Jost Bürgi's Artificium of 1586 in modern view, an ingenious algorithm for calculating tables of the sine function.

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Abstract

In the years of 1586 to 1592 the Swiss instrument maker and mathematician Jost Bürgi devised and documented an ingenious algorithm for efficiently and precisely calculating tables of the sine function. The manuscript *Fundamentum Astronomiæ* explaining this method and Bürgi's tables had been considered as lost, but have been rediscovered in 2013 by Menso Folkerts in the University Library of Wroclaw (Poland). In this paper we explain and discuss Bürgi's algorithm, referred to as *Artificium* or *Kunstweg*, with the tools of modern Linear Algebra. By considering the difference table of the sine function and by using matrices and eigenvalue problems, we develop a theory of the algorithm and discuss the rate of convergence.

Key words: Jost Bürgi, table of sines, differences, matrices, eigenvalue problem, convergence quotient.

1 Introduction

Consider a right angle to be divided into n equal parts. Approximately 1586, Jost Bürgi found an elegant algorithm, referred to as *Artificium*, for efficiently calculating the sines of all subdividing angles, which he documented in *Fundamentum Astronomiæ*, [?]. Bürgi's Artificium algorithm and the resulting table were considered as lost, but both documents had been rediscovered 2013 in the University Library of Wroclaw (Poland) by Menso Folkerts [?], [?], [?]. More recent texts are [?], [?].

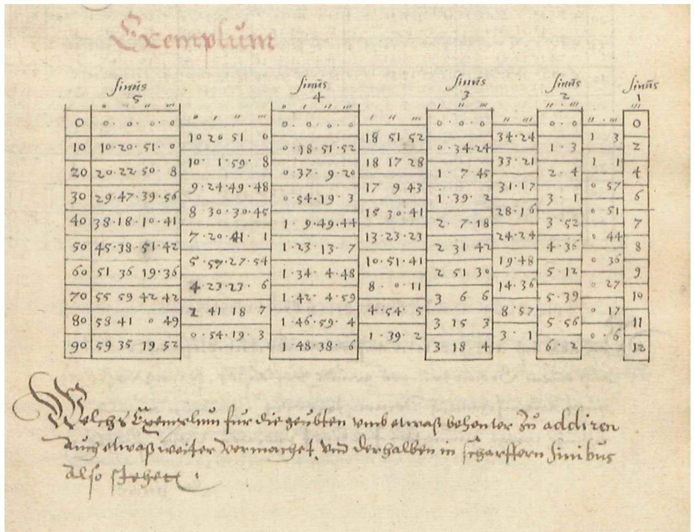

Figure 1: Bürgi's example $n = 9$ from Fundamentum Astronomiæ, numbers in the hexagesimal system.
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2 The Artificium algorithm: Introductory example

Jost Bürgi's Artificium simultaneously finds all sine values

$$\sin\left(\frac{k\,\frac{\pi}{2}}{n}\right), \quad k = 1, \ldots, n, \quad n \in \mathbb{N}, \quad n > 1 \tag{1}$$

by a convergent algorithm that can be pushed to any desired accuracy.

In Fundamentum Astronomiæ, [?], Bürgi uses the example $n = 9$ where, according to the customs of the 16th century, quantities used in astronomy are represented in the hexagesimal number system (Fig. 1). For finding the "sines of all degrees of the right angle" Bürgi suggests $n = 90$.

As an introductory example we will use the simpler case

$$n = 3: \quad \sin(30^o) = \frac{1}{2}, \quad \sin(60^o) = \frac{\sqrt{3}}{2}, \quad \sin(90^o) = 1\,. \tag{2}$$

Bürgi's algorithm generates a table, beginning with the rightmost column **a** and working from right to left. The numbers printed in *italics* are not carried along.

