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Fundamentals of Regularization Theory, with an Outlook to Triple Collision

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Abstract

What is regularization? Why is it useful for studying planetary systems? Although collisions in evolved planetary systems are rare events, the theory of regularization (i.e. the removal of collision singularities) is an important tool for developing good perturbation theories and efficient numerical algorithms. We give an overview of these techniques, including a short discussion of triple collision.

Outline

1. Introduction
2. Perturbation theories
3. Kepler motion
4. Levi-Civita regularization
5. Kustaanheimo-Stiefel (KS) regularization with quaternions
6. The perturbed Kepler problem
7. Appendices
    - Appendix I. Regularizing the restricted 3-body problem
    - Appendix II. MATLAB code for animated Kepler motions
    - Appendix III. An outlook to triple collision

1. Introduction

- A well-founded assumption: Motion of planetary systems follows universal laws: Newtonian gravitation and small perturbations
- Basic ingredients of celestial mechanics:
  - Theory of the unperturbed Kepler motion: A star and one planet
  - Theory of Kepler motion under a small perturbation
- Our Goals:
  - A simple, i.e. linear, perturbation theory of Kepler motion
  - A simple description of the unperturbed Kepler motion
- The common tool for both goals is regularization

Introduction, continued

What is Regularization ?

- Historically, regularization was developed for
  - Investigating the singularities of Kelper motion
  - Describing collisions of two point masses
  - Improving the numerical integration of (near) collision orbits
- References: Sundman 1907, Levi-Civita 1920, Kustaanheimo and Stiefel (KS) 1965, Moser 1970
- For our purpose, regularization according to Levi-Civita or KS has two extremely useful side effects:
  - The differential equations of motion become linear
  - The classical theory of Kepler motion essentially reduces to the much simpler theory of the harmonic oscillator

2. Perturbation theories

Perturbation theories of linear problems are simple !

$$\dot{x}(t) + A(t)\,x(t) - b(t) = \varepsilon f(x,t), \quad \dot{(\ )} = \frac{d}{dt}(\ ),$$

$$x : t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}^n, \quad A(t) \ \text{a time-dependent matrix}$$

Perturbation series (a formal series, to any order):

$$x(t) = x_0(t) + \varepsilon\, x_1(t) + \varepsilon^2\, x_2(t) + \dots$$

The k-th order solution $x_k(t)$ satisfies the linear differential equation

$$\dot{x_k}(t) + A(t)\,x_k(t) = f_{k-1}(t), \quad k = 0, 1, 2, \dots ,$$

where

Perturbation theories, continued

$$f_{-1}(t) := b(t),$$

and $f_0(t), f_1(t), \ldots$ are defined as the coefficients of the formal Taylor series of $f(x,t)$ with respect to ε:

$$\sum_{k=0}^{\infty} \varepsilon^k \, f_k(t) = f\Big(x_0(t) + \varepsilon\, x_1(t) + \varepsilon^2\, x_2(t) + \ldots\,,\; t\Big)\,.$$

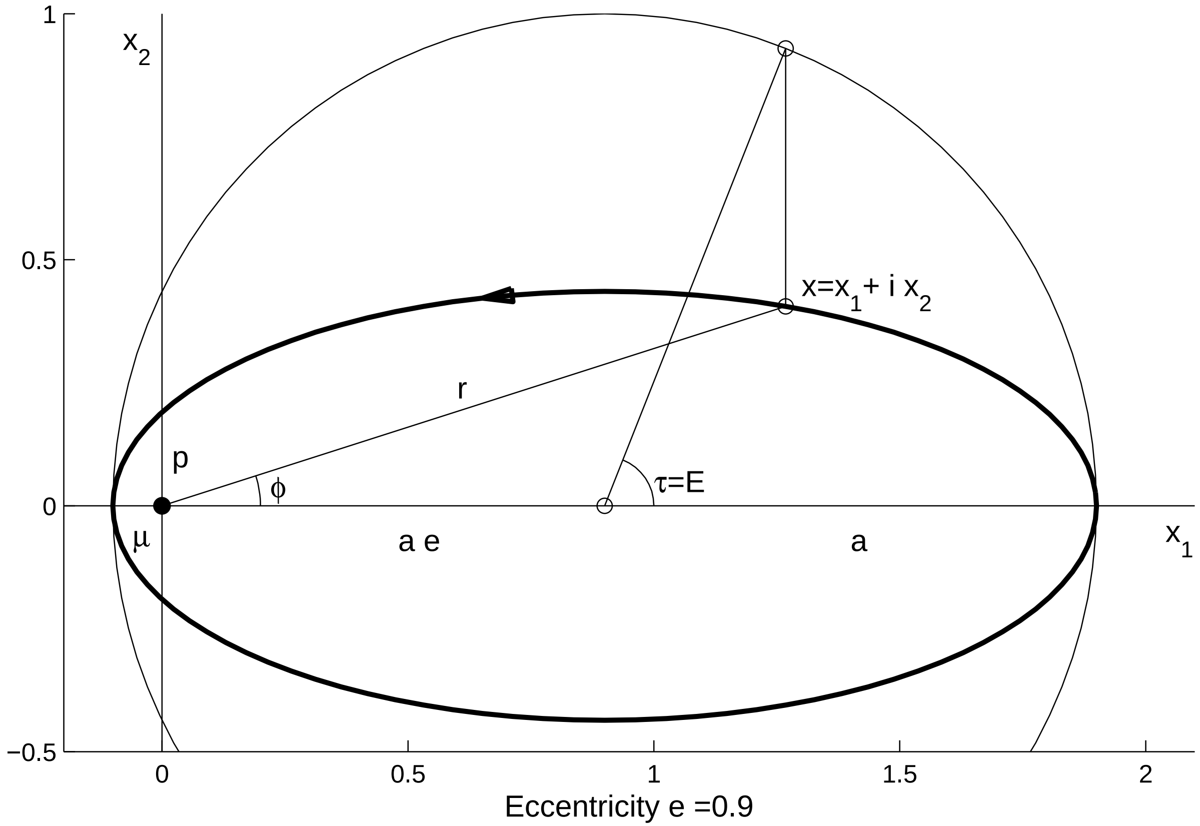
Remark:

The differential equations for $x_k(t)$ are of the type of the unperturbed problem $\varepsilon = 0$; they differ only in the right-hand sides.

3. Planar Kepler Motion

$$\ddot{x} + \mu \frac{x}{|x|^3} = 0\,, \quad x = \binom{x_1}{x_2} \in \mathbb{R}^2 \quad \text{or} \quad x = x_1 + i\,x_2 \in \mathbb{C}$$

$$\dot{(\,)} = \frac{d}{dt}(\,), \quad t = \text{time}, \quad \mu = \text{gravitational parameter}$$



Explicit solution, Kepler formulas

$a =$ major semi-axis $\quad r =$ radial distance $\quad p =$ semi latus rectum

$e =$ eccentricity $\quad \varphi =$ polar angle $\quad E =$ eccentric anomaly

$$\left.\begin{aligned} x_1 &= a\,(e + \cos E) \\ x_2 &= a\sqrt{1-e^2}\cdot \sin E \end{aligned}\right\} \Longrightarrow \quad r = |x| = a\,(1 + e\,\cos E)$$

$$t = \sqrt{\frac{a^3}{\mu}}\cdot (E + e\,\sin E)\,, \qquad \frac{dt}{dE} = \sqrt{\frac{a}{\mu}}\cdot r$$

Further explicit formulas

Orbit in polar coordinates

$$r = \frac{p}{1 - e\,\cos\varphi}\,, \qquad p = a\,(1 - e^2)$$

$$\tan(\frac{\varphi}{2}) = \sqrt{\frac{1-e}{1+e}}\,\tan(\frac{E}{2})$$

Conservation of energy: $\frac{1}{2}|\dot{x}|^2 - \frac{\mu}{r} = -h\,, \quad h = \frac{\mu}{2\,a} > 0$

Conservation of angular momentum: $|x \times \dot{x}| = \sqrt{\mu\, p}$

4. Levi-Civita regularization (planar)

Conformal squaring: $x \;=\; u^2 \in \mathbb{C}$

$$\begin{aligned} x &= x_1 + i\,x_2 = a\,\left(e + \cos E + i\,\sqrt{1-e^2}\,\sin E\right) \quad \Longleftarrow \\ u &= \sqrt{a\,(1+e)}\,\cos(\frac{E}{2}) + i\,\sqrt{a\,(1-e)}\,\sin(\frac{E}{2}) \end{aligned}$$

Regularization procedure

Step 1. Time transformation: $dt = c^{-1}\, r\, d\tau, \quad c > 0$

Case 1: $c = 1$, $\tau =$ fictitious time, Sundman transformation

Case 2: $c = \sqrt{2\,h}\,$, $\tau = E =$ eccentric anomaly

Step 2. Conformal squaring: $x \;=\; u^2 \in \mathbb{C}$

Step 3. Use the energy integral for eliminating u'

4.1. The formal regularization procedure

$$\ddot{x} + \mu \frac{x}{r^3} = 0\,, \qquad \frac{1}{2}\,|\dot{x}|^2 - \frac{\mu}{r} = -h\,, \quad r = |x|$$

Step 1:

$$\frac{d}{dt} = c\,r^{-1}\,\frac{d}{d\tau}\,, \quad \frac{d^2}{dt^2} = c^2\left(r^{-2}\,\frac{d^2}{d\tau^2} + (\frac{c'}{c}\,r - r')\,r^{-3}\,\frac{d}{d\tau}\right)\,, \quad (\;)' = \frac{d}{d\tau}(\;)$$

$$\Longrightarrow\; c^2\Big(r\,x'' + (\frac{c'}{c}\,r - r')\,x'\Big) + \mu\,x = 0\,, \qquad \frac{1}{2}\,c^2\,r^{-2}\,|x'|^2 - \frac{\mu}{r} = -h$$

