LOCAL RRH

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ABSTRACT. In [6] Engeli and Felder describe a generalized Riemann-Roch-Hirzebruch formula to compute the Lefschetz numbers of differential operators on holomorphic vector bundles. Essentially, the trace is computed by taking a cup product of the differential operator with a cocyle in a certain sheaf hypercohomology group. Engeli and Felder give a Dolbeault representative of such a cocycle. In this paper, we construct a Čech representative.

As corollaries, we obtain

(i) a formula for the change of the Lefschetz number when blowing up points.

(ii) a localization formula that expresses the Lefschetz number as the sum over residues at the isolated zeroes of a holomorphic vector field, provided such a field exists. This formula is similar to the Baum-Bott formula for characteristic numbers and reduces to this formula in the special case where the differential operator is the identity.

1. INTRODUCTION

1.1. **The Generalized RRH Formula.** Let M be an n-dimensional compact connected complex manifold and $E \to M$ a holomorphic vector bundle of rank r. Denote by $\mathcal{E}$ the sheaf of holomorphic sections of E and by $\mathcal{D}_E$ the sheaf of holomorphic differential operators on E. Let $D \in \Gamma(M, \mathcal{D})$ be a global holomorphic differential operator. It descends to an operator, also denoted D, on the sheaf cohomology $H^\bullet(M; \mathcal{E})$. The generalized Riemann-Roch-Hirzebruch formula, conjectured by Feigin and Shoikhet and proved by Engeli and Felder in [6], computes the supertrace of this operator as an integral of a differential form.

$$L(D) = \mathrm{str}_{H^\bullet(M;\mathcal{E})}(D) = \int_M \mu \tag{1}$$

The $2n$-form μ on the right hand side is constructed using methods of formal differential geometry. We will explain this construction explicitly in section 3.

Of course, the form μ represents a de Rham cohomology class. This is, in turn, the same as a class in the hypercohomology of the complex of sheaves

$$\Omega^0 \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n$$

of holomorphic forms. Of course, the hypercohomology class defined by the Dolbeault representative μ can also be represented by a Čech representative. The basic result of this paper is an explicit formula for this Čech representative.

1.2. **Applications and Results.** Although the change from a Dolbeault to a Čech representative might seem rather irrelevant, it still allows us to obtain at least two relevant results.

The first is a localization theorem in the spirit of [1].

Key words and phrases.

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Theorem 1. *Suppose there exists a holomorphic vector field ξ on M with isolated zeros $z_1, .., z_k$ and a lifting to E. Then*

$$L(D) = \sum_{j=1}^{p} Res_{z_j} \Omega_j$$

for some locally defined holomorphic n-forms Ω_j. In the notation of sections 2 and 6 these forms are explicitly given by an integral over the n-simplex Δ_n:

$$\Omega_j = \int_{\Delta_n} \tau_{2n}^r(\hat{D}, \omega_\Delta, \dots, \omega_\Delta)$$

The second result concerns the change of the trace $L(D)$ upon blowing up a point of M. We assume that the bundle E is trivial and that $n > 1$. Let $\tilde{M}$ denote the manifold M with the point $m \in M$ blown up. Let $\tilde{D}$ be holomorphic differential operator on $\tilde{M}$. By Hartog's theorem, it defines a differential operator of M. We denote the respective traces of D acting on $H^\bullet(M; \mathcal{D})$ and $\tilde{D}$ acting on $H^\bullet(M; \mathcal{D})$ by $L(D)$ and $L(\tilde{D})$.

Theorem 2.

$$L(D) - L(\tilde{D}) = Res_m \Omega$$

for some locally defined holomorphic n-form Ω. In the notation of sections 2 and 5 these forms are explicitly given by an integral over the n-simplex Δ_n:

$$\Omega = \int_{\Delta_n} \tau_{2n}(\hat{D}, \omega_\Delta, \dots, \omega_\Delta)$$

Remark 3. In both these Theorems the forms Ω depend only on local data. In particular, the change in Lefschetz number upon blowup is computable from the the ∞-jet of D at m alone. For example, one can blow up arbitrarily many points $m_1, m_2, ..$ in arbitrary order, but the change in the Lefschetz number upon blowing up m_j is always the same constant number.