$\sin(0^o)$	*0*		*0*		*0*		*0*		*0*		*0*
		2911		780		209		56		15	
$\sin(30^o)$	2911		780		209		56		15		4
		2131		571		153		41		11	
$\sin(60^o)$	5042		1351		362		97		26		7
		780		209		56		15		4	
$\sin(90^o)$	5822		1560		418		112		30		8
		-780		*-209*		*-56*		*-15*		*-4*	
					...	**f**	**e**	**d**	**c**	**b**	**a**

We briefly describe the algorithm by the following three steps. Step 0 below defines the initial column **a**; Steps 1 and 2 generate columns **b** and **c** to the left, and their repetition generates columns **d** and **e**, etc. The symbol $(\ )'$ means transposition of a vector or a matrix.

0. Initial column: $\mathbf{a} = (a_1, a_2, \ldots, a_n)' \in \mathbb{R}^n$, (almost) arbitrary, for example, but not necessarily, approximations for multiples of sine values, $f \cdot \sin(k\,\frac{\pi}{2}\,/n),\ k = 1, \ldots, n$, rounded to integers. $f \neq 0$ is an arbitrary factor, e.g. $f = 8$ in the introductory example.

1. Next column to the left: $\mathbf{b} = (b_1, b_2, \ldots, b_n)'$ = cumulative sum of the a_j from bottom to top, first $b_n = a_n/2$, then $b_k = b_{k+1} + a_k,\ k = n-1, \ldots, 1$.

2. Further column to the left: $\mathbf{c} = (c_1, c_2, \ldots, c_n)'$ = cumulative sum of the b_k from top to bottom, first $c_1 = b_1$, then $c_k = c_{k-1} + b_k,\ k = 2, \ldots, n$.

Clearly, as a consequence of the use of cumulative sums, the complete Artificium table is the difference table of the leftmost column, with a sign change in columns 3,5,.. . For discussing the details of the algorithm we will therefore look at the difference table of the sine function in Section 3.

Since the initial column **a** is close to a multiple of $\sin(k\,\pi/6)$ and we need to get $\sin(\pi/2) = 1$, it seems natural to normalize the leftmost column by dividing it by its bottom element, with the result (0, 0.5, 0.86602542, 1)'. *This is in fact the final step in the Artificium algorithm.*

For our choice of the initial column in the case $n = 3$ the approximations for for the first element, $\sin(\pi/6)$ happen to be exact. The normalized second elements of the odd columns, $a_2/a_3, c_2/c_3, e_2/e_3, \ldots$ approximate $\sin(2\,\pi/6)$ with increasing accuracy. In the table below, the three lines list (1) the normalized second elements in rational form, (2) the errors, i.e. the differences to $\sin(\pi/3) = \sqrt{3}/2 = 0.86602540$ and (3) the ratios of two consecutive errors.

The errors are nearly in a geometric sequence with an almost constant ratio of consecutive terms. In Section 5 the limit of the ratios (the convergence quotient) will be identified as $\left(2+\sqrt{3}\right)^2 = 13.92820$.

$\ldots, c_2/c_3,\ a_2/a_3$	1351/1560	362/418	97/112	26/30	7/8
$\ldots, a_2/a_3 - \sqrt{3}/2$	2.3724e-7	3.3043e-6	4.6025e-5	6.4126e-4	8.9746e-3
Ratio to next error		13.92823	13.92855	13.93299	13.99526

3 The difference table of the sine function

At the end of the 16th century trigonometry was still in the process of being developed. The trigonometric addition formulas were used in order to simplify multiplications of long numbers. This process, known as *Prosthaphæresis*, is partially attributed to Bürgi, and was also used by him around 1590, [?]. It is based on the identity

$$\cos(\alpha) \cdot \cos(\beta) = \frac{1}{2}\Big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big) \tag{3}$$

and allows to calculate a product by table-look-ups and simple operations (additions and a halving).

At the beginning of the next century Bürgi found a simpler and more effective method for simplifying multiplications: his famous Progreß Tabulen [?], [?], [?], a table of the exponential function $f_n = 1.0001^n,\ n = 0, \ldots, 23027$, published 1620 in Prag. Together with John Napier, who independently published his table of the log-sine function in 1614, Bürgi laid the grounds for logarithmic calculation which remained the basis of all scientific computations for more than three centuries, see also [?].

