Jost Bürgi's Artificium, an Ingenious Algorithm for Calculating Tables of the Sine Function

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Abstract

In the years of 1586 to 1592 the Swiss instrument maker and mathematician Jost Bürgi devised and documented an ingenious algorithm for efficiently and precisely calculating tables of the sine function. The manuscript *Fundamentum Astronomiæ* explaining this method and Bürgi's tables had been considered as lost, but have been rediscovered in 2013 by Menso Folkerts in the University Library of Wroclaw (Poland). In this presentation we explain and discuss Bürgi's algorithm, referred to as *Artificium* or *Kunstweg*, with the tools of modern Linear Algebra. By considering the difference table of the sine function and by using matrices and eigenvalue problems, we develop a theory of the algorithm and discuss the rate of convergence.

Contents

1. Jost Bürgi's Artificium

Jost Bürgi's Artificium algorithm is described in his mathematical text Fundamentum Astronomiæ, written between 1586 and 1592, but only rediscovered 2013 by Menso Folkerts [4,5]. Details are given by, among others, Dieter Launert [6,7], Fritz Staudacher [10] and George Szpiro [11]. The Artificium is an algorithm for calculating

$$\sin\left(\frac{j\,\frac{\pi}{2}}{n}\right), \quad j = 1, \ldots, n, \quad n > 1$$

efficiently and precisely. Bürgi's example is $n = 9$ (see p. 5); he also suggests $n = 90$, *"every degree of the right angle"*. We will use the simpler case

$$n = 3: \quad \sin(30^o) = \frac{1}{2}, \quad \sin(60^o) = \frac{\sqrt{3}}{2}, \quad \sin(90^o) = 1\,.$$

From Fundamentum Astronomiæ

As customary in 16th century astronomy, the hexagesimal number system is used.

Working from right to left, the algorithm generates a table of numbers. The ones printed in red are given for clarification only, they are not carried along.

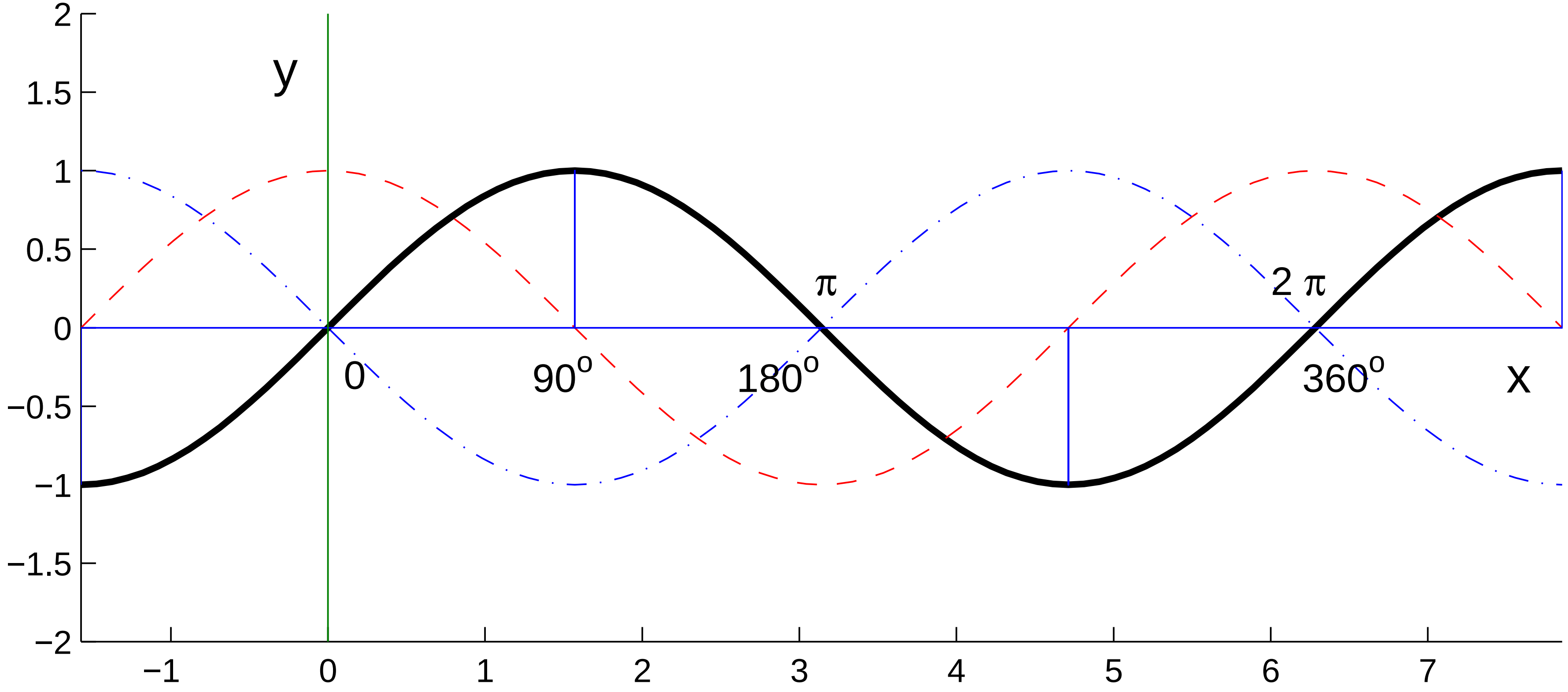
$\sin(0^o)$	0		0		0		0		0		0
		2911		780		209		56		15	
$\sin(30^o)$	2911		780		209		56		15		4
		2131		571		153		41		11	
$\sin(60^o)$	5042		1351		362		97		26		7
		780		209		56		15		4	
$\sin(90^o)$	5822		1560		418		112		30		8
		-780		-209		-56		-15		-4	
					$\dots$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{d}$	$\mathbf{c}$	$\mathbf{b}$	$\mathbf{a}$

