rings A and B, then the *lift* of π to the real spectrum

$$\begin{array}{rccc} \pi^{\mathrm{RSp}}\colon & B^{\mathrm{RSp}} & \longrightarrow & A^{\mathrm{RSp}}; \\ & (\rho,\mathbb{F}) & \longmapsto & (\rho\circ\pi,\mathbb{F}) \end{array}$$

is a continuous map [BCR13, Proposition 7.1.7]. Thus, the following result gives a first motivation to study the real spectrum of rings.

Theorem 3.1.4 ([BCR13, Proposition 7.1.7]). *The functor $(-)^{\mathrm{RSp}}$ is contravariant from the category of commutative rings with unity to the category of compact topological spaces.*

Example 3.1.5 ([BIPP23, Example 2.24 (3)], [BCR13, Example 7.1.4] for $\mathbb{K}=\mathbb{R}$). Let $A=\mathbb{K}[x]$, where $\mathbb{K}\subset\mathbb{R}$ is real closed and x is a variable. For every $u\in\mathbb{R}$, set

$$\begin{aligned} \alpha_u &:= \{ f\in\mathbb{K}[x] \mid f(u)\geq 0 \}, \\ \alpha_{u^+} &:= \{ f\in\mathbb{K}[x] \mid \exists\varepsilon>0, \forall v\in]u,u+\varepsilon[,\ f(v)\geq 0 \}, \\ \alpha_{u^-} &:= \{ f\in\mathbb{K}[x] \mid \exists\varepsilon>0, \forall v\in]u-\varepsilon,u[,\ f(v)\geq 0 \}, \end{aligned}$$

which are prime cones of A. They verify $\alpha_{u^\pm}\subset\alpha_u$, with equality if and only if $u\notin\mathbb{K}$. The following prime cones complete the description of the real spectrum:

$$\begin{aligned} \alpha_{+\infty} &:= \{ f\in\mathbb{K}[x] \mid \exists m\in\mathbb{K}, \forall v\in]m,+\infty[,\ f(v)\geq 0 \}, \\ \alpha_{-\infty} &:= \{ f\in\mathbb{K}[x] \mid \exists m\in\mathbb{K}, \forall v\in]-\infty,m[,\ f(v)\geq 0 \}. \end{aligned}$$

By factoring polynomials, the topology of $\mathbb{K}[x]^{\mathrm{RSp}}$ has a basis of open subsets consisting of the intervals

$$\begin{aligned} \tilde{U}(x-s,-x+t) = [s^+,t^-] &= \{\alpha_u \mid s<u<t\}\cup\{\alpha_{u^-} \mid s<u\leq t\} \\ &\qquad \cup\{\alpha_{u^+} \mid s\leq u<t\}, \\ \tilde{U}(x-s) = [s^+,+\infty] &= \{\alpha_u \mid u>s\}\cup\{\alpha_{u^-} \mid u>s\} \\ &\qquad \cup\{\alpha_{u^+} \mid u\geq s\}\cup\{\alpha_{+\infty}\}, \\ \tilde{U}(-x+t) = [-\infty,t^-] &= \{\alpha_u \mid u<t\}\cup\{\alpha_{u^-} \mid u\leq t\} \\ &\qquad \cup\{\alpha_{u^+} \mid u<t\}\cup\{\alpha_{-\infty}\}, \end{aligned}$$

where $s<t$ are two elements of $\mathbb{R}$. Note that $\mathbb{K}[x]^{\mathrm{RSp}}$ is not a Hausdorff space since α_u belongs to the closure of both α_{u^+} and α_{u^-}.

To define the Archimedean spectrum, we recall the concept of Archimedicity in the context of general real fields.

Definition 3.1.6 ([BIPP23, Definition 2.26])**.** Let $R_1 \subset R_2$ be subrings of an ordered field. The subring R_2 is *Archimedean* over R_1 if every element of R_2 is bounded above by some element of R_1.

Before defining the Archimedean spectrum, we note that this definition allows us to characterize the closed points of the real spectrum in terms of Archimedicity. Recall that by the third item of Proposition 3.1.2, an element $(\rho, \mathbb{F}_\rho) \in A^{\mathrm{RSp}}_{\mathrm{cl}}$ is an equivalence class of ring morphisms to a real closed field $\mathbb{F}_\rho$, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(A)$.

Proposition 3.1.7 ([BIPP23, Proposition 2.27])**.** *In the notation of the third item of Proposition 3.1.2, if $(\rho, \mathbb{F}_\rho)$ is an element of A^{RSp}, then $(\rho, \mathbb{F}_\rho)$ is closed in the spectral topology if and only if $\mathbb{F}_\rho$ is Archimedean over $\rho(A)$.*

Let now A be a $\mathbb{K}$-algebra where $\mathbb{K}$ is a real closed field. The Archimedean spectrum is, informally, a subset of the real spectrum whose real closed fields are Archimedean over the ground field.

Remark 3.1.8. Let A be a $\mathbb{K}$-algebra where $\mathbb{K}$ is a real closed field and $(\rho, \mathbb{F}) \in A^{\mathrm{RSp}}_{\mathrm{cl}}$. The composition of the inclusion $\mathbb{K} \to A$ with $\rho\colon A \to \mathbb{F}$ gives a field morphism $\mathbb{K} \to \mathbb{F}$. Thus $\mathbb{K}$ is a subfield of $\mathbb{F}$. Since every positive element of $\mathbb{K}$ is a square, the order on $\mathbb{F}$ extends the order on $\mathbb{K}$.

Definition 3.1.9. Given $\mathbb{K}$ a real closed field and A a $\mathbb{K}$-algebra, the *Archimedean spectrum* of A is the set

$$A^{\mathrm{RSp}}_{\mathrm{Arch}} := \big\{ (\rho, \mathbb{F}_\rho) \in A^{\mathrm{RSp}} \,\big|\, \mathbb{F}_\rho \text{ is Archimedean over } \mathbb{K} \big\},$$

endowed with the subspace topology from the spectral topology, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(A)$.

In particular, from Proposition 3.1.7, $A^{\mathrm{RSp}}_{\mathrm{Arch}} \subset A^{\mathrm{RSp}}_{\mathrm{cl}}$. In addition, as for the real spectrum, the Archimedean spectrum is a functor.

Proposition 3.1.10. *The functor $(-)^{\mathrm{RSp}}_{\mathrm{Arch}}$ is contravariant from the category of $\mathbb{K}$-algebras to the category of topological spaces.*

Proof. Consider A, B two $\mathbb{K}$-algebras and $\pi\colon A \to B$ a $\mathbb{K}$-algebra morphism. From Theorem 3.1.4,

$$\begin{array}{rccc} \pi^{\mathrm{RSp}}\colon & B^{\mathrm{RSp}} & \longrightarrow & A^{\mathrm{RSp}}; \\ & (\rho, \mathbb{F}_\rho) & \longmapsto & (\rho \circ \pi, \mathbb{F}_\rho) \end{array}$$

is a continuous map. Consider $\pi^{\text{RSp}}_{\text{Arch}}$ the restriction of π^{RSp} to the Archimedean spectrum of B. For an element $(\rho, \mathbb{F}_\rho) \in B^{\text{RSp}}_{\text{Arch}}$, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(B)$, $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}$. So in particular, the image of $\pi^{\text{RSp}}_{\text{Arch}}$ is contained in $A^{\text{RSp}}_{\text{Arch}}$. Hence, $(-)^{\text{RSp}}_{\text{Arch}}$ is a contravariant functor from the category of $\mathbb{K}$-algebras to the category of topological spaces. □

We presented a contravariant functor $(-)^{\text{RSp}}$ from commutative rings with a unit to compact topological spaces. Additionally, we characterized the closed points of these induced topological spaces and introduced the Archimedean spectrum. In the next section, we use this framework, along with the coordinate ring of algebraic sets, to examine functorial properties of the closed points of the real spectrum of algebraic sets.

3.2 Functoriality of the closed points of the real spectrum of algebraic sets

Using the coordinate ring of the $\mathbb{K}$-extension of an algebraic set $V \subset \mathbb{L}^n$ for some real closed fields $\mathbb{K}$ such that $\mathbb{L} \subset \mathbb{K}$, we show that $(-)^{\text{RSp}}_{\text{cl}}$ is a functor sending proper algebraic maps to continuous maps—in the coming section, we apply this concept to the algebraic sets $\mathcal{M}_\Gamma$ and $\widehat{\mathcal{P}^1(n)}$ defined over $\overline{\mathbb{Q}}^r$. For the rest of the subsection, let $\mathbb{L}, \mathbb{K}$ be real closed fields such that $\mathbb{L} \subset \mathbb{K}$. Consider an algebraic set $V \subset \mathbb{L}^n$ and recall that $V(\mathbb{K})$ denotes the $\mathbb{K}$-extension of V. Denote by

$$V(\mathbb{K})^{\text{RSp}} := \mathbb{K}[V]^{\text{RSp}}$$

the real spectrum of $V(\mathbb{K})$, where $\mathbb{K}[V]$ is the coordinate ring of $V(\mathbb{K})$, see Definition 2.2.5. Moreover, we endowed the $\mathbb{K}$-points $V(\mathbb{K})$ with the Euclidean topology coming from the norm $N : \mathbb{K}^n \to \mathbb{K}_{\geq 0}$, as defined in Subsection 2.2.

Remark 3.2.1. The Euclidean topology on $V(\mathbb{K})$ is equivalent to the topology generated by the basis of open sets

$$U(f_1, \ldots, f_p) := \{\, v \in V(\mathbb{K}) \mid f_1(v) > 0, \ldots, f_p(v) > 0 \,\},$$

for $f_1, \ldots, f_p \in \mathbb{K}[V]$, see [BCR13, Subsection 2.1].

Lemma 3.2.2. *Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be algebraic sets. If the coordinate ring of $V(\mathbb{R})$ is $\mathbb{R}[V] = \mathbb{R}[x_1, \ldots, x_n]/I(V)$ and $\pi : V(\mathbb{R}) \to W(\mathbb{R})$ is a proper algebraic map, then there exist constants $c, d \in \mathbb{N}$ such that for every $v \in V(\mathbb{R})$:*

$$|x_i(v)| \leq c \left(1 + N(\pi(v))^2\right)^d .$$

Proof. Since π is proper, for every $w \in W(\mathbb{R})$ the fiber $\pi^{-1}(w) \subset V(\mathbb{R})$ is compact in the Euclidean topology. The map $x_i : V \to \mathbb{R}$ is continuous, so that it attains its maximum on $\pi^{-1}(w)$ by the extreme value theorem. Hence, the map

$$\begin{array}{rccc} \tilde{v}_i \colon & W(\mathbb{R}) & \longrightarrow & \mathbb{R}; \\ & w & \longmapsto & \max \{ x_i(v) \mid v \in \pi^{-1}(w) \} \end{array}$$

is well defined. Consider the graph $X_\pi := \{ (v, w) \in (V \times W)(\mathbb{R}) \mid \pi(v) = w \}$ of π, which is algebraic because V, W, and π are algebraic and $X_{\tilde{v}_i} := \{ (w, t) \in W(\mathbb{R}) \times \mathbb{R} \mid \tilde{v}_i(w) = t \}$ the graph of $\tilde{v}_i$. By the definition of $\tilde{v}_i$, it holds

$$\begin{aligned} \tilde{v}_i(w) = t \quad \text{if and only if} \quad & \exists v \in V(\mathbb{R}),\ \pi(v) = w,\ x_i(v) = t, \\ & \text{and } \forall v' \in V(\mathbb{R}),\ \pi(v') = w \Rightarrow x_i(v') \leq t. \end{aligned}$$

Since all sets and maps involved are semialgebraic, the set

$$X = \left\{ (v, w, t) \in (V \times W)(\mathbb{R}) \times \mathbb{R} \,\middle|\, \begin{array}{l} \pi(v) = w,\ x_i(v) = t, \\ \text{and } \ \forall v' \in V,\ \pi(v') = w \Rightarrow x_i(v') \leq t \end{array} \right\}$$

is semialgebraic. In particular, its projection on $W(\mathbb{R}) \times \mathbb{R}$ is a semialgebraic set [BCR13, Theorem 2.2.1], which identifies with $X_{\tilde{v}_i}$. Thus $\tilde{v}_i$ is semialgebraic. By [BCR13, Proposition 2.6.2], there exist constants $c > 0$ and $d \in \mathbb{N}$ such that

$$|\tilde{v}_i(w)| \leq c \left(1 + N(w)^2\right)^d \quad \text{for all } w \in W(\mathbb{R}).$$

So in particular, $|x_i(v)| \leq c(1 + N(\pi(v))^2)^d$ for every $v \in V(\mathbb{R})$. □

To do computation in the real spectrum, we use the following notation:

Notation 3.2.3 ([BIPP23, Remark 2.36]). Let $\mathbb{L}, \mathbb{K}$ be real closed fields such that $\mathbb{L} \subset \mathbb{K}$, $V \subset \mathbb{L}^n$ an algebraic set, and $(\rho, \mathbb{F}_\rho) \in \mathbb{K}[V]^{\mathrm{RSp}}$, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(\mathbb{K}[x_1, \ldots, x_n])$. If $x_\rho = (\rho(x_1), \ldots, \rho(x_n)) \in V(\mathbb{F}_\rho)$, then

$$(\rho, \mathbb{F}_\rho) = (ev(x_\rho), \mathbb{F}_\rho).$$

Thus, for all $f \in \mathbb{K}[V]$, we note $f(x_\rho) := \rho(f)$.

Theorem 3.2.4. *Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be algebraic sets. If $\pi \colon V(\mathbb{R}) \to W(\mathbb{R})$ is a proper surjective algebraic map, then the image of $\pi_{\mathrm{cl}}^{\mathrm{RSp}}$, the restriction to the closed points of the induced map π^{RSp}, is $W(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$. That is, we have a continuous surjective map*

$$\pi_{\mathrm{cl}}^{\mathrm{RSp}} \colon V(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}} \twoheadrightarrow W(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}.$$

Proof. As in the notation of the third item of Proposition 3.1.2, consider $(\rho, \mathbb{F}_\rho) \in V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(\mathbb{R}[V])$. We first prove that $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[W])$. Consider the coordinate ring

$$\mathbb{R}[V] := \mathbb{R}[x_1, \dots, x_n]/I(V),$$

as in Definition 2.2.5 and the element $v_\rho := (\rho(x_1), \dots, \rho(x_n)) \in V(\mathbb{F}_\rho)$ defined in Notation 3.2.3. By Lemma 3.2.2, there exist constants $c, d \in \mathbb{N}$ such that

$$\text{for every } v \in V(\mathbb{R}) : |x_i(v)| \leq c\left(1 + N(\pi(v))^2\right)^d.$$

So by the Transfer principle (Theorem 2.2.2) the following inequality holds

$$|x_i(v_\rho)| \leq c(1 + N\left(\pi_{\mathbb{F}_\rho}(v_\rho)\right)^2)^d,$$

where $\pi_{\mathbb{F}_\rho}$ is the $\mathbb{F}_\rho$-extension of π. Hence $x_i(v_\rho)$ is bounded by some element of $\rho(\mathbb{R}[W])$. Finally, since $(\rho, \mathbb{F}_\rho)$ is a closed point of the real spectrum, $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[V])$ by Proposition 3.1.7. Hence $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[W])$ so that

$$\mathrm{Im}\left(\pi^{\mathrm{RSp}}_{\mathrm{cl}}\right) \subset W(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Finally, $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$ is surjective by [BIPP23, Lemma 7.6] so that

$$\mathrm{Im}\left(\pi^{\mathrm{RSp}}_{\mathrm{cl}}\right) = W(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

□

Under the right setting, $(-)^{\mathrm{RSp}}_{\mathrm{cl}}$ defines a functor from algebraic sets with proper algebraic maps to compact spaces with continuous maps. In the next section, we use $(-)^{\mathrm{RSp}}_{\mathrm{cl}}$ to define a compactification of any semialgebraic set.

3.3 Constructible sets and real spectrum compactification of semialgebraic sets

This section recalls the definition of the real spectrum compactification of a semialgebraic set. To do so, we use the closed points of the real spectrum and study the constructible subsets of the real spectrum. The Euclidean topology allows us to embed an algebraic set into its Archimedean spectrum.

Theorem 3.3.1 ([BCR13, Proposition 7.1.5] and [BCR13, Proposition 7.1.5])**.** *Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields and $V \subset \mathbb{L}^n$ an algebraic set, then the evaluation map*

$$\begin{array}{rccc} ev\colon & V(\mathbb{K}) & \longrightarrow & V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}; \\ & (v_1, \dots, v_n) & \longmapsto & (ev(v_1, \dots, v_n), \mathbb{K}) \end{array}$$

is a continuous injection from $V(\mathbb{K})$, with its Euclidean topology, to $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ with its spectral topology. Moreover, $V(\mathbb{K})$ is dense in $V(\mathbb{K})^{\mathrm{RSp}}$ and so in its Archimedean spectrum.

Proof. By [BCR13, Proposition 7.1.5], $ev\colon V(\mathbb{K}) \to V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is continuous and injective. By [BIPP23, Corollary 2.32], $V(\mathbb{K})$ is dense in $V(\mathbb{K})^{\mathrm{RSp}}$. □

From Theorem 3.3.1, we define the *real spectrum compactification* of $V(\mathbb{K})$ as the closure of the image of the evaluation map in the spectral topology:

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} := \overline{ev(V(\mathbb{K}))}.$$

We will need in Chapter 5 a stronger statement in the special case when $\mathbb{L} = \overline{\mathbb{Q}}^r$:

Proposition 3.3.2 ([BIPP23, Proposition 2.33]). *Let $V \subset (\overline{\mathbb{Q}}^r)^n$ be an algebraic set. The evaluation map*

$$\begin{array}{rccc} ev\colon & V(\mathbb{R}) & \longrightarrow & V\left(\overline{\mathbb{Q}}^r\right)^{\mathrm{RSp}}_{\mathrm{cl}}; \\ & (v_1,\dots,v_n) & \longmapsto & (ev(v_1,\dots,v_n),\mathbb{R}) \end{array}$$

is a homeomorphism from $V(\mathbb{R})$, with its Euclidean topology, onto its image with the spectral topology. Moreover, $V(\mathbb{R})$ is open and dense in $V(\overline{\mathbb{Q}}^r)^{\mathrm{RSp}}_{\mathrm{cl}}$ and $V(\overline{\mathbb{Q}}^r)^{\mathrm{RSp}}_{\mathrm{cl}}$ is metrizable.

Now that we defined the real spectrum compactification of an algebraic set, we introduce the corresponding construction for semialgebraic sets—such as the character variety. To do so, we first need constructible sets.

Definition 3.3.3 ([BCR13, Definition 7.1.10]). Let A be a commutative ring with a unit. A *constructible subset* of A^{RSp} is a boolean combination of basic open subsets $\tilde{U}(a_1,\dots,a_n)$. That is, obtained from the basic open sets of the real spectrum topology by taking finite unions, finite intersections and complements.

Constructible sets form the essential building blocks of compact sets in the real spectrum topology. Moreover, they offer a correspondence between semialgebraic subsets of an algebraic set V and compact subsets of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. They are therefore essential, as the following result shows.

Proposition 3.3.4 ([BCR13, Corollary 7.1.13, Proposition 7.2.2, and Theorem 7.2.3]). *Let $\mathbb{L},\mathbb{K}$ be real closed fields such that $\mathbb{L} \subset \mathbb{K}$, $V \subset \mathbb{L}^n$ an algebraic set, and S a semialgebraic subset of V.*

1. *Every constructible subset of $V(\mathbb{K})^{\mathrm{RSp}}$ is compact with respect to the spectral topology. Moreover, an open subset of $V(\mathbb{K})^{\mathrm{RSp}}$ is constructible if and only if it is compact.*

2. *There exists a unique constructible set* $\widetilde{S(\mathbb{K})} \subset V(\mathbb{K})^{\mathrm{RSp}}$, *so that*

$$\widetilde{S(\mathbb{K})} \cap V(\mathbb{K}) = S(\mathbb{K}).$$

3. *If* $S(\mathbb{K})$ *is a boolean combination of*

$$U(f_i) = \{\, v \in V(\mathbb{K}) \mid f_i(v) > 0 \,\},$$

where $f_i \in \mathbb{K}[V]$, *then* $\widetilde{S(\mathbb{K})}$ *is the same boolean combination of*

$$\tilde{U}(f_i) = \{\, (\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}} \mid \rho(f_i) > 0 \,\}.$$

4. *The mapping* $S(\mathbb{K}) \mapsto \widetilde{S(\mathbb{K})}$ *is an isomorphism from the boolean algebra of semialgebraic subsets of* $V(\mathbb{K})$ *onto the boolean algebra of constructible subsets of* $V(\mathbb{K})^{\mathrm{RSp}}$.

5. *The semialgebraic set* $S(\mathbb{K})$ *is open (respectively closed) in* $V(\mathbb{K})$ *if and only if* $\widetilde{S(\mathbb{K})}$ *is open (respectively closed) in* $V(\mathbb{K})^{\mathrm{RSp}}$. *Hence, the isomorphism* $S(\mathbb{K}) \mapsto \widetilde{S(\mathbb{K})}$ *induces a bijection from the family of open semialgebraic subsets of* $V(\mathbb{K})$ *onto the family of compact open subsets of* $V(\mathbb{K})^{\mathrm{RSp}}$.

Constructible sets can exhibit surprising behavior. In the following example, the closed points of a constructible set are not the intersection of the closed points of the algebraic set with the constructible set.

Example 3.3.5 ([BIPP23, Example 2.34])**.** Let $\mathbb{A}^1 \subset \overline{\mathbb{Q}}^r$ denotes the affine line, and let $\mathbb{K}$ be a real closed subfield of $\mathbb{R}$. For all $u \in \mathbb{K}$

$$\overline{\{\, \alpha_{u^+} \,\}} = \{\, \alpha_u, \alpha_{u^+} \,\} \text{ and } \overline{\{\, \alpha_{u^-} \,\}} = \{\, \alpha_u, \alpha_{u^-} \,\}.$$

Moreover, for $u \in \mathbb{R} \backslash \mathbb{K}$ it holds $\alpha_{u^+} = \alpha_{u^-} = \alpha_u$, which are closed points of the real spectrum. Thus

$$\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} = \{\alpha_u \mid u \in \mathbb{R}\,\} \cup \{\, \alpha_{\pm\infty} \,\},$$

where $\mathbb{R}$ embeds as an open and dense subset. Semialgebraic subsets of $\mathbb{A}^1(\mathbb{K})$ are finite unions of intervals and half-lines with endpoints in $\mathbb{K}$. For the semialgebraic set $S(\mathbb{K}) := (s,t] \cap \mathbb{K}$ with $s, t \in \mathbb{K}$, it holds

$$\widetilde{S(\mathbb{K})} = (s,t] \cup \{\alpha_{u^\pm} \mid u \in (s,t) \cap \mathbb{K}\,\} \cup \{\, \alpha_{s^+}, \alpha_{t^-} \,\},$$
$$\widetilde{S(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} = S(\mathbb{K}).$$

In particular, $\widetilde{S(\mathbb{K})} \cap \mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ is not compact. However, the closed points of $\widetilde{S(\mathbb{K})}$ are

$$\widetilde{S(\mathbb{K})}_{\mathrm{cl}} = (s,t] \cup \{ \alpha_s{}^+ \} ,$$

which is homeomorphic to a closed segment in $\mathbb{R}$. Note that α_{s^+} is closed in $\widetilde{S(\mathbb{K})}$ but not in $\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}$.

If W is a semialgebraic subset of $V \subset \mathbb{R}^n$, the *real spectrum compactification* $W^{\mathrm{RSp}}_{\mathrm{cl}}$ of W is the set of closed points of $\widetilde{W(\mathbb{R})}$. When W is closed, then

$$W^{\mathrm{RSp}}_{\mathrm{cl}} := \widetilde{W(\mathbb{R})} \cap V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} .$$

Proposition 3.3.6 ([BIPP23, Proposition 2.33]). *Let $V \subset (\overline{\mathbb{Q}}^r)^n$ be an algebraic set and $W \subset V$ a closed semialgebraic subset. The space $W(\mathbb{R})$ is open and dense in the Hausdorff and compact space $W^{\mathrm{RSp}}_{\mathrm{cl}}$.*

Remark 3.3.7. We study the real spectrum compactification of semialgebraic models of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, $\mathcal{M}_\Gamma$ and $\widehat{\mathcal{P}^1(n)}$ defined over $\overline{\mathbb{Q}}^r$, see Subsection 2.3. These models depend on a choice of coordinates. However, every choice of coordinates leads to models that are related by a canonical semialgebraic isomorphism. Consequently, the real spectrum compactifications of both models are homeomorphic [BCR13, Proposition 7.2.8] and the real spectrum compactifications $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ and $\widehat{\mathcal{P}^1(n,\mathbb{R})}^{\mathrm{RSp}}_{\mathrm{cl}}$ are canonical up to homeomorphism.

In Chapter 5, we define a continuous surjection from $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ to $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ motivated, in part, by the following result.

Theorem 3.3.8 ([Bru88a, Proposition 7.2]). *There exists a continuous surjection from the real spectrum compactification of the space of marked hyperbolic structures on a surface to its length spectrum compactification (Thurston compactification).*

Remark 3.3.9. There exists also a continuous surjection between the real spectrum compactification and the Weyl chamber length compactification (which generalizes the length spectrum compactification) of higher rank character varieties, where we replace $\mathrm{PSL}_2(\mathbb{R})$ by $\mathrm{PSL}_n(\mathbb{R})$ in the definition of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, see [BIPP23, Theorem 8.2].

Despite its name, the real spectrum compactification is not necessarily a natural compactification of $V(\mathbb{K})$, where a *natural compactification* of a topological space X is a Hausdorff compact topological space Y such that $X \subset Y$ and X is open and dense in Y.

Example 3.3.10. Consider the affine line $\mathbb{A}^1 \subset \overline{\mathbb{Q}}^r$. By Example 3.1.5, there is an homeomorphism

$$\mathbb{A}^1\left(\overline{\mathbb{Q}}^r\right)^{\mathrm{RSp}}_{\mathrm{cl}} = \overline{\mathbb{Q}}^r[x]^{\mathrm{RSp}}_{\mathrm{cl}} \cong \mathbb{R} \cup \{\pm\infty\}.$$

By density of the transcendental numbers in $\mathbb{R}$, the topological space $\mathbb{A}^1(\overline{\mathbb{Q}}^r)$ is not open in $\mathbb{A}^1(\overline{\mathbb{Q}}^r)^{\mathrm{RSp}}_{\mathrm{cl}}$ such that the real spectrum compactification is not a natural compactification.

We introduced the real spectrum compactification of a semialgebraic set and established the proper framework to demonstrate its functoriality. However, depending on the field over which the algebraic set is defined, this compactification is not always natural. To address this issue, we further investigate the topological properties of the Archimedean spectrum.

3.4 Local compactness of the Archimedean spectrum of algebraic sets

Under suitable conditions on $\mathbb{K}$, the Archimedean spectrum is a locally compact topological space open in $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. This makes $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ a natural compactification of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$. To illustrate this, we compute the Archimedean spectrum of the $\mathbb{K}$-extension of V when $\mathbb{K}$ is Archimedean, as well as the Archimedean spectrum of the affine line over any real closed field.

Definition 3.4.1. An element $b \in \mathbb{K}$ is called *big element*, if for every $c \in \mathbb{K}$, there exists $k \in \mathbb{N}$ that verifies $c < b^k$.

For the remainder of the subsection, $\mathbb{L}$ is a real closed field, $\mathbb{L} \subset \mathbb{K}$ a real closed field with a big element, $V \subset \mathbb{L}^n$ an algebraic set, and we show that $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ is a natural compactification of the topological space $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$.

Remark 3.4.2. For every $(\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$, $\mathbb{F}_\rho$ is a real closed field of finite transcendence degree over $\mathbb{K}$, see Proposition 3.1.2. In particular, if $\mathbb{K}$ has a big element, then $\mathbb{F}_\rho$ also has a big element [Bru88a, Section 5]. Hence, if $\mathbb{K}$ is a real closed field that appears in the boundary of $V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, where $V \subset (\overline{\mathbb{Q}}^r)^n$ is an algebraic set, then the real closed fields that appear in $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ are also real closed fields of finite transcendence degree over $\overline{\mathbb{Q}}^r$ that contain a big element [Bru88a, Section 5].

In this setting, we give a description of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ as an open and dense subset of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ which is a countable union of compact sets.

Theorem 3.4.3. *Let $\mathbb{L}$ be a real closed field, $\mathbb{L} \subset \mathbb{K}$ a real closed field with a big element b, and $V \subset \mathbb{L}^n$ an algebraic set. The Archimedean spectrum of $V(\mathbb{K})$ is an open subset of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ which is a countable union of compact subsets of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. In particular, the space $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is σ-compact and locally compact.*

Proof. Consider the coordinates $x_1, \ldots, x_n$ such that $\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]/I(V)$, where $I(V)$ is the ideal of polynomials vanishing on V. Write $x = (x_1, \ldots, x_n)$ and let $f \in \mathbb{K}[V]$ be an element with coordinate decomposition

$$f(x) := \sum_{I \text{ multiindex}} c_I x^I,$$

where if $I = (i_1, \ldots, i_n)$ for some $i_j \geq 0$, then $x^I := x_1^{i_1} \cdots x_n^{i_n}$. By the Cauchy–Schwartz inequality—a consequence of the Transfer principle (Theorem 2.2.2), it holds for every $v \in \mathbb{K}^n$

$$f(v)^2 = \left(\sum_I c_I v^I \right)^2 \leq \left(\sum c_I^2 \right) \left(\sum v^{2I} \right).$$

Define $g(x) := \sum x_i^2$ so that $v_i^2 \leq g(v)$. With the notation $|I| := \sum i_j$, it holds $v^{2I} \leq g(v)^{|I|}$ and

$$f(v)^2 \leq \left(\sum c_I^2 \right) \left(\sum g(v)^{|I|} \right).$$

Either $g(v) \leq 1$ and

$$f(v)^2 \leq \left(\sum c_I^2 \right) |\mathrm{supp}(f)|,$$

where $\mathrm{supp}(f) := \{ I \text{ multiindex} \mid c_I \neq 0 \}$. Or $g(v) \geq 1$ and

$$f(v)^2 \leq \left(\sum c_I^2 \right) |\mathrm{supp}(f)| g(v)^{d(f)},$$

where $d(f) := \max\{ |I| \mid I \in \mathrm{supp}(f) \}$. In either case, since b is a big element of $\mathbb{K}$, there exists $m \in \mathbb{N}$ such that

$$f(v)^2 \leq b^m \left(1 + g(v)^{d(f)} \right) \quad \forall v \in \mathbb{K}^n.$$

Let $(\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}}$ and x_ρ be the point of $V(\mathbb{F}_\rho)$ defined in Notation 3.2.3. The previous inequality holds in every real closed field. Thus, for every $f \in \mathbb{K}[V]$, there exist $m \in \mathbb{N}$ such that

$$f(x_\rho)^2 \leq b^m \left(1 + g(x_\rho)^{d(f)} \right).$$

So, define for every $k \in \mathbb{N}$, the open set $A_k(\mathbb{K})$ and the closed set $B_k(\mathbb{K})$ as

$$A_k(\mathbb{K}) := \left\{ v \in V(\mathbb{K}) \,\middle|\, \sum v_i^2 < b^k \right\}, \; B_k(\mathbb{K}) := \left\{ v \in V(\mathbb{K}) \,\middle|\, \sum v_i^2 \leq b^k \right\}.$$

Claim: The Archimedean spectrum of $V(\mathbb{K})$ contains $V(\mathbb{K})$ and is an open subset of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ as the following equality holds

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} = \bigcup_{k\in\mathbb{N}} \widetilde{A_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Moreover, $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is a countable union of compact sets as

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} = \bigcup_{k\in\mathbb{N}} \widetilde{B_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Proof: On the one hand, if $(\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$, then $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}$. In particular, it is Archimedean over $\rho(\mathbb{K}[V])$ (see Remark 3.1.8) so that $(\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. Since $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}$ and b is a big element of $\mathbb{K}$, b is also a big element of $\mathbb{F}_\rho$. Thus there exists a natural number k that verifies $g(x_\rho) < b^k$ so that the following inclusions hold

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \subset \bigcup_{k\in\mathbb{N}} \widetilde{A_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} \subset \bigcup_{k\in\mathbb{N}} \widetilde{B_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

On the other hand, consider $(\rho, \mathbb{F}_\rho) \in \widetilde{B_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ for some fixed $k \in \mathbb{N}$. By Proposition 3.1.7, $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{K}[V])$ and we show that $\rho(\mathbb{K}[V])$ is Archimedean over $\mathbb{K}$. Consider $f \in \mathbb{K}[V]$ and $m_1 \in \mathbb{N}$ such that

$$f(x_\rho)^2 < b^{m_1}\left(1 + g(x_\rho)^{d(f)}\right).$$

Since $(\rho, \mathbb{F}_\rho) \in \widetilde{B_k(\mathbb{K})}$ it holds $g(x_\rho) < b^k$. So, there exists $m_2 \in \mathbb{N}$ with

$$f(x_\rho)^2 < b^{m_1+m_2}.$$

Hence $\rho(\mathbb{K}[V])$ is Archimedean over $\mathbb{K}$. Thus, also $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}$ and

$$V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} = \bigcup_{k\in\mathbb{N}} \widetilde{A_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}} = \bigcup_{k\in\mathbb{N}} \widetilde{B_k(\mathbb{K})} \cap V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

In particular, the Archimedean spectrum of an algebraic set is the intersection of a countable union of open constructible sets with $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. Consequently, it forms an open subset of a compact Hausdorff space and, therefore, is locally compact [Mun18, Corollary 29.2]. □

Corollary 3.4.4. *Let $\mathbb{L}$ be a real closed field, $\mathbb{L} \subset \mathbb{K}$ a real closed field with a big element b, and $V \subset \mathbb{L}^n$ an algebraic set. The topological space $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is an open and dense subset of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. In particular, $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ is a natural compactification of $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$.*

Proof. By Theorem 3.4.3, $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is open in $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. Moreover, the evaluation map on elements of $V(\mathbb{K})$ provides a topological embedding of $V(\mathbb{K})$ in $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ as a dense subset (Theorem 3.3.1). Thus $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ is also dense in $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$. □

Proposition 3.4.5. *Let $\mathbb{L}, \mathbb{K}$ be real closed fields such that $\mathbb{L} \subset \mathbb{K}$, with $\mathbb{K}$ is Archimedean over $\mathbb{Z}$, and $V \subset \mathbb{L}^n$ an algebraic set. The evaluation map*

$$\begin{array}{rccc} ev\colon & V(\mathbb{R}) & \longrightarrow & V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}; \\ & (v_1, \ldots, v_n) & \longmapsto & (ev(v_1, \ldots, v_n), \mathbb{R}) \end{array}$$

is an homeomorphism from $V(\mathbb{R})$, with its Euclidean topology, to $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ with its spectral topology. Thus, $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ is a natural compactification of $V(\mathbb{R})$.

