Annoted list of some selected papers

Wendelin Werner.

Pre-SLE results on Brownian intersection exponents.

Asymptotic behaviour of disconnection and non-intersection exponents, Probab. Th. Rel. Fields **108**, 131-152 (1997).
Intersection exponents for planar Brownian motion, with Greg Lawler, Ann. Probab. **27**, 1601-1642 (1999).
Universality for conformally invariant intersection exponents, with G. Lawler, J. Europ. Math. Soc. **2**, 291-328 (2000).

In these three papers, various features of the critical exponents that describe the asymptotic decay of the non-intersection probabilities between two-dimensional Brownian traces are described. In particular, the following items are explained:

- The "cascade relations" that relate all these exponents (these relations were then interpeted by Duplantier in terms of quantum gravity relations).
- The fact that the Brownian exponents should indeed be the same as the exponents that conjecturally describe critical percolation events. This was the locality idea that later turned out to be fruitful in the SLE setting.

The true self-repelling motion.

The true self-repelling motion, with B. Tóth, Probab. Th. Rel. Fields **111**, 375-452 (1998)

This paper constructs a one-dimensional stochastic process of a new type and derives some of its main features: Its path properties are very different from Brownian motion, and its dynamics look at first glance as if they are deterministic. This process is related to some hyperbolic PDEs and it is also provides an example of a random planar space-filling planar path that describes a natural random continuous tree contour.

The hot spots conjecture is false.

A counterexample to the "hot spots" conjecture, with K. Burdzy, Ann. Math. **149**, 309-317 (1999)

In this short paper, the first example of a domain D such that the second eigenfunction of the Neumann Laplacian in D has its maximum in the interior of D (and not on ∂D) is given.

SLE papers, computing exponents and solving the Mandelbrot conjecture.

These four papers are joint with Greg Lawler and Oded Schramm
Values of Brownian intersection exponents I: Half-plane exponents, Acta Math. **187**, 237-273 (2001).
Values of Brownian intersection exponents II: Plane exponents, Acta Math. **187**, 275-308 (2001).
Analyticity of intersection exponents for planar Brownian motion, Acta Math. **189**, 179-201 (2002).
The dimension of the planar Brownian frontier is $4/3$, Math. Res. Lett. **8**, 401-411 (2001).

This sequence of four papers contains among other things:

- Some general considerations about Loewner chains needed in the proofs of the following facts.
- The study of various properties of SLE processes, in particular the locality property of SLE_6 and the radial/chordal equivalence of SLE_6.
- The computation of critical exponents associated with chordal and radial SLE curves.
- The proof of the conjectures giving the values of the Brownian intersection exponents (combining the previous facts with the ideas of our pre-SLE papers).
- An analyticity result about Brownian intersection exponents, allowing to relate them to the Brownian disconnection exponents.
- A proof of Mandelbrot's conjecture that the dimension of the outer boundary of a 2D Brownian path is $4/3$ (combining the last two results).

Percolation exponents.

Critical exponents for two-dimensional percolation, with S. Smirnov, Math. Res. Lett. **8**, 729-744 (2001).
One-arm exponent for critical 2D percolation, with G. Lawler and O. Schramm, Electr. J. Probab. **7**, article no. 2 (2002).

In these two papers, combining SLE_6 computations with Stas Smirnov's proof of conformal invariance of discrete critical percolation and Kesten's scaling relations, all the conjectures about the values of critical exponents for critical and near-critical percolation (for site percolation on the triangular lattice) are proven.

Scaling limit of uniform spanning trees and loop-erased random walks.
Conformal invariance of planar loop-erased random walks and uniform spanning trees, with G. Lawler and O. Schramm, Ann. Prob. **32**, 939-995 (2004).

In this paper, it is shown that the scaling limit of 2D loop-erased random walks and of the uniform spanning tree contours are the SLE_2 and SLE_8 processes respectively. This is the first instance where such a convergence of a discrete model to SLE was described (and the martingale technique introduced there was then also successfully reapplied in subsequent work on percolation or on the Ising model).

Conformal restriction.
Conformal restriction properties. Chordal case, with G. Lawler and O. Schramm, J. Amer. Math. Soc. **16**, 917-955 (2003).

In this paper, the notion of "conformal restriction" is defined and studied. This paper includes:
- A characterization and construction of all sets satisfying chordal conformal restriction.
- The definition of the $\text{SLE}_\kappa(\rho)$ curves, a natural variant of SLE_κ.
- A description of the lack of restriction property of SLE's for $\kappa \neq 8/3$ and a concrete interpretation of the so-called "central charge" in the SLE context.
- A proof of the fact that Brownian outer boundaries are in fact locally exactly $\text{SLE}_{8/3}(\rho)$ curves (which is a stronger statement than the fact that they share the same critical exponents – this stronger feature was not clearly conjectured in the physics literature).

Further papers in that vein include the following:
The conformally invariant measure on self-avoiding loops, J. Amer. Math. Soc. 21, 137-169 (2008)

where a very simple characterization of the unique measure natural measure of self-avoiding loops that satisfies conformal restriction is given. This shows that the three measures describing the outer boundaries of Brownian loops, the outer boundaries of critical percolation cluster scaling limits and $\text{SLE}_{8/3}$ loops are in fact exactly the same.