0. Initial column: $\mathbf{a} = (a_1, a_2, \dots, a_n)' \in \mathbb{R}^n$, (almost) arbitrary, for example, but not necessarily, multiples of the sines to be calculated, $f \cdot \sin(\frac{j\,\pi}{2\,n})$, rounded to integers ($f = 8$ in the above case).

1. Next column to the left: $\mathbf{b} = (b_1, b_2, \dots, b_n)'$ = cumulative sum of the a_j upwards, first $b_n = \frac{a_n}{2}$, then $b_j = b_{j+1} + a_j,\ \ j = n-1, \dots, 1$.

2. Further column to the left: $\mathbf{c} = (c_1, c_2, \dots, c_n)'$ = cumulative sum of the b_j downwards, first $c_1 = b_1$, then $c_j = c_{j-1} + b_j,\ \ j = 2, \dots, n$.

The sine function $y = \sin(x)$



dashed: first derivative = cos(x), dashdot: second derivative = −sin(x)

Continuation to the left

Continuation of the table to the left by repeating Steps 1. and 2. The Artificium actually is a difference table, built from right to left.

The odd columns $\mathbf{a}, \mathbf{c}, \mathbf{e}, \mathbf{g}, \ldots$, normalized by dividing them by their bottom element, approximate $\sin(\frac{j\,\pi}{2\,n})$ with increasing accuracy.

In the following table the data concerning $\sin(60^o)$ are collected:

$\ldots, c_2/c_3,\ a_2/a_3$	1351/1560	362/418	97/112	26/30	7/8
$\ldots, a_2/a_3 - \sqrt{3}/2$	2.3724e-7	3.3043e-6	4.6025e-5	6.4126e-4	8.9746e-3
Ratio to next error		13.92823	13.92855	13.93299	13.99526

The limit of the Ratio will be shown to be $7 + 4\sqrt{3} = 13.92820323$.

2. Differences

Difference tables are efficient tools for tabulating functions with equidistant arguments, e.g. the third powers $f(n) = n^3$:

n	$f(n)$	Δ_1	Δ_2	Δ_3
0	0		0	
		1		6
1	1		6	
		7		6
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		
6	216			

- In this example the third difference is constant
- The top elements and the last column must be known
- Construction of the table by cumulative summation from right to left and from top to bottom
- Additions only!

The difference table of the sine function

Prosthaphæresis (co-invented and used by Bürgi, [8])

$$\cos(\alpha)\cdot\cos(\beta) = \frac{1}{2}\Big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big)$$

Let $\alpha = \frac{\pi}{2} - \frac{y-x}{2}\,, \quad \beta = \frac{y+x}{2}\,;$ this implies

$$\sin y - \sin x = 2\ \sin\left(\frac{y-x}{2}\right)\,\cos\left(\frac{y+x}{2}\right).$$

Difference Table of $f(x) = \sin x$:

$f(x)$	Δ_1	Δ_2
$\sin(x-2\,\delta)$		
	$2\,\sin\delta\cdot\cos(x-\delta)$	
$\sin x$		$-4\,\sin^2\delta\cdot\sin x$
	$2\,\sin\delta\cdot\cos(x+\delta)$	
$\sin(x+2\,\delta)$		

The second difference is proportional to the function value on the same line (with a negative factor)

Bürgi's main result

Remarks

- The tabulation of the cumulative sums is the inverse map of the construction of the difference table
- The initial conditions are a consequence of the symmetries of the sin- and cos-functions at $x = 0$ and $x = 90^o$
- The normalization to $\sin(90^o) = 1$ needs one division per element

Theorem 1:

For (almost) arbitrary initial columns $\mathbf{a} = (a_1, \ldots, a_n)'$ with $n > 1$, the normalized odd columns $a_k/a_n, c_k/c_n, e_k/e_n, \ldots$ converge to $\sin(k\,\frac{\pi/2}{n}),\ k = 1, \ldots, n$.

3. Vectors and Matrices

In the case $n = 3$ of p. 4 we define $\tilde{\mathbf{a}} = (a_1, a_2, \frac{a_3}{2})' = \mathbf{H} \cdot \mathbf{a}$, where $\mathbf{H}$ generally is the diagonal matrix with the n diagonal elements $1, 1, \ldots, 1, \frac{1}{2}$. Now the first two steps of p. 6 are

$$\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{a}}, \qquad \mathbf{c} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{\mathbf{T}} \mathbf{b}\,.$$

Therefore the mapping corresponding to an Artificium step is

$$\boxed{\mathbf{c} = \mathbf{M}\,\mathbf{a} \quad \text{with} \quad \mathbf{M} = \mathbf{T} \cdot \mathbf{T}' \cdot \mathbf{H}\,.}$$

The Bürgi matrix M

The matrix $\mathbf{M} = \mathbf{T} \cdot \mathbf{T}' \cdot \mathbf{H}$ describing the Artificium mapping will be called the Bürgi Matrix. It had already been mentioned by D. Launert and A. Thom [6]. For $n = 5$ we obtain

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0.5 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1.5 \\ 1 & 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 & 2.5 \end{pmatrix} .$$

The matrix T has a simple inverse

$\mathbf{T} \in \mathbb{R}^{n \times n}$ is the lower triangular matrix filled with ones. For later use we derive an alternative representation of $\mathbf{T}$. Let $\mathbf{I}$ be the unit matrix, and let $\mathbf{L}$ be the unit subdiagonal matrix,

$$\mathbf{I} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Then we have

$$\mathbf{T} = \mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \cdots + \mathbf{L}^{n-1} = (\mathbf{I} - \mathbf{L})^{-1}.$$

The Bürgi matrix $\mathbf{M}$ also has a simple inverse

If $\mathbf{a}^{(0)} = \mathbf{a} \in \mathbb{R}^n$ is the initial column and the further odd columns are denoted by $\mathbf{a}^{(1)} = \mathbf{c},\ \mathbf{a}^{(2)} = \mathbf{e},\ \ldots,$ the Artificium algorithm may be written as

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{a}^{(j-1)}, \quad j = 1, 2, \ldots\,,$$

in modern Linear Algebra known as power iteration (von Mises - Geiringer). Assume $\mathbf{M}$ has only one eigenvalue, λ_1, of maximum magnitude. Then the normalized vectors $\mathbf{a}^{(j)}/||\mathbf{a}^{(j)}||$ converge to the corresponding eigenvector $\mathbf{v_1}$ with the property $\mathbf{M}\,\mathbf{v_1} = \mathbf{v_1}\,\lambda_1$.

For solving the eigenvalue problem we consider the inverse of $\mathbf{M}$, which is tridiagonal (use $\mathbf{T}$ of p. 14):

$$\mathbf{M}^{-1} = \mathbf{H}^{-1}\,(\mathbf{I} - \mathbf{L})'\,(\mathbf{I} - \mathbf{L}).$$

The mapping induced by $\mathbf{M}^{-1}$

For $n = 4$ we obtain

$$\mathbf{M}^{-1} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{pmatrix}, \quad \tilde{\mathbf{M}} = \begin{pmatrix} -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \end{pmatrix}.$$

By introducing the supplemented vector $\tilde{\mathbf{x}} = \big(0; \mathbf{x}; x_{n-1}\big)$, exactly modelling the behaviour of the sine funtion at 0 and at $\pi/2$, the mapping induced by $\mathbf{M}^{-1}$ becomes

$$\mathbf{y} = \tilde{\mathbf{M}}\,\tilde{\mathbf{x}}$$

Results

- The Artificium algorithm inverts the formation of the difference table of the sine function in the interval $[0, \frac{\pi}{2}]$ up to an unknown factor
- Bürgi takes care of this factor by normalizing the leftmost column to $\sin(\pi/2) = 1$ by dividig it by its nth element.