Proof. First, Notice that $\mathbb{R}$ is Archimedean over $\mathbb{Z}$ so that ev is well defined. By [BIPP23, Proposition 2.33 (1)], ev is a continous and open injection from $V(\mathbb{R})$ with its Euclidean topology to $V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$ with its spectral topology, see Remark 3.2.1. So it remains to show that ev is a surjection onto the Archimedean spectrum.

Let $(\rho, \mathbb{F}_\rho) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ and consider $\sigma\colon \mathbb{F}_\rho \to \mathbb{R}$ an ordered field monomorphism as in [Hal11, Theorem 3.5]. Then by the fourth item of Proposition 3.1.2

$$(\rho, \mathbb{F}_\rho) = (\sigma\rho, \mathbb{R}) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}.$$

Let $\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]/I(V)$ be the coordinate ring of $V(\mathbb{K})$ as in Definition 2.2.5. Using Notation 3.2.3, it holds $(\sigma\rho(x_1), \ldots, \sigma\rho(x_n)) \in V(\mathbb{R})$ and

$$(ev(\sigma\rho(x_1), \ldots, \sigma\rho(x_n)), \mathbb{R}) = (\sigma\rho, \mathbb{R}) \in V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}.$$

Thus ev is surjective as wanted, hence a homeomorphism. □

Remark 3.4.6. One might wish to generalize this result to the setting of complete Archimedean fields, see [SC21, Section 3]. However, the notion of Archimedicity used to define generalized Hahn fields is stronger than the definition of Archimedicity employed in this text, so that the uniqueness of a complete Archimedean field fails.

As an example of computation, we describe the Archimedean spectrum of $\mathbb{A}^1$ in terms of Dedekind cuts, where a *Dedekind cut* of a real field $\mathbb{L}$ is a partition of $\mathbb{L}$ into two nonempty subsets D_- and D_+ such that D_- is closed downwards and does not contain a greatest element.

Proposition 3.4.7. *For a real closed field $\mathbb{K}$*

$$\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} = \mathbb{K} \cup \left\{ \mathbb{K} = D_- \cup D_+ \;\middle|\; \begin{array}{ll} D_- \neq \varnothing & \textit{has not lowest upper bound,} \\ D_+ \neq \varnothing & \textit{has no greatest lower bound} \end{array} \right\}.$$

Proof. With the notation of the third item of Proposition 3.1.2, let $(\rho, \mathbb{F}_\rho)$ be an element of $\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}$. The coordinate ring $\mathbb{K}[x]$ of $\mathbb{A}^1(\mathbb{K})$ is principal so that the kernel of ρ is either trivial or a principal ideal generated by an irreducible polynomial f. On the one hand, suppose $\ker(\rho) = (f)$. If f is linear, then $\mathbb{F}_\rho = \mathbb{K}$ and ρ is the evaluation on an element in $\mathbb{K}$. Otherwise, since $\mathbb{K}$ is real closed, f is quadratic. However, in this case, $\mathbb{F}_\rho = \mathbb{K}(\sqrt{-1})$ which is not orderable. This is a contradiction with $\mathbb{F}_\rho$ is real closed. On the other hand, ρ is injective and gives an ordering of $\mathbb{K}(x) = \mathrm{Frac}(\mathbb{K}[x])$. Thus

$$\begin{aligned}\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}} &= \mathbb{K} \cup \{\, \text{orderings on } \mathbb{K}(x) \,\},\\ \mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} &= \mathbb{K} \cup \{\, \text{orderings on } \mathbb{K}(x) \text{ Archimedean over } \mathbb{K} \,\}.\end{aligned}$$

<u>Claim:</u> There is a one-to-one correspondence between the orderings of $\mathbb{K}(x)$ which are Archimedean over $\mathbb{K}$ and the Dedekind cuts of $\mathbb{K}$ with no lowest upper bound and no greatest lower bound.
Proof: On the one hand, consider an ordering $\leq$ on $\mathbb{K}(x)$ Archimedean over $\mathbb{K}$ and the sets

$$\begin{aligned} D_- &:= \{\, a \in \mathbb{K} \mid a \leq x \,\},\\ D_+ &:= \{\, a \in \mathbb{K} \mid x \leq a \,\}.\end{aligned}$$

Since the ordering is Archimedean over $\mathbb{K}$, D_+ is nonempty and x^{-1} is bounded from above. So the set D_- is nonempty. Suppose D_- has a lowest upper bound u in $\mathbb{K}$. If u is an element in D_-, then $u \leq x \leq u + \varepsilon$ for every $0 \leq \varepsilon \in \mathbb{K}$. In particular,

$$\varepsilon^{-1} \leq (x-u)^{-1} \text{ for every } 0 \leq \varepsilon \in \mathbb{K},$$

which is a contradiction with the ordering on $\mathbb{K}(x)$ is Archimedean over $\mathbb{K}$. If u is an element in D_+, then $u - \varepsilon \leq x \leq u$ for every $0 \leq \varepsilon \in \mathbb{K}$ so that

$$\varepsilon^{-1} \leq (u-x)^{-1} \text{ for every } 0 \leq \varepsilon \in \mathbb{K}.$$

This is also a contradiction with: the ordering on $\mathbb{K}(x)$ is Archimedean over $\mathbb{K}$. Hence D_- does not have a lowest upper bound, and by a similar reasoning, D_+ does not have a greatest lower bound. Therefore, an Archimedean ordering on $\mathbb{K}(x)$ defines a Dedekind cut of $\mathbb{K}$ with no lowest upper bound and greatest lower bound.

On the other hand, consider a partition $\mathbb{K} = D_- \cup D_+$ where D_- has not lowest upper bound and D_+ has no greatest lower bound. Define an ordering on $\mathbb{K}(x)$ generated by

$$\begin{aligned} a < x &\text{ for every } a \in D_-,\\ x < b &\text{ for every } b \in D_+.\end{aligned}$$

Since $\mathbb{K}$ is real closed, any $g \in \mathbb{K}(x) \backslash \{ 0 \}$ has a decomposition

$$g(x) = \prod_{c_i \in D_-} (x - c_i)^{m_i} \prod_{c_j \in D_+} (x - c_j)^{m_j} \prod_{k=1}^{\ell} h_k(x),$$

where the c_i's are the distinct roots of g, and h_k are irreducible quadratic polynomials. In particular, $0 < h_k$ for every $1 \leq k \leq \ell$ so that

$$0 < g \text{ if and only if } \sum_{c_j \in D_+} m_j \in 2\mathbb{Z}.$$

Equivalently $0 < g$ if and only if there exists $a \in D_-$ such that the restriction $g|_{\mathbb{K}_{>a} \cap D_-}$ is strictly positive. We show that this ordering is Archimedean. By definition of the ordering, the linear terms of the decomposition of g are bounded by some element in the ordered field $\mathbb{K}$. Therefore it remains to prove that the quadratic terms are also bounded in $\mathbb{K}$. Since all h_k are irreducible, there exists $0 < \varepsilon$ with $\varepsilon < \prod_k h_k(a)$ for every $a \in \mathbb{K}$. In particular, there exists $\varepsilon' \in \mathbb{K}$ so that $g^{-1} < \varepsilon'$. This is true for every g in the real field $\mathbb{K}(x)$ so that the ordering is Archimedean. □

Remark 3.4.8. For every non-Archimedean real closed field $\mathbb{K}$, $\mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ does not have the structure of a field. Indeed, the natural addition of the Dedekind completion of a non-Archimedean field is not invertible, see [Hal11, Lemma 3.10].

We introduced the Archimedean spectrum of the $\mathbb{K}$-extension of an algebraic set V and studied one of its natural compactifications given by the closed points of the real spectrum. Before further exploring the Archimedean spectrum using the Berkovich analytification of $V(\mathbb{K})$ in Section 4.2, we first study universal geometric spaces over the real spectrum compactification of $\mathcal{M}_\Gamma$. This allows us to show strong convergence properties within the real spectrum in Subsection 4.1.3. Consequently, we gain a deeper understanding of the dynamical properties of $\mathcal{M}_\Gamma$ by leveraging the local compactness of the Archimedean spectrum.

Chapter 4

Real spectrum compactifications of universal geometric spaces

Section 4.1 constructs universal geometric spaces over $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ and studies properties of the projection map onto $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$. The fibers of the projection map, at the level of the real spectrum, are well organized over $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ and coincide with the Archimedean spectrum, yielding new accessibility results in the real spectrum. In the first subsection, we construct this universal geometric space using the $\mathrm{SL}_n(\mathbb{R})$-action on $\mathbb{P}^{n-1}(\mathbb{R})$. In the final two subsections, we construct a suitable subset of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})_{\mathrm{cl}}^{\mathrm{RSp}}$ which contains $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})}$ and serves as a universal symmetric space over $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$.

Section 4.2 reviews the theory of Berkovich and real analytification, and connects them to the theory of the Archimedean spectrum. We use the canonical homeomorphism between the Archimedean spectrum and the real analytification to construct a continuous and proper map from the Archimedean spectrum to the Berkovich analytification. We characterize its image completely and examine the specific case of the Archimedean spectrum of the projective line. In this setting, the image in the Berkovich analytification is a $\mathrm{PSL}_2(\mathbb{F})$-invariant $\mathbb{R}$-subtree.

4.1 Universal geometric spaces over minimal vectors

In this section, we study the real spectrum compactification of $\mathcal{M}_\Gamma(\mathbb{R})$, the algebraic set of minimal vectors of a finitely generated group Γ in $\mathrm{SL}_n(\mathbb{R})$, its interaction with compact algebraic sets (as the projective space $\mathbb{P}^{n-1}$), and the symmetric space $\widehat{\mathcal{P}^1(n)}$. A key focus is on providing a geometric understanding of the convergence of sequences in the real spectrum. Specifically, we demonstrate that representations on the boundary of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ induce actions of Γ on $\mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ and $\widehat{\mathcal{P}^1(n,\mathbb{K})}^{\mathrm{RSp}}_{\mathrm{Arch}}$, where $\mathbb{K}$ is a well-chosen non-Archimedean real closed field. This motivates the relationship we exhibit in Section 4.2 between the Archimedean spectrum and the Berkovich analytification.

4.1.1 The universal projective space

We examine the lift of the projection map $\pi\colon (V\times W)(\mathbb{R})\to V(\mathbb{R})$ when $V\subset\mathbb{R}^n$ is an algebraic set and $W\subset\mathbb{R}^m$ is a compact algebraic set. We analyze the fibers of $\pi^{\mathrm{RSp}}_{\mathrm{cl}}\colon (V\times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}\to V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ using the Archimedean spectrum and show that the fibers are well organized over $V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. As an application, we construct a universal projective space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ that encodes degeneracies of Γ-actions on $\mathbb{P}^{n-1}(\mathbb{R})$ induced by representations in $\mathcal{M}_\Gamma(\mathbb{R})$.

Remark 4.1.1. Let $\mathbb{L}\subset\mathbb{K}$ be real closed fields and $V\subset\mathbb{L}^n$, $W\subset\mathbb{L}^m$ algebraic sets. From [BCR13, Theorem 2.8.3. (iii)], there exists a natural $\mathbb{K}$-algebra isomorphism

$$\iota:\mathbb{K}[V]\otimes_{\mathbb{K}}\mathbb{K}[W]\cong\mathbb{K}[V\times W]$$

given by $\iota(\sum f_i\otimes g_i)(v,w)=\sum f_i(v)g_i(w)$ for every $(v,w)\in(V\times W)(\mathbb{K})$, see [Sha74, Chapter 1, Subsection 2.2, Example 4, page 17]. In particular, by the universal property of the tensor product, for every $(\rho,\mathbb{F})\in(V\times W)(\mathbb{K})^{\mathrm{RSp}}$, there exists $(\rho^1,\mathbb{F})\in V(\mathbb{K})^{\mathrm{RSp}},(\rho^2,\mathbb{F})\in W(\mathbb{K})^{\mathrm{RSp}}$ such that the following diagram commutes

$$\begin{array}{ccccc}
\mathbb{K}[V] & & \xrightarrow{\qquad\qquad\rho^1\qquad\qquad} & & \searrow \\
\downarrow & & & & \\
\mathbb{K}[V]\otimes_{\mathbb{K}}\mathbb{K}[W] & \xrightarrow{\iota} & \mathbb{K}[V\times W] & \xrightarrow{\rho} & \mathbb{F}, \\
\uparrow & & & & \\
\mathbb{K}[W] & & \xrightarrow{\qquad\qquad\rho^2\qquad\qquad} & & \nearrow
\end{array}$$

where the vertical arrows are the inclusions. Hence, for every $h\in\mathbb{K}[V\times W]$

$$\rho(h)=\sum\rho^1\left(h_i^1\right)\rho^2\left(h_i^2\right)\quad\text{where }\iota^{-1}(h)=\sum h_i^1\otimes h_i^2\in\mathbb{K}[V]\otimes_{\mathbb{K}}\mathbb{K}\left[W\right].$$

We denote by $(\rho^1,\rho^2,\mathbb{F})$ this decomposition of the morphism $(\rho,\mathbb{F})$.

Proposition 4.1.2 ([CR82, Proposition 4.3])**.** *Let $\pi\colon A\to B$ be a ring morphism, and $(\rho,\mathbb{F}_\rho)\in A^{\mathrm{RSp}}$. Then $(\pi^{\mathrm{RSp}})^{-1}(\rho,\mathbb{F}_\rho)$ is homeomorphic to $(\mathbb{F}_\rho\otimes_A B)^{\mathrm{RSp}}$.*

We refer to [CR82, Proposition 2.5] for a description of the homeomorphism.

Theorem 4.1.3. *Let $V \subset \mathbb{R}^n$ be an algebraic set and $W \subset \mathbb{R}^m$ a compact algebraic set. If $\pi\colon (V \times W)(\mathbb{R}) \to V(\mathbb{R})$ is the projection map, then the restriction to the closed points of the induced map π^{RSp} takes its values in $W(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, that is*

$$\pi^{\mathrm{RSp}}_{\mathrm{cl}} : (V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \to V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Moreover, it is continuous, surjective and the fiber of $(\rho, \mathbb{K}_\rho) \in V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ is homeomorphic to $W(\mathbb{K}_\rho)^{\mathrm{RSp}}_{\mathrm{Arch}}$.

Proof. Since $W(\mathbb{R})$ is compact, the projection map is algebraic and proper. Thus, by Theorem 3.2.4, the map $\pi^{\mathrm{RSp}}_{\mathrm{cl}} : (V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \to V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ is well-defined, continuous and surjective.

We show that the fiber of an element $(\rho^1, \mathbb{K}_{\rho^1}) \in V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ is homeomorphic to $W(\mathbb{K}_{\rho^1})^{\mathrm{RSp}}_{\mathrm{Arch}}$. Consider $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ endowed with the subspace topology induced by the spectral topology on $(V \times W)(\mathbb{R})^{\mathrm{RSp}}$. By Proposition 4.1.2, there exists a canonical homeomorphism

$$\begin{aligned}\left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{R}[V]} \mathbb{R}[V \times W]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{R}[V]} \mathbb{R}[V] \otimes_{\mathbb{R}} \mathbb{R}[W]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1}[W]\right)^{\mathrm{RSp}} = W\left(\mathbb{K}_{\rho^1}\right)^{\mathrm{RSp}},\end{aligned}$$

where the second homeomorphism comes from [BCR13, Theorem 2.8.3. (iii)]. We study the intersection of this preimage with $(V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Consider $(\rho, \mathbb{F}_\rho) \in (\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \cap (V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ and its decomposition $(\rho, \mathbb{F}_\rho) = (\rho^1, \rho^2, \mathbb{F}_\rho)$ as in Remark 4.1.1. Consider the coordinate ring

$$\mathbb{R}[V \times W] := \mathbb{R}[x_1, \ldots, x_n]/I(V \times W),$$

as in Definition 2.2.5 and the element $v_\rho := (\rho(x_1), \ldots, \rho(x_n)) \in V(\mathbb{F}_\rho)$ defined in Notation 3.2.3. By Lemma 3.2.2, there exist constants $c, d \in \mathbb{N}$ such that

$$\text{for every } v \in V(\mathbb{R}) : |x_i(v)| \leq c\left(1 + N(\pi(v))^2\right)^d.$$

So by the Transfer principle (Theorem 2.2.2)

$$|x_i(v_\rho)| \leq c(1 + N\left(\pi_{\mathbb{F}_\rho}(v_\rho)\right)^2)^d,$$

such that $\rho(\mathbb{R}[V \times W])$ is Archimedean over $\rho^1(\mathbb{R}[V]) \subset \mathbb{K}_{\rho^1}$. In particular, $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}_{\rho^1}$ so that $(\rho^2, \mathbb{F}_\rho) \in \mathbb{K}_{\rho^1}[W]^{\mathrm{RSp}}_{\mathrm{Arch}}$. Hence

$$\left(\pi^{\mathrm{RSp}}_{\mathrm{cl}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \subset W(\mathbb{K}_{\rho^1})^{\mathrm{RSp}}_{\mathrm{Arch}}.$$

We prove the remaining inclusion. Consider $(\rho^2, \mathbb{F}) \in W(\mathbb{K}^1_\rho)^{\mathrm{RSp}}_{\mathrm{Arch}}$. Then $\rho^2|_{\mathbb{K}_{\rho^1}} \colon \mathbb{K}_{\rho^1} \to \mathbb{F}$ is an ordered field morphism (see Remark 3.1.8). Since $\mathbb{F}$ is Archimedean over $\mathbb{K}_{\rho^1}$ which is Archimedean over $\rho^1(\mathbb{R}[V])$

$$\left(\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \rho^2, \mathbb{F}\right) \in (V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Moreover, $\pi^{\mathrm{RSp}}(\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \rho^2, \mathbb{F}) = (\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \mathbb{F}) = (\rho^1, \mathbb{K}_{\rho^1})$ by Proposition 3.1.2 so that

$$\left(\pi^{\mathrm{RSp}}_{\mathrm{cl}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \cong W(\mathbb{K}_{\rho^1})^{\mathrm{RSp}}_{\mathrm{Arch}}.$$

□

In the setting of Theorem 4.1.3, the fibers over $V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ exhibit good behavior under projection. Informally, we show that if a sequence of elements in $V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ converges to $(\rho, \mathbb{K}_\rho)$, then the corresponding fibers also converge to the fiber of $(\rho, \mathbb{K}_\rho)$. To describe this behavior, we use that π^{RSp} is an open map, which is described in [CR82, Theorem 6.3], though our setting provides a natural proof.

Lemma 4.1.4 ([CR82, Theorem 6.3])**.** *Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields and $V \subset \mathbb{L}^n$, $W \subset \mathbb{L}^m$ algebraic sets. If $\pi \colon (V \times W)(\mathbb{K}) \to V(\mathbb{K})$ is the projection map, then the lift $\pi^{\mathrm{RSp}} \colon (V \times W)(\mathbb{K})^{\mathrm{RSp}} \to V(\mathbb{K})^{\mathrm{RSp}}$ is open.*

Proof. Since for two open sets $U_1, U_2 \subset (V \times W)(\mathbb{K})^{\mathrm{RSp}}$, the equality

$$\pi^{\mathrm{RSp}}(U_1 \cup U_2) = \pi^{\mathrm{RSp}}(U_1) \cup \pi^{\mathrm{RSp}}(U_2)$$

holds, it is enough to show that the image of a basis element

$$\tilde{U}(f_1, \dots, f_p) \subset (V \times W)(\mathbb{K})^{\mathrm{RSp}}$$

is open for any $f_i \in \mathbb{K}[V \times W]$. Consider the open subset $U(f_1, \dots, f_p) = \tilde{U}(f_1, \dots, f_p) \cap (V \times W)(\mathbb{K})$, see the fifth item of Proposition 3.3.4. Since the projection map is open, $\pi(U(f_1, \dots, f_p))$ is an open subset of $V(\mathbb{K})$. In particular, by the Finiteness Theorem [BCR13, Theorem 2.7.2], there exists $g_1^i, \dots, g_{q_i}^i \in \mathbb{K}[V]$ with

$$\pi(U(f_1, \dots, f_p)) = \bigcup_i U\left(g_1^i, \dots, g_{q_i}^i\right).$$

We show that $\pi^{\mathrm{RSp}}(\tilde{U}(f_1, \dots, f_p)) = \cup_i \widetilde{U(g_1^i, \dots, g_{q_i}^i)}$ which is open by Proposition 3.3.4. As in Definition 2.2.5, consider the coordinate rings

$$\mathbb{K}[V] = \mathbb{K}[x_1, \dots, x_n]/I(V) \text{ and } \mathbb{K}[W] = \mathbb{K}[y_1, \dots, y_m]/I(W).$$

Let $(\rho, \mathbb{F}_\rho) \in \tilde{U}(f_1, \ldots, f_p)$ and, as in Notation 3.2.3, its associated element

$$z_\rho = (\rho(x_1), \ldots, \rho(x_n), \rho(y_1), \ldots, \rho(y_m)) \subset (V \times W)(\mathbb{F}_\rho).$$

By definition $z_\rho \in U(f_1, \ldots, f_p)(\mathbb{F}_\rho) := \{ v \in V(\mathbb{F}_\rho) \mid f_i(v) > 0 \quad \forall i \}$. By [BCR13, Proposition 5.2.1]

$$\pi_{\mathbb{F}_\rho}(z_\rho) \in \left(\bigcup_i U\left(g_1^i, \ldots, g_{q_i}^i\right) \right)(\mathbb{F}_\rho) = \bigcup_i U\left(g_1^i, \ldots, g_{q_i}^i\right)(\mathbb{F}_\rho),$$

and by the third item of Proposition 3.3.4

$$\bigcup_i U\left(g_1^i, \ldots, g_{q_i}^i\right)(\mathbb{F}_\rho) \subset \bigcup_i U\left(\widetilde{g_1^i, \ldots, g_{q_i}^i}\right).$$

Hence the following inclusion holds

$$\pi^{\mathrm{RSp}}\left(\tilde{U}(f_1, \ldots, f_p)\right) \subset \bigcup_i U\left(\widetilde{g_1^i, \ldots, g_{q_i}^i}\right).$$

Moreover, for every $w \in \cup_i U(g_1^i, \ldots, g_{q_i}^i)$ there exists $v \in U(f_1, \ldots, f_p)$ such that $\pi(v) = w$. So, by [BCR13, Proposition 5.2.1], for every $w_{\mathbb{F}} \in \cup_i U(g_1^i, \ldots, g_{q_i}^i)(\mathbb{F})$ there exists $v_{\mathbb{F}} \in U(f_1, \ldots, f_p)(\mathbb{F})$ with $\pi(v_{\mathbb{F}}) = w_{\mathbb{F}}$. So

$$\pi^{\mathrm{RSp}}\left(\tilde{U}(f_1, \ldots, f_p)\right) = \bigcup_i U\left(\widetilde{g_1^i, \ldots, g_{q_i}^i}\right),$$

which is an open subset of $V(\mathbb{K})^{\mathrm{RSp}}$. □

In our context, this lemma has an analogue at the level of closed points.

Theorem 4.1.5. *Let $\mathbb{L} \subset \mathbb{K}$ be real closed fields, $V \subset \mathbb{L}^n$, $W \subset \mathbb{L}^m$ algebraic sets, and $\pi\colon (V \times W)(\mathbb{K}) \to V(\mathbb{K})$ the projection map. If there exist open sets $U_k(\mathbb{K}) \subset (V \times W)(\mathbb{K})$ for $k \in \mathbb{N}$ such that the restriction of π^{RSp} to*

$$E := \bigcup_{k \in \mathbb{N}} \widetilde{U_k(\mathbb{K})} \cap (V \times W)(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}}$$

has value in the closed points and is surjective, that is

$$\pi^{\mathrm{RSp}}|_E \colon E \to V(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{cl}},$$

then $\pi^{\mathrm{RSp}}|_E$ is also open.

Proof. Consider an open set $U^E \subset E$, such that there exists an open set $U \subset (V \times W)(\mathbb{K})^{\mathrm{RSp}}$ with $U^E = U \cap E$. Since $\cup_{k\in\mathbb{N}}\widetilde{U_k(\mathbb{K})} \subset (V \times W)(\mathbb{K})^{\mathrm{RSp}}$ is open by Proposition 3.3.4, we suppose $U \subset \cup_{k\in\mathbb{N}}\widetilde{U_k(\mathbb{K})}$. It holds

$$\pi_{\mathrm{cl}}^{\mathrm{RSp}}\left(U^E\right) \subset \pi^{\mathrm{RSp}}(U) \cap V(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}},$$

and we prove the reverse inclusion. Consider an element

$$\left(\rho^1, \mathbb{K}_{\rho^1}\right) \in \pi^{\mathrm{RSp}}(U) \cap V(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}}$$

so that $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \cap U$ is a non-empty open subset of $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ for the subspace topology. By Proposition 4.1.2, there exists a canonical homeomorphism

$$\begin{aligned}\left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{K}[V]} \mathbb{K}[V \times W]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{K}[V]} \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[W]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1}[W]\right)^{\mathrm{RSp}} = W\left(\mathbb{K}_{\rho^1}\right)^{\mathrm{RSp}},\end{aligned}$$

where the second homeomorphism comes from [BCR13, Theorem 2.8.3.(iii)]. Denote by $\varphi\colon (\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \to W(\mathbb{K}_{\rho^1})^{\mathrm{RSp}}$ this homeomorphism. Then

$$\varphi\left(\left(\pi^{\mathrm{RSp}}\right)^{-1}\left(\rho^1, \mathbb{K}_{\rho^1}\right) \cap U\right)$$

is a non-empty open subset of $W(\mathbb{K}_{\rho^1})^{\mathrm{RSp}}$. By Theorem 3.3.1, this open set contains a closed point

$$(\rho^2, \mathbb{K}_{\rho^2}) \in \varphi\left(\left(\pi^{\mathrm{RSp}}\right)^{-1}\left(\rho^1, \mathbb{K}_{\rho^1}\right) \cap U\right) \cap W\left(\mathbb{K}_{\rho^1}\right)_{\mathrm{cl}}^{\mathrm{RSp}}.$$

Since φ is continuous, $\varphi^{-1}(\rho^2, \mathbb{K}_{\rho^2}) = (\psi, \mathbb{F})$ is a closed subset of $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ which is in U. In particular, there exists a closed set $A \subset (V \times W)(\mathbb{K})^{\mathrm{RSp}}$ with

$$(\psi, \mathbb{F}) = A \cap \left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}).$$

Since $(\rho^1, \mathbb{K}_{\rho^1}) \in V(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}}$ and π^{RSp} is continuous, the fiber $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ is closed in $(V \times W)(\mathbb{K})^{\mathrm{RSp}}$. Hence $(\psi, \mathbb{F}) = A \cap (\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ is closed and

$$\begin{aligned}(\psi, \mathbb{F}) &\in (V \times W)(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}} \cap U \cap \left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \\ &= (V \times W)(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}} \cap \bigcup_{k\in\mathbb{N}} \widetilde{U_k(\mathbb{K})} \cap U \cap \left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \\ &= E \cap U \cap \left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}).\end{aligned}$$

Hence

$$\pi_{\mathrm{cl}}^{\mathrm{RSp}}\left(U^E\right) = \pi^{\mathrm{RSp}}(U) \cap V(\mathbb{K})_{\mathrm{cl}}^{\mathrm{RSp}},$$

which is open by Lemma 4.1.4. □

We now present the convergence of the fibers in $(V \times W)(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ in a fully constructive manner.

Theorem 4.1.6. *Consider the same setting as in Theorem 4.1.5. Let $(\rho^1_n, \mathbb{K}_{\rho^1_n}) \subset V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ be a sequence converging to an element $(\rho^1, \mathbb{K}_{\rho^1}) \in V(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. For every element of the fiber $(\rho^1, \rho^2, \mathbb{F}) \in (\pi^{\mathrm{RSp}}_{\mathrm{cl}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ and every open set U in E around this element, there exists $M \in \mathbb{N}$ and*

$$(\rho^1_M, \rho^2_M, \mathbb{F}_M) \in \left(\pi^{\mathrm{RSp}}_{\mathrm{cl}}\right)^{-1} (\rho^1_M, \mathbb{K}_{\rho^1_M}) \cap U.$$

Proof. As in the statement, consider $(\rho^1, \rho^2, \mathbb{F}) \in (\pi^{\mathrm{RSp}}_{\mathrm{cl}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ and $U \subset E$ an open set around it. By Theorem 4.1.5, the map $\pi^{\mathrm{RSp}}|_E$ is open. Thus

$$U' := \pi^{\mathrm{RSp}}|_E(U)$$

is open in $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ and contains $(\rho^1, \mathbb{K}_{\rho^1})$. Since the sequence $(\rho^1_n, \mathbb{K}_{\rho^1_n})$ converges towards $(\rho^1, \mathbb{K}_{\rho^1})$, there exists $M \in \mathbb{N}$ so that $(\rho^1_M, \mathbb{K}_{\rho^1_M}) \in U'$. Moreover, since $U' = \pi^{\mathrm{RSp}}|_E(U)$, there exists $(\rho, \mathbb{F}_\rho) \in U$ such that

$$\pi^{\mathrm{RSp}}|_E\left((\rho, \mathbb{F}_\rho)\right) = \left(\rho^1_M, \mathbb{K}_{\rho^1_M}\right).$$

Hence

$$(\rho, \mathbb{F}_\rho) \in \left(\pi^{\mathrm{RSp}}\right)^{-1} (\rho^1_M, \mathbb{K}_{\rho^1_M}) \cap U$$

is the desired element in the fiber. □

We apply these results to the case $V(\mathbb{K}) = \mathcal{M}_\Gamma(\mathbb{R})$ and $W(\mathbb{K}) = \mathbb{P}^{n-1}(\mathbb{R})$. There exists a natural continuous action of Γ on $\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ given by

$$\gamma.\,(\phi, x) = (\phi, \phi(\gamma)x)\,.$$

Proposition 4.1.7. *The Γ-action on $\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ extends to an action by homeomorphisms on $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}$ preserving $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$.*

Proof. Since $\mathrm{SL}_n(\mathbb{R})$ acts on $\mathbb{P}^{n-1}(\mathbb{R})$ semialgebraically, the graph of the action of one element of Γ on $\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ is semialgebraic. Thus by [BCR13, Proposition 7.2.8], the action extends canonically to an action by homeomorphisms of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}$ preserving $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. □

We now prove that $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ forms a universal geometric space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Informally, we show that $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ behaves similarly to the product $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \times \mathbb{P}^{n-1}(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, even though these spaces are not homeomorphic when endowed with the spectral topology.

Theorem 4.1.8. *If* $\pi\colon \mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}) \to \mathcal{M}_\Gamma(\mathbb{R})$ *is the projection map, then the restriction to the closed points of the induced map* π^{RSp} *takes its values in* $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$*, that is*

$$\pi^{\mathrm{RSp}}_{\mathrm{cl}} : \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}} \to \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Moreover, it is continuous, surjective, Γ*-equivariant, and the fiber of* $(\rho, \mathbb{K}_\rho) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ *is homeomorphic to* $\mathbb{P}^{n-1}(\mathbb{K}_\rho)^{\mathrm{RSp}}_{\mathrm{Arch}}$*.*

Proof. The projective space is identified with the set of orthogonal projections onto subspaces of dimension 1 of $\mathbb{R}^n$ [BCR13, Theorem 3.4.4]

$$\mathbb{P}^{n-1}(\mathbb{R}) := \{\, A \in M_{n\times n}(\mathbb{R}) \mid A^T = A, A^2 = A, \operatorname{tr}(A) = 1 \,\}.$$

It is a compact algebraic set so that by Theorem 4.1.3, the map

$$\pi^{\mathrm{RSp}}_{\mathrm{cl}} : \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}} \to \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$$

is continuous, surjective, and the fiber of $(\rho, \mathbb{K}_\rho) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{K}_\rho)^{\mathrm{RSp}}_{\mathrm{Arch}}$.

We show that $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$ is Γ-equivariant for the Γ-action on $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ described in Proposition 4.1.7, and the trivial action of Γ on $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Let $(\rho, \mathbb{F}_\rho) \in (\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. By Remark 4.1.1, there exist $(\rho^1, \mathbb{F}_\rho) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}$ and $(\rho^2, \mathbb{F}_\rho) \in \mathbb{P}^{n-1}(\mathbb{R})^{\mathrm{RSp}}$ such that

$$\forall h \in \mathbb{R}\left[\mathcal{M}_\Gamma \times \mathbb{P}^{n-1}\right] \quad \rho(h) = \sum \rho^1(h^1_i)\rho^2(h^2_i),$$

where $\iota^{-1}(h) = \sum h^1_i \otimes h^2_i \in \mathbb{R}[\mathcal{M}_\Gamma] \otimes_{\mathbb{R}} \mathbb{R}[\mathbb{P}^{n-1}]$. For $\gamma \in \Gamma$ and $h^1 \in \mathbb{R}[\mathcal{M}_\Gamma]$, the following equalities hold:

$$\begin{aligned}\pi^{\mathrm{RSp}}_{\mathrm{cl}}(\gamma\rho)\left(h^1\right) &= \pi^{\mathrm{RSp}}_{\mathrm{cl}}\left(\rho^1, \rho^2 \circ \gamma\right)\left(h^1\right)\\ &= \rho^1\left(h^1\right)\\ &= \gamma\rho^1\left(h^1\right) = \gamma\pi^{\mathrm{RSp}}_{\mathrm{cl}}(\rho)\left(h^1\right).\end{aligned}$$

Thus, $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$ is Γ-equivariant, as desired. □

Remark 4.1.9. The map $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$ is Γ-equivariant, so its fibers are Γ-invariant. Thus every $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ induces a Γ-action on $\mathbb{P}^{n-1}(\mathbb{K}_\rho)^{\mathrm{RSp}}_{\mathrm{Arch}}$ such that, for $\gamma \in \Gamma$, $(\rho^2, \mathbb{K}_{\rho^2}) \in (\pi^{\mathrm{RSp}}_{\mathrm{cl}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$, and $h \in \mathbb{K}_{\rho^1}[\mathbb{P}^{n-1}]$ we have:

$$\gamma\rho^2(h) = \rho^2(\rho^1(\gamma)h).$$

Finally, we can explicitly characterize the well organization of the fibers of $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$.

Corollary 4.1.10. *Let* $(\rho^1_n, \mathbb{K}_{\rho^1_n}) \subset \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ *be a sequence converging to an element* $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$*. For every element of the fiber* $(\rho^1, \rho^2, \mathbb{F}) \in (\pi^{\mathrm{RSp}}_{\mathrm{cl}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ *and every open set in* $(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ *around this element, there exists* $M \in \mathbb{N}$ *and* $(\rho^1_M, \rho^2_M, \mathbb{F}_M)$ *in* $(\pi^{\mathrm{RSp}}_{\mathrm{cl}})^{-1}(\rho^1_M, \mathbb{K}_{\rho^1_M}) \cap U$*. In particular,* $(\rho^2_M, \mathbb{F}_M)$ *is in* $\mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$*.*

Proof. Consider the open subsets $U_k(\mathbb{R}) := \mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ of $\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})$ for every $k \in \mathbb{N}$ such that with the notation from Theorem 4.1.5

$$E = \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathbb{P}^{n-1}(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Then, $\pi^{\mathrm{RSp}}|_E = \pi^{\mathrm{RSp}}_{\mathrm{cl}}$ and by Theorem 4.1.8, the map $\pi^{\mathrm{RSp}}_{\mathrm{cl}}$ verifies all the conditions from Theorem 4.1.6. Thus, the statement is a consequence of the more general Theorem 4.1.6. □

Remark 4.1.11. By combining $\mathbb{P}^1(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{Arch}} \cong \mathbb{P}^1(\mathbb{R})$ with the accessibility theorem from [BIPP23, Corollary 3.9], we also obtain an accessibility theorem stating that all Γ-actions on $\mathbb{P}^1(\mathbb{F})^{\mathrm{RSp}}_{\mathrm{Arch}}$ arise from Γ-actions on $\mathbb{P}^1(\mathbb{R})$.