Step 2:

$$x = u^2, \quad x' = 2uu', \quad x'' = 2(uu'' + u'^2), \quad r = u\bar{u}, \quad r' = u'\bar{u} + u\bar{u}'$$

$$\Longrightarrow\; c^2\Big(r{\cdot}2uu'' + \underbrace{r\cdot 2u'^2 - u'\bar{u}\cdot 2uu'}_{0} + \frac{c'}{c}\,r{\cdot}2uu' - u\bar{u}'{\cdot}2uu'\Big) + \mu u^2 = 0$$

Energy equation $\Longrightarrow\; \frac{1}{2}\,c^2\,r^{-2}\cdot 4uu'\bar{u}\bar{u}' - \frac{\mu}{r} = -h$ or $2\,c^2\,u'\,\bar{u}' = \mu - r\,h$

The formal regularization procedure, continued

Step 3: Elimination of $u' \bar{u}'$ from the last two lines and division by ru:

$$\boxed{2\,c^2\,u'' + 2\,c\,c'\,u' + h\,u = 0\,, \quad u \in \mathbb{C}}$$

- For h=const, c=const, c'=0: A harmonic oscillator in 2 dimensions, $2\,c^2\,u'' + h\,u = 0.$ Frequency $\omega := c^{-1}\sqrt{h/2}$
- All Kepler formulas may be conveniently derived from the above ODE and the transformation rules
- Initial conditions from a complex square root, e.g.

$$u = \sqrt{x} = \frac{x + |x|}{\sqrt{2\,(\mathrm{Re}\,x + |x|)}}$$

4.2. A natural derivation of the Kepler formulas

1. **Natural choice** $c = \sqrt{2\,h}$ (p. 11, Case 2)
   (a) The frequency of the oscillator of p.13 becomes independent of h
   (b) $u'' + \frac{1}{4}\,u = 0 \quad \Longrightarrow \quad u = A\,\cos(\frac{E}{2}) + i\,B\,\sin(\frac{E}{2})$
   (c) We can hope that E is an angle associated with Kepler motion
2. **Orbit** $x = u^2 = \frac{1}{2}\,(A^2 - B^2) + \frac{1}{2}\,(A^2 + B^2)\,\cos E + i\,A\,B\,\sin E$
   (a) Figure p. 8 (geometry of the ellipse) implies for

$$\begin{aligned} E = 0: &\quad x = A^2 &= a(1+e), &\quad A = \sqrt{a(1+e)} \\ E = \pi: &\quad x = -B^2 &= -a(1-e), &\quad B = \sqrt{a(1-e)} \end{aligned}$$

   (b) $x_1 = a\,(e + \cos E), \quad x_2 = a\,\sqrt{1-e^2}\cdot\sin E$
   follow from (2) and (2a). Furthermore: $r = |x| = a\,(1 + e\,\cos E)$
   (c) Geometric meaning of E is easily inferred from the figure on p. 8

3. **Energy integral**

(a) (1b) implies $4\,|u'|^2 = a\,(1 - e\cos E)$

(b) Combine this with the last equation of p. 12, $2\,c^2\,|u'|^2 = \mu - r\,h$

(c) Use r from (2b) to obtain $\quad 2\,h\,a = \mu, \quad c = \sqrt{\mu/a}$

h (and therefore a) are integrals of motion, independent of E

4. **Time**

(a) P. 11, Case 2 implies $\quad dt = \sqrt{\frac{a}{\mu}}\,a\,(1 + e\cos E)\cdot dE$

(b) Integrate: $t - t_0 = \sqrt{\frac{a^3}{\mu}}\,(E + e\sin E)$ (Kepler's equation)

(c) Kepler's third law follows immediately: $\omega^2\,a^3 = \mu$, where $\omega = 2\,\pi/T$ is the average angular velocity, T=revolution period

5. **Polar coordinates** $r, \varphi: \; x = r\, e^{i\varphi}, \; u = \sqrt{r}\, e^{i\varphi/2}$

   (a) From (1b) and (2a) the classical relation of p. 10 follows directly:

   $$\tan(\tfrac{\varphi}{2}) = \sqrt{\tfrac{1-e}{1+e}}\, \tan(\tfrac{E}{2})$$

   (b) Substitution of this into $r = a\,(1 + e\,\cos E)$ of (2b) by using the identity $\cos E = \frac{1-\tan^2(E/2)}{1+\tan^2(E/2)}$ yields $r = p/(1 - e\,\cos\varphi)$, where $p = a\,(1 - e^2)$ is the semi-latus rectum.