Power iteration

Notation: upper index (j) counts the odd columns:

Initial column $\mathbf{a}^{(0)} := \mathbf{a} \in \mathbb{R}^n$

Further odd columns $\mathbf{a}^{(1)} := \mathbf{c},\ \mathbf{a}^{(2)} := \mathbf{e},\ \ldots$

with components $\mathbf{a}^{(j)} = \left(a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}\right)'$.

The Artificium algorithm

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{a}^{(j-1)}\,, \quad \mathbf{s}^{(j)} = \frac{\mathbf{a}^{(j)}}{a_n^{(j)}}\,, \quad j = 1, 2, \ldots\,,$$

is the well-known power iteration (R. von Mises, Hilda Geiringer, 1929).

Relevant for convergence theory: Eigenvalue problem of $\mathbf{M}$.

4. Eigenvectors and Eigenvalues

The Eigenvalueproblem of $\mathbf{M}$

If the eigenvalue λ_1 of maximum magnitude is simple, the power iteration converges direction-wise to the corresponding eigenvector $\mathbf{v_1}$ satisfying $\mathbf{M}\,\mathbf{v_1} = \mathbf{v_1}\,\lambda_1$.

$\mathbf{M}^{-1}$ has the same eigenvectors as $\mathbf{M}$, but the reciprocal eigenvalues.

The following theorem may easily be proven using elementary trigonometry (p. 10) and the mapping induced by the matrix $\tilde{\mathbf{M}}$ based on $\mathbf{M}^{-1}$, representing calculation of the negative second difference.

Theorem 2. There exists a regular matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M}$ is similar to $\mathbf{D}$, i.e.

$$\mathbf{M}\,\mathbf{V} = \mathbf{V}\,\mathbf{D}\,.$$

The matrix $\mathbf{V} = \big(v_{ki}\big)$ with

$$v_{ki} = \sin\left(k\,(i-\frac{1}{2})\,\frac{\pi}{n}\right), \qquad k,\, i = 1, \ldots, n$$

contains n linearly independent eigenvectors of $\mathbf{M}$ as its columns (i fixed), and $\mathbf{D}$ contains the eigenvalues

$$\lambda_i = \frac{1}{4\,\sin^2\left((i-\frac{1}{2})\,\frac{\pi}{2\,n}\right)} \quad \text{with} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n$$

on its diagonal.

5. Rate of Convergence

Bürgi's normalizations for obtaining approximations x_{kj} of sines:

$$\mathbf{s}^{(j)} = \frac{\mathbf{a}^{(j)}}{a_n^{(j)}} \quad \Longrightarrow \quad s_n^{(j)} = 1$$

Bürgi's Artificium algorithm yields

$$\lim_{j\to\infty} \mathbf{s}^{(j)} = \mathbf{v}_1 \qquad \text{(first eigenvector)}$$

Norm of the error: $e^{(j)} = \left\| \mathbf{s}^{(j)} - \mathbf{v}_1 \right\|_2$

Convergence quotient: $q^{(j)} = \dfrac{e^{(j-1)}}{e^{(j)}}$

Often we have $Q := \lim_{j\to\infty} q^{(j)} = \dfrac{\lambda_1}{\lambda_2}$

Modified initial column $\mathbf{u} = \mathbf{a}$ in the basis of $\mathbf{V}$

According to the first equation of Theorem 2, $\mathbf{M}\,\mathbf{V} = \mathbf{V}\,\mathbf{D}$, the power iteration can be represented in a simpler form:

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{V}\,\mathbf{V}^{-1}\,\mathbf{a}^{(j-1)} \quad \Rightarrow \quad \mathbf{V}^{-1}\,\mathbf{a}^{(j)} = \mathbf{D}\,\mathbf{V}^{-1}\,\mathbf{a}^{(j-1)}\,.$$

Modified iteration vector: $\mathbf{u}^{(j)} = \mathbf{V}^{-1}\,\mathbf{a}^{(j)}$

Modified initial column: $\mathbf{u} = \mathbf{u}^{(0)} = \mathbf{V}^{-1}\,\mathbf{a}^{(0)} = \mathbf{V}^{-1}\,\mathbf{a}$

Therefore: $\mathbf{u}^{(j)} = \mathbf{D}\,\mathbf{u}^{(j-1)}$ or $\boxed{\mathbf{u}^{(j)} = \mathbf{D}^{j}\,\mathbf{u}^{(0)}}$

The rate of convergence depends on the eigenvalues $\lambda_i = \frac{1}{4}/\sin^2\left((i-\frac{1}{2})\,\frac{\pi}{2\,n}\right)$ and on the modified initial column $\mathbf{u}$.