1.3. **Outline of this Paper.** We will heavily use formal differential geometry on complex manifolds. Unfortunately, we know of no single reference which presents this topic in a way suitable for our treatment. Hence we will give a more or less self-contained introduction in section 2.

In section 3 we will provide the yet missing definition of μ in eqn. (1), and rephrase this formula in a more sheaf theoretic language.

In section 4 the explicit formula for the Čech representative of $[\mu]$ that was mentioned in the introduction will be given.

Finally, the sections 5 and 6 are dedicated to the proofs of Theorems 1 and 2.

2. Formal Differential Geometry on Complex Manifolds

There currently exist several slightly different styles and notational conventions in formal differential geometry. We will briefly review the basic notions in the language used in this paper. This section is mostly a simple adaptation of Fedosov's treatise of symplectic manifolds [7]. Apart from some rephrasings, the only new result in this section is Proposition 18.

We will use the following notational convention. A smooth differential form $\boldsymbol{\eta} = \eta + \bar{\eta} \in \Gamma(T^{*1,0} \oplus T^{*0,1})$ on a complex manifold will be denoted in bold letters, and its components $\eta \in \Gamma(T^{*1,0})$ and $\bar{\eta} \in \Gamma(T^{*0,1})$ by plain letters and with a bar respectively. Similarly, let $\boldsymbol{\nabla}$ be a connection on a holomorphic vector bundle over M. It can always be split up into a "holomorphic" and an "antiholomorphic" part

$$\boldsymbol{\nabla} = \nabla + \bar{\nabla}$$

in the sense that for any vector fields $X \in \Gamma(M, T^{1,0}M)$, $\bar{X} \in \Gamma(M, T^{0,1}M)$ we have $\nabla_{\bar{X}} = \bar{\nabla}_X = 0$. As indicated, the smooth connection $\boldsymbol{\nabla}$ will always be printed in bold font, while the plain ∇ denotes only the holomorphic component.[1]

2.1. **The Bundles of Formal Functions and Formal Differential Operators.** Let F be the frame bundle of the holomorphic tangent bundle of M. It is a $GL_n(\mathbb{C})$-principal bundle. Hence we can form the associated bundle (of algebras)

$$G_\infty M := F \times_{GL_n} \mathbb{C}[[y^1, .., y^n]]$$

where the algebra of formal power series $\mathbb{C}[[y^1, .., y^n]]$ is equipped with the obvious $GL_n(\mathbb{C})$-module structure. We call this bundle the bundle of *formal functions on M*.

Similarly, define the tensor product bundle

$$G_\infty E := J_\infty M \times E.$$

which we call the bundle of *formal sections of E*. Naturally, $J_\infty E$ is a bundle of modules over the bundle of algebras $G_\infty M$.

Furthermore, note that the subbundle of formal functions which are at most linear in y can be naturally identified with the bundle $J_1 M$ of 1-jets of holomorphic funtions on M. In particular, there are maps

$$\pi_i : G_\infty^{\leq i} M \to J_i M. \tag{2}$$

for $i = 0, 1$.

Definition 4. A connection $\boldsymbol{\nabla}$ on $G_\infty M$ is *good* if it is compatible with the above structures in the following sense,

- It is a derivation, i.e., for a vector field X and local sections f_1, f_2, we have
$$\boldsymbol{\nabla}_X(f_1 f_2) = (\boldsymbol{\nabla}_X f_1) f_2 + f_1(\boldsymbol{\nabla}_X f_2).$$
- It commutes with the projections (2), i.e., the following diagram commutes for any vector field X and local section f:
$$\begin{array}{ccc} f \in \Gamma(G_\infty M) & \xrightarrow{\pi_1} & (\pi_1 f) \in \Gamma(J_1 M) \\ \downarrow{\scriptstyle \nabla_X} & & \downarrow{\scriptstyle X\cdot} \\ \boldsymbol{\nabla}_X f \in \Gamma(G_\infty M) & \xrightarrow{\pi_0} & (\pi_0 \boldsymbol{\nabla}_X f) = X \pi_1 f \in \Gamma(J_0 M) \end{array}$$
Here the map $X\cdot$ maps the 1-jet (f_0, f_1) to the 0-jet $X f_0 - \iota_X f_1$.