Brownian loop-soup.
The Brownian loop-soup, with G. Lawler, Probab. Th. Rel. Fields **128**, 565-588 (2004).

This paper contains the definition of the Brownian loop-soup, a Poissonian cloud of non-interacting Brownian loops in a domain that turns to be a rather central object in this theory. It is explained here how it can be used to measure the "restriction defect" of SLE processes for $\kappa \neq 8/3$ and how the Brownian loop-soup is related to the scaling limit of the loops that are erased when performing loop-erased random walks.

Near-critical interfaces are not symmetric.
Asymmetry of near-critical percolation interfaces, with P. Nolin, J. Amer. Math. Soc. **22**, 797-819 (2009).

In this paper, a feature that was not predicted in the physics literature is derived: When one observes a piece of a near-critical percolation interface in the scaling limit, one can actually almost surely detect that it is not a critical interface.

Conformal loop ensembles, loop-soup clusters.
Conformal loop ensembles: The Markovian characterization and the construction via loop-soups, with S. Sheffield, Ann. Math. **176**, 1827-1917 (2012)

In this paper, a construction, description and characterization of the Conformal loop ensembles is given. The characterization of this one-parameter familiy of disjoint simple loops in a domain by the natural "CLE property" shows that they are the natural conjectural scaling limits of the collection of interfaces for a number of lattice models. The construction and proof of the conformal invariance of CLE uses in an essential way the Brownian loop-soup. In particular, it is explained that outer boundaries of clusters of "subcritical loop-soups" are exactly such CLE (which in particular provides a construction of SLE processes based on clouds of Brownian loops and excursions only, as announced in the note *SLEs as boundaries of clusters of Brownian loops,* C.R. Acad. Sci. Paris **337**, 481-486 (2003)).

Further papers in different directions related to these CLEs and their construction include:
Decomposition of Brownian Loop-soup clusters, with W. Qian, J. Europ. Math. Soc. 21, 3225-3253, 2019.

In this paper, using somewhat different techniques, a rather surprising decomposition of critical loop-soup clusters via a Poisson point process of Brownian excursions is derived. This enables to relate and unify various features (derived by Schramm-Sheffield, Sheffield-Miller, Le Jan, Dynkin) about the relation between Gaussian Free Field and their level lines, loop-soups occupation times and loop-soup clusters).

Connection probabilities for conformal loop ensembles, with J. Miller, Communications in Mathematical Physics 362 (2018) 415-453.
Here, crossing probabilities of conformal rectangles with various boundary conditions provided by conditioned CLE_κ are computed. This provides in particular a direct purely continuum-based justifications for the conjectured formulas relating κ to the discrete $O(N)$ or FK_q models.

Renormalization.
A simple renormalization flow for FK-percolation models, in Jean-Michel Bismut 65th anniversary volume.
Near-uniform spanning forests and renormalization, with S. Benoist and L. Dumaz, Ann. Probab. (to appear).

In the first of these papers, a simple way to describe a simple Markov process of sets of graphs is outlined, whose stationary measure(s) should describe exactly the critical FK-models in their scaling limits. In the second paper, it is explained how to make this formalism work in the special case of the two-dimensional uniform spanning tree model.

CLE duality, percolation within fractal carpets.
CLE percolations, with J. Miller and S. Sheffield, Forum of Mathematics Pi, Vol. 5, 99 pages, 2017.

This paper establishes the continuous version (and conjectural scaling limit) of the Edwards-Sokal coupling between FK-percolation models and Potts models. This provides a direct relation between the CLE consisting of simple loops and those consisting of non-simple loops. Some main features of this paper include:

- A description and characterization of "critical percolation interfaces" in the random fractal defined by a CLE. This is the first instance of such a continuous conformally invariant percolation model within fractal domains.
- A direct relation between this FK/Potts duality and the Gaussian Free Field. This also enables to shed some light on the relation between CLE and the GFF.

In *Non-simple SLE curves are not determined by their range*, J. Europ. Math. Soc. 22, 669-716, 2020 with the same coauthors, it is shown (among other things) that the above-mentioned percolation within CLE carpets is indeed non-trivial (i.e., random when conditioned on the carpet).

Lecture Notes.
Random planar curves and Schramm-Loewner evolutions, 2002 Saint-Flour summer school, L.N. Math. **1840**, pp. 107-195, Springer, 2004.
This is a survey and introduction to the first batch of results with Greg Lawler and Oded Schramm. This includes in particular a complete self-contained outline of all arguments that come together to derive the proof of Mandelbrot's conjecture and to the determination of the Brownian intersection exponents.

Conformal restriction and related questions, Probability Surveys **2**, 145-190, 2005.
Some recent aspects of conformally invariant systems, Les Houches summer school lecture notes (July 2005), Mathematical statistical physics, 57–99, 2006.
These provide a self-contained presentation of some of the conformal restriction ideas up to 2003/2004, and to the idea of using loop-soup clusters to describe Conformal loop ensembles.

Lectures on two-dimensional critical percolation, IAS-Park City 2007 summer school, 297-360, AMS, 2009.
This is a self-contained presentation explaining all the steps that lead to the determination of critical exponents for critical and near-critical percolation.