We constructed a universal projective space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. This provides a framework for studying the actions induced by elements of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ on $\mathbb{P}^{n-1}(\mathbb{K})$ within $\mathbb{P}^{n-1}(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$. In the following subsections, we extend this approach to analyze the induced actions on the symmetric space $\mathcal{P}^1(n)$. Specifically, we construct a universal symmetric space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$.

4.1.2 Displacement of an action and its associated geometric space

In this subsection, we construct a Γ-invariant open subset E of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ which is a countable union of compact subsets of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ and that contains $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})}$. Moreover, as we see in the next subsection, E surjects continuously onto $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$.

Remark 4.1.12. There exists a natural Γ-action on $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})}$ given by

$$\gamma.\left(\phi, (A, t_1, \dots, t_{n-1})\right) = \left(\phi, \phi(\gamma).(A, t_1, \dots, t_{n-1})\right).$$

To encode the Γ-action on $\widehat{\mathcal{P}^1(n, \mathbb{R})}$ induced by an element in $\mathcal{M}_\Gamma(\mathbb{R})$ and define a universal space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$, we look at elements of the symmetric space within specified balls defined by the displacement function of the representation. Consider

$F = \{ \gamma_1, \ldots, \gamma_s \}$ a finite generating set of Γ and recall that, as in Subsection 2.3, $M_{n\times n}(\mathbb{R})^F$ is equipped with a scalar product

$$\langle (A_1, \ldots, A_s) \ , \ (B_1, \ldots, B_s) \rangle := \sum_{i=1}^{s} \operatorname{tr}\left(A_i^T B_i\right),$$

for every $A_1, \ldots, A_s, B_1, \ldots, B_s \in M_{n\times n}(\mathbb{R})$. Then we define the norm

$$\begin{array}{rccc} \eta\colon & \operatorname{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{R})) & \longrightarrow & \mathbb{R}_{\geq 0}; \\ & \rho & \longmapsto & \langle (\rho(\gamma_1), \ldots, \rho(\gamma_s)) \ , \ (\rho(\gamma_1), \ldots, \rho(\gamma_s)) \rangle \,. \end{array}$$

In particular, η is semialgebraically continuous. Let d_δ be the semialgebraically continuous Cartan multiplicative distance on $\mathcal{P}^1(n,\mathbb{R})$ given in Proposition 2.4.4, and $\hat{d}_\delta$ its extension to $\widehat{\mathcal{P}^1(n,\mathbb{R})}$. Consider for every $k \in \mathbb{N}$

$$\mathcal{U}_k(\mathbb{R}) := \left\{ (\rho, (A,t)) \in \mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \,\middle|\, \hat{d}_\delta(\mathrm{Id}, (A,t)) < \eta(\rho)^k \right\},$$

which is open and semialgebraic in $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$ as both $\hat{d}_\delta$ and η are, see Proposition 2.4.4.

Definition 4.1.13. Define the open subset of $\widehat{(\mathcal{M}_\Gamma(\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R}))}^{\mathrm{RSp}}_{\mathrm{cl}}$ given by

$$E := \bigcup_{k\in\mathbb{N}} \widetilde{\mathcal{U}_k(\mathbb{R})} \cap \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \right)^{\mathrm{RSp}}_{\mathrm{cl}},$$

where $\widetilde{\mathcal{U}_k(\mathbb{R})}$ is the constructible set associated to $\mathcal{U}_k(\mathbb{R})$, see Definition 3.3.3.

Denote by $\|\cdot\|_F$ the word length with respect to the generating set F of Γ.

Lemma 4.1.14. *For every $(\rho, A) \in \mathcal{M}_\Gamma(\mathbb{R}) \times \mathcal{P}^1(n,\mathbb{R})$ and $\gamma \in \Gamma$, it holds*

$$d_\delta(\mathrm{Id}, \rho(\gamma).A) \leq \eta(\rho)^{\frac{n}{2}\|\gamma\|_F} d_\delta(\mathrm{Id}, A).$$

Proof. Using the first item of Proposition 2.4.4 and the $\mathrm{SL}_n(\mathbb{R})$-invariance of the Cartan's multiplicative distance we obtain for every $\gamma \in \Gamma$

$$d_\delta(\mathrm{Id}, \rho(\gamma).A) = d_\delta(\mathrm{Id}, \rho(\gamma_1 \cdots \gamma_\ell).A) \leq \left(\prod_{j=1}^{\ell} d_\delta(\mathrm{Id}, \rho(\gamma_i).\mathrm{Id}) \right) d_\delta(\mathrm{Id}, A), \quad (4.1)$$

where $\|\gamma\|_F = \ell$, $\gamma_i \in F$ for every $1 \leq i \leq \ell$ and $\gamma = \gamma_1 \cdots \gamma_\ell$. We now bound the value of $d_\delta(\mathrm{Id}, \rho(\gamma_i).\mathrm{Id})$ by powers of $\eta(\rho)$. As seen in Subsection 2.4, any $g \in \mathrm{SL}_n(\mathbb{R})$ has a Cartan decomposition $g = kc(g)k'$ where $k, k' \in \mathrm{SO}(n,\mathbb{R})$ and

$$c(g) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ where the } \lambda_i \text{ are ordered so that } \lambda_n \leq \cdots \leq \lambda_1.$$

Then $g^T g = k'^{-1} c(g)^2 k'$ so that $\mathrm{tr}(g^T g) = \lambda_1^2 + \cdots + \lambda_n^2$ and $d_\delta(\mathrm{Id}, g.\mathrm{Id}) = \lambda_1 \lambda_n^{-1}$. Since $\det(g)$ is the product of the λ_i and is equal to one, we obtain $\lambda_1 \leq d_\delta(\mathrm{Id}, g.\mathrm{Id}) \leq \lambda_1^n$ and $\lambda_1^2 \leq \mathrm{tr}(g^T g) \leq n\lambda_1^2$. Hence

$$d_\delta(\mathrm{Id}, g.\mathrm{Id}) \leq \mathrm{tr}\left(g^T g\right)^{\frac{n}{2}}. \tag{4.2}$$

Furthermore, from the inequalities (4.1) and (4.2), we obtain

$$\begin{aligned} d_\delta(\mathrm{Id}, \rho(\gamma).A) &\leq \left(\prod_{j=1}^{\ell} d_\delta(\mathrm{Id}, \rho(\gamma_i).\mathrm{Id})\right) d_\delta(\mathrm{Id}, A) \\ &\leq \left(\prod_{j=1}^{\ell} \mathrm{tr}\left(\rho(\gamma_i)^T \rho(\gamma_i)\right)\right)^{\frac{n}{2}} d_\delta(\mathrm{Id}, A) \\ &\leq \eta(\rho)^{\frac{n}{2}\|\gamma\|_F} d_\delta(\mathrm{Id}, A). \end{aligned}$$

Hence we obtain the desired inequality. ☐

Corollary 4.1.15. *For every $\gamma \in \Gamma$ and $k \in \mathbb{N}$, we have the inclusion $\gamma.\mathcal{U}_k(\mathbb{R}) \subset \mathcal{U}_{k+\frac{n}{2}\|\gamma\|_F}(\mathbb{R})$.*

Corollary 4.1.16. *The Γ-action on $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$ extends to an action by homeomorphisms on $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}$ preserving E.*

Proof. Since $\mathrm{SL}_n(\mathbb{R})$ acts on $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ semialgebraically, the graph of the action of one element of Γ on $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$ is semialgebraic. Therefore, by [BCR13, Proposition 7.2.8], the action extends canonically to a homeomorphism of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}$, preserving $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$. Additionally, by Corollary 4.1.15, this homeomorphism preserves E. ☐

Since the sets $\widetilde{\mathcal{U}_k(\mathbb{R})}$ are open subsets of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}$ for any $k \in \mathbb{N}$, the above defined set E is a locally compact open subset of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ which contains $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$ and on which Γ acts by homeomorphisms.

Remark 4.1.17. As in Theorem 3.4.3, if we replace $\mathcal{U}_k(\mathbb{R})$ in the definition of E by

$$\mathcal{V}_k := \left\{ (\rho, (A,t)) \in \mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \,\middle|\, \hat{d}_\delta(\mathrm{Id}, (A,t)) \leq \eta(\rho)^k \right\},$$

which are closed and semialgebraic in $\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}$. Then

$$E := \bigcup_{k\in\mathbb{N}} \widetilde{\mathcal{U}_k(\mathbb{R})} \cap \Big(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}\Big)^{\mathrm{RSp}}_{\mathrm{cl}} = \bigcup_{k\in\mathbb{N}} \widetilde{\mathcal{V}_k(\mathbb{R})} \cap \Big(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})}\Big)^{\mathrm{RSp}}_{\mathrm{cl}}.$$

so that E is also a countable union of compact subsets of $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$. Thus it is σ-compact.

Using constructible subsets of the real spectrum, we constructed a topological space E on which Γ acts by homeomorphisms. In the following subsection, we study the lift of the projection map $\pi : \mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \to \mathcal{M}_\Gamma(\mathbb{R})$ to the real spectrum, particularly focusing on its restriction to E. This results in a universal symmetric space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$.

4.1.3 The universal symmetric space

In this subsection, we construct a universal symmetric space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ that encodes the degeneracies of Γ-actions on $\widehat{\mathcal{P}^1(n,\mathbb{R})}$ induced by representations in $\mathcal{M}_\Gamma(\mathbb{R})$. In particular, the Archimedean spectrum allows us to describe the fibers of the projection $\pi : \mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)} \to \mathcal{M}_\Gamma$ at the level of the real spectrum, especially its restriction to E. Moreover, we prove that the fibers are well organized over the target space and establish convergence results about them. To do this, we need the following two linear algebra results.

Lemma 4.1.18. *Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$.*

1. *If A is positive semidefinite, then for every $1 \leq i,j \leq n$ it holds*
$$|A_{i,j}| = |\langle Ae_i\ ,\ e_j\rangle| \leq |\lambda_1|.$$
2. *For every $1 \leq \ell \leq n$, we have $\lambda_n^\ell \leq \det(A[\ell])$, where $\det(A[\ell])$ is the ℓ-principal minor of A.*

Proof. For the first item, let $1 \leq i,j \leq n$ such that
$$|A_{i,j}| = |\langle Ae_i\ ,\ e_j\rangle| \leq \max_{\|x\|=1,\|y\|=1} |\langle Ax\ ,\ y\rangle|.$$
Since A is positive semidefinite, by the Rayleigh–Ritz Theorem (see for example [HJ85, Theorem 4.2.2])
$$0 < \max_{\|x\|=1,\|y\|=1} |\langle Ax\ ,\ y\rangle| \leq \lambda_1.$$
Hence $|A_{i,j}| \leq \lambda_1$. The second item is a consequence of the Cauchy's interlacing theorem [HJ85, Theorem 4.3.17]. Indeed, if the eigenvalues of $A[\ell]$ are $\mu_\ell \leq \cdots \leq \mu_1$, then by the Cauchy's interlacing theorem $\lambda_n \leq \mu_i$ for every $1 \leq i \leq n$. Hence
$$\lambda_n^\ell \leq \prod_{i=1}^{\ell} \mu_i = \det(A[\ell])$$
as desired. ☐

Recall that there exists a multiplicative Cartan pseudo-distance $\hat{d}_\delta$ defined on $\widehat{\mathcal{P}^1(n)}$ (Proposition 2.4.4) and that $\eta : \phi \mapsto \sum_{i=1}^{s} \operatorname{tr}(\phi(\gamma_i)^T \phi(\gamma_i))$ defines a scalar product on the representation space $\operatorname{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{R}))$. Moreover, these two maps are semialgebraically continuous, which allows us to extend them using the Transfer principle (Theorem 2.2.2) to a $\mathbb{K}$-multiplicative pseudo-distance on $\widehat{\mathcal{P}^1(n,\mathbb{K})}$ and a scalar product on $\operatorname{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{K}))$ for any real closed field $\mathbb{K}$. This allows us to prove the following theorem using similar techniques as in Subsection 4.1.1

Theorem 4.1.19. *If* $\pi \colon \mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})} \to \mathcal{M}_\Gamma(\mathbb{R})$ *is the projection map, then the restriction to* E *of the induced map* π^{RSp} *takes its values in* $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$*, that is*

$$\pi^{\mathrm{RSp}}|_E : E \to \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Moreover, it is continuous, surjective, Γ*-equivariant, and the fiber of* $(\rho, \mathbb{K}_\rho) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ *is homeomorphic to* $\widehat{\mathcal{P}^1(n,\mathbb{K}_\rho)}^{\mathrm{RSp}}_{\mathrm{Arch}}$.

Proof. As in the notation of the third item of Proposition 3.1.2, consider $(\rho, \mathbb{F}_\rho) \in \widetilde{\mathcal{U}_k(\mathbb{R})} \cap (\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n,\mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ for some $k \in \mathbb{N}$, where $\mathbb{F}_\rho$ is the real closure of the field of fractions of $\rho(\mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}])$. We first prove that $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[\mathcal{M}_\Gamma])$. For every $1 \le i,j \le n$, $1 \le \ell \le n-1$ and $1 \le m \le sn^2$, consider the coordinates $x_{i,j}, y_\ell$ and z_m and the coordinate rings as in Definition 2.2.5

$$\mathbb{R}\left[\widehat{\mathcal{P}^1(n)}\right] = \mathbb{R}[x_{i,j}, y_\ell]/I\left(\widehat{\mathcal{P}^1(n)}\right) \text{ and } \mathbb{R}[\mathcal{M}_\Gamma] = \mathbb{R}[z_m]/I(\mathcal{M}_\Gamma).$$

Consider also the element $w_\rho = (\rho(z_m), \rho(x_{i,j}), \rho(y_\ell)) \in \mathcal{M}_\Gamma(\mathbb{F}_\rho) \times \widehat{\mathcal{P}^1(n,\mathbb{F}_\rho)}$ defined in Notation 3.2.3. With the notations from Proposition 2.4.4, for any $(A, t_1, \dots, t_{n-1}) \in \widehat{\mathcal{P}^1(n,\mathbb{R})}$ where $0 < \lambda_n \le \dots \le \lambda_1$ are the eigenvalues of A

$$d_\delta(\mathrm{Id}, A) = N(\operatorname{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_1/\lambda_n.$$

Thus, using Lemma 4.1.18, we obtain

$$|x_{i,j}(A)| = |\langle Ae_i\ ,\ e_j \rangle| \le \lambda_1 \le \frac{\lambda_1}{\lambda_n} = d_\delta(\mathrm{Id}, A),$$
$$t_\ell^2 = \frac{1}{\det(A[\ell])} \le \lambda_n^{-\ell} \le \left(\frac{\lambda_1}{\lambda_n}\right)^\ell = d_\delta(\mathrm{Id}, A)^\ell.$$

Using that $(\rho, \mathbb{F}_\rho)$ is an element of $\widetilde{\mathcal{U}_k(\mathbb{R})}$, the $\mathbb{F}_\rho$-extensions of $\hat{d}_\delta$ and η verify

$$\hat{d}_\delta(\mathrm{Id}, x_{i,j}(w_\rho), y_\ell(w_\rho)) < \eta(z_m(w_\rho))^k.$$

Hence, by the Transfer principle (Theorem 2.2.2)

$$|x_{i,j}(w_\rho)| \leq \hat{d}_\delta\left(\mathrm{Id}, x_{i,j}(w_\rho), y_\ell(w_\rho)\right) < \eta\left(z_m(w_\rho)\right)^k, \tag{4.3}$$

$$y_\ell(w_\rho)^2 \leq \hat{d}_\delta\left(\mathrm{Id}, x_{i,j}(w_\rho), y_\ell(w_\rho)\right)^\ell < \eta\left(z_m(w_\rho)\right)^{k\ell}, \tag{4.4}$$

so that $x_{i,j}(w_\rho)$ and $y_\ell(w_\rho)$ are bounded by $\eta(z_m(w_\rho))^{k\ell}$ for every $1 \leq i,j \leq n$ and every $1 \leq \ell \leq n-1$. Hence $\rho(\mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}])$ is Archimedean over $\rho(\mathbb{R}[\mathcal{M}_\Gamma])$. Since $(\rho, \mathbb{F}_\rho)$ is a closed point of the real spectrum, $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}])$ by Proposition 3.1.7. Hence $\mathbb{F}_\rho$ is Archimedean over $\rho(\mathbb{R}[\mathcal{M}_\Gamma])$ so that

$$\pi^{\mathrm{RSp}}(E) \subset \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Second, we prove that $\pi^{\mathrm{RSp}}|_E$ is surjective onto $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Let $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Using $\mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}] \cong \mathbb{R}[\mathcal{M}_\Gamma] \otimes \mathbb{R}[\widehat{\mathcal{P}^1(n)}]$ [BCR13, Theorem 2.3.8. iii], we extend ρ^1 trivially on $\mathbb{R}[\mathcal{M}_\Gamma] \otimes \mathbb{R}[\widehat{\mathcal{P}^1(n)}]$ as

$$\begin{array}{rccc} \rho\colon & \mathbb{R}[\mathcal{M}_\Gamma] \otimes \mathbb{R}[\widehat{\mathcal{P}^1(n)}] & \longrightarrow & \mathbb{K}_{\rho^1}; \\ & \sum h_i^1 \otimes h_i^2 & \longmapsto & \sum \rho^1(h_i^1). \end{array}$$

Since $\eta(z_m(w_\rho))$ is a big element of $\mathbb{K}_{\rho^1}$, there exists a natural number k with $\hat{d}_\delta(\mathrm{Id}, x_{i,j}(w_\rho), y_\ell(w_\rho)) < \eta(z_m(w_\rho))^k$ so that $(\rho, \mathbb{K}_{\rho^1}) \in E$. Moreover, for every $f \in \mathbb{R}[\mathcal{M}_\Gamma]$, as in Remark 4.1.1

$$\pi^{\mathrm{RSp}}_{\mathrm{cl}}\left(\rho, \mathbb{K}_{\rho^1}\right)(f) = f \circ \pi(\rho^1(z_m), 1) = f(\rho^1(z_m)) = \rho^1(f),$$

so that $\pi^{\mathrm{RSp}}_{\mathrm{cl}}|_E$ is surjective.

Third, we show that $\pi^{\mathrm{RSp}}|_E$ is Γ-equivariant for the trivial action of Γ on $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ and the Γ-action on $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ described in Corollary 4.1.16. Let $(\rho, \mathbb{F}_\rho) \in (\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$. By Remark 4.1.1, there exist elements $(\rho^1, \mathbb{F}_\rho) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}$ and $(\rho^2, \mathbb{F}_\rho) \in \widehat{\mathcal{P}^1(n, \mathbb{R})}^{\mathrm{RSp}}$ such that, for every $h \in \mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n, \mathbb{R})}]$

$$\rho(h) = \sum \rho^1(h_i^1)\rho^2(h_i^2) \quad \text{where } \iota^{-1}(h) = \sum h_i^1 \otimes h_i^2 \in \mathbb{R}[\mathcal{M}_\Gamma] \otimes_{\mathbb{R}} \mathbb{R}\left[\widehat{\mathcal{P}^1(n)}\right].$$

Thus, for $\gamma \in \Gamma$ and $h^1 \in \mathbb{R}[\mathcal{M}_\Gamma]$, the following equalities hold:

$$\begin{aligned} \pi^{\mathrm{RSp}}_{\mathrm{cl}}(\gamma\rho)\left(h^1\right) &= \pi^{\mathrm{RSp}}_{\mathrm{cl}}\left(\rho^1, \rho^2 \circ \gamma\right)\left(h^1\right) \\ &= \rho_1\left(h^1\right) \\ &= \gamma\rho^1\left(h^1\right) = \gamma\pi^{\mathrm{RSp}}_{\mathrm{cl}}(\rho)\left(h^1\right). \end{aligned}$$

Hence, $\pi_{\mathrm{cl}}^{\mathrm{RSp}}$ is Γ-equivariant, as desired.

Finally, we show that the fiber in E of an element $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ is homeomorphic to $\widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}_{\mathrm{Arch}}^{\mathrm{RSp}}$. Consider $(\pi^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ endowed with the subspace topology induced by the spectral topology on $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}$. By Proposition 4.1.2, there exists a canonical homeomorphism

$$\begin{aligned}\left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{R}[\mathcal{M}_\Gamma]} \mathbb{R}\left[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}\right]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1} \otimes_{\mathbb{R}[\mathcal{M}_\Gamma]} \mathbb{R}\left[\mathcal{M}_\Gamma\right] \otimes_{\mathbb{R}} \mathbb{R}\left[\widehat{\mathcal{P}^1(n)}\right]\right)^{\mathrm{RSp}} \\ &\cong \left(\mathbb{K}_{\rho^1}\left[\widehat{\mathcal{P}^1(n)}\right]\right)^{\mathrm{RSp}} = \widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}^{\mathrm{RSp}},\end{aligned}$$

where the second homeomorphism comes from [BCR13, Theorem 2.8.3. (iii)]. We examine the intersection of this preimage with E. Consider

$$(\rho, \mathbb{F}_\rho) \in \left(\pi^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \cap E$$

and its decomposition $(\rho, \mathbb{F}_\rho) = (\rho^1, \rho^2, \mathbb{F}_\rho)$, as in Remark 4.1.1. By Equations 4.3 and 4.4 above, $\rho(\mathbb{R}[\mathcal{M}_\Gamma \times \widehat{\mathcal{P}^1(n)}])$ is Archimedean over $\rho^1(\mathbb{R}[\mathcal{M}_\Gamma]) \subset \mathbb{K}_{\rho^1}$. In particular, $\mathbb{F}_\rho$ is Archimedean over $\mathbb{K}_{\rho^1}$ so that $(\rho^2, \mathbb{F}_\rho) \in \mathbb{K}_{\rho^1}[\widehat{\mathcal{P}^1(n)}]_{\mathrm{Arch}}^{\mathrm{RSp}}$. Hence

$$\pi^{\mathrm{RSp}}|_E^{-1}(\rho, \mathbb{F}_\rho) \subset \widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}_{\mathrm{Arch}}^{\mathrm{RSp}}.$$

We prove the remaining inclusion. Consider $(\rho^2, \mathbb{K}_{\rho^2}) \in \widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}_{\mathrm{Arch}}^{\mathrm{RSp}}$. Then $\rho^2|_{\mathbb{K}_{\rho^1}} : \mathbb{K}_{\rho^1} \to \mathbb{K}_{\rho^2}$ is an ordered field morphism by Remark 3.1.8. Then

$$\left(\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \rho^2, \mathbb{K}_{\rho^2}\right) \in \left(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})}\right)_{\mathrm{cl}}^{\mathrm{RSp}}$$

as $\mathbb{K}_{\rho^2}$ is Archimedean over $\mathbb{K}_{\rho^1}$ which is Archimedean over $\rho^1(\mathbb{R}[\mathcal{M}_\Gamma])$. Moreover, $\pi^{\mathrm{RSp}}(\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \rho^2, \mathbb{K}_{\rho^2}) = (\rho^2|_{\mathbb{K}_{\rho^1}} \circ \rho^1, \mathbb{K}_{\rho^2}) = (\rho^1, \mathbb{K}_{\rho^1})$ so

$$\left(\pi_{\mathrm{cl}}^{\mathrm{RSp}}\right)^{-1}(\rho^1, \mathbb{K}_{\rho^1}) \cong \widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}_{\mathrm{Arch}}^{\mathrm{RSp}}.$$

□

Remark 4.1.20. The map $\pi_{\mathrm{cl}}^{\mathrm{RSp}}$ is Γ-equivariant, so its fibers are Γ-invariant. For $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$, this induces a Γ-action on $\widehat{\mathcal{P}^1(n, \mathbb{K}_{\rho^1})}_{\mathrm{Arch}}^{\mathrm{RSp}}$ such that, for $\gamma \in \Gamma$, $(\rho^2, \mathbb{K}_{\rho^2}) \in (\pi_{\mathrm{cl}}^{\mathrm{RSp}})^{-1}(\rho^1, \mathbb{K}_{\rho^1})$, and $h \in \mathbb{K}_{\rho^1}[\widehat{\mathcal{P}^1(n)}]$ we have:

$$\gamma \rho^2(h) = \rho^2(\rho^1(\gamma) h).$$

As in Subsection 4.1.1, fibers over points in the real spectrum compactification of $\mathcal{M}_\Gamma(\mathbb{R})$ behave well with respect to the projection. Informally, we show that for a sequence of elements in $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ that converges to some element $(\rho, \mathbb{K}_\rho)$, the fibers of the elements in the sequence converge to the fiber of the element $(\rho, \mathbb{K}_\rho)$. This highlights some similar behavior between $(\mathcal{M}_\Gamma(\mathbb{R}) \times \widehat{\mathcal{P}^1(n, \mathbb{R})})^{\mathrm{RSp}}_{\mathrm{cl}}$ and $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \times \widehat{\mathcal{P}^1(n, \mathbb{R})}^{\mathrm{RSp}}_{\mathrm{cl}}$.

Corollary 4.1.21. *Let $(\rho^1_n, \mathbb{K}_{\rho^1_n}) \subset \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ be a sequence converging to an element $(\rho^1, \mathbb{K}_{\rho^1}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. For every element of the fiber $(\rho^1, \rho^2, \mathbb{F}) \in (\pi^{\mathrm{RSp}}|_E)^{-1}(\rho^1, \mathbb{K}_{\rho^1})$ and every open set in E around this element, there exists $M \in \mathbb{N}$ and $(\rho^1_M, \rho^2_M, \mathbb{F}_M)$ in the intersection $(\pi^{\mathrm{RSp}}|_E)^{-1}(\rho^1_M, \mathbb{K}_{\rho^1_M}) \cap U$.*

Proof. By Theorem 4.1.19$\pi^{\mathrm{RSp}}|_E$ verifies all the conditions from Theorem 4.1.6. Thus, the statement is a consequence of the more general Theorem 4.1.6. □

We constructed a universal symmetric space over $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. This provides a framework for studying the actions induced by elements of $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ on $\widehat{\mathcal{P}^1(n, \mathbb{K})}$ within $\widehat{\mathcal{P}^1(n, \mathbb{K})}^{\mathrm{RSp}}_{\mathrm{Arch}}$. Furthermore, an element $(\rho, \mathbb{K}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ induces an action of Γ on the nonstandard symmetric space $\mathcal{P}^1(n, \mathbb{K})$ and its associated building $B_\mathbb{K}$, see [BIPP23, Section 5] and [App24, Theorem 8.1]. This action of Γ on both spaces is well-described by the actions induced by sequences $(\rho_n, \mathbb{R}) \in \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ that converge to $(\rho, \mathbb{K})$. However, unlike the Archimedean spectrum, $\mathcal{P}^1(n, \mathbb{K})$ and $B_\mathbb{K}$ are not locally compact. In contrast, $\widehat{\mathcal{P}^1(n, \mathbb{K})}^{\mathrm{RSp}}_{\mathrm{Arch}}$ provides a locally compact space containing $\widehat{\mathcal{P}^1(n, \mathbb{K})}$, offering a suitable framework to study the geometry and dynamics of elements in $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$. Consequently, the next section focuses on exploring the relationship between the Archimedean spectrum of an algebraic set and its Berkovich analytification, using real analytification as an intermediary step.

4.2 The Archimedean spectrum and the analytification of an algebraic set

The Berkovich analytification offers a framework for applying complex analysis techniques to the study of algebraic sets defined over ultrametric fields. Similarly, the real analytification provides its real counterpart for studying algebraic sets defined over non-Archimedean real fields using analytic methods. In this section, we explore the identification between the Archimedean spectrum and the real analytification of an algebraic set. We further examine its relationship with the Berkovich analytification. Specifically, we compute the image of the Archimedean spectrum within the Berkovich analytification and derive actions on $\mathbb{R}$-trees.

4.2.1 Berkovich and real analytifications of algebraic sets

We present the material necessary to study the real and Berkovich analytifications applied to our context. For a more detailed study of real and Berkovich analytifications, we refer to the texts [Ber90], [BR10], and [JSY22].

In this subsection, $\mathbb{K}$ is a real closed field non-Archimedean over the reals with a non-trivial absolute value $|\cdot|_{\mathbb{K}}$ which is compatible with the order. That is, $|\cdot|_{\mathbb{K}}\colon \mathbb{K} \to \mathbb{R}$ is a non-Archimedean *absolute value* if for every $h, k \in \mathbb{K}$:

- $|k|_{\mathbb{K}} \geq 0$ with equality if and only if $k = 0$,
- $|hk|_{\mathbb{K}} = |h|_{\mathbb{K}}|k|_{\mathbb{K}}$,
- $|h + k|_{\mathbb{K}} \leq \max\{\, |h|_{\mathbb{K}}, |k|_{\mathbb{K}}\,\}$.

Also, A is a $\mathbb{K}$-algebra and $V = \mathrm{Spec}(A)$ is an affine $\mathbb{K}$-variety.

Remark 4.2.1. In the context of subsection 4.2.3, $\mathbb{L} \subset \mathbb{K}$ are real closed fields where $\mathbb{K}$ has a big element b, and $V \subset \mathbb{L}^n$ is an algebraic set. Then $A = \mathbb{K}[V]$ is a $\mathbb{K}$-algebra and $V(\mathbb{K}) = \mathrm{Spec}(\mathbb{K}[V])$ is an affine $\mathbb{K}$-variety. Moreover, for every $h \in \mathbb{K}$ the two subsets of $\mathbb{Q}$

$$\left\{ \frac{m}{n} \,\middle|\, b^m \leq h^n, n \in \mathbb{N}_{\geq 0}, m \in \mathbb{Z} \right\} \text{ and } \left\{ \frac{m'}{n'} \,\middle|\, b^{m'} \geq h^{n'}, n' \in \mathbb{N}_{\geq 0}, m' \in \mathbb{Z} \right\}$$

define a Dedekind cut of $\mathbb{Q}$. Hence the two subsets above define a real number denoted $\log_b(h)$. Then

$$\nu(h) := e^{-\log_b |h|}, \text{ where } |h| := \max\{\, h, -h\,\}$$

is a non-trivial *order compatible absolute value* on $\mathbb{K}$ [Bru88a, Section 5].

A *multiplicative seminorm* on a $\mathbb{K}$-algebra A is a map $\eta\colon A \to \mathbb{R}_{\geq 0}$ such that

- $\eta(k) = |k|_{\mathbb{K}}$ for every $k \in \mathbb{K}$,
- $\eta(fg) = \eta(f)\eta(g)$ for every $f, g \in A$,
- $\eta(f + g) \leq \max\{\, \eta(f), \eta(g)\,\}$ for every $f, g \in A$.

A *signed multiplicative seminorm* on a $\mathbb{K}$-algebra A is a map $\eta\colon A \to \mathbb{R}$ such that

- $\eta(k) = sgn(k)|k|_{\mathbb{K}}$ for every $k \in \mathbb{K}$,
- $\eta(fg) = \eta(f)\eta(g)$ for every $f, g \in A$,
- $\min\{\, \eta(f), \eta(g)\,\} \leq \eta(f + g) \leq \max\{\, \eta(f), \eta(g)\,\}$ for every $f, g \in A$.

We now define the Berkovich analytification and the real analytification of an affine $\mathbb{K}$-variety.

Definition 4.2.2 ([Ber90, Definition 1.5.1] and [JSY22, Definition 3.3])**.** Let A be a $\mathbb{K}$-algebra. The *Berkovich analytification* of $V = \operatorname{Spec}(A)$ is the topological space

$$\mathcal{M}(A) := \{\, \eta\colon A \to \mathbb{R}_{\geq 0} \,|\, \eta \text{ is a multiplicative seminorm} \,\},$$

with the coarsest topology that makes the evaluation map

$$\begin{array}{ccc} \mathcal{M}(A) & \longrightarrow & \mathbb{R}_{\geq 0}; \\ \eta & \longmapsto & \eta(f), \end{array}$$

continuous for every element f in A.

Similarly, the *real analytification* of V is

$$\mathcal{M}_{\mathrm{R}}(A) := \{\, \eta\colon A \to \mathbb{R} \,|\, \eta \text{ is a signed multiplicative seminorm} \,\},$$

with the coarsest topology that makes the evaluation map

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{R}}(A) & \longrightarrow & \mathbb{R}; \\ \eta & \longmapsto & \eta(f), \end{array}$$

continuous for every element f in A.

Both the Berkovich and the real analytifications have definitions in terms of ideals in the affine variety and absolute value on the residue field. This gives a relation between the spectrum of an algebraic set and its analytification.

Proposition 4.2.3 ([Ber90, Remark 3.4.2])**.** *The Berkovich analytification of $V = \operatorname{Spec}(A)$ is in bijective correspondence with*

$$V^{\mathrm{an}} := \left\{ (p, |\cdot|_p) \;\middle|\; \begin{array}{l} p \textit{ is a prime ideal in } A, \\ |\cdot|_p \textit{ an absolute value on } \operatorname{Frac}(A/p) \textit{ extending } |\cdot|_{\mathbb{K}} \end{array} \right\}.$$

Moreover, if V^{an} is endowed with the coarsest topology such that

$$\begin{array}{cccc} \mathit{supp}\colon & V^{\mathrm{an}} & \longrightarrow & V; \\ & (p, |\cdot|_p) & \longmapsto & p \end{array}$$

is continuous and the map

$$\begin{array}{ccc} \mathit{supp}(U)^{-1} \subset V^{\mathrm{an}} & \longrightarrow & \mathbb{R}_{\geq 0}; \\ (p, |\cdot|_p) & \longmapsto & |f|_p, \end{array}$$

is continuous for every open $U \subset V$ and every regular map f on U, then the bijection is a homeomorphism.

Proposition 4.2.4 ([JSY22, Proposition 3.4]). *The real analytification of $V = \mathrm{Spec}(A)$ is in bijective correspondence with the set*

$$V_{\mathrm{R}}^{\mathrm{an}} := \left\{ (p, |\cdot|_p, <_p) \;\middle|\; \begin{array}{l} p \text{ a prime ideal in } A, \\ |\cdot|_p \text{ an absolute value on } \mathrm{Frac}(A/p) \text{ extending } |\cdot|_{\mathbb{K}}, \\ <_p \text{ an order on } \mathrm{Frac}(A/p) \text{ compatible with } |\cdot|_{\mathbb{K}} \end{array} \right\}.$$

Furthermore, if the set of triples is endowed with the coarsest topology such that

$$\begin{array}{rccc} supp\colon & V_{\mathrm{R}}^{\mathrm{an}} & \longrightarrow & V; \\ & (p, |\cdot|_p, <_p) & \longmapsto & p \end{array}$$

is continuous and the map

$$\begin{array}{ccc} supp(U)^{-1} \subset V_{\mathrm{R}}^{\mathrm{an}} & \longrightarrow & \mathbb{R}; \\ (p, |\cdot|_p, <_p) & \longmapsto & \mathrm{sgn}_p(f)|f|_p, \end{array}$$

is continuous for every open $U \subset V$ and every regular map f on U, then the bijection is a homeomorphism.