The eigenvalues (p. 20) satisfy $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. With $\mathbf{V}^{-1} = \frac{2}{n}\,\mathbf{V}'\,\mathbf{H}$, the rth component u_r of the modified initial column becomes $(r = 1, 2, \ldots, n)$:

$$u_r = \frac{2}{n} \sum_{i=1}^{n}{}' \sin\left((r - \tfrac{1}{2})\, i\, \tfrac{\pi}{n}\right) a_i, \qquad \Sigma' : \text{last term with half weight}$$

The initial column $\mathbf{a}$ must be chosen such that $u_1 \neq 0$. The statement of p. 21 on Q holds, if also $u_2 \neq 0$.

Theorem 3

Let $r \geq 2$ be the smallest index with $u_r \neq 0$. Then $Q = \lim_{j\to\infty} q^{(j)} = \dfrac{\lambda_1}{\lambda_r}$.

6. Examples

The cases $n = 3$ and $n = 9$ are the introductory example and Bürgi's example yielding $r = 3$. The case $n = 4$ is one of the many examples with $r = 2$ and rather slow convergence $Q < 9$, where integer initial columns are difficult to find or do not exist.

n=3: $$\mathbf{a} = (4,7,8)', \quad u_2 = \tfrac{2}{3}\left(1 \cdot a_1 + 0 \cdot a_2 - 1 \cdot \frac{a_3}{2}\right) = 0, \; r = 3 \;\Rightarrow$$
$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(75^o)}{\sin^2(15^o)} = 7 + 4\sqrt{3} = 13.92820$$

n=4: $$\mathbf{a} = (4,7,9,10)', \quad u_2 = 0.20111, \quad r = 2 \quad \Rightarrow$$
$$Q_2 = \frac{\lambda_1}{\lambda_2} = \frac{\sin^2(33.75^o)}{\sin^2(11.25^o)} = 8.10973$$

n=9: $$\mathbf{a} = (2,4,6,7,8,9,10,11,12)', \quad u_2 = 0, \quad r = 3 \quad \Rightarrow$$
$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(25^o)}{\sin^2(5^o)} = 23.51281$$

n=15: $$\mathbf{a} = (1,2,4,5,6,7,8,9,10,10,11,11,12,12,12)', \; u_2 = u_3 = 0,$$
$$r = 4 \;\Rightarrow\quad Q_4 = \frac{\lambda_1}{\lambda_4} = \frac{\sin^2(21^o)}{\sin^2(3^o)} = 46.88760$$

These two examples considering values of n divisible by 15, $n = 15\,m$, were found by Grégoire Nicollier [9]. They are characterized by initial columns with only a few non-zero elements. The last example shows a remarkable initial column leading to $r = 6$ and $Q_6 \approx 121$, however only with irrational components (involving the golden ratio $\phi = \frac{\sin 42^o + \sin 78^o}{\sin 6^o + \sin 66^o} = 2\,\cos 36^o = \frac{1+\sqrt{5}}{2} = 1.618034$).

n=15 m: $a_k = 1$ if $k = 2\,m$ or $k = 10\,m$ or $k = 12\,m\,,\;\; a_k = 0$ otherwise,

$r = 4 \;\Rightarrow \quad Q_4 \approx 49$, e.g. $Q_4 =$ 48.94 for n=90 (Nicollier)

n=15 m: $a_k = 1$ if $k = m$ or $k = 11\,m,\;\; a_k = \phi$ if $k = 7\,m$ or $k = 13\,m$,

$a_k = 0$ otherwise, $\quad r = 6 \;\Rightarrow$

$Q_6 = \dfrac{\lambda_1}{\lambda_6} \approx 121$, goes back to $Q_4 \approx 49$ after a few steps if ϕ is only approximated, e.g. by $\phi \approx \frac{13}{8}$ (Grégoire Nicollier, Sion [9])

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