Similarly, a connection $\boldsymbol{\nabla}^E$ on $G_\infty E$ is *compatible* with the good connection $\boldsymbol{\nabla}$ on $G_\infty M$ if it preserves the module structure, i.e.,

$$\boldsymbol{\nabla}^E_X(fs) = (\boldsymbol{\nabla}_X f)s + f(\boldsymbol{\nabla}^E_X s)$$

for all vector fields X, sections f of $G_\infty M$ and sections s of $G_\infty E$.

To understand the meaning of these two conditions more concretely, pick a local coordinate system z^j on M. Then the first condition means that the connection one-form $\boldsymbol{A}$ of $\boldsymbol{\nabla} = d + \boldsymbol{A}$ takes values in the formal vector fields

$$\boldsymbol{A} = dz^l(a^i_l + a^i_{l,j} y^j + a^i_{l,jk} y^j y^k + \dots)\frac{\partial}{\partial y^i} + d\bar{z}^l(\bar{a}^i_l + \bar{a}^i_{l,j} y^j + \dots)\frac{\partial}{\partial y^i} \tag{3}$$

and the second means that the constant term is minus the identity

$$a^i_l = -\delta^i_l \qquad \bar{a}^i_l = 0$$

[1]Of course, this does not mean that the connection coefficients of ∇ are holomorphic, just that they have no $d\bar{z}$-component.

The compatibility condition on $\boldsymbol{\nabla}^E$ means that

$$\boldsymbol{\nabla}^E = \boldsymbol{\nabla}_0^E + \boldsymbol{\nabla} \otimes \mathbb{1}_E + \boldsymbol{E}$$

for some connection $\boldsymbol{\nabla}_0^E$ on E and some form $\boldsymbol{E}$ taking values in $G_\infty M \otimes \mathrm{End}(E)$.

Remark 5. Note that a connection $\boldsymbol{\nabla}$ on $G_\infty M$ automatically defines an affine connection $\boldsymbol{\nabla}_0$ on TM. Concretely, the connection coefficients are given by the linear part of (3), i.e., $a^i_{l,j}$ and $\bar{a}^i_{l,j}$. Conversely, since $G_\infty M$ is an associated bundle to the frame bundle, any affine connection defines a (non-good) connection on $G_\infty M$. In the local frame, it is simply obtained by letting $a^i_{l,j}$ equal the affine connection coefficients and setting all other a's to zero.

In the same manner, the constant, non-vector field term of the connection coefficients of $\boldsymbol{\nabla}^E$ defines a connection on E.

Convention: In this paper, we will deal exclusively with good connections on $G_\infty M$ and connections on $G_\infty E$ compatible with good connections. Furthermore, we will assume that the connections $\boldsymbol{\nabla}_0$ and $\boldsymbol{\nabla}_0^E$ defined as in the previous remark, have the form

$$\boldsymbol{\nabla}_0 = \nabla_0 + \bar{\partial} \qquad\qquad \boldsymbol{\nabla}_0^E = \nabla_0^E + \bar{\partial}.$$

We call these connections simply connections. In particular, when we talk about a *connection* $\boldsymbol{\nabla}^E$ on $G_\infty E$, we imply that there is a good connection $\boldsymbol{\nabla}$ on $G_\infty M$ with which it is compatible.