These two spaces represent distinct structures of algebraic varieties. However, they are connected by the following proposition.

Proposition 4.2.5 ([JSY22, Lemma 3.9]). *The map*

$$\begin{array}{ccc} V_{\mathrm{R}}^{\mathrm{an}} & \longrightarrow & V^{\mathrm{an}}; \\ (p, |\cdot|_p, <_p) & \longmapsto & (p, |\cdot|_p), \end{array}$$

is a proper continuous map of topological spaces.

Both spaces are crucial in the study of analytic properties of ultrametric spaces. Although ultrametric spaces are totally disconnected, their Berkovich analytification offers a method to embed them within a uniquely path connected space.

Proposition 4.2.6 ([Ber90, Theorem 3.4.8]). *The Berkovich analytification of $V = \mathrm{Spec}(A)$ is*

- *a locally compact Hausdorff space which is locally contractible.*
- *uniquely path connected if and only if V, with the Zariski topology, is connected.*

Moreover, there exists a canonical embedding from V to V^{an} and V is dense in its Berkovich analytification.

The real analytification also exhibits intriguing characteristics similar to those of the Berkovich analytification.

Proposition 4.2.7 ([JSY22, Proposition 3.6, Corollary 3.18])**.** *If V is an affine $\mathbb{K}$-variety, then $V^{\mathrm{an}}_{\mathrm{R}}$ is a Hausdorff space. Moreover, there exists a canonical embedding from V to V^{an} and V is dense in its real analytification.*

While the real analytification provides a topological space that retains the real structure of the studied algebraic set, it does not preserve all the favorable topological properties of the Berkovich analytification. As a result, the analytic study of real algebraic sets becomes more challenging.

Proposition 4.2.8 ([JSY22, Theorem 3.20])**.** *Let $F : [0, 1] \to V^{an}_{R}$ be a continuous map, where V is an affine $\mathbb{K}$-variety. Then F is constant. In particular, the real analytification of an affine $\mathbb{K}$-variety is totally path disconnected.*

Moreover, using hyperfield theory [Jun21], one can show that both the Berkovich and real analytifications are functors from $\mathbb{K}$-algebras to topological spaces.

Proposition 4.2.9 ([Jun21, Proposition 5.5])**.** *The Berkovich and the real analytifications are contravariant functors* $\mathrm{Hom}_{\mathbb{K}}(\cdot, \mathbb{T})$ *and* $\mathrm{Hom}_{\mathbb{K}}(\cdot, \mathbb{RT})$ *from $\mathbb{K}$-algebras to topological spaces.*

We introduced two analytifications with desirable topological properties for affine algebraic varieties over non-Archimedean real closed fields. In the next subsection, we study in more detail the relationship between the Archimedean spectrum, the real analytification and the Berkovich analytification.

4.2.2 Archimedean spectrum and real analytification

The Archimedean spectrum of an algebraic set is homeomorphic to its real analytification [JSY22]. We use this homeomorphism to deduce topological properties of the Archimedean spectrum. Moreover, we describe the image of the Archimedean spectrum of a $\mathbb{K}$-algebra in its Berkovich analytification.

Theorem 4.2.10 ([JSY22, Theorem 3.17])**.** *Let $\mathbb{K}$ be a real closed field with a non-trivial order compatible absolute value, A a $\mathbb{K}$-algebra and $V = \mathrm{Spec}(A)$ an affine $\mathbb{K}$-variety. There exists a canonical map from $A^{\mathrm{RSp}}_{\mathrm{Arch}}$ to V^{an}_{R}, which is a homeomorphism (this map is ψ_1 in the proof of Theorem 4.2.14).*

A direct consequence of Theorem 4.2.10 and Theorem 3.4.3 is the local compactness of the real analytification.

Corollary 4.2.11. *The real analytification of an affine $\mathbb{K}$-variety is a countable union of compact sets. Thus, it is σ-compact and locally compact.*

Furthermore, the real spectrum of an affine $\mathbb{K}$-variety is metrizable under a countability condition on $\mathbb{K}$ [BIPP23, Proposition 2.33]. This gives a real counterpart to the metrizability theorem, stated in [HLP15, Remark 1.5] for the Berkovich analytification, which is a direct consequence of Theorem 4.2.10.

Corollary 4.2.12. *Let $\mathbb{K}$ be a real closed field with a non-trivial absolute value and V an affine $\mathbb{K}$-variety. If $\mathbb{K}$ is countable, then the real analytification $V_{\mathrm{R}}^{\mathrm{an}}$ is metrizable.*

The following result is a direct consequence of Proposition 4.2.8.

Corollary 4.2.13. *The Archimedean spectrum is a functor from $\mathbb{K}$-algebras to Hausdorff and totally path disconnected topological spaces.*

In particular, the Archimedean spectrum of the $\mathbb{K}$-extension of an algebraic set is totally disconnected. Finally, we describe the image of the Archimedean spectrum of a $\mathbb{K}$-algebra in its Berkovich analytification using the maps from Theorem 4.2.10 and Proposition 4.2.5.

Theorem 4.2.14. *The image of $\psi\colon A_{\mathrm{Arch}}^{\mathrm{RSp}} \to \mathcal{M}(A)$ is*

$$\left\{\eta \in \mathcal{M}(A) \,\middle|\, \eta\left(f_1^2 + \cdots + f_q^2\right) = \max_i \eta\left(f_i^2\right) \quad \forall f_1, \ldots, f_q \in A \right\}.$$

Proof. We consider two maps. The first one is the homeomorphism from Theorem 4.2.10:

$$\begin{array}{cccccc} \psi_1 : & A_{\mathrm{Arch}}^{\mathrm{RSp}} & \to & \operatorname{Spec}(A)_{\mathrm{R}}^{\mathrm{an}} & \to & \operatorname{Spec}(A)^{\mathrm{an}}, \\ & \alpha & \mapsto & \left(supp\left(\alpha\right), \leq_\alpha, |\cdot|_\alpha\right) & \mapsto & \left(supp(\alpha), |\cdot|_\alpha\right). \end{array}$$

Here, as in Proposition 3.1.2, $supp(\alpha)$ is the prime ideal $\alpha \cap -\alpha$ associated to the prime cone α, $\leq_\alpha$ is the order on $\operatorname{Frac}(A/supp(\alpha))$ such that $0 \leq_\alpha \overline{f}/\overline{g}$ if $fg \in \alpha$ and

$$\left|\overline{f}\right|_\alpha := \sup \left\{ |k|_{\mathbb{K}} \,\middle|\, k \in \mathbb{K}_{\geq 0},\, k \leq_\alpha sgn\left(\overline{f}\right) \cdot \overline{f} \right\}$$

is an absolute value on the fraction field [JSY22, Lemma 3.13]. This absolute value is well defined because the real closure of the fraction field of $\operatorname{Frac}(A/supp(\alpha))$ is Archimedean over $\mathbb{K}$. The second map

$$\begin{array}{cccc} \psi_2 : & \operatorname{Spec}(A)^{\mathrm{an}} & \to & \mathcal{M}(A), \\ & \left(supp\left(\alpha\right), |\cdot|_\alpha\right) & \mapsto & |\cdot|_\alpha^s \end{array}$$

where $|\cdot|_\alpha^s : f \mapsto |\overline{f}|_\alpha$ defines the homeomorphism from Proposition 4.2.3. We study the image of $A_{\mathrm{Arch}}^{\mathrm{RSp}}$ in $\mathcal{M}(A)$ using the composition $\psi := \psi_2 \circ \psi_1$. For $f_1, \ldots, f_q \in A$, $\alpha \in A_{\mathrm{Arch}}^{\mathrm{RSp}}$ a prime cone, and by positivity of a sum of squares, it holds

$$\begin{aligned} \psi(\alpha)\left(f_1^2 + \cdots + f_q^2\right) &= \left|f_1^2 + \cdots + f_q^2\right|_\alpha^s \\ &= \sup \left\{ |k_1 + \cdots + k_q|_{\mathbb{K}} \,\middle|\, k_i \in \mathbb{K}_{\geq 0},\ k_i \leq_\alpha \overline{f_i}^2,\ 1 \leq i \leq q \right\}. \end{aligned}$$

The absolute values on $\mathbb{K}$ is non-Archimedean. Thus, the following inequality holds

$$|k_1 + \cdots + k_q|_{\mathbb{K}} \leq \max_{1\leq i\leq q} |k_i|_{\mathbb{K}}.$$

Moreover, the inequality is an equality. Indeed, all the k_i are positive and the non-Archimedean absolute value on $\mathbb{K}$ is compatible with the order. Hence

$$|k_1 + \cdots + k_q|_{\mathbb{K}} \geq |k_i|_{\mathbb{K}}$$

for every $1 \leq i \leq q$. Hence $\psi(\alpha)\left(f_1^2 + \cdots + f_q^2\right) = \max_i |f_i^2|_\alpha^s$ so that

$$\operatorname{Im}(\psi) \subset \left\{\eta \in \mathcal{M}(A) \,\middle|\, \eta\left(f_1^2 + \cdots + f_q^2\right) = \max_i \eta\left(f_i^2\right)\right\}.$$

We prove the inverse inclusion. Let $\eta : A \to \mathbb{R}$ be a $\mathbb{K}$-seminorm so that $\eta(f_1^2 + \cdots + f_q^2) = \max_i \eta(f_i^2)$ for every $f_1, \ldots, f_q \in A$. This seminorm defines a $\mathbb{K}$-absolute value on the real closure of the fraction field of A/I_η via

$$\left|\overline{f}\right| = \eta(f),$$

where I_η is the support of the seminorm η. Suppose $\operatorname{Frac}(A/I_\eta)$ is not orderable. Then, there exists $f_1, \ldots, f_n, g_1, \ldots, g_n \in A$ with $g_1, \ldots, g_n \notin I_\eta$ such that

$$-1 = \sum_{i=1}^{n} \left(\frac{\overline{f_i}}{\overline{g_i}}\right)^2 \text{ or equivalently } -\left(\prod_{i=1}^{n} \overline{g_i}\right)^2 = \sum_{i=1}^{n} \left(\prod_{j\neq i} \overline{f_i}\overline{g_j}\right)^2.$$

Thus, using the seminorm η, we obtain

$$\begin{aligned} 0 = \eta(0) &= \eta\left(\sum_{i=1}^{n}\left(\prod_{j\neq i} f_i g_j\right)^2 + \left(\prod_{i=1}^{n} g_i\right)^2\right) \\ &= \max\left\{\max_i \left\{\eta\left(\prod_{j\neq i} f_i g_j\right)^2\right\}, \eta\left(\prod_{i=1}^{n} g_i\right)^2\right\} > 0. \end{aligned}$$

The last inequality holds because every $g_i \notin I_\eta$ so that the seminorm of their product is no zero. Now, consider an order $\leq_\eta$ on $\operatorname{Frac}(A/I_\eta)$ and its associated absolute value defined by

$$\left|\overline{f}\right|_\eta := \sup\left\{|k|_{\mathbb{K}} \,\middle|\, k \in \mathbb{K}_{\geq 0},\ k \leq_\eta \operatorname{sgn}\left(\overline{f}\right) \cdot \overline{f}\right\}.$$

This absolute value is compatible with the order on the fraction field so that $(I_\eta, \leq_\eta, |\cdot|_\eta)$ defines an element in $A^{\mathrm{RSp}}_{\mathrm{Arch}}$. Hence

$$\operatorname{Im}(\psi) = \left\{\eta \in \mathcal{M}(A) \,\middle|\, \eta\left(f_1^2 + \cdots + f_q^2\right) = \max_i \eta\left(f_i^2\right) \quad \forall f_1, \ldots, f_q \in A\right\}.$$

□

Definition 4.2.15. An element in the image of $\psi\colon A^{\mathrm{RSp}}_{\mathrm{Arch}} \to \mathcal{M}(A)$ is called a *real element* of $\mathcal{M}(A)$.

Corollary 4.2.16. *The image T of $A^{\mathrm{RSp}}_{\mathrm{Arch}}$ inside $\mathcal{M}(A)$ is closed.*

Proof. For a fixed finite collection $f_1, \ldots, f_q \in A$, define the map

$$\begin{array}{rccc} \varphi_{f_1,\ldots,f_q}\colon & \mathcal{M}(A) & \longrightarrow & \mathbb{R}; \\ & \eta & \longmapsto & \eta\left(f_1^2 + \cdots + f_q^2\right) - \max\left\{\, \eta\left(f_1^2\right), \ldots, \eta\left(f_q^2\right) \right\}. \end{array}$$

By Definition 4.2.2, this map is continuous, so that

$$Z_{f_1,\ldots,f_q} := \left\{\eta \in \mathcal{M}(A) \,\middle|\, \varphi_{f_1,\ldots,f_q}(\eta) = 0 \right\}$$

is closed for every $f_1, \ldots, f_q \in A$. Hence, by Theorem 4.2.14

$$T = \bigcap_{q \geq 1} \bigcap_{f_1,\ldots,f_q \in A} Z_{f_1,\ldots,f_q},$$

which is a closed subset of $\mathcal{M}(A)$. □

We studied the connections between the Archimedean spectrum, the real analytification, and the Berkovich analytification. In particular, we provided a description of the image of the Archimedean spectrum in the Berkovich analytification. In the next subsection, we apply this connection to the projective line to derive actions on $\mathbb{R}$-trees from these results.

4.2.3 Application to the universal projective line

We prove that the image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ in $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ is a $\mathrm{PSL}_2(\mathbb{K})$-invariant closed uniquely path connected subset, where $\mathbb{K}$ is a non-Archimedean real closed field with a big element. In particular, the image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{K})$ in $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{K})$ is a $\mathbb{R}$-tree. To show this, we first describe the $\mathbb{R}$-tree structure of $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ for any non-Archimedean field $\mathbb{F}$ with an absolute value. For further details, we refer the reader to [Ber90, Subsection 4.2], [BR10, Section 2] and [DF19, Section 1].

Let $\mathbb{F}$ be an algebraically closed non-Archimedean complete field with an absolute value $|\cdot|_{\mathbb{F}}$. Following Definition 4.2.2, the space $\mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ has a natural partial order defined by

$$\eta_1 \leq \eta_2 \text{ if and only if } \eta_1(f) \leq \eta_2(f) \text{ for every } f \in \mathbb{F}[x].$$

With this order, any two elements $\eta_1, \eta_2 \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ admit a unique least upper bound $\eta_1 \vee \eta_2 \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$, see Example 4.2.20. By viewing $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ as the one point compactification $\mathbb{A}^1(\mathbb{F})^{\mathrm{an}} \cup \{\, \infty \,\}$, we extend the partial order from $\mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ to

$\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ by setting $\eta \leq \infty$ for every $\eta \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ [BR10, Section 2]. For any two elements $\eta_1, \eta_2 \in \mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$, define the unique path joining them by

$$\begin{aligned}\ell(\eta_1,\eta_2) := &\left\{ \eta_3 \in \mathbb{P}^1(\mathbb{F})^{\mathrm{an}} \,\middle|\, \eta_1 \leq \eta_3 \leq \eta_1 \vee \eta_2 \right\} \\ &\cup \left\{ \eta_3 \in \mathbb{P}^1(\mathbb{F})^{\mathrm{an}} \,\middle|\, \eta_2 \leq \eta_3 \leq \eta_1 \vee \eta_2 \right\}.\end{aligned}$$

Remark 4.2.17. We refer to [BR10, Subsection 2.2] for a description of $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ in terms of $\mathbb{F}$-multiplicative seminorms on the homogeneous coordinate ring of $\mathbb{P}^1(\mathbb{F})$, which provides a more natural approach for studying the action of $\mathrm{PGL}_2(\mathbb{F})$ on $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$.

For $v \in \mathbb{F}$ and $r \in \mathbb{R}$, set $D(v,r) := \{ w \in \mathbb{A}^1(\mathbb{F}) \mid |w-v|_{\mathbb{F}} \leq r \}$.

Theorem 4.2.18 ([Ber90, Example 1.4.4. page 18])**.** *For every $\eta \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$, there exists a nested sequence of closed disks*

$$\mathbb{A}^1(\mathbb{F}) \supseteq D(v_1,r_1) \supseteq D(v_2,r_2) \supseteq \cdots,$$

where $v_i \in \mathbb{A}^1(\mathbb{F})$ and $r_i \in \mathbb{R}_{\geq 0}$ such that for every $f \in \mathbb{F}[x]$

$$\eta(f) = \lim_{i\to\infty} \sup_{w \in D(v_i,r_i)} |f(w)|_{\mathbb{F}}.$$

Two nested sequences define the same element if and only if

- *each has a nonempty intersection, and their intersections are the same,*
- *both have empty intersection, and the sequences are cofinal.*

Remark 4.2.19 ([Ber90, Example 1.4.4. page 18])**.** If an element $\eta \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ corresponds to a nested sequence of closed disks

$$\mathbb{A}^1(\mathbb{F}) \supseteq D(v_1,r_1) \supseteq D(v_2,r_2) \supseteq \cdots,$$

then we denote η by $(D(v_i,r_i))_i$. In the following, we use the notation

$$\mathcal{H}\left(\mathbb{A}^1(\mathbb{F})\right) := \left\{ D(v,r) \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}} \,\middle|\, v \in \mathbb{F},\ r \in \mathbb{R}_{>0} \right\}.$$

Moreover, for every $f \in \mathbb{F}[x]$

$$\sup_{w \in D(v,r)} |f(w)|_{\mathbb{F}} = \max_n |a_n|_{\mathbb{F}} r^n,$$

where $f(x) = \sum_n a_n(x^n - v)$ is the expansion of f with center v.

Example 4.2.20. With the notation from Theorem 4.2.18, the unique least upper bound of $D(v_1, r_1)$ and $D(v_2, r_2) \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ is

$$D(v_1, \max\{ r_1, r_2, |v_2 - v_1|_{\mathbb{F}} \}).$$

So the unique path between $D(v_1, r_1)$ and $D(v_2, r_2) \in \mathbb{A}^1(\mathbb{F})^{\mathrm{an}}$ corresponds to

$$\begin{aligned}\ell(D(v_1, r_1), D(v_2, r_2)) = &\{ D(v_1, r) \mid r_1 \leq r \leq \max\{ |v_2 - v_1|_{\mathbb{F}}, r_2 \} \} \\ &\cup \{ D(v_2, s) \mid r_2 \leq s \leq \max\{ |v_2 - v_1|_{\mathbb{F}}, r_1 \} \},\end{aligned}$$

where we use the strong triangular inequality to argue that every point in $D(v, r)$ is the center of the closed disk $D(v, r)$.

Through the identification $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}} = \mathbb{A}^1(\mathbb{F})^{\mathrm{an}} \cup \{ \infty \}$, Theorem 4.2.18 allows us to further study the topology on $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$

Definition 4.2.21. Identifying $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ with $\mathbb{A}^1(\mathbb{F})^{\mathrm{an}} \cup \{ \infty \}$, define the *open and closed Berkovich discs*

$$\begin{aligned}\mathcal{D}(v, r)^- &:= \{ \eta \in \mathbb{A}^1(\mathbb{F})^{an} \mid \eta(x - v) < r \}, \\ \mathcal{D}(v, r) &:= \{ \eta \in \mathbb{A}^1(\mathbb{F})^{an} \mid \eta(x - v) \leq r \},\end{aligned}$$

and we view them as subsets of $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$.

Proposition 4.2.22 ([BR10, Proposition 2.7])**.** *A basis for the open sets of* $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ *is given by the sets of the form*

$$\mathcal{D}(v, r)^-, \quad \mathcal{D}(v, r)^- \backslash \bigcup_{i=1}^{M} \mathcal{D}(v_i, r_i), \quad \textit{and} \quad \mathbb{P}^1(\mathbb{F})^{\mathrm{an}} \backslash \bigcup_{i=1}^{M} \mathcal{D}(v_i, r_i),$$

where $v, v_i \in \mathbb{F}$ *and* $r, r_i \in \mathbb{R}_{>0}$.

Remark 4.2.23 ([BR10, Lemma 2.9])**.** Let $(D(v_i, r_i))_i$ be an element in $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$ associated to the $\mathbb{F}$-multiplicative seminorm η and

$$U := \mathcal{D}(v, r)^- \text{ or } U := \mathcal{D}(v, r)^- \backslash \bigcup_{i=1}^{M} \mathcal{D}(w_i, s_i)$$

a basis open set around $(D(v_i, r_i))_i$. By Theorem 4.2.18, for every $1 \leq i \leq M$

$$\eta(x - v) = \lim_n \sup_{z \in D(v_n, r_n)} |z - v|_{\mathbb{F}} \text{ and } \eta(x - w_i) = \lim_n \sup_{z \in D(v_n, r_n)} |z - w_i|_{\mathbb{F}}$$

such that there exists $N \in \mathbb{N}$ with $D(v_n, r_n) \in U$ for every $n \geq N$. Hence, the sequence of discs $D(v_i, r_i)$ converges in the Berkovich topology to the element $(D(v_i, r_i))_i$. If U is of the form $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}} \backslash \cup_{i=1}^{M} \mathcal{D}(w_i, s_i)$, a similar argument proves the convergence so that $\mathcal{H}(\mathbb{A}^1(\mathbb{F}))$ is dense in $\mathbb{P}^1(\mathbb{F})^{\mathrm{an}}$.

Lemma 4.2.24 ([BR10, Lemma 2.10])**.** *With the definition of paths from above,* $\mathbb{P}^1(\mathbb{F})^{\text{an}}$ *is uniquely path connected.*

This property can be studied further as follows. Using the notation from Theorem 4.2.18, define the diameter function

$$\begin{array}{rccc} \text{diam}\colon & \mathbb{A}^1(\mathbb{F})^{\text{an}} & \longrightarrow & \mathbb{R}_{\geq 0}; \\ & (D(v_i, r_i))_i & \longmapsto & \lim_i r_i. \end{array}$$

This definition is independent of the choice of the nested sequence. As in [BR10, Subsection 2.7], we obtain a well defined distance

$$\begin{array}{rccc} d^{\text{an}}\colon & \left(\mathbb{P}^1(\mathbb{F})^{\text{an}} \backslash \mathbb{P}^1(\mathbb{F})\right)^2 & \longrightarrow & \mathbb{R}_{\geq 0}; \\ & ((D(v_i, r_i)), (D(w_j, s_j)) & \longmapsto & \log\left(\frac{\text{diam}(D(v_i,r_i) \vee D(w_j,s_j))^2}{\text{diam}(D(v_i,r_i))\text{diam}(D(w_j,s_j))}\right). \end{array}$$

We extend d^{an} to a pseudo-metric on $\mathbb{P}^1(\mathbb{F})^{\text{an}}$ by: for every $D(v_i, r_i), D(w_i, s_i) \in \mathbb{P}^1(\mathbb{F})^{\text{an}}$

$$\begin{array}{ll} d^{\text{an}}(D(v_i, r_i), D(w_j, s_j)) = \infty & \text{if } D(v_i, r_i) \neq D(w_j, s_j) \\ d^{\text{an}}(D(v_i, r_i), D(w_j, s_j)) = 0 & \text{otherwise.} \end{array}$$

Remark 4.2.25. The topology on $\mathbb{P}^1(\mathbb{F})^{\text{an}}$ induced by d^{an} is stronger than the Berkovich topology.

Theorem 4.2.26 ([DF19, Lemma 1.3])**.** *The metric* d^{an} *turns* $\mathbb{P}^1(\mathbb{F})^{\text{an}} \backslash \mathbb{P}^1(\mathbb{F})$ *into a complete* $\mathbb{R}$*-tree on which* $\text{PSL}_2(\mathbb{F})$ *acts by isometries.*

Finally, by applying Galois theory, this construction extends to non-algebraically closed fields, see [DF19, Subsection 1.6]. Let $\mathbb{K}$ be a non-Archimedean real closed field with an absolute value and $\mathbb{F} = \mathbb{K}(\sqrt{-1})$ its algebraic closure. The Galois group $\text{Gal}(\mathbb{F}/\mathbb{K})$ acts on $\mathbb{F}$ by complex conjugation. Thus, $\text{Gal}(\mathbb{F}/\mathbb{K})$ acts by isometries on $\mathbb{P}^1(\mathbb{F})^{\text{an}}$ as

$$(D(v_i, r_i))_i \mapsto (D(\overline{v_i}, r_i))_i.$$

By [DF19, page 525], $\mathbb{P}^1(\mathbb{K})^{\text{an}}$ is then homeomorphic to $\mathbb{P}^1(\mathbb{F})^{\text{an}}/\text{Gal}(\mathbb{F}/\mathbb{K})$.

Proposition 4.2.27 ([DF19, Lemma 1.6])**.** *The set of fixed points of the action of* $\text{Gal}(\mathbb{F}/\mathbb{K})$ *on* $\mathbb{P}^1(\mathbb{F})^{\text{an}}$ *is the closure of the convex hull of* $\mathbb{P}^1(\mathbb{K})$*.*

Lemma 4.2.28 ([DF19, Proposition 1.7])**.** *The space* $\mathbb{P}^1(\mathbb{K})^{\text{an}} \backslash \mathbb{P}^1(\mathbb{K})$*, with the metric induced from* $\mathbb{P}^1(\mathbb{F})^{\text{an}} \backslash \mathbb{P}^1(\mathbb{F})$*, is a complete* $\mathbb{R}$*-tree on which* $\text{PSL}_2(\mathbb{K})$ *acts by isometries.*

We show that the image T of $\psi\colon \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}\backslash\mathbb{P}^1(\mathbb{K}) \to \mathbb{P}^1(\mathbb{K})^{\mathrm{an}}\backslash\mathbb{P}^1(\mathbb{K})$ is a $\mathbb{R}$-subtree which is $\mathrm{PSL}_2(\mathbb{K})$-invariant. This will follow from the description of the image of $\mathbb{A}^1(\mathbb{K})^{\mathrm{an}}_{\mathrm{R}}$ in $\mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$.

Example 4.2.29 ([JSY22, Example 3.12])**.** An element $\eta \in \mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$ is real, see Definition 4.2.15, if it can be represented by a sequence $(D(v_i, r_i))_i \subset \mathbb{A}^1(\mathbb{F})^{an}$ such that $D(v_i, r_i) \cap \mathbb{A}^1(\mathbb{K}) \neq \varnothing$ for every i.

Theorem 4.2.30. *The image T_1 of $\psi_A\colon \mathbb{A}^1(\mathbb{K})^{\mathrm{an}}_{\mathrm{R}} \to \mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$ is uniquely path connected.*

Proof. Let $D(v, r), D(w, s)$ be elements of T_1, where $v, w \in \mathbb{K}(\sqrt{-1})$, $r, s \in \mathbb{R}_{\geq 0}$, and denote by $|\cdot|_{\mathbb{K}(\sqrt{-1})}$the absolute value on $\mathbb{K}(\sqrt{-1})$ defined by

$$|k + h\sqrt{-1}|_{\mathbb{K}(\sqrt{-1})} = |\sqrt{k^2 + h^2}|_{\mathbb{K}}.$$

By Example 4.2.29

$$D(v, r) \cap \mathbb{A}^1(\mathbb{K}) \neq \varnothing \neq D(w, s) \cap \mathbb{A}^1(\mathbb{K}).$$

In particular, any $D(v, r')$ such that $r \leq r' \leq \max\{\, r, |v - w|_{\mathbb{K}(\sqrt{-1})}\,\}$ verifies $D(v, r') \cap \mathbb{A}^1(\mathbb{K}) \neq \varnothing$ and any $D(w, s')$ such that $s \leq s' \leq \max\{\, s, |v - w|_{\mathbb{K}(\sqrt{-1})}\,\}$ verifies $D(s, s') \cap \mathbb{A}^1(\mathbb{K}) \neq \varnothing$. Hence the unique path $\ell(D(v, r), D(w, s))$ in $\mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$ is contained in T_1. Moreover, by Example 4.2.29, any element $(D(v_i, r_i))_i \in T_1$ verifies

$$D(v_i, r_i) \cap \mathbb{A}^1(\mathbb{K}) \neq \varnothing \quad \text{for every } i \in \mathbb{N}.$$

So, by Proposition 4.2.22 and Remark 4.2.23, the real point $(D(v_i, r_i))_i$ is the limit of the disks $D(v_i, r_i)$, which are real. In particular, every element of T_1 is contained in the closure of $\mathcal{H}(\mathbb{A}^1(\mathbb{F}))$. Thus, T_1 is a connected subset of a uniquely path connected set. Hence, T_1 is a uniquely path connected subspace of $\mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$. □

Corollary 4.2.31. *The image of $\psi\colon \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \to \mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ is uniquely path connected.*

Proof. By the Chinese remainder Theorem, the coordinate ring of $\mathbb{P}^1(\mathbb{K}) = \mathbb{A}^1(\mathbb{K}) \cup \{\,\infty\,\}$ is the product $\mathbb{K}[\mathbb{A}^1] \times \mathbb{K}$. By [DST19, Subsection 13.4.1]

$$\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} = \left(\mathbb{K}\left[\mathbb{A}^1\right] \times \mathbb{K}\right)^{\mathrm{RSp}}_{\mathrm{Arch}} = \mathbb{A}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \cup \{\,\infty\,\},$$

so we extend continuously $\psi_A\colon \mathbb{A}^1(\mathbb{K})^{\mathrm{an}}_{\mathrm{R}} \to \mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$ from Theorem 4.2.30 to

$$\psi\colon \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \to \mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$$

by setting $\psi(\{\,\infty\,\}) = \{\,\infty\,\}$. Since $\{\,\infty\,\}$ is in the closure of the real elements of $\mathbb{A}^1(\mathbb{K})^{\mathrm{an}}$, the image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ in $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ is a connected subset of a uniquely path connected space. Hence, it is uniquely path connected. □

Corollary 4.2.32. *Let $\mathbb{K}$ be a non-Archimedean real closed field with a big element. The image of $\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{K})$ inside $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{K})$ is a closed, $\mathrm{PSL}_2(\mathbb{K})$-invariant $\mathbb{R}$-subtree of the space $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$.*

Proof. As in Corollary 4.2.31, consider $\psi \colon \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \to \mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$. The map ψ sends element of $\mathbb{P}^1(\mathbb{K}) \subset \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}}$ to elements of $\mathbb{P}^1(\mathbb{K}) \subset \mathbb{P}^1(\mathbb{K})^{\mathrm{an}}$ [JSY22, Example 3.12]. Hence, it induces a continuous map

$$\Psi \colon \mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{K}) \to \mathbb{P}^1(\mathbb{K})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{K}).$$

Since $\{ \infty \}$ is the limit of real elements and the image of ψ is uniquely path connected, the image of Ψ is uniquely path connected. In particular,

$$T := \Psi \left(\mathbb{P}^1(\mathbb{K})^{\mathrm{RSp}}_{\mathrm{Arch}} \backslash \mathbb{P}^1(\mathbb{K}) \right)$$

is a $\mathbb{R}$-subtree of $\mathbb{P}^1(\mathbb{K})^{\mathrm{an}} \backslash \mathbb{P}^1(\mathbb{K})$, which is closed by Corollary 4.2.16. We now show that T is $\mathrm{PSL}_2(\mathbb{K})$-invariant. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{K})$$

and $f \in \mathbb{K}[x]$. We homogenize $f(x)^2 = \sum_{i=0}^{2n} \alpha_i x^i$ by setting

$$g(x, y) = \sum_{i=0}^{2n} \alpha_i x^i y^{2n-i} \in \mathbb{K}[x, y].$$

The induced action of A on the homogenous polynomial is $Ag(x, y) = g(ax+b, cy+d)$ such that

$$Af(x)^2 = Ag(x, x)(cx + d)^{-2n}.$$

In particular, $Af(x)^2 = g(ax + b, cx + d)(cx + d)^{-2n} = h(x)^2$ for some $h \in \mathbb{K}[x]$. Hence, for $\eta \in T$ and $f_1, \ldots, f_k \in \mathbb{K}[x]$

$$\begin{aligned} A\eta \left(f_1^2 + \cdots + f_k^2 \right) &= \eta \left(Af_1^2 + \cdots + Af_k^2 \right) \\ &= \eta \left(h_1^2 + \cdots + h_k^2 \right) \\ &= \max_{1 \leq i \leq k} \eta \left(h_i^2 \right) = \max_{1 \leq i \leq k} A\eta \left(f_i^2 \right). \end{aligned}$$

Hence, $A\eta \in T$ so that T is $\mathrm{PSL}_2(\mathbb{K})$-invariant. □

Chapter 5

Real spectrum and oriented Gromov equivariant compactifications

Section 5.1 recalls the definitions of cyclic orders and oriented $\mathbb{R}$-trees. We remind the construction of the oriented Gromov equivariant compactification $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ of the character variety $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ using the oriented Gromov equivariant topology, as in [Wol11]. We prove that this topology is first countable, and describe the Γ-actions by orientation preserving isometry on oriented $\mathbb{R}$-trees as Γ-actions by isometries on asymptotic cones of the hyperbolic plane endowed with a limit orientation.

Section 5.2 uses a description of elements in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ as representations of Γ in $\mathrm{PSL}_2(\mathbb{F})$ for some well-chosen real closed field $\mathbb{F}$, and a description of the standard orientation on $\mathbb{H}^2$ via a semialgebraic equation to associate to every element of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ a Γ-action by isometries preserving the orientation on an oriented $\mathbb{R}$-tree. We finally use an accessibility result from [BIPP23] involving Robinson fields to characterize the constructed oriented $\mathbb{R}$-trees using asymptotic cones of the hyperbolic plane endowed with a limit orientation.

Finally, Section 5.3 constructs a continuous surjection from $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ to $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ using the above descriptions of Γ-actions by orientation preserving isometries on $\mathbb{R}$-trees as limits of Γ-actions on the oriented hyperbolic plane, and a density argument.

5.1 Oriented Gromov equivariant compactification

This section reviews the construction of the oriented Gromov equivariant compactification $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ of the character variety $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ as introduced by Wolff in [Wol11]. We provide the necessary notation to define oriented $\mathbb{R}$-trees, and recall that points of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ are Γ-actions by orientation preserving isometries on oriented $\mathbb{R}$-trees. We recall the construction of the oriented Gromov equivariant topology and its invariance under small perturbations. Finally, we use the theory of asymptotic cones to describe the Γ-actions on oriented $\mathbb{R}$-trees that appear on $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ as limits of Γ-actions on the oriented hyperbolic plane.