6. **Angular momentum integral**

   (a) From (2) there follows $|x \times \frac{dx}{dE}| = \sqrt{a\,p}\; r$

   (b) With the time defined by $r\,dE = \sqrt{\mu/a}\,dt$ the constance of the angular momentum is obtained: $|x \times \dot{x}| = \sqrt{\mu\,p}$

5. The Kustaanheimo-Stiefel (KS) regularization (spatial)

$$\ddot{x} + \mu \frac{x}{r^3} = 0\,, \qquad \frac{1}{2}\,|\dot{x}|^2 - \frac{\mu}{r} = -h\,, \quad r = |x|\,, \quad x \in \mathbb{R}^3$$

Step 2: The KS transformation (Hopf map)

$$u = (u_0, u_1, u_2, u_3)^T \in \mathbb{R}^4 \mapsto x = (x_0, x_1, x_2)^T \in \mathbb{R}^3$$

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 - u_2^2 + u_3^2 \\ x_1 &= 2\,(u_0\,u_1 - u_2\,u_3) \\ x_2 &= 2\,(u_0\,u_2 + u_1\,u_3) \end{aligned}$$

with the bilinear differential relation

$$u_3\,du_0 - u_2\,du_1 + u_1\,du_2 - u_0\,du_3 = 0$$

5.1. A few references

1. W.R. Hamilton, 1844, Philosophical Magazine **25**, 489-495.
2. T. Levi-Civita, 1920, Acta Math. **42**, 99-144.
3. H. Hopf, 1931, Math. Ann. **104**; Selecta Heinz Hopf 38-63, 1964.
4. P. Kustaanheimo, 1964, Ann. Univ. Turku, Ser. AI **73**.
5. P. Kustaanheimo, E. Stiefel, 1965, J. R. Ang. Math. **218**, 204-219.
6. E. Stiefel, G. Scheifele, 1971, Springer, 301 pp.
7. A. Deprit, A. Elipe, S. Ferrer, 1994, Cel.Mech. and DA **58**, 151-201.
8. M. D. Vivarelli, 1983, Cel. Mech **29**, 45-50.
9. J. Vrbik, 1994, Can. J. Phys. **72**, 141-146.
10. J. Waldvogel, 2006, Cel. Mech. and Dyn. Astr. **95**, 201-212.

5.2. Quaternions

W. R. Hamilton (1844): On quaternions, or a new system of imaginaries in algebra. Philos. Mag. **25**, 489-495.

Three independent imaginary units, i, j, k, satisfying

$$i^2 = j^2 = k^2 = -1$$

$$i\,j = -j\,i = k, \quad j\,k = -k\,j = i, \quad k\,i = -i\,k = j\,.$$

The object $u = u_0 + i\,u_1 + j\,u_2 + k\,u_3$ with $u_l \in \mathbb{R}$ is called a quaternion, $u \in \mathbb{U}$. The above multiplication rules and vector space addition define the quaternion algebra:

- Multiplication is non-commutative in general, but $u\,c = c\,u \;\forall\; c \in \mathbb{R}$
- Multiplication is associative, $(u\,v)\,w = u\,(v\,w)$

5.3. Miscellaneous definitions and properties

Conjugation:	$\bar{u} = u_0 - i\, u_1 - j\, u_2 - k\, u_3$
	$\bar{\bar{u}} = u$
Real quaternion:	A quaternion $u \in \mathbb{U}$ is real, $u \in \mathbb{R}$,
	if and only if $u = \bar{u}$
Modulus $\lvert u\rvert$:	$u\, \bar{u} = \bar{u}\, u = \lvert u\rvert^2 = \sum_{l=0}^{3} u_l^2$
	u commutes with $\bar{u}$
Conjugation of a product:	$\overline{u\, v} = \bar{v}\, \bar{u}$

5.4. KS regularization with quaternions

a) Preliminaries

Let $u = u_0 + i\,u_1 + j\,u_2 + k\,u_3$

Definition: $\boxed{u^\star = u_0 + i\,u_1 + j\,u_2 - k\,u_3}$ "star conjugation"

We have: $u^\star = k^{-1}\,\bar{u}\,k = -k\,\bar{u}\,k$

Properties: $(u^\star)^\star = u$

$|u^\star|^2 = |u|^2$

$(u\,v)^\star = v^\star\,u^\star$

KS regularization, continued

b) The KS transformation

Consider the map $\boxed{x = u\,u^\star}$ with $u \in \mathbb{U}$

Due to $x^\star = (u^\star)^\star\, u^\star = x$ we identically have $x_3 = 0$; therefore the quaternion $x = x_0 + i\,x_1 + j\,x_2$ may be associated with a vector $\in \mathbb{R}^3$.

In components:

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 - u_2^2 + u_3^2 \\ x_1 &= 2\,(u_0\,u_1 - u_2\,u_3) \\ x_2 &= 2\,(u_0\,u_2 + u_1\,u_3) \end{aligned}$$

The KS transformation or Hopf map (p. 17) !