Finally, we will need the bundles of *formal differential operators*

$$D_\infty M = F \times_{GL_n} \mathbb{C}[[y^1, .., y^n]][\frac{\partial}{\partial y^1}, .., \frac{\partial}{\partial y^n}]$$

on M and on E

$$D_\infty E = D_\infty M \otimes \mathrm{End}(E).$$

They are bundles of non-commutative algebras acting naturally on $G_\infty M$ and $G_\infty E$. We will denote the sub-bundles of formal differential operators of degree p in $\frac{\partial}{\partial y}$ and degree q in y by

$$D_\infty^{p,q} M.$$

Furthermore, any good connection $\boldsymbol{\nabla}$ on $G_\infty M$ uniquely extends to $D_\infty M$ in such a way that it remains a derivation. In a local frame as before the connection on $D_\infty M$ is given by

$$d + [\boldsymbol{A}, \cdot]$$

where, also as before, $\boldsymbol{A}$ is the connection one-form in this frame, i.e., $\boldsymbol{\nabla} = d + \boldsymbol{A}$. In the same way, a compatible connection

$$\boldsymbol{\nabla}^E = \boldsymbol{\nabla} + \boldsymbol{nabla}_0^E + \boldsymbol{E}$$

on $G_\infty E$ extends to a connection on $D_\infty E$.

2.2. **Flat Connections.** The bundles $G_\infty M$ and $G_\infty E$ are more complicated than M and E, but have a decisive virtue: they possess many flat connections. Even better, the space of flat connections has a very simple structure:

Proposition 6. *The space of flat connections on G_∞ is isomorphic to the space of pairs*

$$(\nabla_0, \Phi)$$

where ∇_0 is a torsion free affine connection on M and Φ is a formal vector field of cubic or higher order. The latter is a section of a sub-bundle

$$D_\infty^{1,\geq 3} M$$

of the bundle of formal differential operators $D_\infty M$.

The space of flat connections $\boldsymbol{\nabla}^E$ on E compatible with a given good flat connection on $G_\infty M$ is isomorphic to the space of pairs

$$(\nabla_0^E, \Phi^E)$$

where ∇_0^E is a connection on E and Φ^E is a section of

$$G_\infty^{\geq 2} M \otimes End(E).$$

Here the term connection *is used as stated in the Convention in the previous section.*

Observe that the above proposition shows that the flat connections form an affine space. However, the dependence of the connection coeffiecients on the given data (∇_0, Φ) is highly nonlinear. We postpone the proof of this proposition to section 2.4.

Remark 7 (Geometric Meaning)*.* Later we will see explicitly that picking a flat good connection is equivalent to picking at every point $m \in M$ a (jet of a) parameterization of a neighborhood of m by $T_m M$.

2.3. **Lifting Functions to Flat Sections.** The idea of formal differential geometry is that one can lift questions about the holomorphic functions on M (or sections of E) to the flat sections of $G_\infty M$ (or $G_\infty E$), given a flat connection $\boldsymbol{\nabla}$.

Proposition 8. *Let $\boldsymbol{\nabla}$ be a (possibly only smooth) good flat connection on $G_\infty M$. Then the sheaf of flat sections of $G_\infty M$ is isomorphic to the structure sheaf $\mathcal{O}(M)$ of M as a sheaf of algebras.*

Let $\boldsymbol{\nabla}^E$ be a flat connection of $G_\infty E$. Then the sheaf of flat sections of $G_\infty E$ is isomorphic to the sheaf of holomorphic sections $\mathcal{E}$ of E.

Furthermore, the above isomorphism maps the module structure of $\mathcal{O}(M)$ on $\mathcal{E}$ to the the natural module structure of the flat sections of $G_\infty M$ on the flat sections of $G_\infty E$.

To see that this proposition is true, we first need to show that any holomorphic function f can be lifted to a flat section $\hat{f}$ of $G_\infty M$, such that $\hat{f}$ evaluated at $y = 0$ on the fiber above each point equals f. The equations to be solved are hence

$$\nabla \hat{f} = 0 \tag{4}$$

$$\bar{\nabla} \hat{f} = 0 \tag{5}$$

$$\hat{f}(y = 0) = f \tag{6}$$

The first equation together with the third already defines $\hat{f}$ completely:

Lemma 9. *If the following holds:*

$$\nabla \hat{f} = 0$$

$$\hat{f}(y = 0) = 0$$

then $\hat{f} = 0$.