In the following, we will use the tools of modern mathematics for explaining and discussing the Artificium algorithm. By putting $\alpha = \big(\pi - (y-x)\big)/2$, $\beta = (y+x)/2$ we obtain

$$\sin y - \sin x = 2\ \sin\left(\frac{y-x}{2}\right)\ \cos\left(\frac{y+x}{2}\right), \tag{4}$$

an identity useful for investigating the *difference table* of the sine function $f(x) = \sin x$ (Fig. 2). Using three arguments and $2\,\delta > 0$ as their mutual difference we obtain the difference table with first and second differences Δ_1 and Δ_2:

$$\begin{array}{lll} f(x) & \Delta_1 & \Delta_2 \\ \hline \sin(x-2\,\delta) & & \\ & 2\ \sin\delta\cdot\cos(x-\delta) & \\ \sin x & & -4\ \sin^2\delta\cdot\sin x \\ & 2\ \sin\delta\cdot\cos(x+\delta) & \\ \sin(x+2\,\delta) & & \end{array} \tag{5}$$

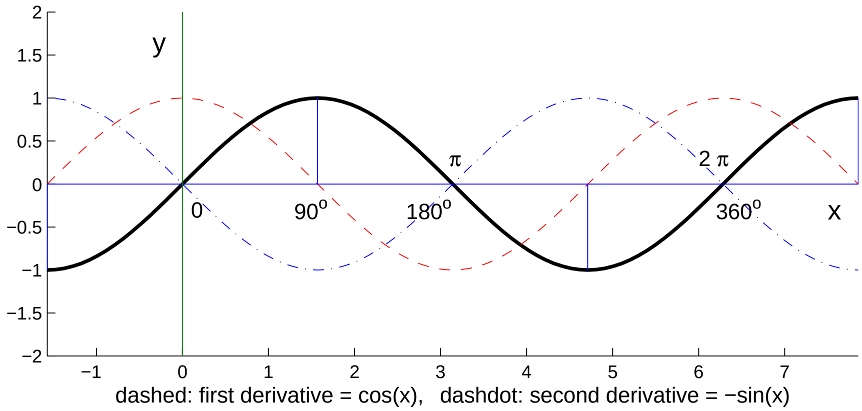

Figure 2: The function $y = f(x) = \sin(x)$, dashed: first derivative $f'(x) = \cos(x)$ dashdot: 2nd derivative $f''(x) = -\sin(x)$

There immediately follows: *The second difference is proportional (with a negative factor) to the value of the sine function on the same line.* Bürgi might have observed this theoretically and experimentally as well, perhaps with equidistant angles in more than three rows and numerical sine approximations.

The inverse operation of calculating the difference table is forming the cumulative sums from right to left, from bottom to top in the odd columns to compensate the omission of the negative sign of Δ_2 in Equ. (**??**). Since calculating differences of almost equal numbers results in a loss of accuracy (is numerically unstable), Bürgi could have hoped that the inverse process would result in a stable and convergent algorithm.

The initial conditions for the cumulative sums take care of the symmetries of the sine und cosine functions at $x = 0$ and $x = 90^o$, see the entries of the introductory example printed in *italics*. The columns at their boundaries behave like odd functions, the odd columns $\mathbf{a}, \mathbf{c}, \ldots$ at $x = 0$, the even columns $\mathbf{b}, \mathbf{d}, \ldots$ at $x = 90^o$. This results in the rules of Step 2 and 1: For the odd columns the initial value on top is always 0 (not carried along), for the even column $\mathbf{b}$ we would have to define $b_{n+1} = -b_n$, which implies $b_n = a_n/2$, taking into account that column $\mathbf{a}$ is the negative difference of column $\mathbf{b}$.

Obviously, the Artificium algorithm finds the sine values only up to an (unknown) factor. To satify $\sin(90^o) = 1$ every element of the leftmost column needs to be normalized by dividing it by its bottom element. This is summarized in Theorem 1 below. In the remaining sections a proof will be given in several steps.