Modulus: $|x| = \sqrt{x\,\bar{x}} = \sqrt{u\,(u^\star\,\bar{u^\star})\,\bar{u}} = \sqrt{|u^\star|^2\,|u|^2} = |u|^2$

KS regularization, continued

c) Fibration instead of inverse map

Find all u with $u\,u^\star = x = x_0 + i\,x_1 + j\,x_2$

First step: Particular solution v with $v = v^\star = v_0 + i\,v_1 + j\,v_2$, in analogy to p. 13 for the complex case,

$$v = \frac{x + |x|}{\sqrt{2\,(x_0 + |x|)}}\,.$$

Second step: All solutions u of $u\,u^\star = x$ are given by

$$u = v\,e^{k\,\varphi} = v\,(\cos\varphi + k\,\sin\varphi)\,, \quad \varphi \in \mathbb{R}$$

Proof (sketch): $u\,u^\star = v\,e^{k\,\varphi}\,e^{-k\,\varphi}\,v^\star = v\,v^\star = x$ □

KS regularization, continued

d) Differentiation

We have

$$dx = du\ u^\star + u\ du^\star\ .$$

The bilinear relation of KS (p. 17),

$$2\,(u_3\,du_0 - u_2\,du_1 + u_1\,du_2 - u_0\,du_3) = 0\,,$$

may be written as the commutator relation

$$u\ du^\star - du\ u^\star = 0\ ;$$

therefore

$$\boxed{dx = 2\ u\ du^\star = 2\ du\ u^\star}$$

Remark: With the above bilinear relation the tangential map of $u \mapsto u\,u^\star$ behaves like a map in a commutative algebra.

KS regularization, continued

e) The formal regularization procedure

Step 1 yields, as on p. 12 $(c = 1):\quad r\,x'' - r'\,x' + \mu\,x = 0$ (*)

Energy relation, as on p. 12:

$$\frac{1}{2}\,r^{-2}\cdot 4\,u\,(u^{\star\prime}\,\bar{u}^{\star\prime})\,\bar{u} - \frac{\mu}{r} = -h \qquad \Longrightarrow \qquad 2\,|u'|^2 = \mu - r\,h$$

Step 2: Differentiation yields

$$\begin{aligned} x &= u\,u^\star & r &= u\,\bar{u} \\ x' &= 2\,u\,u^{\star\prime} & r' &= u'\,\bar{u} + u\,\bar{u}' \\ x'' &= 2\,u\,u^{\star\prime\prime} + 2\,u'\,u^{\star\prime} \end{aligned}$$

The non-commutative computations

Substitution into (*) yields

$$(u\,\bar{u})\,(2\,u\,u^{\star\prime\prime}) + (u\,\bar{u})\ \cdot \underbrace{\ (2\,u\,'u^{\star\prime}) - (u'\,\bar{u}\ }_{\substack{2(u\bar{u})u'u^{\star\prime} - 2u'(\bar{u}u)u^{\star\prime} \\ = 0}} \ + \ u\,\bar{u}')\,2\,\underbrace{u\,u^{\star\prime}}_{u'\,u^{\star}} + \mu\,u\,u^{\star} \ = \ 0$$

Together with the energy relation, after left-multiplication by $r^{-1}\,u^{-1}$ and star-conjugation:

$$\boxed{2\,u'' + h\,u = 0\,, \quad u \in \mathbb{U}}$$

in perfect formal agreement with the planar case, p. 13.

6. The perturbed Kepler problem

$$\ddot{x} + \mu \frac{x}{|x|^3} = \varepsilon f(x,t)\,,$$

where $x \in \mathbb{R}^3$ is the position vector, $f : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$ is a perturbing function, and ε is a small parameter. Alternatively, the symbols $x, f \in \mathbb{U}$ will be considered as quaternions with a vanishing k-component.

Regularization

Step 1 of p. 25, again with $c = 1$: $\quad r\, x'' - r'\, x' + \mu\, x = r^3\, \varepsilon\, f(x,t)$

Energy relation as on p. 25: $\quad 2\,|u'|^2 = \mu - r\, h$

Substitution as on p. 26: $\quad \cdots = u\, \bar{u}\, r^2\, \varepsilon\, f(x,t)$

Finally: Left-multiplication by $r^{-1}\, u^{-1}$ and star conjugation yields

Summary

Equivalent regularized system:

$$\begin{aligned} 2\,u'' + h\,u &= r\,\varepsilon f(x,t)\,\bar{u}^\star\,, && r = |u|^2, \quad (\,)' = \tfrac{d}{d\tau} \\ t' &= r\,, && x = u\,u^\star \\ h' &= -\varepsilon\,\langle\, x', f(x,t)\,\rangle && \text{or} \quad h = r^{-1}\,(\mu - 2\,|u'|^2) \end{aligned}$$

Remark

Introducing the osculating eccentric anomaly E by $dE = \sqrt{2\,h}\,d\tau$ transforms the first differential equation into

$$4\,u'' + u = \frac{\varepsilon}{h}\,\Big(r\,f(x,t)\,\bar{u}^\star + 2\,\langle\, x', f(x,t)\,\rangle\,u'\Big)\,,$$

a perturbed harmonic oscillator with frequency $\omega = \frac{1}{2}$. Here $(\,)' = \frac{d}{dE}$.