5.1.1 Cyclic orders and oriented $\mathbb{R}$-trees

Following [Wol11], we recall the definitions of cyclic orders, which formalizes orientations on general sets, and of oriented $\mathbb{R}$-trees. For $\mathbb{R}$-trees with extendible segments, this is equivalent to defining a coherent cyclic order on the visual boundary of the $\mathbb{R}$-tree. Finally, we examine the space of minimal actions by orientation preserving isometries on $\mathbb{R}$-trees.

Definition 5.1.1. Let Ω be a set. A *cyclic order* on Ω is a function $o\colon \Omega^3 \to \{-1, 0, 1\}$ such that

1. for every $z_1, z_2, z_3 \in \Omega$: $o(z_1, z_2, z_3) = 0$ if and only if $\mathrm{Card}\{z_1, z_2, z_3\} \leq 2$,
2. for every $z_1, z_2, z_3 \in \Omega$,: $o(z_1, z_2, z_3) = o(z_2, z_3, z_1) = -o(z_1, z_3, z_2)$,
3. for every $z_1, z_2, z_3, z_4 \in \Omega$, if $o(z_1, z_2, z_3) = 1 = o(z_1, z_3, z_4)$, then
$$o(z_1, z_2, z_4) = 1.$$

A cyclic order is equivalent to defining an alternating 2-cocycle from Ω^3 to $\{-1, 0, 1\}$ which is zero if and only if the triple of points of Ω is not composed of 3 distinct points. Cyclic orders allow for the definition of orientations on $\mathbb{R}$-trees, thereby introducing the concept of oriented $\mathbb{R}$-trees.

Definition 5.1.2. Let $\kappa > 0$, X be a $\mathrm{CAT}(-\kappa)$ space and $P \in X$. A *germ of oriented segments* at P in X is an equivalence class of nondegenerate oriented segments based at P for the following equivalence relation: two oriented segments P-x, P-y are equivalent if and only if

$$\exists \varepsilon > 0 \text{ such that } P\text{-}x|_{[0,\varepsilon)} = P\text{-}y|_{[0,\varepsilon)}.$$

Denote by $\mathcal{G}_X(P)$ the set of germs of oriented segments at P in X and by $[P\text{-}x]$ an equivalence class of oriented segment.

Remark 5.1.3. In [Wol11], the space X is more generally a Gromov hyperbolic space. Since $\mathbb{H}^2$ and $\mathbb{R}$-trees are CAT$(-\kappa)$ spaces, the restriction to CAT$(-\kappa)$ enables a simpler definition of the visual boundary of X later on.

Definition 5.1.4. An *orientation of an* $\mathbb{R}$*-tree* T is the data, for every $P \in T$, of a cyclic order $or(P)$ defined on $\mathcal{G}_T(P)$. An $\mathbb{R}$-tree equipped with an orientation is called an *oriented* $\mathbb{R}$*-tree*. Denote by $\mathrm{Or}(T)$ the set of orientations on T.

Definition 5.1.5. Let (T, or) and (T', or') be two oriented $\mathbb{R}$-trees and h an isometry between T and T'. Then, h defines at each point $P \in T$ a bijection $\mathcal{G}h_P \colon \mathcal{G}_T(P) \to \mathcal{G}_{T'}(h(P))$. We say that h *preserves the orientation* of T if for every $P \in T$ and every triple of germs of oriented segments $([x], [y], [z]) \in \mathcal{G}_T(P)$

$$or'(h(P))(\mathcal{G}h_P([x]), \mathcal{G}h_P([y]), \mathcal{G}h_P([z])) = or(P)([x], [y], [z]).$$

That is, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_T(P)^3 & \xrightarrow{\quad or(P) \quad} & \{-1, 0, 1\}. \\ \big\downarrow {\scriptstyle (\mathcal{G}h_P)^3} & \nearrow {\scriptstyle or'(h(P))} & \\ \mathcal{G}_{T'}(h(P))^3 & & \end{array}$$

The set of isometries of T which preserve the orientation *or* forms a subgroup of $\mathrm{Isom}(T)$, denoted $\mathrm{Isom}_{\mathrm{or}}(T)$. Finally, an action of a group on an oriented $\mathbb{R}$-tree T *preserves the orientation* if it takes its values in the orientation preserving group of isometries of T.

As in [Wol11, Page 1269], if $P \in T$, denote by $\mathrm{Trip}(P)$ the set of pairwise distinct triples of germs of oriented segments starting at P, and set $\mathrm{Trip}(T) := \cup_{P \in T} \mathrm{Trip}(P)$. Let $[P\text{-}x], [P\text{-}y], [P\text{-}z]$ be three pairwise distinct germs of oriented segments starting at P. The corresponding element in $\mathrm{Trip}(P)$ is denoted by $\mathrm{Trip}(P, x, y, x)$ and called *germs of tripods* of T. With this notation, an orientation of T is a function $or \colon \mathrm{Trip}(T) \to \{\, -1, 1 \,\}$ verifying:

$$or(\mathrm{Trip}(P, x, y, z)) = or(\mathrm{Trip}(P, z, x, y)) = -or(\mathrm{Trip}(P, x, z, y)),$$
$$\text{if } or(\mathrm{Trip}(P, x, y, z)) = 1 = or(\mathrm{Trip}(P, x, z, w)) \text{ then } or(\mathrm{Trip}(P, x, y, w)) = 1.$$

Then $\mathrm{Or}(T)$ is a closed subspace of $\{\, -1, 1 \,\}^{\mathrm{Trip}(T)}$ when endowed with the product topology. In the next subsection, we use a correspondence between an $\mathbb{R}$-tree T endowed with an orientation, and T endowed with a specific cyclic order on its visual boundary $\partial_\infty T$. This correspondence holds for $\mathbb{R}$-trees with extendible segments.

Definition 5.1.6. An $\mathbb{R}$-tree has *extendible segments* if every oriented segment is the initial segment of some ray.

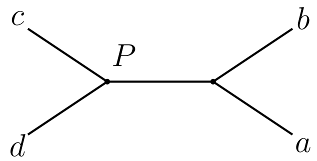
If r is a ray in T, its initial segment defines a germ of oriented segment.

Definition 5.1.7. A *germ of rays* in T is an equivalence class of rays, for the relation of defining the same germ of oriented segment.

To avoid any ambiguity regarding equivalence classes, we use the fact that X is a CAT($-\kappa$) space to identify the visual boundary $\partial_\infty X$ with the set of geodesic rays starting at P. This correspondence is a bijection (see [BH13, Proposition II.8.2]).

Definition 5.1.8. Let T be an $\mathbb{R}$-tree. We say that a cyclic order $\mathbf{o}$ on $\partial_\infty T$ is *coherent* if for every $P \in T$ and every pairwise distinct triple $([a],[b],[c])$ of germs of rays starting at P, the element $\mathbf{o}(a,b,c)$ does not depend on the chosen representatives a, b, c of $[a],[b],[c]$.

Example 5.1.9 ([Wol11, page 1269])**.** For instance, in the following configuration



a total cyclic order $\mathbf{o}$ on the boundary $\{a, b, c, d\}$ is coherent if and only if:

$$\mathbf{o}(a,b,c) = \mathbf{o}(a,b,d) \quad \text{and} \quad \mathbf{o}(a,c,d) = \mathbf{o}(b,c,d).$$

The set of coherent cyclic orders on $\partial_\infty T$ is a subspace of $\{-1,0,1\}^{(\partial_\infty T)^3}$, which we endow with the product topology. With this topology, one can show the correspondence between cyclic orders on $\partial_\infty T$ and orientation on T.

Remark 5.1.10 ([Wol11, Page 1270])**.** For every nondegenerate triple $(a,b,c) \in (\partial_\infty T)^3$, the intersection of the rays is a point denoted

$$P_{abc} := a\text{-}b \cap b\text{-}c \cap a\text{-}c \in T.$$

Proposition 5.1.11 ([Wol11, Proposition 3.8])**.** *Let T be an $\mathbb{R}$-tree with extendible segments. The map* Push: $\mathrm{Or}(T) \to \{-1,0,1\}^{(\partial_\infty T)^3}$ *that sends $or \in \mathrm{Or}(T)$ to*

$$\mathrm{Push}(or)\colon (\partial_\infty T)^3 \longrightarrow \{-1,0,1\};$$
$$(a,b,c) \longmapsto \begin{cases} 0 & \textit{if } \mathrm{Card}(a,b,c) \leq 2, \\ or([P_{abc}\text{-}a],\ [P_{abc}\text{-}b],\ [P_{abc}\text{-}c]) & \textit{otherwise} \end{cases}$$

is a homeomorphism onto its image, where $[P_{abc}\text{-}a]$ denotes the germ of oriented segments defined by the ray between P_{abc} and a.

Let Γ be a finitely generated group with finite generating set F. The action of Γ by isometries on an $\mathbb{R}$-tree T is *minimal* if T has no Γ-invariant subtree distinct from $\varnothing$ and T. Using [Wol11, Proposition 3.9], the following quotient up to equivariant isometry preserving the orientation is a set [Wol11, Page 1273]:

$$\mathcal{T}' := \left\{ (\phi, T, \mathbf{o}) \;\middle|\; \begin{array}{l} T \text{ an } \mathbb{R}\text{-tree not reduced to a point,} \\ \phi\colon \Gamma \to \mathrm{Isom}_{\mathrm{or}}(T) \text{ a minimal action, and} \\ \mathbf{o} \text{ a coherent cyclic order on } \partial_\infty T \end{array} \right\} / \sim .$$

We also define its subset:

$$\mathcal{T}^o := \left\{ [\phi, T, \mathbf{o}] \in \mathcal{T}' \;\middle|\; \begin{array}{l} \min\limits_{x\in T} \max\limits_{\gamma\in F} \mathrm{dist}_T\left(x, \phi(\gamma)x\right) = 1 \text{ and} \\ \text{if } \exists a \in \partial_\infty T \text{ with } \phi(\Gamma)a = a, \text{ then } T \cong \mathbb{R} \end{array} \right\}.$$

Remark 5.1.12. A Γ-invariant minimal $\mathbb{R}$-tree not reduced to a point is the union of the translation axes of the hyperbolic elements in the image of Γ [MS84, Pau89]. In particular, such an $\mathbb{R}$-tree has extendible segments [Pau89, Lemma 4.3]. Thus, an orientation on a Γ-invariant minimal tree is equivalent to endow the $\mathbb{R}$-tree with a coherent cyclic order on its visual boundary by Proposition 5.1.11.

Wolff refines the Gromov–Hausdorff equivariant topology by incorporating the orientation of the hyperbolic plane. The resulting oriented Gromov equivariant topology on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$ uses the identification of $\mathrm{PSL}_2(\mathbb{R})$ with the group of orientation preserving isometries of $\mathbb{H}^2$. A key feature of this topology is that cyclic orders of triples of points are stable under small perturbations, both in $\mathbb{H}^2$ and in oriented $\mathbb{R}$-trees [Wol11, Subsection 3.2.2].

5.1.2 Construction of the oriented Gromov equivariant compactification

Let X be either an oriented $\mathbb{R}$-tree with extendible segments (so with a coherent cyclic order on its boundary) or $\mathbb{H}^2$, the ball (Poincaré) model of the hyperbolic plane, with its standard orientation (counterclockwise orientation on $\partial_\infty\mathbb{H}^2 \cong \mathbb{S}^1$) and a metric $\mathrm{dist}_{\mathbb{H}^2}$ proportional to its standard metric. Denote by $\delta(X)$ its best hyperbolicity constant and by $\mathbf{o}$ the cyclic order on $\partial_\infty X$. Following [Wol11], we recapitulate the construction of the oriented Gromov equivariant topology. We show that this topology is first countable on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$ and that the standard orientation on $\mathbb{H}^2$ is described by a semialgebraic equation.

Lemma 5.1.13. *If* sgn *denotes the sign function on* $\mathbb{R}$*, then*

$$\begin{array}{rccc} o_{\mathbb{R}}\colon & (\mathbb{S}^1)^3 & \longrightarrow & \{-1, 0, 1\}; \\ & (z_1, z_2, z_3) & \longmapsto & \mathrm{sgn}\left(\det(z_2 - z_1, z_3 - z_2)\right) \end{array}$$

defines a cyclic order on $\mathbb{S}^1$*. In coordinates, if* $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{S}^1$*, then*

$$o_{\mathbb{R}}((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \operatorname{sgn}((y_3 - y_2)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_2)).$$

Moreover, for every $\mathbb{F}$ *real closed,* $o_{\mathbb{R}}$ *extends to a cyclic order on* $\mathbb{S}^1(\mathbb{F})$ *which is* $\mathrm{PSL}_2(\mathbb{F})$*-invariant.*

Remark 5.1.14. This formula describes the counterclockwise orientation on $\mathbb{S}^1$. It is independent of the base point chosen for the identification $\partial_\infty \mathbb{H} \cong \mathbb{S}^1$, as shown by the argument in the proof of [Wol11, Lemma 3.15].

Proof. It is a direct computation to verify that $o_{\mathbb{R}}$ satisfies the second item of Definition 5.1.1. That is, for every $(z_1, z_2, z_3) \in (\mathbb{S}^1)^3$ it holds

$$o_{\mathbb{R}}(z_1, z_2, z_3) = o_{\mathbb{R}}(z_2, z_3, z_1) = -o_{\mathbb{R}}(z_1, z_3, z_2).$$

If $\mathrm{Card}\{ z_1, z_2, z_3 \} \leq 2$ then $o_{\mathbb{R}}(z_1, z_2, z_3) = 0$ is a direct computation. Suppose $o_{\mathbb{R}}(z_1, z_2, z_3) = 0$ for some $(z_1, z_2, z_3) \in (\mathbb{S}^1)^3$ such that

$$\det(z_2 - z_1, z_3 - z_2) = 0.$$

If one column is 0, without loss of generality $z_2 - z_1 = 0$, then $\mathrm{Card}\{ z_1, z_2, z_3 \} \leq 2$. Otherwise, there exists $\lambda \in \mathbb{R}^*$ such that $z_2 - z_1 = \lambda(z_3 - z_2)$. That is

$$z_2 = \frac{1}{1+\lambda} z_1 + \frac{\lambda}{1+\lambda} z_3.$$

Thus z_2 is on the line that connects z_1 to z_3 in $\mathbb{R}^2$. But this line intersects the circle in at most two points, so $\mathrm{Card}\{ z_1, z_2, z_3 \} \leq 2$. Hence $o_{\mathbb{R}}(z_1, z_2, z_3) = 0$ if and only if $\mathrm{Card}\{ z_1, z_2, z_3 \} \leq 2$, and $o_{\mathbb{R}}$ satisfies the first item of Definition 5.1.1.

We verify that $o_{\mathbb{R}}$ satisfies the last item of Definition 5.1.1. Consider four elements $z_1, z_2, z_3, z_4 \in \mathbb{S}^1$, such that $o_{\mathbb{R}}(z_1, z_2, z_3) = 1 = o_{\mathbb{R}}(z_1, z_3, z_4)$, and we verify that $o_{\mathbb{R}}(z_1, z_2, z_4) = 1$. Define the continuous functions

$$f \colon \mathbb{S}^1 \backslash \{ z_1, z_3 \} \to \{ -1, 1 \} \quad r \mapsto o_{\mathbb{R}}(z_1, r, z_3),$$
$$g \colon \mathbb{S}^1 \backslash \{ z_1, z_2 \} \to \{ -1, 1 \} \quad r \mapsto o_{\mathbb{R}}(z_1, z_2, r).$$

Since $o_{\mathbb{R}}$ is continuous as a composition of continuous functions, the function f is continuous. Since $f(z_2) = 1$ and $f(z_4) = -1$, the two elements z_2 and z_4 are in different connected components of $\mathbb{S}^1 \backslash \{ z_1, z_3 \}$. Thus, z_4 and z_3 are in a same connected component of $\mathbb{S}^1 \backslash \{ z_1, z_2 \}$. Hence, by continuity of g

$$1 = g(z_4) = g(z_3).$$

That is $o_{\mathbb{R}}(z_1, z_2, z_4) = 1$ as desired so that $o_{\mathbb{R}}$ is a cyclic order on $\mathbb{S}^1$. The second part of the statement is a direct consequence of the Transfer principle (Theorem 2.2.2) as $o_{\mathbb{R}}$ is described by a semialgebraic equation and the determinant is invariant by the $\mathrm{PSL}_2(\mathbb{R})$-action. □

Lemma 5.1.15 ([Wol11, Lemma 3.10])**.** *Let X be an oriented $\mathbb{R}$-tree or $\mathbb{H}^2$. If $x_1, x_2, x_3 \in X$, then there exists a unique $P \in X$ which minimizes the function $x \mapsto \mathrm{dist}_X(x, x_1) + \mathrm{dist}_X(x, x_2) + \mathrm{dist}_X(x, x_3)$. Moreover, the map $X^3 \to X$ that sends $(x_1, x_2, x_3) \mapsto P$ is continuous. We call P the* center of the triple.

For $r \geq 0$, denote by $V(r) \subset X^3$ the set of $(x_1, x_2, x_3) \in X^3$ such that for every permutation (i, j, k) of $(1, 2, 3)$, the Gromov product verifies

$$2(x_i, x_k)_{x_j} := \mathrm{dist}_X(x_i, x_j) + \mathrm{dist}_X(x_j, x_k) - \mathrm{dist}_X(x_i, x_k) > 2r.$$

Lemma 5.1.16 ([Wol11, Lemma 3.12])**.** *For every $(x_1, x_2, x_3) \in V(6\delta(X))$, the center of the tripod $P \notin \{ x_1, x_2, x_3 \}$.*

This lemma allows to control the shape of tripods and specifically to avoid their degeneration under small perturbations. We now define a rigid notion of orientation, and of subsets of X that come in the same order.

Notation 5.1.17 ([Wol11, Subsection 3.2.2])**.** Given (x_1, x_2, x_3) in $V(6\delta(X))$, along with P the center of the tripod, denote by

$$\mathfrak{or}(x_1, x_2, x_3) := \mathbf{o}(a_1, a_2, a_3),$$

where a_i is a ray based at P and passing through x_i for $i \in \{ 1, 2, 3 \}$. This is a well defined quantity which does not depend on the chosen ray.

Remark 5.1.18 ([Wol11, Page 1274])**.** In $\mathbb{H}^2$, there exists a single ray a_i, up to parametrization, based at P and passing through x_i. From above, the three rays a_1, a_2, a_3 define three points on $\partial_\infty \mathbb{H}^2$ and the quantity $\mathfrak{or}(x_1, x_2, x_3)$ is well defined for $\mathbf{o}$ the cyclic order on $\partial_\infty \mathbb{H}^2$ as defined in Lemma 5.1.13. For every $P \in \mathbb{H}^2$, by the uniqueness of a_i passing through P and x_i, the set $\mathcal{G}_{\mathbb{H}^2}(P)$ is naturally in bijective correspondence with $\partial_\infty \mathbb{H}^2$. In particular, a cyclic order on $\partial_\infty \mathbb{H}^2$ defines a cyclic order on $\mathcal{G}_{\mathbb{H}^2}(P)$.

Notation 5.1.19. For the rest of this text, denote an element of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$, be it a class of actions on $\mathbb{H}^2$ or on oriented $\mathbb{R}$-trees, by $[\phi, X, \mathbf{o}]$ or $[\phi, X, or]$. Denote also $\mathfrak{or}\colon V(6\delta(X)) \to \{ -1, 1 \}$ the function defined in Notation 5.1.17 and *or* the induced orientation on X (a cyclic order on $\mathcal{G}_X(P)$ for every $P \in X$). That is, if X is an oriented $\mathbb{R}$-tree with extendible segments

$$or = \mathrm{Push}^{-1}(\mathbf{o}),$$

see Proposition 5.1.11, and if $X = \mathbb{H}^2$, then *or* is the standard orientation on $\mathbb{H}^2$, see Remark 5.1.18.

Let $[\phi, X, \mathbf{o}], [\phi', X', \mathbf{o}'] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$. Let $\varepsilon > 0$, Q be a finite subset of Γ, and $K = (x_1, \ldots, x_p) \subset X$, $K' = (x'_1, \ldots, x'_p) \subset X'$ be finite sequences. As in [Pau89], the sequence K' is a *Q-equivariant ε-approximation* of K if for every $g, h \in Q$, and every $i, j \in \{1, \ldots, p\}$ it holds

$$|\mathrm{dist}_X(\phi(g)x_i, \phi(h)x_j) - \mathrm{dist}_{X'}\left(\phi'(g)x'_i, \phi'(h)x'_j\right)| \leq \varepsilon.$$

Remark 5.1.20 ([Wol11, Remark 3.18])**.** Suppose that X, X' are either the hyperbolic plane or an $\mathbb{R}$-tree, $\alpha, \varepsilon_1, \varepsilon_2 > 0$, and $x_1, x_2, x_3 \in X$, $x'_1, x'_2, x'_3 \in X'$ verify

$$\begin{aligned}&(x_1, x_2, x_3) \in V(6\delta(X) + \alpha), \quad |\delta(X) - \delta(X')| \leq \varepsilon_1 \quad \text{and}\\ &|\mathrm{dist}_X(x_i, x_j) - \mathrm{dist}_{X'}\left(x'_i, x'_j\right)| \leq \varepsilon_2 \quad \forall i, j \in \{1, 2, 3\}.\end{aligned}$$

Then $(x'_1, x'_2, x'_3) \in V(6\delta(X') + \alpha - 6\varepsilon_1 - 3\varepsilon_2)$. Therefore, the sets $V(r)$ provide open conditions, robust under ε-approximations guaranteeing that we can consider the orientations defined by triples of points.

Definition 5.1.21. Let $[\phi, X, \mathbf{o}], [\phi', X', \mathbf{o}']$ be two elements in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$. Let $\varepsilon > 0$, Q be a finite subset of Γ with $e \in Q$, and $K = (x_1, \ldots, x_p) \subset X$, $K' = (x'_1, \ldots, x'_p) \subset X'$ be finite sequences. We say that K' is an *oriented Q-equivariant ε-approximation* of K if K' is a Q-equivariant ε-approximation of K and

$$\mathfrak{or}(x_i, x_j, x_k) = \mathfrak{or}'\left(x'_i, x'_j, x'_k\right)$$

for every $(x_i, x_j, x_k) \in V(6\delta(X) + 9\varepsilon)$.

If $[\phi, X, \mathbf{o}] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$, $\varepsilon > 0$, $K = (x_1, \ldots, x_p) \subset X$ is a finite sequence, and $Q \subset \Gamma$ a finite subset containing e, then denote by

$$U_{K,\varepsilon,Q}(\phi, X, \mathbf{o})$$

the subset of $[\phi', X', \mathbf{o}'] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$ such that X' contains an oriented Q-equivariant ε-approximation of K.

Proposition 5.1.22 ([Wol11, Proposition 3.20])**.** *The subsets defined above*

$$U_{K,\varepsilon,Q}(\phi, X, \mathbf{o}) \subset \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$$

form a basis of open sets for the oriented Gromov equivariant topology*.*

On $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, the oriented Gromov equivariant topology is equivalent to the compact-open topology [Wol11, Proposition 3.2.1]. The *oriented Gromov* *equivariant compactification* is the closure of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ in the topological space $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}^o$, which is denoted by

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}.$$

It is a natural compactification, in the sense that, $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is open an dense in the Hausdorff compact space $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ and the action of $\mathrm{Out}(\Gamma)$ on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ extends continuously to an action of $\mathrm{Out}(\Gamma)$ on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ [Wol11, Theorem 3.23].

Remark 5.1.23. Throughout this section, the hyperbolic plane is equipped with a fixed orientation, its standard one. The novelty compared to the length spectrum compactification is that the representations take values in the orientation preserving isometry group of X. Furthermore, the equivalence relation defining $\mathcal{T}'$ in Subsection 5.1.1 specifically involves quotients by orientation preserving isometries, rather than arbitrary isometries. This distinction is crucial for preserving the orientation structure in our construction.

The following proposition adapts [Pau87, Proposition 1.6], incorporating a remark from the proof of [Wol07, Théorème 5.4.6]. An ε-approximation Q-equivariant between two metric spaces X, X' is a logical relation $\mathcal{R}$ within $X \times X'$ which is surjective [Pau87, Definition 1]. Then, an ε-approximation Q-equivariant between X and X' is *closed* if it is closed as a subset of $X \times X'$.

Proposition 5.1.24. *If Γ is countable, then $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ is first countable. That is, each element of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ has a countable basis of open neighborhoods.*

Proof. Let $[\phi, X, \mathbf{o}] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ and (K_n) a sequence of finite subsequences in X, increasing for the inclusion, such that their union is dense in X and Γ-invariant. We show that the open sets

$$\left\{ (U_{K_n,r,Q}(\phi, X, \mathbf{o})) \;\middle|\; Q \text{ a finite subset of } \Gamma \text{ containing } e,\ n \in \mathbb{N},\ r \in \mathbb{Q}_{>0} \right\}$$

form a basis of open neighborhoods of $[\phi, X, \mathbf{o}]$ in the oriented Gromov equivariant topology. Let K be a finite subsequence of X, Q a finite subset of Γ containing e, and $\varepsilon > 0$. Cover K with a finite number of open balls of radius $\varepsilon/12$. By density of the union of the K_n inside X, the centers of each of these balls is at distance smaller than $\varepsilon/12$ from a point in one of the K_n. Denote by

$$K' \subset K_n$$

the finite subsequence of X formed by these points in K_n. Then K is contained in the $\varepsilon/6$-neighborhood of K', and K' is contained in the $\varepsilon/6$-neighborhood of K. Furthermore, by Γ-invariance of the union of the K_n, the union $K' \cup QK'$ is contained in one of the K_j, for j sufficiently large. Using [Pau87, Remarque 1.4, Remarque 1.5], there exists a closed Q-equivariant ε-approximation

$$K'' \text{ of } K,$$

where K'' is contained in K_j. Let r be a rational number with $0 < r < \varepsilon$ and we show that

$$U_{K_j,r,Q}(\phi, X, \mathbf{o}) \subset U_{K,2\varepsilon,Q}(\phi, X, \mathbf{o}).$$

Let $[\phi^*, X^*, \mathbf{o}^*] \in U_{K_j,r,Q}(\phi, X, \mathbf{o})$. There exists an r-approximation Q-equivariant between $K^* \subset X^*$ and K_j. In particular, there is an r-approximation Q-equivariant between a finite subset $A \subset K^*$ and K''. By the last remark in the proof of [Pau87, Lemme 1.2], there exists an $(r + \varepsilon)$-approximation between A and K. Since K and K_j are subsets of the same oriented space, they come in the same order. Hence $[\phi^*, X^*, \mathbf{o}^*] \in U_{K,2\varepsilon,Q}(\phi, X, \mathbf{o})$ as wanted. □

We described the oriented Gromov equivariant topology of character varieties using orientations on $\mathbb{R}$-trees. In the next subsection, we characterize the orientation on $\mathbb{R}$-trees appearing in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ as ultralimits of orientations on the hyperbolic plane.

5.1.3 Description via asymptotic cones

This subsection describes elements of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ as ultralimits of sequences in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. Building on [Pau89], we show that Γ-actions by isometry preserving the orientation on oriented $\mathbb{R}$-trees can be characterized as ultralimits of Γ-actions on asymptotic cones equipped with an ultralimit orientation. For any $\phi \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, define the *displacement of the representation* as

$$\lg(\phi) := \inf_{x \in \mathbb{H}^2} \max_{\gamma \in F} \mathrm{dist}_{\mathbb{H}^2}\left(x, \phi(\gamma)x\right),$$

where $\mathbb{H}^2$ denotes the ball (Poincaré) model of the hyperbolic plane equipped with its standard metric, F is a finite generating set of Γ and $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ via Möbius transformations. We rely on a key property of reductive homomorphisms.

Proposition 5.1.25 ([Par12, Proposition 18])**.** *An element $\phi \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is reductive if and only if the infimum of the displacement function is achieved. That is, there exists $x_0 \in \mathbb{H}^2$ so that*

$$\max_{\gamma \in F} \mathrm{dist}_{\mathbb{H}^2}\left(x_0, \phi(\gamma)x_0\right) \leq \max_{\gamma \in F} \mathrm{dist}_{\mathbb{H}^2}\left(x, \phi(\gamma)x\right) \quad \forall x \in \mathbb{H}^2.$$

Notation 5.1.26. Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, (ϕ_k) a sequence in $\mathrm{Hom}_{\mathrm{red}}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ such that $\lg(\phi_k) \to \infty$, $*_k \in \mathbb{H}^2$ an element achieving the infimum of the displacement function of ϕ_k for every $k \in \mathbb{N}$, and $T := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lg(\phi_k)), (*_k)\right)$. The sequence $(\phi_k) \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ induces a Γ-action by isometries on T via

$$\phi^{\mathfrak{u}}(\gamma)([z_k]) = [\phi_k(\gamma)z_k] \quad \forall \gamma \in \Gamma, \text{ and } \forall [z_k] \in T.$$

We call $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$ the *ultralimit representation*, see Subsection 2.1 and [Pau09, Page 434].

Proposition 5.1.27 ([Pau09, Page 434]). *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $[\phi_k, \mathbb{H}^2]$ a sequence in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ such that $\lg(\phi_k) \to \infty$, and $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$. If $*_k \in \mathbb{H}^2$ is an element in $\mathbb{H}^2$ achieving the infimum of the displacement function of ϕ_k for every $k \in \mathbb{N}$ and $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lg(\phi_k)), (*_k)\right)$, then*

$$\lim_{\mathfrak{u}} \left(\phi_k, \mathbb{H}^2\right) = (\phi^{\mathfrak{u}}, T^{\mathfrak{u}})$$

in the Gromov–Hausdorff equivariant topology. That is, for every $\varepsilon > 0$, Q a finite subset of Γ with $e \in Q$, and $K = ([x_k^1], \ldots, [x_k^p]) \subset T^{\mathfrak{u}}$ a finite sequence, for $\mathfrak{u}$-almost every $k \in \mathbb{N}$, the sequence

$$K' = \left(x_k^1, \ldots, x_k^p\right) \subset \mathbb{H}^2$$

is a Q-equivariant ε-approximation of K.

Remark 5.1.28 ([Pau09, Page 437]). If a sequence of Γ-actions by isometries on $\mathbb{H}^2$ converges, in the Gromov–Hausdorff equivariant topology, to an $\mathbb{R}$-tree equipped with a Γ-action without a global fixed point, then it also converges in the Gromov–Hausdorff equivariant topology to every invariant subtree, and in particular to its unique minimal invariant subtree.

We now specialize this convergence to account for an orientation.

Remark 5.1.29. Let X be $\mathbb{H}^2$ or an $\mathbb{R}$-tree and $[\sigma]$ a germ of oriented segments at $P \in X$. Consider $\sigma \in [\sigma]$ with endpoints P and x. The oriented segment P-x defines the same germ of oriented segment $[\sigma] = [P\text{-}x]$. So, every germ of oriented segments is represented by an element $x \in X$. It does not depend on the choice of the representative σ or $x \in \mathrm{Im}(\sigma)$. Indeed, for two oriented segments $\sigma, \sigma' \in [\sigma]$, by definition of the equivalence class of a germ, there exists

$$x \in (\mathrm{Im}(\sigma) \cap \mathrm{Im}(\sigma')) \setminus \{ P \}.$$

Then $[P\text{-}x] = [\sigma] = [\sigma']$ in $\mathcal{G}_X(P)$.

In the following definition of $or^{\mathfrak{u}}([P_k])$, for convenience, we denote by $x \in \mathcal{G}_X(P)$ the germ of oriented segments $[P\text{-}x]$, and we use the notation $[P\text{-}x]_{\mathcal{G}}$ to avoid confusion when necessary.

Theorem 5.1.30. *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $(\lambda_k) \subset \mathbb{R}$ a sequence such that $\lambda_k \to \infty$, $(*_k) \subset \mathbb{H}^2$ a sequence, and $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}(\mathbb{H}^2, (\lambda_k), (*_k))$. Consider $[P_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$ and $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$ the set of germs of oriented segments at $[P_k]^{\mathfrak{u}}$. The map $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$:*

$$\begin{array}{rcl} (\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}}))^3 & \longrightarrow & \{-1,0,1\}; \\ ([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}) & \longmapsto & \begin{cases} 0 & \textit{if } \mathrm{Card}\{ [x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}} \} \leq 2, \\ \lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) & \textit{otherwise}, \end{cases} \end{array}$$

is a cyclic order on $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$, where $or(P_k)$ is the cyclic order on $\mathcal{G}_{\mathbb{H}^2}(P_k)$ given by the standard orientation on $\mathbb{H}^2$. Hence $or^{\mathfrak{u}}$ defines an orientation on $T^{\mathfrak{u}}$.

Proof. We first show that $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ is well defined and independent of the choice of representatives, which is the main part of the proof. Any segment in $T^{\mathfrak{u}}$ is the ultralimit of a sequence of segments in $\mathbb{H}^2$ by [DK18, Corollary 11.38]. Suppose the germ of oriented segments $[[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}}$ is represented by two sequences $([P_k\text{-}x_k]_{\mathcal{G}})$ and $([P'_k\text{-}x'_k]_{\mathcal{G}})$. As in Remark 5.1.29, we may assume

$$[x_k]^{\mathfrak{u}} = [x'_k]^{\mathfrak{u}} \in T^{\mathfrak{u}} \quad \text{and} \quad [P_k]^{\mathfrak{u}} = [P'_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}.$$

If $\mathrm{Card}\{ [[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}}, [[P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}, [[P_k]^{\mathfrak{u}}\text{-}[z_k]^{\mathfrak{u}}]_{\mathcal{G}} \} \leq 2$, then at least two of the germs of oriented segments are equal. Without loss of generality, suppose

$$[[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}.$$

As in Remark 5.1.29, there exists

$$[w_k]^{\mathfrak{u}} \in \mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}) \cap \mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}) \backslash \{ [P_k]^{\mathfrak{u}} \}.$$

Since $[x_k]^{\mathfrak{u}} = [x'_k]^{\mathfrak{u}}$ and $[P_k]^{\mathfrak{u}} = [P'_k]^{\mathfrak{u}}$

$$[[P'_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P_k]^{\mathfrak{u}}\text{-}[w_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P'_k]^{\mathfrak{u}}\text{-}[x'_k]^{\mathfrak{u}}]_{\mathcal{G}}.$$

Hence $\mathrm{Card}\{ [[P'_k]^{\mathfrak{u}}\text{-}[x'_k]^{\mathfrak{u}}]_{\mathcal{G}}, [[P'_k]\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}, [[P'_k]\text{-}[z_k]^{\mathfrak{u}}]_{\mathcal{G}} \} \leq 2$ so that

$$or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})\left([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}\right) = 0 = or^{\mathfrak{u}}([P'_k]^{\mathfrak{u}})\left([x'_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}\right).$$

Suppose all three germs are distinct. On the one hand, suppose that

$$\lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) = 1 \quad \text{and} \quad \lim_{\mathfrak{u}} or(P'_k)(x'_k, y_k, z_k) = 0.$$

Since $\{-1,0,1\}$ is discrete, the second limit implies $or(P'_k)(x'_k, y_k, z_k) = 0$ for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $or(P_k)$ is a cyclic order, without loss of generality

$$[P'_k\text{-}x'_k]_{\mathcal{G}} = [P'_k\text{-}y_k]_{\mathcal{G}}$$

for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $\mathbb{H}^2$ is uniquely geodesic, assume without loss of generality that $\mathrm{Im}(P'_k\text{-}x'_k) \subset \mathrm{Im}(P'_k\text{-}y_k)$ for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $T^{\mathfrak{u}}$ is uniquely geodesic

$$\mathrm{Im}([P'_k]^{\mathfrak{u}}\text{-}[x'_k]^{\mathfrak{u}}) \subset \mathrm{Im}([P'_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}})$$

so that $[[P'_k]^{\mathfrak{u}}\text{-}[x'_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P'_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}$. This is a contradiction with the three germs of oriented segments being distinct.