Theorem 1. For (almost) arbitrary initial columns $\mathbf{a} = (a_1, \ldots, a_n)'$ with $n > 1$, the normalized odd columns $a_k/a_n, c_k/c_n, e_k/e_n, \ldots$ converge to $\sin(k\,\frac{\pi/2}{n}),\ k = 1, \ldots, n.$ $\square$

4 Vectors and Matrices

We will use modern Linear Algebra in order to prove Theorem 1. First, the Artificium algorithm will be described in terms of vectors and matrices. Beginning with the column vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)' \in \mathbb{R}^n$, we also introduce the vector

$$\tilde{\mathbf{a}} = (a_1, a_2, \ldots, a_{n-1}, \frac{a_n}{2})' = \mathbf{H} \cdot \mathbf{a}\,, \tag{6}$$

where $\mathbf{H} \in \mathbb{R}^{n \times n}$ is the diagonal matrix with diagonal elements $1, 1, \ldots, 1, \frac{1}{2}$. Furthermore, let $\mathbf{T}$ be the lower triangular matrix $\mathbf{T} = \big(t_{kj}\big) \in \mathbb{R}^{n \times n}$, filled with ones, $t_{kj} = 1$ if $k \geq j$, 0 otherwise. Then the columns $\mathbf{b}, \mathbf{c}$ of the Artificium table may be written as

$$\mathbf{b} = \mathbf{T}'\,\tilde{\mathbf{a}}, \quad \mathbf{c} = \mathbf{T}\,\mathbf{b}\,. \tag{7}$$

Therefore, the combined Steps 1 and 2 of the Artificium algorithm are

$$\mathbf{c} = \mathbf{M}\,\mathbf{a} \quad \text{with} \quad \mathbf{M} = \mathbf{T} \cdot \mathbf{T}' \cdot \mathbf{H}\,; \tag{8}$$

$\mathbf{M}$ will be called the **Bürgi matrix**, it had already been mentioned by Folkerts/Launert/Thom [?]. E.g., for $n = 5$ Equ. (**??**) yields

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0.5 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1.5 \\ 1 & 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 & 2.5 \end{pmatrix} \in \mathbb{R}^{5 \times 5}. \tag{9}$$

The matrix $\mathbf{T}$ has a simple inverse: by introducuing the unit matrix $\mathbf{I} \in \mathbb{R}^{n \times n}$ and the unit subdiagonal matrix $\mathbf{L} \in \mathbb{R}^{n \times n}$, $\mathbf{T}$ may be written as a Taylor series,

$$\mathbf{T} = \mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \cdots + \mathbf{L}^{n-1} = (\mathbf{I} - \mathbf{L})^{-1}. \tag{10}$$

Therefore, also $\mathbf{M}$ has a simple inverse; Equ. (**??**) yields

$$\mathbf{M}^{-1} = \mathbf{H}^{-1}\,(\mathbf{I} - \mathbf{L})'\,(\mathbf{I} - \mathbf{L}) \tag{11}$$

for the inverse of the Bürgi matrix, e.g. for $n = 5$:

$$\mathbf{M}^{-1} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}. \tag{12}$$

In summary, the mapping induced by $\mathbf{M}^{-1}$,

$$\mathbf{y} = \mathbf{M}^{-1}\,\mathbf{x} \quad \text{with} \quad \mathbf{x} = \big(x_1, x_2, \ldots, x_n\big)' \tag{13}$$

generates the negative second differences of the vector $\tilde{\mathbf{x}} = \big(0; \mathbf{x}; x_{n-1}\big)$, where the supplemented components $\tilde{x}_0$ and $\tilde{x}_{n+1}$ exactly model the behaviour of the sine funtion at 0 and at $\pi/2$. Therefore, the Artificium algorithm inverts the formation of the difference table of the sine function in the interval $[0, \frac{\pi}{2}]$ up to an unknown factor. Bürgi takes care of this factor by normalizing the leftmost column to $\sin(\pi/2) = 1$ by dividig it by its nth element.