Osculating elements

Model equation: $4\,u'' + u = g,\ g$ a small perturbation, $u \in \mathbb{R}$

With $v := 2\,u'$ the model equation may be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g/2 \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

A matrix solution $U(E)$ of $U' = A\,U$ (unperturbed problem):

$$U(E) = \exp(A\,E) = \begin{pmatrix} \cos(\frac{E}{2}) & \sin(\frac{E}{2}) \\ -\sin(\frac{E}{2}) & \cos(\frac{E}{2}) \end{pmatrix}$$

Variation of the constant

We seek a solution of the form

$$\begin{pmatrix} u(E) \\ v(E) \end{pmatrix} = U(E) \begin{pmatrix} \alpha(E) \\ \beta(E) \end{pmatrix} ,$$

where $\alpha(E), \beta(E)$ are the (orbital) elements. Substitute this and its derivative into the matrix differential equation of p. 29 and solve for the derivatives of the elements:

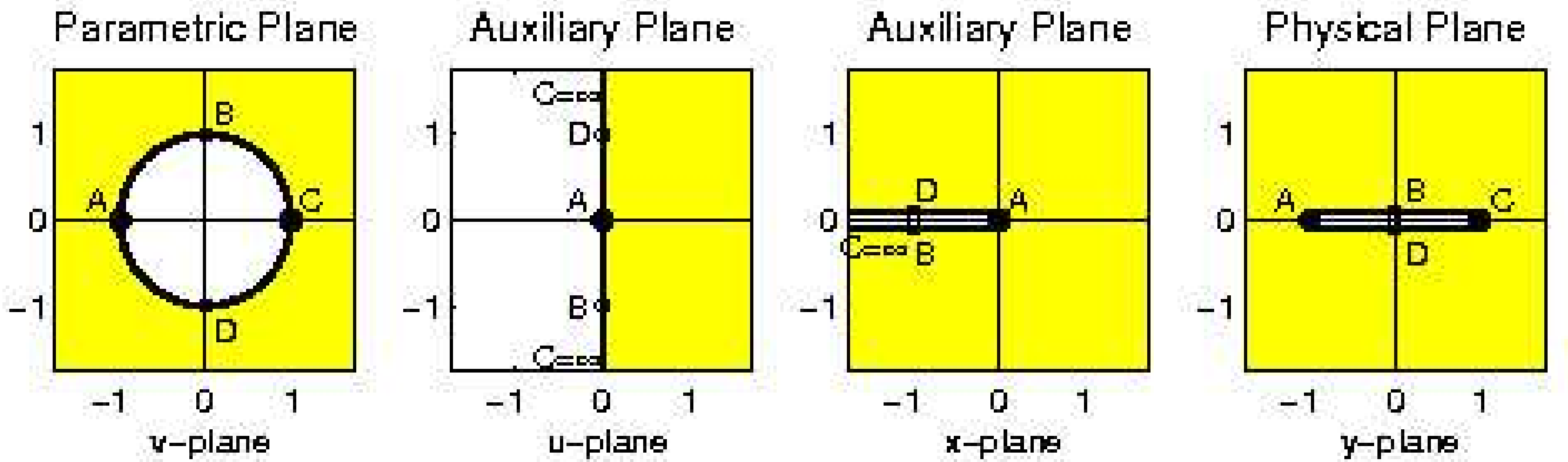
$$\begin{aligned} \frac{d\vec{\alpha}}{dE} &= -\frac{\vec{g}}{2} \cdot \sin(\frac{E}{2}) \\ \frac{d\vec{\beta}}{dE} &= \frac{\vec{g}}{2} \cdot \cos(\frac{E}{2}) \ . \end{aligned}$$

Here we have used vector symbols in order to indicate that the above equations not only hold for scalars $\alpha, \beta, g \in \mathbb{R}$, but also for vectors $\vec{\alpha}, \vec{\beta}, \vec{g} \in \mathbb{R}^n, \ n \in \mathbb{N}$.