The proof will follow easily from the construction of $\hat{f}$. Equation (4) has the explicit form

$$0 = \nabla f = \nabla_0 f - \delta f + \omega f.$$

Here ∇_0 is a connection on $G_\infty M$, associated to the affine connection ∇_0. The operator δ is in local coordinates defined by

$$\delta = dz^i \frac{\partial}{\partial y^i}. \tag{7}$$

The formal vector field valued form ω is the holomorphic part of the so called the *Maurer-Cartan form* and is at least quadratic in y, i.e., a section of $(T^{*1,0}M) \otimes D_\infty^{1,\geq 2}M$.

We now want to solve (4) recursively, order by order in y. As done by Fedosov, this computation can be neatly organized using the following operators

$$\delta^* = y^i \iota_i = y^i \iota_{\frac{\partial}{\partial z^i}} \tag{8}$$

$$\delta^{-1} = G^{-1}\delta^* \tag{9}$$

Here G is the gradation operator whose value is (holomorphic form degree)+(degree in y), i.e.,

$$G(y^{i_1}\cdots y^{i_p}\frac{\partial}{\partial y^{j_1}}\cdots\frac{\partial}{\partial y^{j_q}}dz^{k_1}\cdots dz^{k_r}d\bar{z}^{l_1}\cdots d\bar{z}^{l_s}) = (p+r)y^{i_1}\cdots y^{i_p}\frac{\partial}{\partial y^{j_1}}\cdots\frac{\partial}{\partial y^{j_q}}dz^{k_1}\cdots dz^{k_r}d\bar{z}^{l_1}\cdots d\bar{z}^{l_s}.$$

The following Lemma shows that δ^{-1} is indeed similar to an inverse of δ.

Lemma 10.

$$\{\delta, \delta^*\} = G \tag{10}$$

$$\left\{\delta, \delta^{-1}\right\} = 1 - \Pi_{00} \tag{11}$$

(12)

where Π_{00} is the projector onto the component of zero holomorphic form- and zero y-degree.

Proof. □

Applying δ^{-1} to both sides of (4) we obtain

$$\hat{f} = \delta^{-1}\left(\nabla_0 \hat{f} + \omega\hat{f}\right). \tag{13}$$

Since the operator δ^{-1} raises degree in y by one, and since ∇_0 and ω $O(y^2)$ do not lower the y-degree, this equation constitutes a recursive definition of $\hat{f}$. In particular, if the constant part vanishes, $\hat{f} = 0$, proving Lemma 9.

Lemma 11. *The section $\hat{f}$ of $G_\infty M$ defined above satisfies (4) and (5).*

Proof. The statement of Lemma 9 remains true for a section valued *form* $\hat{\eta}$, as long as $\delta^{-1}\eta = 0$. This can be easily checked by a similar calculation as above. Hence, to show eqn. (5), it suffices to show that

$$\nabla\nabla\hat{f} = 0$$
$$\delta^{-1}\nabla\hat{f} = 0$$
$$(\nabla\hat{f})(y=0) = 0.$$

The first equation is trivially satisfied by the flatness of ∇. The third equation follows from the second. The l.h.s. of the second in turn reads

$$\delta^{-1}\delta^{-1}\delta\left(\nabla_0\hat{f} + \omega\hat{f}\right)$$

where we applied δ to (13), used (11) and inserted. The above expression vanishes since $(\delta^{-1})^2 = 0$.

For the antiholomorphic part, eqn. (5), the argument is simpler. Obviously

$$\nabla\bar{\nabla}\hat{f} = \bar{\nabla}\nabla\hat{f} = 0$$

by the above and flatness of $\boldsymbol{\nabla}$. Furthermore

$$\bar{\nabla}\hat{f}(y=0) = \bar{\partial} f = 0$$

is just the analyticity of f. □

Remark 12. One can also define the lift $\hat{f}$ of a smooth function f by the definition (13). This lift will satisfy $\nabla\hat{f} = 0$, but not $\bar{\nabla}\hat{f} = 0$. The lift $\hat{f}$ can be thought of as the lift of the "function" (∞-jet) defined by the not necessarily convergent Taylor series

$$\sum_I \frac{z^I}{I!}\frac{\partial f}{\partial z^I}$$

of f.