5 Eigenvectors and Eigenvalues

We now consider the sequence of the odd columns (from right to left) of the Artificium table, and use the upper index j to count the number of iteration steps: $\mathbf{a}^{(0)} := \mathbf{a} \in \mathbb{R}^n$ for the initial column and $\mathbf{a}^{(1)} := \mathbf{c}$, $\mathbf{a}^{(2)} := \mathbf{e}$, ... for the further odd columns, with components $\mathbf{a}^{(j)} = \big(a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}\big)'$. Then the algorithm including the normalization may be written as

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{a}^{(j-1)}\,, \quad \mathbf{s}^{(j)} = \frac{\mathbf{a}^{(j)}}{a_n^{(j)}}\,, \quad j = 1, 2, \ldots\,, \tag{14}$$

where $\mathbf{s}^{(j)} = \big(s_1^{(j)}, s_2^{(j)}, \ldots, s_n^{(j)}\big)'$ is the vector of the jth approximations for the sine values. This is the well-known *power iteration*, published 1929 by Richard von Mises (1883-1953) and Hilda Geiringer (wife, 1893-1973). It is closely related to the eigenvalues λ_i ($i = 1, \ldots, n$) of $\mathbf{M}$ and the corresponding eigenvectors $\mathbf{v_j}$. If the eigenvalue λ_1 of maximum magnitude is simple, the power iteration converges direction-wise to the corresponding eigenvector $\mathbf{v_1}$ satisfying $\mathbf{M}\,\mathbf{v_1} = \mathbf{v_1}\,\lambda_1$.

In order to discuss the convergence speed of the power iteration (**??**) we need to solve the eigenvalue problem of the Bürgi matrix $\mathbf{M}$.

Theorem 2. There exists a regular matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M}$ is similar to $\mathbf{D}$, i.e.

$$\mathbf{M}\,\mathbf{V} = \mathbf{V}\,\mathbf{D}\,. \tag{15}$$

The matrix $\mathbf{V} = \big(v_{ki}\big)$ with

$$v_{ki} = \sin\Big(k\,(i-\frac{1}{2})\,\frac{\pi}{n}\Big), \quad k,\, i = 1,\ldots,n \tag{16}$$

contains n linearly independent eigenvectors of $\mathbf{M}$ as its columns (i fixed), and $\mathbf{D}$ contains the eigenvalues

$$\lambda_i = \frac{1}{4\,\sin^2\big((i-\frac{1}{2})\,\frac{\pi}{2\,n}\big)} \quad \text{with} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n \tag{17}$$

on its diagonal. □

Proof. Multiplying the eigenvector condition Equ. (**??**) from the left with $\mathbf{M^{-1}}$ and from the right with $\mathbf{D^{-1}}$ yields $\mathbf{M^{-1}\,V} = \mathbf{V\,D^{-1}}$, i.e. $\mathbf{M^{-1}}$ has the same eigenvectors as $\mathbf{M}$, but the reciprocal eigenvalues. We will therefore first consider the simpler eigenvalue problem of $\mathbf{M^{-1}}$.

Consider now the image $\mathbf{w} = \big(w_1, w_2, \ldots, w_n\big)'$ of the ith column $\mathbf{v}_i = \big(v_{1i}, v_{2i}, \ldots, v_{ni}\big)'$ of $\mathbf{V}$ under the mapping induced by $\mathbf{M^{-1}}$. Our goal is to take advantage of the tridiagonal, almost periodic stucture of $\mathbf{M^{-1}}$ seen in Equ. (**??**). Using the abbreviation $\omega_i = (i-\frac{1}{2})\,\frac{\pi}{n}$ and observing $v_{0i} = 0$ and $v_{n+1,i} = v_{n-1,i}$ we obtain

$$w_k = -\sin\big((k-1)\omega_i\big) + 2\,v_{ki} - \sin\big((k+1)\omega_i\big)\,, \quad k = 1,2,\ldots,n\,.$$