Appendix I. Regularizing the restricted 3-body problem

- Motion of a massless particle under the gravitational force of two heavy primaries moving on circular orbits
- Use a rotating coordinate system, traditionally centered at the center of mass, such that the primaries are fixed at the complex positions $-\mu$ and $1-\mu$, respectively $(0 < \mu < 1)$
- We discuss the simultaneous regularization of both types of collisions. G. D. Birkhoff (1915): time transformation (Step 1): $dt = r_1\, r_2\, d\tau$, where $r_1,\ r_2$ are the distances of the particle from the primaries
- Coordinate transformation in 2 dimensions (Step 2): A sequence of conformal mappings $v \in \mathbb{C} \mapsto u \mapsto x \mapsto y$, where for simplicity we choose $y \in \mathbb{C}$ as a normalized physical coordinate with the primaries located at $y = -1$ and $y = 1$

The Joukowsky-Birkhoff transformation



$$v \mapsto u = \frac{v+1}{v-1} \qquad u \mapsto x = u^2 \qquad x \mapsto y = \frac{x+1}{x-1}$$

Composition of the three mappings

Two dimensions:

$$y = \frac{\left(\frac{v+1}{v-1}\right)^2 + 1}{\left(\frac{v+1}{v-1}\right)^2 - 1} \qquad \text{or} \qquad y = \frac{1}{2}\left(v + \frac{1}{v}\right)$$

Three dimensions: $v, u, x, y \in \mathbb{U}, \quad x = x^\star,\ y = y^\star$

$$v \mapsto u = 1 + 2\,(v-1)^{-1}\,, \quad u \mapsto x = u\,u^\star\,, \quad x \mapsto y = 1 + 2\,(x-1)^{-1}$$

Composition yields: $\boxed{y = 1 + (v^\star - 1)\,(v^\star + v)^{-1}\,(v - 1)}$

Proof

$$(*) \qquad y = 1 + 2\left(u\,u^\star - 1\right)^{-1} \qquad \text{with} \qquad u = 1 + 2\,(v-1)^{-1}$$

Because of

$$\begin{aligned} u\,u^\star - 1 &= (u-1)\,(u^\star-1) + u - 1 + u^\star - 1 \\ &= 4\,(v-1)^{-1}\,(v^\star-1)^{-1} + 2\,(v-1)^{-1} + 2\,(v^\star-1)^{-1} \end{aligned}$$

we write $(*)$ as

$$y = 1 + 2\,(v^\star-1)\,\underbrace{(v^\star-1)^{-1}\left(u\,u^\star-1\right)^{-1}(v-1)^{-1}}_{D^{-1}}\,(v-1)\,.$$

$$\begin{aligned} \Longrightarrow\; D &= (v-1)\Big(4\,(v-1)^{-1}(v^\star-1)^{-1} + 2(v-1)^{-1} + 2(v^\star-1)^{-1}\Big)(v^\star-1) \\ &= 4 + 2\,(v^\star-1) + 2\,(v-1) = 2\,(v+v^\star)\,, \end{aligned}$$

whence the statement follows.

Conclusions (so far)

- The “language” of quaternions allows for a concise formalism for developing the Kustaanheimo-Stiefel theory of regularization of the perturbed spatial Kepler problem. This, in turn, is the basis of modern, efficient perturbation theories of the Kepler problem.

- The use of ”star-conjugation”,

$$u = u_0 + i\,u_1 + j\,u_2 + k\,u_3\,, \quad u^\star = u_0 + i\,u_1 + j\,u_2 - k\,u_3\,,$$

  yields a spatial regularization theory in perfect formal agreement with Levi-Civita’s planar regularization using complex numbers.

- Both collision types in the spatial restricted problem of three bodies may be regularized simultaneously by means of a generalization of the Joukowsky-Birkhoff mapping, elegantly representable in terms of quaternions.

Appendix II: MATLAB code for animated Kepler motions

```
function ind = kepler(e,N,Nrev)        % Save as kepler.m
% e=eccentricity, N steps/revol.   Call: e.g., kepler(0.8,128,4)
% Nrev revol's of Kepler motion in phys.coord (Nrev/2 in reg.c.)
a=1; mu=1; b=a*sqrt(1-e^2);
a1=sqrt(a*(1+e)); b1=sqrt(a*(1-e)); n1=sqrt(a^3/mu);
N2=2*N; t=n1*[0:N2]'*4*pi/N2; E=t-e*sin(t); E0=1+0*E;
tol=sqrt(eps); ind=0; while any(abs(E-E0)>tol), ind=ind+1;
               E0=E; E=E-(E+e*sin(E)-t)./(1+e*cos(E)); end;
x = a*(e+cos(E)); y = b*sin(E);
u = a1*cos(E/2);  v = b1*sin(E/2);
X = x(1); Y = y(1); U = u(1); V = v(1);  LW='LineWidth';
clf; subplot(2,1,1); p = plot(x,y,'b',X,Y,LW,2);
axis equal; hold on; plot(0,0,'ko',LW,6,'MarkerSize',6);
```

```
title(strcat('Kepler motion and Levi-Civita regularization, ',...
      'e = ', num2str(e)),'FontSize',15);
xlabel('Physical system','FontSize',12);
subplot(2,1,2); q = plot(u,v,'c',U,V,LW,2);
axis equal; hold on; plot(0,0,'ko',LW,6,'MarkerSize',6);
xlabel('Regularized system','FontSize',12);
set(p(2),LW,6,'Marker','o','MarkerSize',6,'Color','r');
set(q(2),LW,6,'Marker','o','MarkerSize',6,'Color','m');
for k=1:ceil(Nrev/2),
  for k=1:N2,
    set(p(2),'Xdata',x(k),'Ydata',y(k));
    set(q(2),'Xdata',u(k),'Ydata',v(k));
    pause(.0015);
  end
end % Template by Peter Arbenz, Computational Science, ETH Zurich
```

Appendix III: Outlook to triple collision

Fundamental difference:

Binary collision: Continuous dependence on the initial point in phase space. The family of nearby close-encounter orbits varies smoothly.