2.4. **Constructing Flat Connections I: Fedosov's Way.** We now want to prove Proposition 6 by explicitly solving the Maurer-Cartan equations

$$\boldsymbol{\nabla}^2 = 0.$$

All the constructions in this section will be coordinate-independent. Nevertheless, concrete expressions of the tensors involved will be given in some fixed local coordinate system z^j to facilitate both notation and understanding.

The flat good connection we want to construct must have the form

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_0 - \delta + \boldsymbol{\omega}.$$

Here again, ∇_0 is the connection on $G_\infty M$, associated to the given affine connection ∇_0. The operator δ is defined by (7) as in the prvious section. The formal vector field valued form $\boldsymbol{\omega} = \omega + om\bar{e}ga$ is called the *Maurer-Cartan form* and is at least quadratic in y, i.e., a section of $T^*M)\otimes D_\infty^{1,\geq 2}M$. Our goal is to explicitly compute this form $\boldsymbol{\omega}$.

In terms of these quantities, the Maurer-Cartan equation becomes

$$(14)\quad 0 = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}_0^2 + [\boldsymbol{\nabla}_0, \delta] + [\delta, \boldsymbol{\omega}] + [\boldsymbol{\nabla}_0, \boldsymbol{\omega}] + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] = \boldsymbol{R} + \boldsymbol{T} + \delta\boldsymbol{\omega} + \boldsymbol{\nabla}_0\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}].$$

Here we again abused notation and denoted by ∇_0 also the associated connection on $D_\infty M$ and by δ the operator on $D_\infty M$ defined by taking the commutator with δ. Furthermore, the linear and constant formal vector fields $\boldsymbol{R} = R + \bar{R}$ and $\boldsymbol{T}$ are related to the curvature form $\boldsymbol{R}^i_j$ and the torsion form $\boldsymbol{T}^i$ of ∇_0 as follows

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}^i_j y^j \frac{\partial}{\partial y^i} \\ \boldsymbol{T} &= \boldsymbol{T}^i \frac{\partial}{\partial y^i}. \end{aligned}$$

Note that $\boldsymbol{T}$ is the only contribution of order y^0 to (14), and hence has to vanish. This means that the affine connection on M determined by a good flat connection on $G_\infty M$ is always torsion free.

Note also that (14) is an equation of holomorphic two-forms and hence can be decomposed into three distinct equations, one on the holomorphic cotangent bundle $\Omega^{2,0}(M)$, one mixed equation on $\Omega^{1,1}(M)$ and one antiholomorphic on $\Omega^{0,2}(M)$. The first two should be seen as a *definition* of the holomorphic and antiholomorphic part of ω, whereas the last is a nontrivial *relation* fulfilled by these two parts.

As in the previous section, we first want to exploit the "definition" and solve the holomorphic and mixed components of (14) order by order in y. This yields a recursive definition of ω. This computation can again be neatly organized using the operators δ^{-1} from (8). Applying δ^{-1} to both sides of (14) we obtain for the holomorphic part

$$\omega = \delta^{-1}\left(R + \nabla_0\omega + \frac{1}{2}[\omega,\omega]\right) + \delta(\delta^{-1}\omega). \tag{15}$$

In this equation $\delta^{-1}\omega$ can be freely chosen, we set

$$\delta^{-1}\omega = \Phi.$$

Now, since δ^{-1} raises the degree in y by one, we see that (15) recursively determines all orders of ω.

The antiholomorphic part *om$\bar{e}$ga* is then related to the curvature of ∇ by

$$\nabla\bar{\omega} = -\left[\bar{\nabla}_0, \nabla\right].$$

This equation also can be solved uniquely, in the same way that the equation $\nabla\hat{f} = 0$ could be solved in the preceding section when lifting functions.