The addition formula of the sine function yields $w_k = 2\,(1-\cos\omega_i)\,v_{ki}$. Therefore, the ith column of $\mathbf{V}$ is in fact an eigenvector, and the corresponding eigenvalue of $\mathbf{M^{-1}}$ is $2\,(1-\cos\omega_i) = 4\,\sin^2(\omega_i/2)$. This directly yields Equ. (**??**) for the eigenvalues of $\mathbf{M}$. □

6 Rate of Convergence

We will now investigate the power iteration for finding the eigenvector $\mathbf{v_1}$ of $\mathbf{M}$ associated with the unique eigenvalue λ_1 of maximum magnitude, with the goal of proving its convergence. We will also get results on the speed of convergence. As a consequence of of Bürgi's normalizing division, Equ. (**??**) implies $s_{nj} = 1$. On the other hand, Equ. (**??**) with $k = n,\ i = 1$ implies $v_{n1} = 1$, i.e. the eigenvector $\mathbf{v_1}$ as stated in Theorem 2, satisfies the same normalization as Bürgi's approximations $\mathbf{s}_j$. We can therefore restate Theorem 1 as

$$\lim_{j\to\infty} \mathbf{s}^{(j)} = \mathbf{v}_1 = \big(v_{11}, \ldots, v_{n1}\big)' \;\text{ with }\; v_{k1} = \sin\big(k\,\frac{\pi/2}{n}\big),\; k = 1,\ldots,n\,,$$

or equivalently by introducing the error vector $\mathbf{e}^{(j)} = \big(e_1^{(j)}, \ldots, e_n^{(j)}\big)'$:

$$e_k^{(j)} = s_k^{(j)} - \sin\big(k\,\frac{\pi/2}{n}\big) \to 0 \quad \text{as} \quad j \to \infty\,. \tag{18}$$

Power iteration becomes much more transparent if the matrix $\mathbf{M}$ is represented in the coordinate system of its linearly independent eigenvectors. To do so, write Equ. (**??**) as

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{V}\,\mathbf{V}^{-1}\,\mathbf{a}^{(j-1)}\,.$$

Multiplying both sides from the left by $\mathbf{V}^{-1}$ and using Equ. (**??**) yields

$$\mathbf{u}^{(j)} = \mathbf{D}\,\mathbf{u}^{(j-1)} \quad \text{with} \quad \mathbf{u}^{(j)} = \mathbf{V}^{-1}\,\mathbf{a}^{(j)} \;\;\text{or}\;\; \mathbf{a}^{(j)} = \mathbf{V}\,\mathbf{u}^{(j)}\,. \tag{19}$$

Compared to Equ. (**??**), the repeated multiplication is done here simply by the diagonal matrix $\mathbf{D}$. The vector $\mathbf{u}^{(j)}$ can now be stated explicitly as

$$\mathbf{u}^{(j)} = \mathbf{D}^j\,\mathbf{u}^{(0)} \quad \text{with} \quad \mathbf{u}^{(0)} = \mathbf{V}^{-1}\,\mathbf{a}^{(0)}\,, \tag{20}$$

where the jth power $\mathbf{D}^j$ of $\mathbf{D}$ is also diagonal. $\mathbf{u}^{(0)}$ will be called the *modified initial vector;* the matrix $\mathbf{V}^{-1}$ needed here is given by

$$\mathbf{V}^{-1} = \frac{2}{n}\,\mathbf{V}'\,\mathbf{H}\,, \tag{21}$$

which is easily verified with elementary trigonometry by calculating $\mathbf{V}\,\mathbf{V}'$ using the explicit definition of $\mathbf{V}$ in Equ. (**??**).