On the other hand, suppose:

$$\lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) = 1 \quad \text{and} \quad \lim_{\mathfrak{u}} or(P'_k)(x'_k, y_k, z_k) = -1.$$

By the independence of the choice of base point for the orientation on $\mathbb{H}^2$ (Remark 5.1.14), this is equivalent to

$$\lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) = 1 \quad \text{and} \quad \lim_{\mathfrak{u}} or(P_k)(x'_k, y_k, z_k) = -1.$$

Consider the rays a_k, a'_k, b_k, c_k passing through P_k and x_k, x'_k, y_k, z_k respectively. By Lemma 5.1.13, the above conditions translate to

$$o_{\mathbb{R}}(a_k, b_k, c_k) = 1 \quad \text{and} \quad o_{\mathbb{R}}(a'_k, b_k, c_k) = -1$$

for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Thus $b_k \in \mathrm{arc}(a_k, a'_k)$ and $c_k \in \mathrm{arc}(a'_k, a_k)$, where $\mathrm{arc}(a'_k, a_k)$ is the oriented segment of $\mathbb{S}^1$ going in the counterclockwise direction. Without loss of generality, the length of $\mathrm{arc}(a_k, a'_k)$ is smaller or equal to the length of $\mathrm{arc}(a'_k, a_k)$. Then, every neighborhood of a_k containing a'_k in the shadow topology on $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ also contains b_k [DK18, Subsection 11.11]. That is, there exists $b^*_k \in \mathrm{Im}(b_k)$ such that

$$\mathrm{dist}_{\mathbb{H}^2}(b^*_k, x_k) \leq \mathrm{dist}_{\mathbb{H}^2}(x_k, x'_k)$$

for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $[x_k]^{\mathfrak{u}} = [x'_k]^{\mathfrak{u}}$, it holds $[x_k]^{\mathfrak{u}} = [b^*_k]^{\mathfrak{u}}$. Thus $[b^*_k]^{\mathfrak{u}} \in (\mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}) \cap \mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}})) \backslash \{ [P_k]^{\mathfrak{u}} \}$ so that

$$[[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}.$$

This is a contradiction with the three germs of oriented segments being distinct so that

$$\lim_{\mathfrak{u}} or(P_k)(x_k, y_k, z_k) = \lim_{\mathfrak{u}} or(P'_k)(x'_k, y_k, z_k).$$

A similar argument proves that $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ does not depend on the choice of $[y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}$ such that $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ does not depend on the choice of representatives.

We show that $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ defines a cyclic order. If $\mathrm{Card}\{ [x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}} \} \leq 2$, then by definition $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}) = 0$. Suppose $[x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}$ are distinct germs of oriented segments in $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$ and verify

$$or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}) = 0.$$

Since $\{ -1, 0, 1 \}$ is discrete, $or(P_k)(x_k, y_k, z_k) = 0$ for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $or(P_k)$ is a cyclic order, without loss of generality

$$[P_k\text{-}x_k]_{\mathcal{G}} = [P_k\text{-}y_k]_{\mathcal{G}}$$

for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $\mathbb{H}^2$ is uniquely geodesic, assume without loss of generality that $\mathrm{Im}(P_k\text{-}x_k) \subset \mathrm{Im}(P_k\text{-}y_k)$ for $\mathfrak{u}$-almost every $k \in \mathbb{N}$. Since $T^{\mathfrak{u}}$ is uniquely geodesic

$$\mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}) \subset \mathrm{Im}([P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}})$$

so that $[[P_k]^{\mathfrak{u}}\text{-}[x_k]^{\mathfrak{u}}]_{\mathcal{G}} = [[P_k]^{\mathfrak{u}}\text{-}[y_k]^{\mathfrak{u}}]_{\mathcal{G}}$. This is a contradiction with the three germs of oriented segments being distinct. Hence

$$\mathrm{Card}\{ [x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}} \} \leq 2$$

so that $or^{\mathfrak{u}}([P_k])$ satisfies the first item of Definition 5.1.1. Similarly, if $[x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}$, and $[z_k]^{\mathfrak{u}}$ are germs of oriented segments in $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$, then for $\mathfrak{u}$-almost every k

$$or(P_k)(x_k, y_k, z_k) = or(P_k)(y_k, z_k, x_k) = -or(P_k)(x_k, z_k, y_k).$$

Thus, by definition of the ultralimit

$$\begin{aligned} or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})([x_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}) &= or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})([y_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}, [x_k]^{\mathfrak{u}}) \\ &= -or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})([x_k]^{\mathfrak{u}}, [z_k]^{\mathfrak{u}}, [y_k]^{\mathfrak{u}}). \end{aligned}$$

so that $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ verifies the second item of Definition 5.1.1. The third axiom follows from the same reasoning. Therefore, $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$ defines a cyclic order, and hence $or^{\mathfrak{u}}$ defines an orientation on $T^{\mathfrak{u}}$.

□

Theorem 5.1.31. *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $[\phi_k, \mathbb{H}^2, or]$ a sequence in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ such that $\lg(\phi_k) \to \infty$, $*_k$ an element in $\mathbb{H}^2$ achieving the infimum of the displacement function of ϕ_k for every $k \in \mathbb{N}$, and $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$. If $T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}$ is the $\phi^{\mathfrak{u}}$-invariant minimal subtree inside the asymptotic cone $\mathrm{Cone}^{\mathfrak{u}}(\mathbb{H}^2, (\lg(\phi_k)), (*_k))$, then*

$$\lim_{\mathfrak{u}}[\phi_k, \mathbb{H}^2, or] = [\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, or^{\mathfrak{u}}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}},$$

where $or^{\mathfrak{u}}$ is the restriction to $T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}$ of the ultralimit orientation defined in Theorem 5.1.30.

Proof. Let $(U_{K_n,\varepsilon_n,Q_n}(\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, or^{\mathfrak{u}}))_{n\in\mathbb{N}}$ be a countable basis of open neighborhoods in the oriented Gromov equivariant topology around $(\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, or^{\mathfrak{u}})$. Consider as in Notation 5.1.17 the functions

$$\mathfrak{or}^{\mathfrak{u}}\colon \begin{array}{ccc} V(0) & \longrightarrow & \{ -1, 1 \}; \\ (x,y,z) & \longmapsto & or^{\mathfrak{u}}(P)(x,y,z) \end{array} \quad \text{and} \quad \mathfrak{or}\colon \begin{array}{ccc} V(6\delta(\mathbb{H}^2)) & \longrightarrow & \{ -1, 1 \}; \\ (x,y,z) & \longmapsto & or(P)(x,y,z), \end{array}$$

where P is the center of the tripod $\mathrm{Trip}(x, y, z)$ as defined in Proposition 5.1.15. For $m \in \mathbb{N}$, suppose $K_m := ([x_k^{1,m}]_k, \dots, [x_k^{\ell_m,m}]_k) \subset T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}$. By Proposition 5.1.27, for every $g, h \in Q_m$ and every $i, j \in \{1, \dots, \ell_m\}$

$$\left| \frac{\mathrm{dist}_{\mathbb{H}^2}\left(\phi_k(g) x_k^{i,m}, \phi_k(h) x_k^{j,m}\right)}{\lg(\phi_k)} - \mathrm{dist}_{T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}}\left(\phi^{\mathfrak{u}}(g)\left[x_k^{i,m}\right]^{\mathfrak{u}}, \phi^{\mathfrak{u}}(h)\left[x_k^{j,m}\right]^{\mathfrak{u}}\right) \right| \leq \varepsilon_m,$$

for $\mathfrak{u}$-almost every k. Moreover, by definition of the ultralimit, for $\mathfrak{u}$-almost every k and every triple of elements $([x_k^{i,m}]^{\mathfrak{u}}, [x_k^{j,m}]^{\mathfrak{u}}, [x_k^{p,m}]^{\mathfrak{u}}) \in K_m$ which satisfies $([x_k^{i,m}]^{\mathfrak{u}}, [x_k^{j,m}]^{\mathfrak{u}}, [x_k^{p,m}]^{\mathfrak{u}}) \in V(9\varepsilon_m)$

$$\mathfrak{or}^{\mathfrak{u}}\left(\left[x_k^{i,m}\right]^{\mathfrak{u}}, \left[x_k^{j,m}\right]^{\mathfrak{u}}, [x_k^{p,m}]^{\mathfrak{u}}\right) = \mathfrak{or}\left(x_k^{i,m}, x_k^{j,m}, x_k^{p,m}\right).$$

Thus, $\mathfrak{u}$-almost every $[\phi_k, \mathbb{H}^2, \mathbf{o}]$ is in the open set $U_{K_n, \varepsilon_n, Q_n}(\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, \mathbf{o}^{\mathfrak{u}})$. Hence $[\phi_k, \mathbb{H}^2, \mathbf{o}]$ converges to $[\phi^{\mathfrak{u}}, T^{\mathfrak{u}}_{\phi^{\mathfrak{u}}}, \mathbf{o}^{\mathfrak{u}}]$ along the ultrafilter $\mathfrak{u}$. □

We recalled the construction of the oriented Gromov equivariant compactification of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ and characterized the boundary elements as Γ-actions by orientation preserving isometries on asymptotic cones equipped with a limit orientation. In the next section, we associate to every element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ a Γ-action on an oriented $\mathbb{R}$-tree by orientation preserving isometries. Our goal is to use the theory developed in this section to compare the two compactifications in the final section.

5.2 Oriented $\mathbb{R}$-trees associated to elements of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$

We associate to each element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ a Γ-action by isometries on an $\mathbb{R}$-tree, following [Bru88a]. In the first subsection, we endow this $\mathbb{R}$-tree with an orientation using the order structure of real closed fields. In the second subsection, we show that this orientation is well described as an ultralimit of orientations, similarly to Subsection 5.1.3.

5.2.1 Description via the non standard hyperbolic plane

This section associates to each element $(\rho, \mathbb{F}_\rho) \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ a Γ-action on an oriented $\mathbb{R}$-tree. Following the approach in [Bru88a], one constructs a Γ-action on an $\mathbb{R}$-tree T, and use [Bru88c] to describe the set of germs of oriented segments at any point $P \in T$. This allows us to define an orientation on T and so, a Γ-action by isometries preserving the orientation on T.

Definition 5.2.1. Given a representation $\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F})$, the real closed field $\mathbb{F}$ is *ϕ-minimal* if ϕ can not be $\mathrm{PSL}_2(\mathbb{F})$-conjugated into a representation $\phi'\colon \Gamma \to \mathrm{PSL}_2(\mathbb{L})$, where $\mathbb{L} \subset \mathbb{F}$ is a proper real closed subfield.

If $\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F})$ is reductive, denoted $(\phi, \mathbb{F})$, and if $\mathbb{F}$ is real closed, a minimal real closed field $\mathbb{F}_\phi \subset \mathbb{F}$ always exists and is unique [BIPP23, Corollary 7.9]. If $\mathbb{F}_1$ and $\mathbb{F}_2$ are two real closed fields, we say that two representations $(\phi_1, \mathbb{F}_1)$ and $(\phi_2, \mathbb{F}_2)$ are equivalent if there exists a real closed field morphism $\psi\colon \mathbb{F}_1 \to \mathbb{F}_2$ such that

$$\psi \circ \phi_1 = g\phi_2 g^{-1} \quad \text{for some } g \in \mathrm{PSL}_2(\mathbb{F}_2).$$

Theorem 5.2.2 ([BIPP23, Theorem 1.1 and Corollary 7.9])**.** *Elements in the boundary of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ are in bijective correspondence with equivalence classes of pairs $[\phi, \mathbb{F}]$, where*

$$\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F}),$$

is reductive and $\mathbb{F}$ is real closed, non-Archimedean and ϕ-minimal. Moreover $\mathbb{F}$ is of finite transcendence degree over $\overline{\mathbb{Q}}^r$.

An element in the equivalence class is a *representative* of $[\phi, \mathbb{F}]$. Because of the finite transcendence degree condition over $\overline{\mathbb{Q}}^r$, not all real closed fields in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ occur. Recall from Definition 3.4.1, that a real closed field $\mathbb{K}$ has a big element b if for every $h \in \mathbb{K}$

$$\exists k \in \mathbb{N} \text{ such that } h < b^k.$$

For every $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, the ϕ-minimal real closed field $\mathbb{F}$ is of finite transcendence degree over $\overline{\mathbb{Q}}^r$. In particular, $\mathbb{F}$ has a big element b [Bru88a, Section 5]. So, as in Section 4.2, define for every $h \in \mathbb{F}$ the two subsets of $\mathbb{Q}$

$$\left\{ \frac{m}{n} \,\middle|\, b^m \le h^n, n \in \mathbb{N}_{\ge 0}, m \in \mathbb{Z} \right\} \text{ and } \left\{ \frac{m'}{n'} \,\middle|\, b^{m'} \ge h^{n'}, n' \in \mathbb{N}_{\ge 0}, m' \in \mathbb{Z} \right\}$$

define a Dedekind cut of $\mathbb{Q}$. Hence the two subsets above define a real number denoted $\log_b(h)$. The function $-\log_b\colon \mathbb{F} \to \mathbb{R}$ is a non-trivial *order compatible valuation* on $\mathbb{F}$ [Bru88a, Section 5]. Denote by $\Lambda := \log_b(\mathbb{F})$ the *valuation group* of $\mathbb{F}$, which is an Abelian divisible subgroup of $\mathbb{R}$ [Bru88a, Section 8]. Denote by $\hat{\mathbb{F}}$ the *valuation completion* of $\mathbb{F}$, which is also real closed [Bru88c, Page 91].

Remark 5.2.3. The Robinson field $\mathbb{R}^{\mathfrak{u}}_\mu$, where $\mathfrak{u}$ is any non-principal ultrafilter and μ any infinite element or $\mathbb{R}^{\mathfrak{u}}$ (Example 2.2.3), is a real closed field with big element μ. Moreover, the valuation group of $\mathbb{R}^{\mathfrak{u}}_\mu$ associated to $\log_\mu$ is $\mathbb{R}$.

To associate to $[\phi, \mathbb{F}]$ an $\mathbb{R}$-tree, we first need to recall some definitions of Λ-metric spaces, see [Chi01]. A set X together with a function $\mathrm{dist}_X \colon X \times X \to \Lambda$ is a Λ-*metric space* if dist_X is positive definite, symmetric and satisfies the triangle inequality. A Λ-*segment* σ is the image of a Λ-isometric embedding $\eta \colon \{t \in \Lambda \colon r \leq t \leq s\} \to X$ for some $r \leq s \in \Lambda$. The set of *endpoints of* σ is $\{\eta(r), \eta(s)\}$. A *germ of oriented* Λ-*segments* at P in X is an equivalence class of nondegenerate oriented Λ-segments based at P for the following equivalence relation: two oriented Λ-segments σ, σ' are equivalent if and only if

$$\exists \varepsilon > 0 \text{ such that } \sigma|_{[0,\varepsilon)} = \sigma'|_{[0,\varepsilon)}.$$

Definition 5.2.4. A Λ-*tree* is a Λ-metric space X satisfying:

- For all $x, y \in X$ there is a Λ-segment σ with endpoints x, y.
- For all Λ-segments σ, σ' whose intersection $\sigma \cap \sigma' = \{x\}$ consists of one common endpoint x of both Λ-segments, the union $\sigma \cup \sigma'$ is a Λ-segment.
- For all Λ-segments σ, σ' with a common endpoint x, the intersection $\sigma \cap \sigma'$ is a Λ-segment with x as one of its endpoints.

By [Chi01, Chapter 2, Lemma 1.1], Λ-segments in a Λ-tree are unique and we write $(x\text{-}y)_\Lambda$ the unique Λ-segment with endpoints $x, y \in X$.

Following the construction in [Bru88c], given $[\phi, \mathbb{F}] \in \partial \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, define the non-Archimedean hyperbolic plane over the ϕ-minimal real closed field $\mathbb{F}_\phi$ as the set

$$\mathbb{H}^2(\mathbb{F}) := \{\, x + iy \in \mathbb{F}[i] \mid x^2 + y^2 < 1 \,\} \subset \mathbb{F}[i],$$

where i is such that $i^2 = -1$. As in the real case, $\mathbb{H}^2(\mathbb{F})$ admits a pseudo-distance [Bru88c, Page 92]

$$\begin{array}{rccc} \mathrm{dist}_{\mathbb{H}^2(\mathbb{F})} \colon & (\mathbb{H}^2(\mathbb{F}))^2 & \longrightarrow & \Lambda; \\ & (z, z') & \longmapsto & \log_\beta \left(\frac{\|1 - (\overline{z} z')\|^2}{(1 - \|z\|^2)(1 - \|z'\|^2)} \right), \end{array}$$

where $\overline{z}$ is the complex conjugation of $z \in \mathbb{F}[i]$, and for any $z = x + iy \in \mathbb{F}[i]$:

$$\|z\| := \sqrt{x^2 + y^2} \in \mathbb{F}.$$

Consider the equivalence relation on $\mathbb{H}^2(\mathbb{F})$ which identifies z with z' if and only if $\mathrm{dist}_{\mathbb{H}^2(\mathbb{F})}(z, z') = 0$ and denote the projection map

$$\pi \colon \mathbb{H}^2(\mathbb{F}) \to T\mathbb{F} := \mathbb{H}^2(\mathbb{F}) / \{\, \mathrm{dist}_{\mathbb{H}^2(\mathbb{F})} = 0 \,\}.$$

Proposition 5.2.5 ([Bru88c, Theorem 28]). *The Λ-metric space $T\mathbb{F}$ is a Λ-tree.*

Moreover, if we consider the action of $\mathrm{PSL}_2(\mathbb{F})$ by Möbius transformation on $\mathbb{H}^2(\mathbb{F})$, then it passes to an action by isometries on $T\mathbb{F}$.

Proposition 5.2.6 ([BIPP23, Theorem 7.15]). *Let $(\phi, \mathbb{F})$ be a representative of a point in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ such that $\mathbb{F}$ is ϕ-minimal. Then, ϕ induces a Γ-action by isometries on $T\mathbb{F}$, which does not have a global fixed point.*

Since the valuation group Λ is a divisible subgroup of $\mathbb{R}$, it is dense inside $\mathbb{R}$. So, as a special case of the base-change functor defined in [Chi01, Theorem 4.7], define the $\mathbb{R}$-tree

$$T\mathbb{F}^{\mathrm{sc}} := \bigcup_{\sigma\ \Lambda\text{-segment in } T\mathbb{F}} \overline{\sigma},$$

where $\overline{\sigma}$ is the unique metric completion of the Λ-segment σ in $T\mathbb{F}$. The $\mathbb{R}$-tree $T\mathbb{F}^{\mathrm{sc}}$ is unique up to isometry and $T\mathbb{F}$ embeds isometrically in $T\mathbb{F}^{\mathrm{sc}}$.

Proposition 5.2.7 ([Chi01, Theorem 4.7 and Corollary 4.9]). *Every $(\phi, \mathbb{F})$ representing an element in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ induces a Γ-action by isometries on the* segment completion $T\mathbb{F}^{\mathrm{sc}}$ *of the Λ-tree $T\mathbb{F}$. Moreover, $T\mathbb{F}^{\mathrm{sc}}$ is an $\mathbb{R}$-tree and the Γ-action is without fixed point.*

Denote by

$$T_\phi \subset T\mathbb{F}^{\mathrm{sc}}$$

the unique, up to isometry, ϕ-invariant minimal subtree [Pau89, Proposition 2.4]. Then, as in Remark 5.1.12, T_ϕ has extendible segments. To endow T_ϕ with an orientation in the sense of Definition 5.1.4, we describe the germs of oriented Λ-segments in $T\mathbb{F}$ as described in [Bru88c]. Denote by

$$\mathcal{O} := \{h \in \mathbb{F} \mid \log_b(|h|) \leq 0\},$$

the valuation ring of $\mathbb{F}$, where $|h| = \max\{h, -h\}$, and its maximal ideal

$$\mathcal{J} := \{h \in \mathbb{F} \mid \log_b(|h|) < 0\}.$$

The quotient field $\mathbb{F}_{\mathcal{O}} := \mathcal{O}/\mathcal{J}$ inherits an order such that $\mathcal{O} \to \mathbb{F}_{\mathcal{O}}$ is order preserving [Bru88c, Page 90] and $\mathbb{F}_{\mathcal{O}}$ is real closed [BP17, Example 5.2]. As the segments in $T\mathbb{F}^{\mathrm{sc}}$ are in one to one correspondence with the Λ-segments in $T\mathbb{F}$ by construction, the following result holds.

Proposition 5.2.8 ([Bru88c, Corollary 40]). *Let $p \in T\mathbb{F}^{\mathrm{sc}}$. The germs of oriented segments at p in $T\mathbb{F}^{\mathrm{sc}}$ correspond bijectively with points on the circle*

$$\mathbb{S}^1(\mathbb{F}_{\mathcal{O}}) := \left\{x + iy \in \mathbb{F}_{\mathcal{O}}[i] \,\middle|\, x^2 + y^2 = 1\right\}.$$

Remark 5.2.9. If $p \in T\mathbb{F}^{\mathrm{sc}} \setminus T\mathbb{F}$, then by construction, $\mathcal{G}_{T\mathbb{F}^{\mathrm{sc}}}(p)$ consists of two elements. Thus, a cyclic order on $\mathcal{G}_{T\mathbb{F}^{\mathrm{sc}}}(p)$ is trivial. Therefore, in the rest of this text, whenever we refer to a cyclic order on $\mathcal{G}_{T\mathbb{F}^{\mathrm{sc}}}(p)$, we assume that $p \in T\mathbb{F} \subset T\mathbb{F}^{\mathrm{sc}}$.

The following remark summarizes the construction of the correspondence from Proposition 5.2.8. We will use it later to show that certain isometries are orientation preserving.

Remark 5.2.10 ([Bru88c, Page 102])**.** Let $\hat{\mathbb{F}}$ denote the valuation completion of $\mathbb{F}$. The field inclusion $\mathbb{F} \to \hat{\mathbb{F}}$ induces an isometry $T\hat{\mathbb{F}} \to T\mathbb{F}$ [Bru88c, Corollary 26], which we use to define the projection

$$\pi \colon \mathbb{H}^2\big(\hat{\mathbb{F}}\big) \to T\mathbb{F}.$$

Given $p \in T\mathbb{F}$, choose an element $P \in \pi^{-1}(p)$. Any $[p\text{-}y] \in \mathcal{G}_{T\mathbb{F}}(p)$ is represented by a segment $(p\text{-}y)_\Lambda$, which is the image under π of some segment $(P\text{-}Y)_\mathbb{F} \subset \mathbb{H}^2(\hat{\mathbb{F}})$. Thus the transitive $\mathrm{SO}(2, \hat{\mathbb{F}})$-action on the set of oriented $\mathbb{F}$-segments at P passes via π to a transitive $\mathrm{SO}(2, \hat{\mathbb{F}})$-action on the set of oriented Λ-segments at p. The stabilizer of the germ of oriented segments $[p\text{-}y]$ is the kernel of

$$\mathrm{SO}\big(2, \hat{\mathbb{F}}\big) \to \mathrm{SO}(2, \mathbb{F}_{\hat{\mathcal{O}}}) = \mathrm{SO}(2, \mathbb{F}_{\mathcal{O}}).$$

Brumfiel shows using this construction that the germs of oriented Λ-segments at p in $T\mathbb{F}$ is in bijection with $\mathrm{SO}(2, \mathbb{F}_{\mathcal{O}})$, which is also in bijection with $\mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$.

Corollary 5.2.11. *The $\mathbb{F}_{\mathcal{O}}$-extension of the cyclic order $o_\mathbb{R}$ (from Lemma 5.1.13) defines a cyclic order on $\mathcal{G}_{T\mathbb{F}}(p) \cong \mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$ for every $p \in T\mathbb{F}^{\mathrm{sc}}$. Thus an orientation on $T\mathbb{F}^{\mathrm{sc}}$ denoted $or_{\mathcal{O}}$.*

Remark 5.2.12. Consider $p \in T\mathbb{F}$, $[p\text{-}y_1], [p\text{-}y_2], [p\text{-}y_3] \in \mathcal{G}_{T\mathbb{F}}(p)$ distinct, $\pi_{\mathcal{O}} \colon \hat{\mathcal{O}} \to \mathbb{F}_{\hat{\mathcal{O}}} = \mathbb{F}_{\mathcal{O}}$ the reduction morphism, and

$$\pi'_{\mathcal{O}} \colon \mathbb{S}^1\big(\hat{\mathbb{F}}\big) \to \mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$$

the induced map, using that $\mathrm{SO}(2, \hat{\mathbb{F}}) = \mathrm{SO}(2, \hat{\mathcal{O}})$ and $\mathrm{SO}(2, \mathbb{F}_{\hat{\mathcal{O}}}) = \mathrm{SO}(2, \mathbb{F}_{\mathcal{O}})$ [Bru88c, Page 102]. By Proposition 5.2.8, $[p\text{-}y_1]$, $[p\text{-}y_2]$, $[p\text{-}y_3]$ correspond bijectively to elements of $\mathbb{S}^1(\mathbb{F}_{\mathcal{O}})$ which we denote by y_1^c, y_2^c, y_3^c. For $P \in \pi^{-1}(p)$, one can consider three germs of oriented segments $[P\text{-}Y_1], [P\text{-}Y_2], [P\text{-}Y_3] \in \mathcal{G}_{\mathbb{H}^2(\hat{\mathbb{F}})}(P)$ with corresponding elements of $\mathbb{S}^1(\hat{\mathbb{F}})$ denoted by Y_1^c, Y_2^c, Y_3^c such that

$$\pi'_{\mathcal{O}}(Y_i^c) = y_i^c \quad \forall i \in \{1, 2, 3\}.$$

By the identification from Remark 5.2.10, and because the morphism $\pi_{\mathcal{O}}\colon \hat{\mathcal{O}} \to \mathbb{F}_{\mathcal{O}}$ is order preserving, it holds

$$\begin{aligned} o_{\mathbb{F}_{\mathcal{O}}}(\pi'_{\mathcal{O}}(Y_1^c), \pi'_{\mathcal{O}}(Y_2^c), \pi'_{\mathcal{O}}(Y_3^c)) &= \operatorname{sgn}_{\mathbb{F}_{\mathcal{O}}} \det(\pi'_{\mathcal{O}}(Y_2^c) - \pi'_{\mathcal{O}}(Y_1^c), \pi'_{\mathcal{O}}(Y_3^c) - \pi'_{\mathcal{O}}(Y_2^c)) \\ &= \operatorname{sgn}_{\mathbb{F}_{\mathcal{O}}} \det(\pi_{\mathcal{O}}(Y_2^c - Y_1^c), \pi_{\mathcal{O}}(Y_3^c - Y_2^c)) \\ &= \operatorname{sgn}_{\hat{\mathbb{F}}} \det(Y_2^c - Y_1^c, Y_3^c - Y_2^c) \\ &= o_{\hat{\mathbb{F}}}(Y_1^c, Y_2^c, Y_3^c). \end{aligned}$$

Notation 5.2.13. Since T_ϕ is an $\mathbb{R}$-subtree of $T\mathbb{F}^{\mathrm{sc}}$, the orientation $or_{\mathcal{O}}$ on $T\mathbb{F}^{\mathrm{sc}}$ restricts to an orientation on T_ϕ. For every $p \in T_\phi$, denote by

$$or_\phi(p)$$

the cyclic order on the germs of oriented segments at p in T_ϕ and by or_ϕ the induced orientation.

It remains to show that the oriented $\mathbb{R}$-tree (T_ϕ, or_ϕ) associated with $[\phi, \mathbb{F}]$ does not depend of the choice of representatives, for the equivalence class defined in Theorem 5.2.2. To this end, we will use the following theorem. The isometry it involves is known from [BIPP23, Proof of Corollary 5.19]; our contribution is to show that this isometry preserves the orientation.

Theorem 5.2.14. *Let $\psi\colon \mathbb{F}_1 \to \mathbb{F}_2$ be an ordered field morphism between two real closed fields and*

$$\phi_1\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F}_1) \quad \textit{and} \quad \phi_2\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F}_2)$$

two representations such that

$$\psi_\phi \circ \phi_1 = g\phi_2 g^{-1} \textit{ for some } g \in \mathrm{PSL}_2(\mathbb{F}_2),$$

where ψ_ϕ is the natural inclusion of $\mathrm{PSL}_2(\mathbb{F}_1)$ in $\mathrm{PSL}_2(\mathbb{F}_2)$ given by ψ. Let b be a big element of $\mathbb{F}_1$ and suppose that $\psi(b)$ is a big element of $\mathbb{F}_2$. Then there exists a Γ-equivariant isometry $(\phi_1, T_{\phi_1}, or_{\phi_1}) \to (\phi_2, T_{\phi_2}, or_{\phi_2})$ which preserves the orientation.

Proof. As defined above, consider the pseudo-distance $\mathrm{dist}_{\mathbb{H}^2(\mathbb{F}_1)}$ and $\mathrm{dist}_{\mathbb{H}^2(\mathbb{F}_2)}$ induced by the big elements b and $\psi(b)$ respectively. By definition of the pseudo-distances, the field morphism ψ induces $\psi_B\colon \mathbb{H}^2(\mathbb{F}_1) \to \mathbb{H}^2(\mathbb{F}_2)$ that sends $x + iy \in \mathbb{H}^2(\mathbb{F}_1)$ to $\psi(x) + i\psi(y) \in \mathbb{H}^2(\mathbb{F}_2)$, which is pseudo-distance preserving. Denote by

$$\begin{array}{rccc} \psi_T\colon & T\mathbb{F}_1 & \longrightarrow & T\mathbb{F}_2; \\ & [x+iy] & \longmapsto & [\psi(x) + i\psi(y)] \end{array}$$

the induced isometric embedding between the associated Λ-trees (as done in [BIPP23, Corollary 5.19]). Denote still by $\psi_T\colon T\mathbb{F}_1^{\mathrm{sc}} \to T\mathbb{F}_2^{\mathrm{sc}}$ the unique continuous extension of ψ_T. Since ψ is a morphism and $\mathrm{PSL}_2(\mathbb{F}_1)$, $\mathrm{PSL}_2(\mathbb{F}_2)$ act by Möbius transformations on $T\mathbb{F}_1$, $T\mathbb{F}_2$ respectively, then for every $\gamma \in \Gamma$

$$\psi_T(\phi_1(\gamma)z) = \psi_\phi(\phi_1)(\gamma)\psi_T(z) \quad \forall z \in T\mathbb{F}_1^{\mathrm{sc}}.$$

We show that $g^{-1}\psi_T$ induces a Γ-equivariant isometry between T_{ϕ_1} and T_{ϕ_2}. Denote by $A_{\phi_1(\gamma)}$ the translation axis of $\phi_1(\gamma)$. Since ψ_T is an isometric embedding, by [Pau89, Theorem 1.2], we obtain

$$\begin{aligned}\psi_T\left(A_{\phi_1(\gamma)}\right) &= \{\psi_T(z) \in T\mathbb{F}_2 \,|\, \mathrm{dist}_{T\mathbb{F}_1}(z, \phi_1(\gamma)z) = \mathrm{lg}(\phi_1(\gamma))\,\} \\ &= \{\psi_T(z) \in T\mathbb{F}_2 \,|\, \mathrm{dist}_{T\mathbb{F}_2}(\psi_T(z), \psi_\phi(\phi_1)(\gamma)\psi_T(z)) = \mathrm{lg}(\psi_\phi(\phi_1)(\gamma)\,\}\,.\end{aligned}$$

Hence $\psi_T(A_{\phi_1(\gamma)}) \subset A_{\psi_\phi(\phi_1)(\gamma)}$. By [Pau89, Proposition 2.4]

$$g^{-1}\psi_T(T_{\phi_1}) = \bigcup_{\gamma\in\Gamma} g^{-1}\psi_T(A_{\phi_1(\gamma)}) \subset \bigcup_{\gamma\in\Gamma} g^{-1}A_{\psi_\phi(\phi_1)(\gamma)}.$$

Finally, using [Pau89, Remark 1.4]

$$\bigcup_{\gamma\in\Gamma} g^{-1}A_{\psi_\phi(\phi_1)(\gamma)} = \bigcup_{\gamma\in\Gamma} A_{g\psi_\phi(\phi_1)(\gamma)g^{-1}} = \bigcup_{\gamma\in\Gamma} A_{\phi_2(\gamma)} = T_{\phi_2}.$$

Since $\mathrm{PSL}_2(\mathbb{F}_2)$ acts by isometries on T_{ϕ_2}, we obtain that

$$g^{-1}\psi_T\colon T_{\phi_1} \to T_{\phi_2}$$

is an isometric embedding. In addition, for any $z \in T_{\phi_1}$ and any $\gamma \in \Gamma$

$$\begin{aligned}g^{-1}\psi_T(\phi_1(\gamma)z) &= g^{-1}\psi_\phi(\phi_1)(\gamma)\psi_T(z) \\ &= g^{-1}\psi_\phi(\phi_1)(\gamma)gg^{-1}\psi_T(z) \\ &= \phi_2(\gamma)g^{-1}\psi_T(z),\end{aligned}$$

so that $g^{-1}\psi_T$ is Γ-equivariant. In particular, $g^{-1}\psi_T(T_{\phi_1})$ is a Γ-invariant $\mathbb{R}$-subtree of T_{ϕ_2}. By uniqueness of the minimal Γ-invariant $\mathbb{R}$-subtree, up to isometry, $g^{-1}\psi_T$ is a Γ-equivariant isometry.