We still need to show is that the $\bar{\omega}$ thus defined satisfies the antiholomorphic part of the Maurer-Cartan equations, namely

$$\bar{\partial}_0\bar{\omega} + \frac{1}{2}[\bar{\omega},\bar{\omega}] = 0 \tag{16}$$

Lemma 13. *Eqn. (16) holds.*

Proof. The equation obviously holds up to order y^0, since all occuring terms are $O(y^2)$. But by uniqueness of the lifting (describe) it is sufficient to show that both sides are annihilated by ∇.

$$\begin{aligned}
\nabla(\bar{\partial}\bar{\omega} + \frac{1}{2}[\bar{\omega},\bar{\omega}]) &= \left\{\nabla, \bar{\partial}\right\}\bar{\omega} - \bar{\partial}\nabla\bar{\omega} + [\nabla\bar{\omega}, \bar{\omega}] \\
&= -\bar{\partial}\nabla\bar{\omega} \\
&= -\left[\bar{\partial}, \nabla\bar{\omega}\right] \\
&= \left[\bar{\partial}, \left\{\nabla, \bar{\partial}\right\}\right] = 0
\end{aligned}$$

The third equality is simply a change of notation, we switched from seeing $\bar{\omega}$ as a formal vector field valued form to seeing it as an operator on the formal functions. In this notation, the last equality becomes an obvious consequence of $\bar{\partial}^2 = 0$. □

Remark 14. Note that it is here that we use that $\bar{\nabla}_0 = \bar{\partial}$. If we worked with a connection with arbitrary antiholomorphic part $\bar{\nabla}_0$ instead, we would have seen that the above calculation yields the constraint $\bar{\nabla}_0^2 = 0$. In fact, the reason for demanding that $\bar{\nabla}_0 = \bar{\partial}$ is to circumvent this additional complication.

Hence the first part of Proposition 6 has been proved.

Similarly, any compatible connection on $G_\infty E$ has the form

$$\nabla^E = \nabla_0^E + \nabla \otimes \mathbb{1}_E + E.$$

The Maurer-Cartan equation then reads

$$0 = (\nabla^E)^2 = (\nabla_0^E)^2 + \nabla_0^E E + \nabla E + \frac{1}{2}[E,E] = F + \nabla_0^E E + \nabla_0 E + \delta E + \omega E + \frac{1}{2}[E,E] \tag{17}$$

Here F is simply the curvature form of ∇_0^E. We want to solve this equation for E. As before, applying δ^{-1} to both sides we obtain

$$E = \delta^{-1}\left(F + (\nabla_0 + \nabla_0^E)E + \omega E + \frac{1}{2}[E,E]\right) + \delta(\delta^{-1}E). \tag{18}$$

Here again $\delta^{-1}E$ can be freely chosen, we set

$$\delta^{-1}E = \Phi^E.$$

Apart from this, (18) constitutes an explicit recursive formula for E.

Almost the same calculation as above shows that the antiholomorphic part $\bar{E}$ satisfies the $\Omega^{0,2}$ part of the Maurer-Cartan euation. Hence Proposition 6 has been proved.

Example 15. For a K"ahler manifold M we can explicitly sove the Maurer Cartan equations (14) for $\delta^{-1}\omega = 0$. Let the connection A be the Levi Civita connection. Then it is easily checked that

$$\begin{aligned} \omega &= 0 \\ \overline{\omega} &= \sum_{j\geq 0} (\delta^{-1}\nabla_0)^j \delta^* \bar{R}. \end{aligned}$$

is the unique solution satisfying

$$\delta^{-1}\omega = 0.$$

Similarly, if the connection ∇_0^E comes from a Hermitian metric on the bundle, we obtain

$$\begin{aligned} E &= 0 \\ \bar{E} &= \sum_{j\geq 0} \left[\delta^{-1}(\nabla_0 + \nabla_0^E)\right]^j \delta^* \bar{F}. \end{aligned}$$

Remark 16. The $T^{*0,2}$-component of the Maurer Cartan equation, i.e.,

$$\overline{\partial}\overline{\omega} + \frac{1}{2} = 0$$

states that $\overline{\partial} + \overline{\omega}$ defines a P_∞ structure on $ST^{1,0}M \otimes \Omega^{0,1}M[-1]$. See [9].