To summarize, the jth approximations $a_k^{(j)}$ of the Artificium algorithm may be represented by a set of explicit expressions, originating from (**??**), (**??**), (**??**) above and using the eigenvectors v_{ki} (**??**) and the eigenvalues λ_i (**??**). Choose the initial column $\mathbf{a}^{(0)} \in \mathbb{R}^n$ and evaluate

$$u_i^{(0)} = \frac{2}{n}\sum_{k=1}^{n}{}'\sin\Big(k\,(i-\tfrac{1}{2})\,\frac{\pi}{n}\Big)\,a_k^{(0)}\,,\;\; \Sigma' : \text{last term with half weight} \tag{22}$$

$$u_i^{(j)} = u_i^{(0)} \cdot \lambda_i^{\,j}\,, \quad i = 1, \ldots, n\,, \quad \lambda_i = \frac{1}{4\,\sin^2\big((i-\frac{1}{2})\,\frac{\pi}{2\,n}\big)} \tag{23}$$

$$a_k^{(j)} = \sum_{i=1}^{n}\sin\Big(k\,(i-\tfrac{1}{2})\,\frac{\pi}{n}\Big)\;u_i^{(j)}\,, \quad k = 1, \ldots, n\,. \tag{24}$$

Therefore, the rate of convergence depends on the eigenvalues λ_i as well as on the choice of the initial column $\mathbf{a}^{(0)}$.

The initial column $\mathbf{a}^{(0)}$ needs to be chosen such that $u_1^{(0)} \neq 0$, i.e. $\mathbf{a}^{(0)}$ needs to have a non-vanishing component in the direction of $\mathbf{v}_1$. In the introductory example of Section 2 the vector $\mathbf{a}^{(0)} = \mathbf{v}_2 = (1, 0, -1)'$ in the direction of the second eigenvector would be an unhappy choice. In exact arithmetic, power iteration yields only vectors of the same direction. A practical realization in Matlab (precision 15 digits) would save the situation: after 37 iterations the first eigenvector is finally reached.

To begin with, assume in addition to $u_1^{(0)} \neq 0$ also $u_2^{(0)} \neq 0$. As a consequence of the inequality in Equ. (**??**) the first term ($i = 1$) of the sum for $a_k^{(j)}$ in Equ. (24) approximates $a_k^{(j)}$, and the second term ($i = 2$) approximates the error $e_k^{(j)}$ for large j:

$$\begin{array}{lclclcl} a_k^{(j)} & = & u_1^{(0)}\, \lambda_1{}^j \, \sin\big(k\, \dfrac{\pi}{2\,n}\big) & + & u_2^{(0)}\, \lambda_2{}^j \, \sin\big(k\, \dfrac{3\,\pi}{2\,n}\big) & + & \ldots \\ a_n^{(j)} & = & u_1^{(0)}\, \lambda_1{}^j & - & u_2^{(0)}\, \lambda_2{}^j & + & \ldots\,. \end{array} \tag{25}$$

The ratio $s_k^{(j)}$ of the two expressions is

$$s_k^{(j)} = \frac{a_k^{(j)}}{a_n^{(j)}} = \sin\big(k\, \frac{\pi}{2\,n}\big) + e_k^{(j)}\,,$$

where the error (**??**) is approximately

$$e_k^{(j)} = \frac{u_2^{(0)}}{u_1^{(0)}} \left(\frac{\lambda_2}{\lambda_1}\right)^j \left(\sin\big(k\, \frac{\pi}{2\,n}\big) + \sin\big(k\, \frac{3\,\pi}{2\,n}\big)\right) + \ \ldots\,. \tag{26}$$

To generalize this particular case, assume now that $u_2^{(0)} = 0$ to be also possible. Then we have the following theorem based on the modified initial column $\mathbf{u}^{(0)}$:

Theorem 3. Consider an Artificium with the modified initial column $\mathbf{u}^{(0)} = \big(u_1^{(0)}, \ldots, u_n^{(0)}\big)$ of (**??**) with $u_1^{(0)} \neq 0$. Let $r \geq 2$ be smallest index with $u_r^{(0)} \neq 0$. Then we have as a generalization of Equ. (**??**)

$$e_k^{(j)} = \frac{u_r^{(0)}}{u_1^{(0)}} \left(\frac{\lambda_r}{\lambda_1}\right)^j \left((-1)^r \, \sin\big(k\, \frac{\pi}{2\,n}\big) + \sin\left(k\, \frac{(2\,r-1)\,\pi}{2\,n}\right)\right) + \ \ldots\,. \tag{27}$$