Triple collision: No continuous dependence on the initial point in phase space. The family of nearby close-encounter orbits varies erratically. Close-encounter orbits may be very complicated.

Simplest model for triple collision: The planar problem of three bodies

Coordinates: $x_j \in \mathbb{R}^2,\ x_1 + x_2 + x_3 = 0,$ masses: $m_j > 0,\ j = 1, 2, 3$

$$
\begin{aligned}
\ddot{x}_1 &= m_2\,(x_2 - x_1)\,|x_2 - x_1|^{-3} + m_3\,(x_3 - x_1)\,|x_3 - x_1|^{-3} \\
\ddot{x}_2 &= m_3\,(x_3 - x_2)\,|x_3 - x_2|^{-3} + m_1\,(x_1 - x_2)\,|x_1 - x_2|^{-3} \\
\ddot{x}_3 &= m_1\,(x_1 - x_3)\,|x_1 - x_3|^{-3} + m_2\,(x_2 - x_3)\,|x_2 - x_3|^{-3}
\end{aligned}
$$

Blow-up properties

Summary of the equations of motion, $x = \big(x_1,\ x_2,\ x_3\big) \in \mathbb{R}^6$:

$$\frac{d^2x}{dt^2} = F(x), \quad t = \text{time},$$

where the function $F:\ \mathbb{R}^6 \mapsto \mathbb{R}^6$ is homogeneous of degree **-2** :

$$F(\rho\, x) = \rho^{-2}\, F(x) \quad \text{for all} \quad \rho \neq 0\,.$$

Blow-up transformation.

Let $\rho \ll 1$ be a fixed small blow-up factor. With x, t being the (small) physical coordinates and $\tilde{x}$, $\tilde{t}$ as the blown-up coordinates (order 1) we define

$$x = \rho\, \tilde{x}\,, \quad t = \rho^{3/2}\, \tilde{t}\,.$$

Blow-up properties, continued

The equations of motion are invariant under the blow-up transformation:

$$\boxed{\frac{d^2\tilde{x}}{d\tilde{t}^{\,2}} = F(\tilde{x})}\,. \quad \text{Check !}$$

Let the blown-up motion $\tilde{x}(\tilde{t})$ be free of collisions, then its velocity is bounded,

$$\left\| \frac{d\tilde{x}}{d\tilde{t}} \right\| = O(\rho^0)\,.$$

This implies

$$\left\| \frac{dx}{dt} \right\| = \rho^{-1/2} \left\| \frac{d\tilde{x}}{d\tilde{t}} \right\| = O(\rho^{-1/2})\,.$$

Reference: Richard McGehee in Inventiones mathematicae **27** (1974), 191-227: Triple collision in the collinear three-body problem.

Reference solutions, Lagrange and Euler

Homothetic solutions, $x(t) = r(t)\,\xi\,,$ where

- $r(t)$ describes a rectilinear Kepler motion
- ξ describes a central configuration: equilateral triangle (Lagrange) or three points on a line, depending on the masses (Euler)

lead to sharp triple collisions, which may be blown up indefinitely, hence

$$\rho \quad \text{arbitrarily small} \quad \Longrightarrow \left\| \frac{dx}{dt} \right\| = O(\rho^{-1/2})\,, \quad \textbf{unbounded !}$$

Orbit in blown-up system corresponds to a full 3BP with zero energy.

Remark. The general (sharp) triple collision was described in terms of convergent series by C. L. Siegel: Der Dreierstoß, Ann. Math. **42** (1941), 127-168.

The energy integral

$$h \;=\; \frac{1}{2}\sum_{j=1}^{3} m_j\,\dot{x}_j^2 \;-\; \sum_{1\le j<k\le 3} \frac{m_j\,m_k}{|x_j - x_k|} \;=\; \text{const}$$

Applying the blow-up transformation (p. 39) yields

$$h = \rho^{-1}\,\tilde{h} \qquad \text{or} \qquad \tilde{h} = \rho\,h\,.$$

- The close triple encounter $(\rho \to 0)$ has the complexity of the three-body problem with zero energy, $\tilde{h} \to 0$.
- (Lagrangean) close triple encounters require (triangular) triple parabolic initial conditions at $\tilde{t} \to -\infty$.
- Typical final state as $\tilde{t} \to +\infty$: hyperbolic-elliptic, a fast escaping binary (arbitrarily large velocity). In particular cases, a triple parabolic escape may result. Then the system emerges from the close triple encounter with bounded velocities.