We show that $g^{-1}\psi_T$ is orientation preserving. As in Remark 5.2.12, consider $p \in T\mathbb{F}_1$ and $y_1^c, y_2^c, y_3^c \in \mathcal{G}_{T\mathbb{F}_1}(p) \cong \mathbb{S}^1(\mathbb{F}_{\mathcal{O}_1})$ distinct, $P \in \pi^{-1}(p)$ and three elements $Y_1^c, Y_2^c, Y_3^c \in \mathcal{G}_{\mathbb{H}^2(\mathbb{F}_1)}(P)$ such that $\pi'_{\mathcal{O}_1}(Y_i^c) = y_i^c$ for every $i \in \{1,2,3\}$. Since $\mathrm{PSL}_2(\mathbb{F}_2)$ preserves the cyclic orders $or_{\mathbb{F}_2}(\psi_B(P))$ and $or_{\mathcal{O}_2}(\psi_T(p))$, it is enough to prove that ψ_T preserves the orientation. Since ψ_B is distance preserving there exist group morphisms ψ_B^c, ψ_T^c induced by ψ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SO}(2,\mathbb{F}_1) & \xrightarrow{\psi_B^c} & \mathrm{SO}(2,\mathbb{F}_2) \\ {\scriptstyle \pi'_{\mathcal{O}_1}}\downarrow & & \downarrow{\scriptstyle \pi'_{\mathcal{O}_2}} \\ \mathrm{SO}(2,\mathbb{F}_{\mathcal{O}_1}) & \xrightarrow[\psi_T^c]{} & \mathrm{SO}(2,\mathbb{F}_{\mathcal{O}_2}). \end{array}$$

Using the identification $\mathrm{SO}(2,\mathbb{F}_*) \cong \mathbb{S}^1(\mathbb{F}_*)$ for $* \in \{1, 2, \mathcal{O}_1, \mathcal{O}_2\}$, it holds

$$\begin{aligned} or_{\mathcal{O}_2}(\psi_T(p))(\psi_T^c(y_1^c), \psi_T^c(y_2^c), \psi_T^c(y_3^c)) &= o_{\mathbb{F}_2}(\psi_B^c(Y_1^c), \psi_B^c(Y_2^c), \psi_B^c(Y_3^c)) \\ &= \mathrm{sgn}_{\mathbb{F}_2}\mathrm{det}_{\mathbb{F}_2}(\psi(Y_2^c - Y_1^c), \psi(Y_3^c - Y_2^c)) \\ &= \mathrm{sgn}_{\mathbb{F}_2}\psi\left(\mathrm{det}_{\mathbb{F}_1}(Y_2^c - Y_1^c, Y_3^c - Y_2^c)\right) \\ &= \mathrm{sgn}_{\mathbb{F}_1}\left(\mathrm{det}_{\mathbb{F}_1}(Y_2^c - Y_1^c, Y_3^c - Y_2^c)\right) \\ &= or_{\mathcal{O}_1}(p)(y_1^c, y_2^c, y_3^c), \end{aligned}$$

where the first equality holds because the above diagram is commutative, the second because ψ_B^c is induced by ψ, the third because ψ is a homomorphism, and the fourth because ψ preserves the order. Thus, $g^{-1}\psi_T$ is an orientation preserving Γ-equivariant isometry. □

For two representatives $(\phi_1, \mathbb{F}_1), (\phi_2, \mathbb{F}_2)$ of $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))_{\mathrm{cl}}^{\mathrm{RSp}}$, there exists a real closed field morphism $\psi\colon \mathbb{F}_1 \to \mathbb{F}_2$ and $g \in \mathrm{PSL}_2(\mathbb{F}_2)$ such that

$$\psi_\phi \circ \phi_1 = g\phi_2 g^{-1}.$$

By Theorem 5.2.14, there exists an orientation preserving Γ-equivariant isometry between T_{ϕ_1} and T_{ϕ_2} equipped with their orientations or_{ϕ_1} and or_{ϕ_2} respectively. Thus the associated oriented $\mathbb{R}$-tree to an element of $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))_{\mathrm{cl}}^{\mathrm{RSp}}$ is canonical.

Lemma 5.2.15. *Each element $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))_{\mathrm{cl}}^{\mathrm{RSp}}$ defines a canonical Γ-action by orientation preserving isometries on a ϕ-minimal oriented $\mathbb{R}$-tree (ϕ, T_ϕ, or_ϕ).*

5.2.2 Description via asymptotic cones

This subsection describes the Γ-action on an oriented $\mathbb{R}$-tree induced by an element $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))_{\mathrm{cl}}^{\mathrm{RSp}}$ as an ultralimit of Γ-actions on $\mathbb{H}^2$, using asymptotic cones and the ultralimit orientation. To do so, we first recall, as in [BIPP23, Section 7], that $[\phi, \mathbb{F}]$ is represented by a representation of Γ in $\mathrm{PSL}_2(\mathbb{R}_\mu^{\mathfrak{u}})$, where $\mathbb{R}_\mu^{\mathfrak{u}}$ is a Robinson field as described in Example 2.2.3.

Definition 5.2.16. Given $\mathfrak{u}$ a non-principal ultrafilter, the sequence of scales (μ_k) is *well adapted* to a sequence of representations $\phi_k \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ if there exists $c_1, c_2 \in \mathbb{R}_{>0}$ such that in the ultraproduct $\mathbb{R}^{\mathfrak{u}}$

$$c_1(\mu_k) \leq \sum_{\gamma \in F} \left(\mathrm{tr}\left(\phi_k(\gamma)\phi_k(\gamma)^T\right)\right) \leq c_2(\mu_k).$$

If only the second inequality holds, then the sequence (μ_k) is *adapted.*

Let $\mathfrak{u}$ be a non-principal ultrafilter and (μ_k) a sequence of scalars adapted to the sequence of representations $\phi_k \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$. Denote by $\phi^{\mathfrak{u}}_\mu \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$ the $(\mathfrak{u}, \mu)$*-limit representation*

$$\phi^{\mathfrak{u}}_\mu(\gamma) := \begin{pmatrix} (\phi_k(\gamma)^{1,1})_k & (\phi_k(\gamma)^{1,2})_k \\ (\phi_k(\gamma)^{2,1})_k & (\phi_k(\gamma)^{2,2})_k \end{pmatrix} \in \mathrm{PSL}_2\left(\mathbb{R}^{\mathfrak{u}}_\mu\right).$$

Theorem 5.2.17 ([BIPP23, Theorem 7.16])**.** *Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $(\phi_k, \mathbb{R})_k \in \mathcal{M}_\Gamma(\mathbb{R})$ and $(\phi^{\mathfrak{u}}_\mu, \mathbb{R}^{\mathfrak{u}}_\mu)$ its $(\mathfrak{u}, \mu)$-limit representation for an adapted sequence of scales $\mu := (\mu_k)$. Then:*

- *$\phi^{\mathfrak{u}}_\mu$ is reductive, and*
- *if μ is well adapted, infinite, and $\mathbb{F}_{\phi^{\mathfrak{u}}_\mu}$ denotes the $\phi^{\mathfrak{u}}_\mu$-minimal field, then $(\phi^{\mathfrak{u}}_\mu, \mathbb{R}^{\mathfrak{u}}_\mu)$ is $\mathrm{SO}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$-conjugate to a representation $(\phi, \mathbb{F}_{\phi^{\mathfrak{u}}_\mu})$ that represents an element in $\partial \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$.*

Conversely, any $(\phi, \mathbb{F})$ representing an element in $\partial \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ arises in this way. More precisely, for any non-principal ultrafilter $\mathfrak{u}$ and any sequence of scales μ giving an infinite element, there exist an order preserving field morphism $\psi \colon \mathbb{F} \to \mathbb{R}^{\mathfrak{u}}_\mu$ and a sequence of homomorphisms $(\phi_k, \mathbb{R})_k \in \mathcal{M}_\Gamma(\mathbb{R})$ for which μ is well adapted and such that $\psi \circ \phi$ and $\phi^{\mathfrak{u}}_\mu$ are $\mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$-conjugate.

To keep the notation from [BIPP23], we use the alternative model of $\mathbb{H}^2$ defined in Section 2.4:

$$\mathcal{P}^1(2, \mathbb{R}) := \{\, A \in M_{2\times 2}(\mathbb{R}) \,|\, \det(A) = 1, A \text{ is symmetric and positive definite} \,\}.$$

Endow $\mathcal{P}^1(2, \mathbb{R})$ with its semialgebraic multiplicative Cartan distance d_δ (Proposition 2.4.4) and its associated distance $\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R})} = \log d_\delta$. The $\mathbb{R}^{\mathfrak{u}}_\mu$-extension of d_δ induces a pseudo-distance $\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}}_\mu)} = \log_\mu (d_\delta)_{\mathbb{R}^{\mathfrak{u}}_\mu}$ on $\mathcal{P}^1(2, \mathbb{R}^{\mathfrak{u}}_\mu)$ using the transfer principle (Theorem 2.2.2). The group $\mathrm{PSL}_2(\mathbb{R})$ acts by congruence on $\mathcal{P}^1(2, \mathbb{R})$:

$$g.x = gxg^T \quad \forall x \in \mathcal{P}^1(2, \mathbb{R}),\ \forall g \in \mathrm{PSL}_2(\mathbb{R})$$

and preserves the distance. Similarly, $\mathrm{PSL}_2(\mathbb{R}^{\mathfrak{u}}_\mu)$ acts by congruence on $\mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}}_\mu)$ and preserves the pseudo-distance [BIPP23, Subsection 5.1].

Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathbb{N}$, $(\lambda_k)\subset\mathbb{R}$ a sequence such that $\lambda_k\to\infty$, $\mu_k=e^{\lambda_k}$ for every $k\in\mathbb{N}$, $\mu=(\mu_k)_{k\in\mathbb{N}}$, $\mathbb{R}^{\mathfrak{u}}_\mu$ the Robinson field associated to $\mathfrak{u}$, and denote by

$$\begin{aligned}
&\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}} := \mathcal{P}^1(2,\mathbb{R})^{\mathbb{N}}/\sim,\\
&\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda := \left\{ [x_k]\in\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}} \;\middle|\; \frac{\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R})}(x_k,\mathrm{Id})}{\lambda_k} \text{ is } \mathfrak{u}\text{-bounded} \right\},
\end{aligned}$$

where $(x_k)\sim(y_k)$ if the two sequences coincide $\mathfrak{u}$-almost surely. Note that if we endow $\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda$ with the pseudo-distance

$$\begin{array}{rccc}
\mathrm{dist}^{\mathfrak{u}}\colon & \mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda\times\mathcal{P}^1(2.\mathbb{R})^{\mathfrak{u}}_\lambda & \longrightarrow & \mathbb{R};\\
& ([x_k],[y_k]) & \longmapsto & \lim_{\mathfrak{u}}\frac{\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R})}(x_k,y_k)}{\lambda_k},
\end{array}$$

then $\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda/\{\,\mathrm{dist}_{\mathfrak{u}}=0\,\}=\mathrm{Cone}^{\mathfrak{u}}\left(\mathcal{P}^1(2,\mathbb{R}),(\lambda_k),(\mathrm{Id})\right)$, see Subsection 2.1.

Theorem 5.2.18 ([BIPP23, Theorem 5.10 and Lemma 5.12])**.** *With the above notation, the map*

$$\begin{array}{rccc}
\Psi_{\mathcal{P}^1}\colon & \mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda & \longrightarrow & \mathcal{P}^1\left(2,\mathbb{R}^{\mathfrak{u}}\right);\\
& \left(\left\{\,x^{i,j}\,\right\}^{i,j}_k\right)_k & \longmapsto & \left\{\,\left(x^{i,j}_k\right)_k\,\right\}^{i,j}
\end{array}$$

induces an isometry between $T_{\mathcal{P}^1}\mathbb{R}^{\mathfrak{u}}_\mu := \mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}}_\mu)/\{\,\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}}_\mu)}=0\,\}$ *and the asymptotic cone* $T^{\mathfrak{u}}_{\mathcal{P}^1} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathcal{P}^1(2,\mathbb{R}),(\lambda_k),(\mathrm{Id})\right)$.

Remark 5.2.19. The induced isometry is obtained using the commutative diagram:

$$\begin{array}{ccc}
\mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_\lambda & \longrightarrow & \mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}})\cap(\mathcal{O}_\mu)^3\\
\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_\mu}\\
 & & \mathcal{P}^1\left(2,\mathbb{R}^{\mathfrak{u}}_\mu\right)\\
\Big\downarrow & & \Big\downarrow\\
T^{\mathfrak{u}}_{\mathcal{P}^1} & \xrightarrow{\quad\eta\quad} & T_{\mathcal{P}^1}\mathbb{R}^{\mathfrak{u}}_\mu,
\end{array}$$

where $\mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}})$ is a semialgebraic subset of $(\mathbb{R}^{\mathfrak{u}})^3$ and π_μ is induced by the quotient map $\mathcal{O}_\mu\to\mathbb{R}^{\mathfrak{u}}_\mu$, see Example 2.2.3.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{P}^1(2,\mathbb{R})$ by isometries and the stabilizer of $\mathrm{Id}\in\mathcal{P}^1(2,\mathbb{R})$ is the subgroup $\mathrm{SO}(2)$. Thus both $\mathbb{H}^2$ and $\mathcal{P}^1(2,\mathbb{R})$ are models for the symmetric space of $\mathrm{SL}_2(\mathbb{R})$. Hence up to rescaling the distance on $\mathbb{H}^2$, there exists a $\mathrm{SL}_2(\mathbb{R})$-equivariant isometry $\mathbb{H}^2\to\mathcal{P}^1(2,\mathbb{R})$ which is algebraic [Ebe85, Page 134].

Remark 5.2.20. In Section 5.1 and 5.2, $\mathbb{H}^2$ is endowed with a metric proportional to its standard one. In particular, we endow $\mathbb{H}^2$ with a metric such that it is isometric to $\mathcal{P}^1(2,\mathbb{R})$ endowed with $\mathrm{dist}_{\mathcal{P}^1(2,\mathbb{R})}$, see [BIPP23, Subsection 5.2].

As in Subsection 2.1, the isometry $\mathbb{H}^2 \to \mathcal{P}^1(2,\mathbb{R})$ leads, on the one hand, to a $\mathrm{SL}_2(\mathbb{R})$-equivariant isometry $(\mathbb{H}^2)^{\mathfrak{u}}_{\lambda} \to \mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_{\lambda}$ and, on the other hand, since it is semialgebraic, to a $\mathrm{SL}_2(\mathbb{R})$-equivariant isometry $\mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}})\cap(\mathcal{O}_\mu)^3 \to \mathbb{H}^2(\mathbb{R}^{\mathfrak{u}})\cap(\mathcal{O}_\mu)^3$ to give:

Corollary 5.2.21. *With the above notation, the map*

$$\begin{array}{rccc} \Psi\colon & (\mathbb{H}^2)^{\mathfrak{u}}_{\lambda} & \longrightarrow & \mathbb{H}^2\left(\mathbb{R}^{\mathfrak{u}}\right); \\ & (x+iy)_k & \longmapsto & (x_k)+i(y_k) \end{array}$$

induces an isometry between the asymptotic cone $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lambda_k), (0)\right)$ *and* $\mathbb{H}^2(\mathbb{R}^{\mathfrak{u}}_{\mu})/\{\,\mathrm{dist}_{\mathbb{H}^2(\mathbb{R}^{\mathfrak{u}}_{\mu})} = 0\,\} =: T\mathbb{R}^{\mathfrak{u}}_{\mu}$. *Moreover, with the notations from Theorem 5.2.17, the isometry is* Γ*-equivariant for the induced* Γ*-actions by* $\phi^{\mathfrak{u}} = \lim_{\mathfrak{u}} \phi_k$ *on* $T^{\mathfrak{u}}$ *and by* $\phi^{\mathfrak{u}}_{\mu}$ *on* $T\mathbb{R}^{\mathfrak{u}}_{\mu}$.

Proof. The first part of the corollary is a consequence of Theorem 5.2.18 and the above mentioned isometries between the models of the hyperbolic plane [Ebe85, Page 134]. It gives the commutative diagram:

$$\begin{array}{ccccccc}
(\mathbb{H}^2)^{\mathfrak{u}}_{\lambda} & \longrightarrow & \mathcal{P}^1(2,\mathbb{R})^{\mathfrak{u}}_{\lambda} & \longrightarrow & \mathcal{P}^1(2,\mathbb{R}^{\mathfrak{u}})\cap(\mathcal{O}_\mu)^3 & \longrightarrow & \mathbb{H}^2(\mathbb{R}^{\mathfrak{u}})\cap(\mathcal{O}_\mu)^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \mathcal{P}^1\left(2,\mathbb{R}^{\mathfrak{u}}_{\mu}\right) & & \\
 & & & & \downarrow & & \\
T^{\mathfrak{u}} & \longrightarrow & T^{\mathfrak{u}}_{\mathcal{P}^1} & \xrightarrow{\ \eta\ } & T_{\mathcal{P}^1}\mathbb{R}^{\mathfrak{u}}_{\mu} & \longrightarrow & T\mathbb{R}^{\mathfrak{u}}_{\mu}.
\end{array}$$

The Γ-equivariance is a direct consequence of the $\mathrm{SL}_2(\mathbb{R})$-equivariance of the isometry $\mathbb{H}^2 \to \mathcal{P}^1(2,\mathbb{R})$ and [BIPP23, Lemma 5.12]. □

With the notations from Theorem 5.2.18 and $(\mathbb{R}^{\mathfrak{u}}_{\mu})_{\mathcal{O}} := \mathcal{O}_\mu/\mathcal{J}_\mu$, endow $T\mathbb{R}^{\mathfrak{u}}_{\mu}$ with the orientation coming from the orientation on $\mathbb{S}^1((\mathbb{R}^{\mathfrak{u}}_{\mu})_{\mathcal{O}})$, which we denote by $or_{(\mathbb{R}^{\mathfrak{u}}_{\mu})_{\mathcal{O}}}$, see Subsection 5.2.1 and Corollary 5.2.11. Endow also $T^{\mathfrak{u}} := \mathrm{Cone}^{\mathfrak{u}}\left(\mathbb{H}^2, (\lambda_k), (0)\right)$ with the ultralimit orientation defined in Theorem 5.1.30. That is, if $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$ denotes the germs of oriented segments at $[P_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$, then the map $or^{\mathfrak{u}}([P_k]^{\mathfrak{u}})$:

$$\begin{array}{ccc} \left(\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})\right)^3 & \longrightarrow & \{\,-1,0,1\,\}; \\ ([x_k]^{\mathfrak{u}},[y_k]^{\mathfrak{u}},[z_k]^{\mathfrak{u}}) & \longmapsto & \begin{cases} 0 & \text{if } \mathrm{Card}\{\,[x_k]^{\mathfrak{u}},[y_k]^{\mathfrak{u}},[z_k]^{\mathfrak{u}}\,\} \leq 2, \\ \lim_{\mathfrak{u}} or(P_k)(x_k,y_k,z_k) & \text{otherwise,} \end{cases} \end{array}$$

which is a cyclic order on $\mathcal{G}_{T^{\mathfrak{u}}}([P_k]^{\mathfrak{u}})$, where $or(P_k)$ is the cyclic order at $\mathcal{G}_{\mathbb{H}^2}(P_k)$ given by the standard orientation on $\mathbb{H}^2$.

Theorem 5.2.22. *With the above defined orientations on $T^{\mathfrak{u}}$ and $T\mathbb{R}^{\mathfrak{u}}_{\mu}$, the Γ-equivariant isometry $\Psi\colon T^{\mathfrak{u}} \to T\mathbb{R}^{\mathfrak{u}}_{\mu}$ from Corollary 5.2.21 is orientation preserving.*

Proof. Let $p_{\mathfrak{u}} := [(p^1+ip^2)_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$ and consider three germs of oriented segments in $\mathcal{G}_{T^{\mathfrak{u}}}(p_{\mathfrak{u}})$ represented by $x_{\mathfrak{u}} := [(x^1+ix^2)_k], y_{\mathfrak{u}} := [(y^1+iy^2)_k]^{\mathfrak{u}}, z_{\mathfrak{u}} := [(z^1+iz^2)_k]^{\mathfrak{u}} \in T^{\mathfrak{u}}$ at the same distance to $p_{\mathfrak{u}}$. For the projection map $\pi\colon \mathbb{H}^2(\mathbb{R}^{\mathfrak{u}}_{\mu}) \to T\mathbb{R}^{\mathfrak{u}}_{\mu}$, consider

$$\begin{aligned}
(\mathcal{X}_k) &:= \left[X_k^1\right]^{\mathfrak{u}} + i\left[X_k^2\right]^{\mathfrak{u}} \in \pi^{-1}\left(\left[x_k^1\right]^{\mathfrak{u}} + i\left[x_k^2\right]^{\mathfrak{u}}\right) = \pi^{-1}\left(\Psi\left(x_{\mathfrak{u}}\right)\right),\\
(\mathcal{Y}_k) &:= \left[Y_k^1\right]^{\mathfrak{u}} + i\left[Y_k^2\right]^{\mathfrak{u}} \in \pi^{-1}\left(\left[y_k^1\right]^{\mathfrak{u}} + i\left[y_k^2\right]^{\mathfrak{u}}\right) = \pi^{-1}\left(\Psi\left(y_{\mathfrak{u}}\right)\right),\\
(\mathcal{Z}_k) &:= \left[Z_k^1\right]^{\mathfrak{u}} + i\left[Z_k^2\right]^{\mathfrak{u}} \in \pi^{-1}\left(\left[z_k^1\right]^{\mathfrak{u}} + i\left[z_k^2\right]^{\mathfrak{u}}\right) = \pi^{-1}\left(\Psi\left(z_{\mathfrak{u}}\right)\right)
\end{aligned}$$

with the right distance to $(\mathcal{P}_k) := [P_k^1]^{\mathfrak{u}} + i[P_k^2]^{\mathfrak{u}} \in \pi^{-1}([p_k^1]^{\mathfrak{u}} + i[p_k^2]^{\mathfrak{u}}) = \pi^{-1}(\Psi(p_{\mathfrak{u}}))$ so that they correspond to elements of $\mathbb{S}^1(\mathbb{R}^{\mathfrak{u}}_{\mu})$, see Remark 5.2.12. Then

$$\begin{aligned}
or_{(\mathbb{R}^{\mathfrak{u}}_{\mu})_{\mathcal{O}}}(\Psi(p_{\mathfrak{u}}))(\Psi(x_{\mathfrak{u}}), \Psi(y_{\mathfrak{u}}), \Psi(z_{\mathfrak{u}})) &= or_{\mathbb{R}^{\mathfrak{u}}_{\mu}}((\mathcal{P}_k))((\mathcal{X}_k), (\mathcal{Y}_k), (\mathcal{Z}_k))\\
&= \mathrm{sgn}_{\mathbb{R}^{\mathfrak{u}}_{\mu}} \det{}_{\mathbb{R}^{\mathfrak{u}}_{\mu}}((\mathcal{Y}_k) - (\mathcal{X}_k), (\mathcal{Z}_k) - (\mathcal{Y}_k))\\
&= \mathrm{sgn}_{\mathbb{R}^{\mathfrak{u}}_{\mu}}[\det(Y_k - X_k, Z_k - Y_k)]^{\mathfrak{u}}\\
&= \mathrm{sgn}_{\mathbb{R}^{\mathfrak{u}}_{\mu}}[or_{\mathbb{R}}(P_k)(X_k, Y_k, Z_k)]^{\mathfrak{u}}\\
&= \lim_{\mathfrak{u}} or_{\mathbb{R}}(P_k)(X_k, Y_k, Z_k),
\end{aligned}$$

where $X_k = (X^1 + iX^2)_k \in \mathbb{H}^2$ for every $k \in \mathbb{N}$ and similarly for Y_k and Z_k. The third equality holds because of the field operations of $\mathbb{R}^{\mathfrak{u}}_{\mu}$ and the last one because of the definition of the order on $\mathbb{R}^{\mathfrak{u}}_{\mu}$. Finally, since Ψ preserves the distance (Theorem 5.2.18), both $[(X^1 + iX^2)_k] = [(x^1 + x^2)_k] \in T^{\mathfrak{u}}$ so that

$$\lim_{\mathfrak{u}} or_{\mathbb{R}}(P_k)(X_k, Y_k, Z_k) = \lim_{\mathfrak{u}} or_{\mathbb{R}}(p)\left(\left(x^1 + ix^2\right)_k, \left(y^1 + iy^2\right)_k, \left(z^1 + iz^2\right)_k\right)$$

and Ψ is orientation preserving. □

We associated a Γ-action by isometries preserving the orientation on an oriented $\mathbb{R}$-tree to every element in $\partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. Moreover, the constructed orientation is natural and described by ultralimits of orientation on $\mathbb{H}^2$. This is partly due to the fact that the orientation on the circle is described by a semialgebraic equation and the naturalness of the real spectrum compactification in terms of asymptotic methods. In the next and final section, we use the results obtained so far to construct a continuous surjection from the real spectrum compactification to the oriented compactification of the character variety.

5.3 A continuous surjection between both compactifications

In this section we construct a continuous surjective map from $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ to $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$. By Theorem 5.2.2, every $[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ is represented by a reductive representation

$$\phi\colon \Gamma \to \mathrm{PSL}_2(\mathbb{F}),$$

where $\mathbb{F}$ is real closed, ϕ-minimal, non-Archimedean, and of finite transcendence degree over $\overline{\mathbb{Q}}^r$. In Subsection 5.2.1, we constructed an oriented ϕ-minimal $\mathbb{R}$-tree (ϕ, T_ϕ, or_ϕ) which does not depend on the choice of representative in the class $[\phi, \mathbb{F}]$, see Theorem 5.2.14 and Lemma 5.2.15.

Definition 5.3.1. This construction gives the following map:

$$\begin{aligned} \beth\colon \quad \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} &\longrightarrow \qquad \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cup \mathcal{T}'; \\ [\phi, \mathbb{F}] \qquad &\longmapsto \quad [\phi, X, or_\phi] = \begin{cases} [\phi, \mathbb{H}^2, or] & \text{if } \mathbb{F} = \mathbb{R} \\ [\phi, T_\phi, or_\phi] & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 5.3.2. *The map $\beth$ defined in Definition 5.3.1 is sequentially continuous for sequences in the interior of the character variety. That is, for every non-principal ultrafilter $\mathfrak{u}$, and for every sequence $[\phi_k, \mathbb{R}] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ which converges to $[\phi, \mathbb{F}] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ in the real spectrum topology, then*

$$\beth[\phi, \mathbb{F}] = \lim_{\mathfrak{u}} \beth[\phi_k, \mathbb{R}] = \lim_{\mathfrak{u}} \left[\phi_k, \mathbb{H}^2, or\right]$$

in the oriented Gromov equivariant topology.

Proof. Since the real spectrum topology and the oriented Gromov equivariant topology are equivalent on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$, it suffices to treat the case

$$[\phi, \mathbb{F}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}.$$

Let $\mathfrak{u}$ be a non-principal ultrafilter and consider a sequence of representatives $(\phi_k, \mathbb{R}) \in \mathcal{M}_\Gamma(\mathbb{R})$ of $[\phi_k, \mathbb{R}]$ such that $0 \in \mathbb{H}^2$ minimizes the displacement function of ϕ_k for every $k \in \mathbb{N}$. Since $\mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}}$ is compact, the sequence $(\phi_k, \mathbb{R})$ admits a $\mathfrak{u}$-limit that we denote by $(\phi, \mathbb{F})$. By [BIPP23, Proposition 7.5], the projection map

$$p\colon \mathcal{M}_\Gamma(\mathbb{R}) \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$$

extends continuously to a map $p^{\mathrm{RSp}}_{\mathrm{cl}}\colon \mathcal{M}_\Gamma(\mathbb{R})^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, and since $p^{\mathrm{RSp}}_{\mathrm{cl}}(\phi_k, \mathbb{R})$ converges to $[\phi, \mathbb{F}]$, it holds

$$p^{\mathrm{RSp}}_{\mathrm{cl}}(\phi, \mathbb{F}) = [\phi, \mathbb{F}].$$

Because $(\phi_k, \mathbb{R})$ converges to a boundary element in $\partial\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$, it holds $\lg(\phi_k) \to \infty$. Denote by μ the sequence of scalars $(e^{\lg \phi_k})_k$, which is well adapted to (ϕ_k) by [BIPP23, Lemma 5.13], and consider $(\phi_\mu^{\mathfrak{u}}, \mathbb{R}_\mu^{\mathfrak{u}})$.

Claim: The elements $(\phi_\mu^{\mathfrak{u}}, \mathbb{R}_\mu^{\mathfrak{u}})$ and $(\phi, \mathbb{F})$ are equivalent in $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$.

Proof: Suppose $(\phi_\mu^{\mathfrak{u}}, \mathbb{R}_\mu^{\mathfrak{u}}) \neq (\phi, \mathbb{F}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$. Since $\mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$ is Hausdorff, there exist two open sets U, U' with $U \cap U' = \varnothing$ such that

$$\left(\phi_\mu^{\mathfrak{u}}, \mathbb{R}_\mu^{\mathfrak{u}}\right) \in U \quad \text{and} \quad (\phi, \mathbb{F}) \in U'.$$

Since $\lim_{\mathfrak{u}}(\phi_k, \mathbb{R}) = (\phi, \mathbb{F})$, it follows that $(\phi_k, \mathbb{R}) \in U'$ $\mathfrak{u}$-almost surely. Let $f_1, \ldots, f_m \in \mathbb{R}[\mathcal{M}_\Gamma]$ such that $U = \tilde{U}(f_1, \ldots, f_m)$. By the description of the bijection in Theorem 5.2.2 (see [BIPP23, Proposition 6.3])

$$\phi_\mu^{\mathfrak{u}}(f_i) = [f_i(\phi_k)]^{\mathfrak{u}} \in \mathbb{R}_\mu^{\mathfrak{u}}.$$

In particular, for every $i \in \{1, \ldots, m\}$

$$\begin{aligned} \phi_\mu^{\mathfrak{u}}(f_i) > 0 \text{ if and only if } & f_i(\phi_k) > 0 \ \mathfrak{u}\text{-almost surely} \\ \text{if and only if } & \phi_k \in \tilde{U}(f_i). \end{aligned}$$

Thus $\phi_k \in U \cap U'$ $\mathfrak{u}$-almost surely, which is a contradiction $U \cap U' = \varnothing$ so that $(\phi_\mu^{\mathfrak{u}}, \mathbb{R}_\mu^{\mathfrak{u}}) = (\phi, \mathbb{F}) \in \mathcal{M}_\Gamma(\mathbb{R})_{\mathrm{cl}}^{\mathrm{RSp}}$

By Theorem 5.2.2, there exists an order preserving field morphism

$$\psi \colon \mathbb{F} \to \mathbb{R}_\mu^{\mathfrak{u}}$$

such that $\psi \circ \phi$ and $\phi_\mu^{\mathfrak{u}}$ are $\mathrm{PSL}_2(\mathbb{R}_\mu^{\mathfrak{u}})$-conjugate. Thus, by Theorem 5.2.14, there exists a Γ-equivariant isometry

$$\psi_{\mathbb{F}} \colon (\phi, T_\phi, or_\phi) \to \left(\phi_\mu^{\mathfrak{u}}, T_{\phi_\mu^{\mathfrak{u}}}, or_{\phi_\mu^{\mathfrak{u}}}\right)$$

which is orientation preserving. By Theorem 5.2.22, there exists

$$\Psi \colon T\mathbb{R}_\mu^{\mathfrak{u}} \to T^{\mathfrak{u}}$$

a Γ-equivariant isometry (for the Γ actions induced by $\phi_\mu^{\mathfrak{u}}$ and $\lim_{\mathfrak{u}} \phi_k$, see Corollary 5.2.21), which is orientation preserving for the orientations $or_{(\mathbb{R}_\mu^{\mathfrak{u}})_{\mathcal{O}}}$ on $T\mathbb{R}_\mu^{\mathfrak{u}}$ and $or^{\mathfrak{u}}$ on $T^{\mathfrak{u}}$. Since $T_{\phi_\mu^{\mathfrak{u}}} \subset T\mathbb{R}_\mu^{\mathfrak{u}}$, we obtain a Γ-equivariant orientation preserving isometric embedding

$$\psi_{\mathfrak{u}} := \Psi \circ \psi_{\mathbb{F}} \colon (\phi, T_\phi, or_\phi) \to (\phi^{\mathfrak{u}}, T^{\mathfrak{u}}, or^{\mathfrak{u}}).$$

Since the $\phi^{\mathfrak{u}}$-invariant minimal subtree of $T^{\mathfrak{u}}$ is unique, up to isometry [Pau89, Proposition 2.4], it is contained in $\psi_{\mathfrak{u}}(T_\phi)$, so that $\psi_{\mathfrak{u}}$ induces a Γ-equivariant orientation preserving isometry $(\phi, T_\phi, or_\phi) \to (\phi^{\mathfrak{u}}, (T^{\mathfrak{u}})_\phi, or^{\mathfrak{u}})$, where $(T^{\mathfrak{u}})_\phi$ is the $\phi^{\mathfrak{u}}$-minimal invariant subtree of $T^{\mathfrak{u}}$. Hence, up to Γ-equivariant orientation preserving isometry

$$\beth[\phi, \mathbb{F}] = [\phi^{\mathfrak{u}}, (T^{\mathfrak{u}})_\phi, or^{\mathfrak{u}}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}.$$

Finally, by Theorem 5.1.31

$$\lim_{\mathfrak{u}} (\phi_k, \mathbb{H}^2, or) = [\phi^{\mathfrak{u}}, (T^{\mathfrak{u}})_\phi, or^{\mathfrak{u}}] \in \partial\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}.$$

Since $\mathfrak{u}$ is arbitrary, the map $\beth$ defined in Definition 5.3.1 is sequentially continuous for sequences in the interior of the character variety. □

Lemma 5.3.3. *The map $\beth$ from Definition 5.3.1 is continuous.*

Proof. Let $[\phi_k, \mathbb{F}_k] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ be a sequence that converges to $[\phi, \mathbb{F}] \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$. Since $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ is metrizable by the first item of [BIPP23, Proposition 2.33], consider a countable basis of open neighborhoods $(B([\phi, \mathbb{F}], 1/N))_N \subset \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ around $[\phi, \mathbb{F}]$, where $B([\phi, \mathbb{F}], 1/N)$ denotes the open ball of radius $1/N$ centered at $[\phi, \mathbb{F}]$ for a fixed metric defining the spectral topology. Since $([\phi_k, \mathbb{F}_k])_k$ converges to $[\phi, \mathbb{F}]$, for every $N \in \mathbb{N}$, there exists $n_1(N) > N$ such that

$$\forall m > n_1(N) \quad [\phi_m, \mathbb{F}_m] \in B\left([\phi, \mathbb{F}], \frac{1}{N}\right).$$

Suppose, for contradiction, that $(\beth[\phi_k, \mathbb{F}_k])_k$ does not converge to $\beth[\phi, \mathbb{F}]$. That is, there exists $U \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ open such that $\beth[\phi, \mathbb{F}] \in U$ and

$$\forall N \in \mathbb{N},\ \exists n(N) \geq n_1(N) \quad \text{with} \quad \beth[\phi_{n(N)}, \mathbb{F}_{n(N)}] \notin U.$$

Thus the function $n\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and the sequence $([\phi_{n(N)}, \mathbb{F}_{n(N)}])_N$ verifies:

$$[\phi_{n(N)}, \mathbb{F}_{n(N)}] \in B\left([\phi, \mathbb{F}], \frac{1}{N}\right) \quad \text{and} \quad \beth[\phi_{n(N)}, \mathbb{F}_{n(N)}] \notin U.$$

Since $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ is compact and Hausdorff, it is normal. In particular, with $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}} \backslash U$ closed and $\beth[\phi, \mathbb{F}] \in U$, there exist $U_1, U_2 \subset \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ open such that

$$\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}} \backslash U \subset U_1, \quad \beth[\phi, \mathbb{F}] \in U_2, \quad \text{and} \quad U_1 \cap U_2 = \varnothing.$$

By the density of $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ in $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$, for every $N \in \mathbb{N}$, there exists a sequence $([\phi^m_{n(N)}, \mathbb{R}])_m \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ such that

$$\lim_m \left[\phi^m_{n(N)}, \mathbb{R}\right] = [\phi_{n(N)}, \mathbb{F}_{n(N)}]$$

in the spectral topology. By Lemma 5.3.2

$$\lim_m \beth[\phi^m_{n(N)}, \mathbb{R}] = \beth[\phi_{n(N)}, \mathbb{F}_{n(N)}] \in U_1.$$

Hence, there exists $\ell_1(N) \in \mathbb{N}$ with $\beth[\phi^m_{n(N)}, \mathbb{R}] \in U_1$ for all $m \geq \ell_1(N)$. Up to considering a subsequence, we assume that ℓ_1 is strictly increasing with N, and define the sequence

$$\left(\left[\phi^{\ell_1(N)}_{n(N)}, \mathbb{R}\right]\right)_N \in \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})).$$

Since $[\phi_{n(N)}, \mathbb{F}_{n(N)}] \in B([\phi, \mathbb{F}], 1/N)$ is open, there exists $\ell(N) \geq \ell_1(N)$ with

$$\forall m \geq \ell(N) \quad [\phi^m_{n(N)}, \mathbb{R}] \in B\left([\phi, \mathbb{F}], \frac{1}{N}\right).$$

Hence the sequence $([\phi^{\ell(N)}_{n(N)}, \mathbb{R}])_N$ verifies

$$\lim_N \left[\phi^{\ell(N)}_{n(N)}, \mathbb{R}\right] = [\phi, \mathbb{F}], \quad \text{and} \quad \beth\left[\phi^{\ell(N)}_{n(N)}, \mathbb{R}\right] \in U_1$$

for every $N \in \mathbb{N}$. This is a contradiction with Lemma 5.3.2. Thus

$$\lim_{\mathfrak{u}} \beth[\phi_k, \mathbb{F}_k] = \beth[\phi, \mathbb{F}]$$

so that $\beth$ is sequentially continuous. Finally, $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}}$ is metrizable by the first item of [BIPP23, Proposition 2.33], so that sequential continuity implies continuity. Hence $\beth$ is continuous. □

Theorem 5.3.4. *The map* $\beth\colon \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}} \to \Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{O}}$ *from Definition 5.3.1 is a continuous surjection.*

Proof. From Lemma 5.3.3, $\beth$ is continuous and is the identity on $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. Therefore $\beth(\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}})$ is a compact set that contains $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. Since the oriented Gromov equivariant topology is Hausdorff (Subsection 5.1.2), $\beth(\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R}))^{\mathrm{RSp}}_{\mathrm{cl}})$ is in particular closed. Because the character variety is dense within its oriented compactification, we have the inclusion

$$\Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right)^{\mathrm{O}} \subset \beth\left(\Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}}\right).$$

Moreover, by density of the character variety within its real spectrum compactification and continuity of $\beth$, the space $\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})) = \beth(\Xi(\Gamma, \mathrm{PSL}_2(\mathbb{R})))$ is dense in the image of $\beth$. Hence

$$\beth\left(\Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right)^{\mathrm{RSp}}_{\mathrm{cl}}\right) = \Xi\left(\Gamma, \mathrm{PSL}_2(\mathbb{R})\right)^{\mathrm{O}},$$

so that $\beth$ is continuous and surjective. □

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Chapter 6

Annexe: a homogeneous incomplete real tree with complete segments

This chapter is based on the article [ADRFJ24]. The text reproduced here is the author's manuscript version, which may differ from the final published version due to editorial modifications.