We define the *convergence quotient* of the Artificium as the ratio $q_k^{(j)} = e_k^{(j-1)}/e_k^{(j)}$ of two consecutive errors. In our first-term approach the *limiting convergence quotient*,

$$Q_r = \lim_{j\to\infty} q_k^{(j)} = \frac{\lambda_1}{\lambda_r}\,, \quad r \geq 2\,, \tag{28}$$

independent of j and k, is a good approximation for $q_k^{(j)}$. For large n we have $Q_r \approx (2\,r-1)^2$.

7 Examples

In the list below we give a few typical examples of initial columns and convergence quotients. For simplicity the upper index of the initial column is suppressed: $a_k^{(0)} = a_k$.

The case $n = 3$ is the introductory example of Section 2; the case $n = 9$ below is Bürgi's example. Both cases yield $r = 3$. If n is a multiple of 3, $n = 3\,m$, integer initial columns leading to $r = 3$ and Q_3 bounded by 25 are not difficult to find; certainly Bürgi had a good intuition.

The case $n = 4$ is one of the many examples with $r = 2$, where integer initial columns are difficult to find or do not exist. Then the convergence is fairly slow, $Q_2 \approx 9$. For $n = 15$ we found an integer initial column close to a multiple of the sines to be calculated, yielding $r = 4$ and $Q_4 = 46.9$.

The final two examples considering values of n divisible by 15, $n = 15\,m$, were found by Grégoire Nicollier [**?**]. They are characterized by initial columns with only a few non-zero elements. The last example shows a remarkable initial column leading to $r = 6$ and $Q_6 \approx 121$, however only with irrational components (involving the golden ratio $\phi = (1+\sqrt{5})/2 = 1.618034$).

n=3: $\mathbf{a} = (4, 7, 8)', \quad u_2 = \frac{2}{3}\left(1 \cdot a_1 + 0 \cdot a_2 - 1 \cdot \frac{a_3}{2}\right) = 0, \ r = 3 \ \Rightarrow$

$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(75^o)}{\sin^2(15^o)} = 7 + 4\sqrt{3} = 13.92820$$

n=4: $\mathbf{a} = (4, 7, 9, 10)', \quad u_2 = 0.20111, \quad r = 2 \quad \Rightarrow$

$$Q_2 = \frac{\lambda_1}{\lambda_2} = \frac{\sin^2(33.75^o)}{\sin^2(11.25^o)} = 8.10973$$

n=9: $\mathbf{a} = (2, 4, 6, 7, 8, 9, 10, 11, 12)', \quad u_2 = 0, \quad r = 3 \quad \Rightarrow$

$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(25^o)}{\sin^2(5^o)} = 23.51281$$

n=15: $\mathbf{a} = (1, 2, 4, 5, 6, 7, 8, 9, 10, 10, 11, 11, 12, 12, 12)', \ u_2 = u_3 = 0,$

$$r = 4 \ \Rightarrow \quad Q_4 = \frac{\lambda_1}{\lambda_4} = \frac{\sin^2(21^o)}{\sin^2(3^o)} = 46.88760$$

n=15 m: $a_k = 1$ if $k = 2\,m$ or $k = 10\,m$ or $k = 12\,m$, $a_k = 0$ otherwise, $r = 4 \Rightarrow \quad Q_4 \approx 49$, e.g. $Q_4 = 48.94$ for n=90 (Nicollier)

n=15 m: $a_k = 1$ if $k = m$ or $k = 11\,m$, $a_k = \phi$ if $k = 7\,m$ or $k = 13\,m$, $a_k = 0$ otherwise, $\quad r = 6 \ \Rightarrow$

$Q_6 = \dfrac{\lambda_1}{\lambda_6} \approx 121$, goes back to $Q_4 \approx 49$ after a few steps if ϕ is only approximated, e.g. by $\phi \approx \dfrac{8}{5}$ (Grégoire Nicollier, Sion)

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