AN INCOMPLETE REAL TREE WITH COMPLETE SEGMENTS

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Abstract. Let $\mathbb{F}$ be the field of real Puiseux series and $\mathcal{T}_{\mathbb{F}}$ the $\mathrm{PSL}(2,\mathbb{F})$–homogeneous $\mathbb{Q}$–tree defined by Brumfiel. We show that completing all the segments of $\mathcal{T}_{\mathbb{F}}$ does not result in a complete metric space.

1. Introduction

Actions on real trees, or more generally on Λ–trees, appear in various ways in the study of degenerations of isotopy classes of marked hyperbolic structures on surfaces, see for example [Bru88a, MS84, MS91], and more recently [BIPP21a, BIPP23]. If Λ is a dense subgroup of $(\mathbb{R},+)$ and $\mathcal{T}$ is a Λ–tree, we can define the real tree

$$\mathcal{T}^{\mathrm{sc}} := \bigcup_{s \text{ segment in } \mathcal{T}} \overline{s},$$

where $\overline{s}$ is the metric completion of the segment s in $\mathcal{T}$. Then the Λ–tree $\mathcal{T}$ isometrically embeds in $\mathcal{T}^{\mathrm{sc}}$. The construction of $\mathcal{T}^{\mathrm{sc}}$ is a special case of the base-change functor defined in [Chi01, Chapter 2, Section 4], which is generalized in [SS12] to affine Λ–buildings.

In this article we give an example of a $\mathbb{Q}$–tree $\mathcal{T}$, as defined by Brumfiel [Bru88b], for which $\mathcal{T}^{\mathrm{sc}}$ is not a complete metric space. For this let $\mathbb{K}$ be a non-Archimedean valued real closed field with value group $\Lambda < \mathbb{R}$, and $\mathcal{T}_{\mathbb{K}}$ its associated Λ–tree as in [Bru88b], see Section 2 for the definition. We call $\mathcal{T}_{\mathbb{K}}^{\mathrm{sc}}$ the *segment completion tree* associated to $\mathbb{K}$.

Theorem 1. *Let $\mathbb{F}$ be the field of real Puiseux series with $\mathbb{Q}$–valuation, and $\mathcal{T}_{\mathbb{F}}$ its associated $\mathbb{Q}$–tree. Then $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$ is not metrically complete.*

This gives an example of a $\mathbb{Q}$–tree with a transitive $\mathrm{PSL}(2,\mathbb{F})$–action by isometries, whose segment completion is not complete. The action extends to a continuous action on $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$ by isometries.

The result as well as the main idea of the proof is perhaps well-known to those working in the field, but the authors are not aware of an explicit example in the literature. We give a geometric proof of Theorem 1 by constructing a Cauchy sequence in $\mathcal{T}_{\mathbb{F}}$ that does not converge in $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$. This sequence is contained in infinitely many different isometric copies of $\mathbb{Q}$ in $\mathcal{T}_{\mathbb{F}}$. Intuitively it can be thought of as a sequence with infinitely many branching points, see Figure 3. A written note of Anne Parreau inspired the

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explicit Cauchy sequence not converging in $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$. The authors expanded on her example and proved that this sequence does not converge.

Actions on affine Λ–buildings appear in the study of degenerations of (higher rank) Teichmüller representations [BIPP21a, BIPP21b, BIPP23, Par00]. Their metric completions are CAT(0) for a suitable metric, see [BIPP21b, Proposition 10]. A natural question is whether such completions are obtained by completing every apartment. The example of Theorem 1 could answer this question in the negative.

We finish the introduction with two small remarks. First, the completeness of general $\mathbb{R}$–trees has been addressed in [CMSP08]. Second, [BT72, Theorem 7.5.3] and [MSSS13] characterize the completeness of Bruhat–Tits buildings, which are examples of $\mathbb{R}$–trees and affine $\mathbb{R}$–buildings. For $\mathrm{SL}_2(\mathbb{K})$, we expect the Bruhat–Tits building to be isomorphic to the segment completion tree associated to $\mathbb{K}$; a question which will be addressed in upcoming work. Theorem 1 would then follow from [BT72, Theorem 7.5.3], since the field of real Puiseux series is not spherically complete. However this isometry is in our knowledge not known.

After introducing the necessary background in Section 2, we prove Theorem 1 in Section 3. We thank Anne Parreau for her support and for pointing us to the right chapter in [BT72]. We are thankful to Marc Burger, Anne Parreau and Beatrice Pozzetti for constructive discussions and feedback. We appreciate the valuable feedback of an anonymous referee.

2. Preliminaries

An ordered field $\mathbb{K}$ is called *real closed* if every positive element is a square and every polynomial of odd degree has a root. It is *non-Archimedean* if there is an element that is larger than any $n \in \mathbb{N} \subseteq \mathbb{K}$. For an ordered abelian group Λ, a Λ–*valuation* of $\mathbb{K}$ is a map $v\colon \mathbb{K} \to \Lambda \cup \{\infty\}$, that satisfies $v(a) = \infty$ if and only if $a = 0$, $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$ with equality if $v(a) \neq v(b)$, for all $a, b \in \mathbb{K}$. It is *order-compatible* if for $0 \leq a \leq b$ one has $\nu(b) \leq \nu(a)$. For the remainder of this article we work with the following valued real closed field $\mathbb{F}$, where $\Lambda = \mathbb{Q}$. The field of *real Puiseux series* is the set

$$\mathbb{F} := \left\{ \sum_{k=-\infty}^{k_0} c_k X^{\frac{k}{m}} \;\middle|\; k_0, m \in \mathbb{Z},\, m > 0,\, c_k \in \mathbb{R},\, c_{k_0} \neq 0 \right\},$$

with the order such that X is larger than any real number. For $a \in \mathbb{F}$ denote by $\mathbb{F}_{>a}$ and $\mathbb{F}_{\geq a}$ the elements of $\mathbb{F}$ that are larger and larger or equal to a in this order respectively. The real Puiseux series form a non-Archimedean, real closed extension of the ordered field of rational functions in one variable $\mathbb{R}(X)$ with the compatible order [BCR98, Example 1.3.6.b)]. The *logarithm*

$$\log\colon \mathbb{F}_{>0} \to \mathbb{Q}, \qquad \sum_{k=-\infty}^{k_0} c_k X^{\frac{k}{m}} \mapsto \frac{k_0}{m}$$

is order preserving, and $-\log|\cdot|$ is an order-compatible $\mathbb{Q}$–valuation of $\mathbb{F}$ by setting $\log(0) := -\infty$, where $|a| := \max\{a, -a\}$ denotes the $\mathbb{F}$–valued absolute value on $\mathbb{F}$. In particular we have for all $a, b \in \mathbb{F}_{>0}$, $\log(ab) = \log(a) + \log(b)$ and $\log(a+b) \leq \max\{\log(a), \log(b)\}$ with equality if $\log(a) \neq \log(b)$.

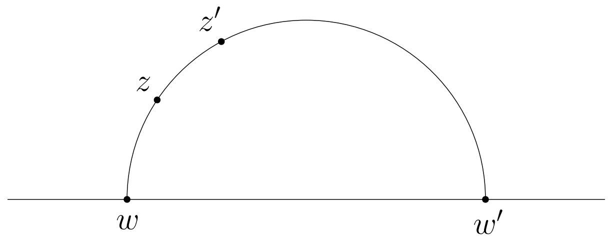

FIGURE 1. Two points z and z' in $\mathbb{H}_{\mathbb{F}}$ determine a quadruple (w, z, z', w'), to which a cross-ratio $\mathrm{CR}(w, z, z', w') \in \mathbb{F}_{\geq 1}$ can be associated.

Following the construction in [Bru88b], we define the *non-Archimedean hyperbolic plane over* $\mathbb{F}$ as the set

$$\mathbb{H}_{\mathbb{F}} := \{x + iy \mid x, y \in \mathbb{F}, y > 0\} \subseteq \mathbb{F}[i],$$

where i is such that $i^2 = -1$. Mimicking the real case, there is a pseudo-distance d defined on $\mathbb{H}_{\mathbb{F}}$ as follows. Given $z = x+iy$ and $z' = x'+iy'$ in $\mathbb{H}_{\mathbb{F}}$, consider the unique $\mathbb{F}$*–line* passing through them, that is, either the vertical ray going through these two points if $x = x'$, or the half-circle through z and z' whose center is on the "real axis" $\{y = 0\}$. Denote the endpoints of the $\mathbb{F}$–line by w and w' such that w, z, z', w' appear in this order on the $\mathbb{F}$–line, see Figure 1. Since $\mathbb{F}$ is real closed, we can define the cross-ratio CR on $\mathbb{H}_{\mathbb{F}}$ analogously as for the real hyperbolic plane by

$$\mathrm{CR}(w, z, z', w') := \frac{\|z - w'\|_{\mathbb{F}}}{\|z - w\|_{\mathbb{F}}} \cdot \frac{\|z' - w\|_{\mathbb{F}}}{\|z' - w'\|_{\mathbb{F}}} \in \mathbb{F}_{\geq 1},$$

where $\|u\|_{\mathbb{F}} := \sqrt{u\bar{u}} \in \mathbb{F}_{\geq 0}$ for $u = u_1 + iu_2 \in \mathbb{F}[i]$ and $\bar{u} = u_1 - iu_2$. Note that $u\bar{u} \geq 0$ and since $\mathbb{F}$ is real closed this element has a positive square root in $\mathbb{F}$. Brumfiel [Bru88b, Equation (18)] showed that

$$d(z, z') := \log \mathrm{CR}(w, z, z', w') \in \mathbb{R}_{\geq 0}$$

is a pseudo-distance on $\mathbb{H}_{\mathbb{F}}$. Note that two points can have distance zero because log is not injective. By the properties of log, we obtain the following description of the pseudo-distance, which we use from now on, see [Bru88b, Equation (19)].

Remark 2.1. Let $z = x + iy$ and $z' = x' + iy'$ be points in $\mathbb{H}_{\mathbb{F}}$, then we have

$$\begin{aligned} d(z, z') &= \log\left(\frac{(x - x')^2 + y^2 + {y'}^2}{yy'}\right) \\ &= \max\left\{\log\left(\frac{(x - x')^2}{yy'}\right), \log\left(\frac{y}{y'}\right), \log\left(\frac{y'}{y}\right)\right\}. \end{aligned}$$

Consider the equivalence relation on $\mathbb{H}_{\mathbb{F}}$ which identifies z with z' if and only if $d(z, z') = 0$ and denote by π the projection

$$\pi : \mathbb{H}_{\mathbb{F}} \to \mathbb{H}_{\mathbb{F}}/\{d = 0\} =: \mathcal{T}_{\mathbb{F}}.$$

In [Bru88b, Theorem (28)], Brumfiel showed that $\mathcal{T}_{\mathbb{F}}$ is a Λ–tree, with $\Lambda = \mathbb{Q}$. We now define Λ–trees following [Chi01].

Let Λ be an ordered abelian group. A set X together with a function $d\colon X \times X \to \Lambda$ is a Λ*–metric space* if d is positive definite, symmetric and satisfies the triangle inequality. The group Λ is itself a Λ–metric space by defining $d(a,b) := |b-a| := \max\{a-b, b-a\}$. A *segment* s is the image of a Λ–isometric embedding $\varphi\colon [a,b]_\Lambda = \{t \in \Lambda\colon a \le t \le b\} \to X$ for some $a \le b \in \Lambda$, and the set of *endpoints of* s is $\{\varphi(a), \varphi(b)\}$. We call the image of an isometric embedding of Λ a Λ*–geodesic*. A Λ*–tree* is a Λ–metric space satisfying the following three properties:

(a) For all $x, y \in X$ there is a segment s with endpoints x, y.
(b) For all segments s, s' whose intersection $s \cap s' = \{x\}$ consists of one common endpoint x of both segments, the union $s \cup s'$ is a segment.
(c) For all segments s, s' with a common endpoint x, the intersection $s \cap s'$ is a segment with x as one of its endpoints.

By [Chi01, Lemma II.1.1], segments in a Λ–tree are unique. We write $[x,y]$ for the unique segment with endpoints $x, y \in X$. We use the following statement later.

Lemma 2.2. *Let X be a Λ–tree and $x, y, z \in X$. Then $[y,z] \subseteq [x,y] \cup [x,z]$.*

Proof. By axiom (c) of Λ–trees there is a point $r \in X$ with $[x,r] = [x,y] \cap [x,z]$. Since then $[r,y] \cap [r,z] = \{r\}$, we have by axiom (b) (and uniqueness of segments) that $[y,z] = [y,r] \cup [r,z] \subseteq [x,y] \cup [x,z]$. □

3. Proof of Theorem 1

This section is concerned with the proof of Theorem 1, which states that the segment completion tree $\mathcal{T}_\mathbb{F}^{\mathrm{sc}}$ associated to the field of real Puiseux series $\mathbb{F}$, is not metrically complete. We construct an explicit Cauchy sequence $(\pi(p_n))_{n\in\mathbb{N}}$ in $\mathcal{T}_\mathbb{F}$, whose isometric embedding does not converge in $\mathcal{T}_\mathbb{F}^{\mathrm{sc}}$.

Let us first define a sequence of rational exponents t_n to define points $p_n := a_n + ib_n \in \mathbb{H}_\mathbb{F}$ whose projections $\pi(p_n) \in \mathcal{T}_\mathbb{F}$ will form the Cauchy sequence that does not converge. Let $t_0 := 0 \in \mathbb{Q}$, $a_0 := 0$, $b_0 := 1 \in \mathbb{F}$ and for $n \geq 1$ define

$$\begin{aligned} t_n &:= \sum_{k=1}^{n} -\frac{1}{2^k} = -\frac{2^n - 1}{2^n} = -1 + \frac{1}{2^n} \in \mathbb{Q}, \\ a_n &:= \sum_{k=1}^{n} X^{t_k} = X^{t_1} + X^{t_2} + \cdots + X^{t_n} \in \mathbb{F}_{>0}, \\ b_n &:= X^{t_n} \in \mathbb{F}_{>0}. \end{aligned}$$

We note that t_n is a monotonically decreasing sequence converging to -1, a_n is monotonically increasing and b_n is monotonically decreasing. Figure 2 contains a schematic picture of the sequence $(p_n = a_n + ib_n)_{n\in\mathbb{N}}$ in $\mathbb{H}_\mathbb{F}$.

Lemma 3.1. *The sequence $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}_\mathbb{F}$.*

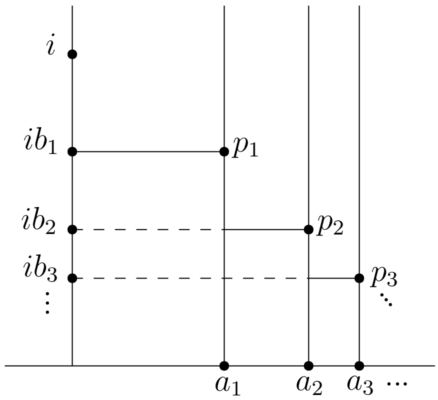

FIGURE 2. The sequence $(p_n)_{n\in\mathbb{N}}$ in the upper half plane $\mathbb{H}_\mathbb{F}$.

Proof. Let $n \leq m$. Then $t_m \leq t_n$, and, using the distance formula from Remark 2.1, we obtain

$$\begin{aligned} d(p_n, p_m) &= \log\left(\frac{(a_m - a_n)^2 + b_n^2 + b_m^2}{b_n b_m}\right) \\ &= \log\left(\left(\sum_{k=n+1}^{m} X^{t_k}\right)^2 + \left(X^{t_n}\right)^2 + \left(X^{t_m}\right)^2\right) - \log\left(X^{t_n}X^{t_m}\right) \\ &= 2t_n - (t_n + t_m) = t_n - t_m \to 0 \qquad (\text{as } n, m \to \infty) \end{aligned}$$

and thus $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

The following lemma is a general fact about the distance function in $\mathbb{H}_\mathbb{F}$.

Lemma 3.2. *For every x, x' in $\mathbb{F}$, there exists $y \in \mathbb{F}_{>0}$ such that*

$$d\left(x + iy, x' + iy\right) = 0.$$

Furthermore, for all $y \in \mathbb{F}_{>0}$ with $d(x + iy, x' + iy) = 0$ and all $t \geq 0$, it holds that $d(x + i(y + t), x' + i(y + t)) = 0$.

Proof. Define $y := X^{\log|x-x'|} > 0$ so that, using Remark 2.1, we obtain

$$d(x + iy, x' + iy) = \max\left\{\log\left(\left(x - x'\right)^2\right) - \log X^{2\log|x-x'|},\, 0\right\} = 0.$$

The second equality follows from $\log(y + t) \geq \log(y)$ and that hence the expression $\log((x - x')^2) - \log(X^{2\log|x-x'|} + t)$ is negative. □

Intuitively Lemma 3.2 tells us that two vertical $\mathbb{F}$–lines in $\mathbb{H}_\mathbb{F}$ are identified from some point on in $\mathcal{T}_\mathbb{F}$, forming an infinite tripod. In Lemma 3.3, we refine this statement for the sequence of vertical $\mathbb{F}$–lines $\ell_n := a_n + i\mathbb{F}_{>0}$. The situation is illustrated in Figure 3.

Lemma 3.3. *Let $n \in \mathbb{N}$, $b \in \mathbb{F}_{>0}$. Consider the vertical $\mathbb{F}$–lines ℓ_n, ℓ_{n+1}.*

(i) *If $\log(b) \geq \log(b_{n+1}) = t_{n+1}$, then $d(a_n + ib, a_{n+1} + ib) = 0$.*
(ii) *If $\log(b) < \log(b_{n+1}) = t_{n+1}$, then all points in ℓ_{n+1} have non-zero distance to $a_n + ib$.*

Proof.

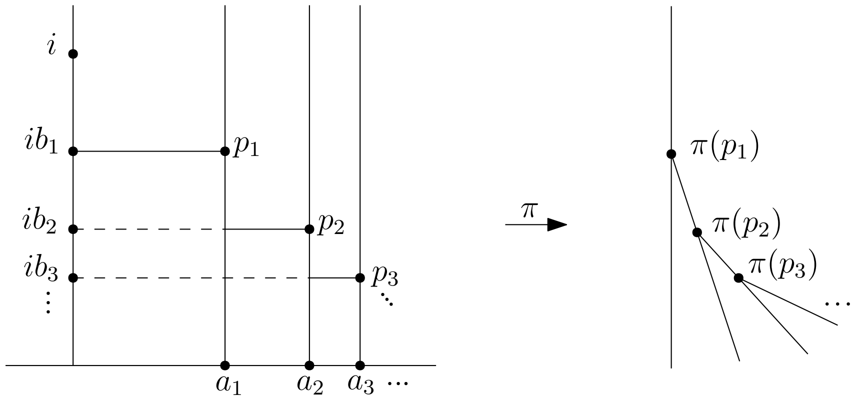

FIGURE 3. The points $p_n \in \ell_n \subseteq \mathbb{H}_{\mathbb{F}}$ correspond to branching points $\pi(p_n)$ of $\pi(\ell_{n-1})$ and $\pi(\ell_n)$ in $\mathcal{T}_{\mathbb{F}}$.

(i) If $\log(b) = \log(b_{n+1})$, we use $\log(a_{n+1} - a_n) = \log(b_{n+1})$ to see that

$$d(a_n + ib, a_{n+1} + ib) = \max\left\{\log\left(\frac{(a_{n+1} - a_n)^2}{b_{n+1}^2}\right), 0\right\} = 0.$$

If $\log(b) > \log(b_{n+1})$, then also $b > b_{n+1}$ and the statement follows from the second part of Lemma 3.2.

(ii) Let $a_{n+1} + ib'$ be a point in ℓ_{n+1} for some $b' \in \mathbb{F}_{>0}$. First, assume $b' < b_{n+1} = X^{t_{n+1}}$, in which case $\log(bb') < 2t_{n+1}$ holds. Then

$$\begin{aligned} d(a_n + ib, a_{n+1} + ib') &= \log\left(\frac{(a_{n+1} - a_n)^2 + b^2 + b'^2}{bb'}\right) \\ &= \log\left(X^{2t_{n+1}} + b^2 + b'^2\right) - \log(bb') \\ &= \log\left(X^{2t_{n+1}}\right) - \log(bb') > 0. \end{aligned}$$

Second, assume $b' \geq b_{n+1} = X^{t_{n+1}}$. We use (i) to see that $d(a_n + ib',$ $a_{n+1} + ib') = 0$. Then, using $d(a_n + ib, a_n + ib') = \log(b') - \log(b) > 0$, we conclude

$$\begin{aligned} d\left(a_n + ib, a_{n+1} + ib'\right) &= d\left(a_n + ib, a_{n+1} + ib'\right) + d\left(a_n + ib', a_{n+1} + ib'\right) \\ &\geq d\left(a_n + ib, a_n + ib'\right) > 0. \end{aligned}$$

□

Lemma 3.3 implies that for all $n \in \mathbb{N}$ the $\mathbb{Q}$–geodesics $\pi(\ell_n)$ and $\pi(\ell_{n+1})$ in $\mathcal{T}_{\mathbb{F}}$ branch off at the point $\pi(p_{n+1})$. By Lemma 3.1, $(p_n)_{n\in\mathbb{N}}$ and hence also $(\pi(p_n))_{n\in\mathbb{N}}$ are Cauchy sequences. The next proposition shows that $(\pi(p_n))_{n\in\mathbb{N}}$ does not converge in the $\mathbb{R}$–tree $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$. Intuitively, the sequence $(\pi(p_n))_{n\in\mathbb{N}}$ does not stay in $\pi(\ell_m)$ for any $m \in \mathbb{N}$, and hence the limit point is not contained in the completion of any of the $\mathbb{Q}$–geodesics $\pi(\ell_m)$. In fact, the following proposition shows that it is not contained in the completion of any $\mathbb{Q}$–geodesic in $\mathcal{T}_{\mathbb{F}}$. The sequence $(\pi(p_n))_{n\in\mathbb{N}} \subseteq \mathcal{T}_{\mathbb{F}}$ is a Cauchy sequence by Lemma 3.1 and thus Theorem 1 follows from the following proposition.

Proposition 3.4. *The sequence $(\pi(p_n))_{n\in\mathbb{N}}$ does not converge in $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$.*

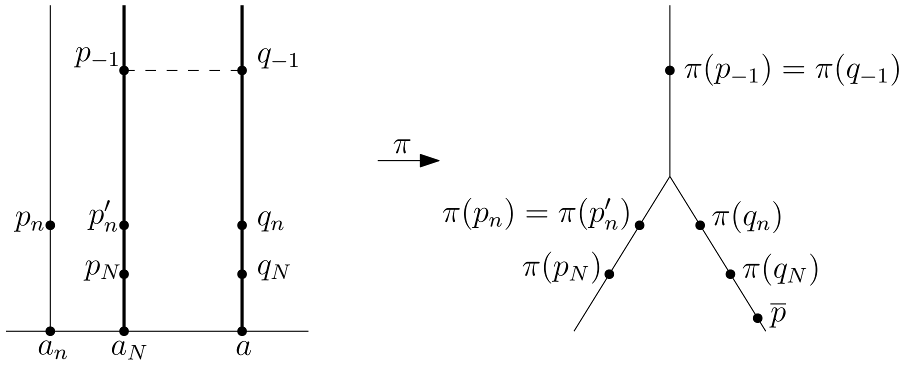

FIGURE 4. The setup to prove $\pi(p_n) = \pi(q_n)$ in Step 2. The tripod on the right is the image under π of the two vertical lines in bold on the left.

Proof. We assume by contradiction that $\pi(p_n)$ converges to some $\overline{p} \in \mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$. Since $\mathcal{T}_{\mathbb{F}}^{\mathrm{sc}}$ is by definition the union of the completions of the segments of $\mathcal{T}_{\mathbb{F}}$, we can find a segment s in $\mathcal{T}_{\mathbb{F}}$, such that $\overline{p}$ lies in its completion $\overline{s}$.

Step 1. Our first goal is to show that we may assume that s is contained in the image of a vertical $\mathbb{F}$–line ℓ of $\mathbb{H}_{\mathbb{F}}$. The segment s has two endpoints in the tree $\mathcal{T}_{\mathbb{F}}$. Choose preimages $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in \mathbb{H}_{\mathbb{F}}$ so that $\pi(z_1)$ and $\pi(z_2)$ are the endpoints of s. By Lemma 3.2, there is a large enough $y \in \mathbb{F}_{>0}$ with $d(x_1 + iy, x_2 + iy) = 0$, hence $\pi(x_1 + iy) = \pi(x_2 + iy) \in \mathcal{T}_{\mathbb{F}}$. Lemma 2.2 implies that $s = [\pi(z_1), \pi(z_2)] \subseteq [\pi(x_1 + iy), \pi(z_1)] \cup [\pi(x_2 + iy), \pi(z_2)]$. We conclude that at least one of the completions of the segments $[\pi(z_1), \pi(x_1 + iy)]$ or $[\pi(z_2), \pi(x_2 + iy)]$ has to contain $\overline{p}$. Without loss of generality $\overline{p}$ lies in the completion of $[\pi(z_1), \pi(x_1 + iy)]$. Set $a := x_1 \in \mathbb{F}$ so that the vertical $\mathbb{F}$–line $\ell := a + i\mathbb{F}_{>0}$ is such that $\overline{p}$ lies in the completion of $\pi(\ell)$.

Step 2. We define points $q_n = a + iX^{t_n} \in \ell \subseteq \mathbb{H}_{\mathbb{F}}$ and claim that $d(p_n, q_n) = 0$ for all $n \in \mathbb{N}$, see Figure 4. To show this, fix n and let $N \in \mathbb{N}$ be large enough so that $d(\pi(p_N), \overline{p}) < d(\pi(p_n), \overline{p})$. Consider the vertical $\mathbb{F}$–line $\ell_N = a_N + i\mathbb{F}_{>0} \subseteq \mathbb{H}_{\mathbb{F}}$ that contains both p_N and the point $p'_n = a_N + iX^{t_n}$. By Lemma 3.3 (i), $d(p_n, p'_n) = 0$ and hence $\pi(p_n) = \pi(p'_n) \in \pi(\ell_N)$. By Lemma 3.2 we can find points $p_{-1} = a_N + ib \in \ell_N$ and $q_{-1} = a + ib \in \ell$ with $d(p_{-1}, q_{-1}) = 0$. We distinguish two cases, and show that in fact the second case cannot occur.

Case 1: $\pi(p_n) \in \pi(\ell)$. Then there exists $y \in \mathbb{F}_{>0}$ such that $\pi(a + iy) = \pi(p_n) \in \pi(\ell)$. But this means nothing else than $d(p_n, a + iy) = 0$. We use $p_n = a_n + iX^{t_n}$ and Remark 2.1 to get

$$0 = d(p_n, a + iy) = \max\left\{\log\left(\frac{(a - a_n)^2}{X^{t_n} y}\right), t_n - \log(y), \log(y) - t_n\right\}.$$

Hence $\log y = t_n$ so that we conclude

$$d(p_n, q_n) = \max\left\{\log\left(\frac{(a - a_n)^2}{X^{2t_n}}\right), 0\right\} = d(p_n, a + iy) = 0.$$

Case 2: $\pi(p_n) \in \pi(\ell_N) \setminus \pi(\ell)$. The goal is to show that in fact this case cannot occur. Observe that $\pi(p_N) \in \pi(\ell_N) \setminus \pi(\ell)$, since otherwise

$\pi(p_N) = \pi(q_N)$ as in Case 1, and the second part of Lemma 3.2 would then imply that $\pi(p_n) = \pi(q_n) \in \pi(\ell)$. By Lemma 2.2 we have for every $r \in \pi(\ell)$, $[\pi(p_N), \pi(p_{-1})] \subseteq [r, \pi(p_N)] \cup [r, \pi(p_{-1})]$. Since $\pi(p_n) \in [\pi(p_N), \pi(p_{-1})] \setminus \pi(\ell)$ and $[r, \pi(p_{-1})] \subseteq \pi(\ell)$, we have $\pi(p_n) \in [\pi(p_N), r]$ and hence

$$d(\pi(p_N), \pi(p_n)) + d(\pi(p_n), r) = d(\pi(p_N), r).$$

By density of $\mathbb{Q}$ in $\mathbb{R}$, choose $r \in \pi(\ell)$ close to $\overline{p}$ in the sense that $d(r, \overline{p}) < d(p_n, p_N)$. Without loss of generality we may assume that $d(\pi(p_N), r) < d(\pi(p_N), \overline{p})$. We then have

$$\begin{aligned} d(\pi(p_n), \overline{p}) &\leq d(\pi(p_n), r) + d(r, \overline{p}) < d(\pi(p_n), r) + d(\pi(p_n), \pi(p_N)) \\ &= d(\pi(p_N), r) < d(\pi(p_N), \overline{p}) < d(\pi(p_n), \overline{p}), \end{aligned}$$

where the last inequality comes from the choice of N. This is a contradiction.

Step 3. Next we would like to get a bound on $\log|a - a_n|$. We use Remark 2.1 to obtain

$$\begin{aligned} 0 = d(p_n, q_n) &= \max\left\{\log\left(\frac{(a - a_n)^2}{X^{2t_n}}\right), 0\right\} \\ &= \max\left\{2\left(\log|a - a_n| - t_n\right), 0\right\}, \end{aligned}$$

which implies $\log|a - a_n| \leq t_n$.

Step 4. Recall that $a_n = \sum_{j=1}^{n} X^{t_j}$ and $t_j = (1 - 2^j)/2^j$ is monotonically decreasing in j. By definition of $\mathbb{F}$, there exist $m \in \mathbb{N}$, $k_0 \in \mathbb{Z}$ and $c_k \in \mathbb{R}$ such that $a \in \mathbb{F}$ can be written as

$$a = \sum_{k=-\infty}^{k_0} c_k X^{\frac{k}{m}}.$$

Let us also write for every $n \in \mathbb{N}$

$$a - a_n = \sum_{k=-\infty}^{k_{0,n}} d_{k,n} X^{\frac{k}{m_n}},$$

for $m_n \in \mathbb{N}$, $k_{0,n} \in \mathbb{Z}$ and $d_{k,n} \in \mathbb{R}$, such that $k_{0,n} := \max\{k \in \mathbb{Z} \colon d_{k,n} \neq 0\}$. From Step 3 we obtain that

$$\frac{k_{0,n}}{m_n} = \log|a - a_n| \leq t_n.$$

Thus for every $n \in \mathbb{N}$ we have

$$a - a_n = \sum_{k=-\infty}^{k_0} c_k X^{\frac{k}{m}} - \sum_{j=1}^{n} X^{t_j} = \sum_{k=-\infty}^{k_{0,n}} d_{k,n} X^{\frac{k}{m_n}},$$

where $k_{0,n}/m_n$ is less or equal than t_n. Since the sequence $(t_n)_{n \in \mathbb{N}}$ is monotonically decreasing, it means that the series expansion for a contains the terms X^{t_j} for all $j \in \{1, \ldots, n-1\}$ which cancel the terms X^{t_j} in a_n. In other words $mt_j \in \mathbb{Z}$, which is equivalent to $m/2^j \in \mathbb{Z}$ for all $j \in \{1, \ldots, n-1\}$ by definition of t_j. As this holds for every $n \in \mathbb{N}$, we obtain a contradiction, since there is no $m \in \mathbb{Z}$ such that $m/2^j \in \mathbb{Z}$ for all $j \in \mathbb{N}$. ☐

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2023	"An incomplete real tree with complete segments." Joint with R. Appenzeller, L. De Rosa and X. Flamm. *Innovations in Incidence Geometry: Algebraic, Topological and Combinatorial.*

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2024	"The real spectrum and the oriented Gromov–Hausdorff compactifications of character varieties."