2.5. **Constructing Flat Connections II: Formal Exponential Maps.** We saw in section 2.1 that the components in $G_\infty M$ of degree 0 and 1 in y can be canonically identified with the bundle of 1-jets of holomorphic functions J_1M. It is easy to see that this isomorphism can be non-canonically extended to the bundle of ∞-jets $J_\infty M$. Picking a good flat connection on $G_\infty M$ is equivalent to picking one such extension

$$G_\infty M \cong J_\infty M. \tag{19}$$

The map from right to left ("horizontal to vertical") associates to the jet f at m its flat lift. It is given explicitly by the recursion

$$\hat{f}(m) = -\delta^{-1}\left(\nabla_0 \hat{f} + \omega \hat{f}\right)(m)$$

which involves only the derivatives of f at m as it should be.

The map from the left to the right ("vertical to horizontal") is given by parallel transporting $\hat{f}$ to a neighborhood of m and then taking the jet of the y^0-component.

To further discuss the correspondence between local coordinates and the flat good connection we introduce the following definition by Kapranov [9].

Definition 17. The *bundle $Exp_\infty M$ of formal exponential maps* is the bundle whose fiber at the point m is the affine space of jets at 0 of holomorphic mappings

$$\phi : T_mM \to M$$

such that

(i) $\phi(0) = m$.
(ii) $d\phi(0) = \mathbb{1}_{\mathrm{End}\, T_m E}$.

Picking a section of $Exp_\infty M$ is also equivalent to picking a particular isomorphism

$$G_\infty M \cong J_\infty M. \tag{20}$$

by composition. Hence, by the aforementioned, equivalent to picking a flat connection of $G_\infty M$. The mapping between flat connections and sections of $Exp_\infty M$ can be spelled out in detail as follows: Given a flat connection ∇ we have, we can map the linear functions y^j in the fiber of $G_\infty M$ over $m \in M$ to jets of functions over m by (19). They define local coordinates and can be inverted to yield the desired parameterization.

In the other direction, it is easiest to work in local coordinates z^j. Let the section of $Exp_\infty M$ in these cordinates be given by functions $y \in T_z M \mapsto \phi^j(z,y)$. We want to find a connection

$$\nabla = d + A$$

such that for any holomorphic function $f(z)$ the image of its jet under (20) is constant:

$$0 = (d+A)(f\circ\phi) = \frac{\partial f}{\partial z^j}\left(\frac{\partial \phi^j}{\partial z^i}dz^i + \frac{\partial \phi^j}{\partial y^i}A^i\right).$$

This equation must hold for any f, hence we can deduce that

$$A^i = -\left[\left(\frac{\partial \phi}{\partial y}\right)^{-1}\right]^i_j \frac{\partial \phi^j}{\partial z^k}dz^k.$$

We also saw in Proposition 6 that the space of flat connections was isomorphic to the affine space of pairs (∇_0, Φ), where ∇_0 is a torsion-free affine connection and Φ is a section of $D_\infty^{1,\geq 3}$. By the above, also the sections of $Exp_\infty M$ are in one-to-one correspondence to these pairs. We can state several facts about this correspondence.

Proposition 18. *The pair $(\nabla_0, 0)$ corresponds to the section of $Exp_\infty M$ given by holomorphic exponential coordinates wrt. the connection ∇_0.*

For the proof, we need the following Lemma, interesting in its own right.

Lemma 19. *For the special choice $\delta^{-1}\omega = 0$ the lift of a function f is given by*

$$\hat{f} = \sum_{j\geq 0} (\delta^{-1}\nabla_0)^j f. \tag{21}$$

Proof. Write

$$\omega = dz^k \omega_k^l \frac{\partial}{\partial y^l}.$$

Then the equality $\delta^{-1}\omega = 0$ means that $y^k\omega_k^l = 0$ for all l. Hence the second term on the rhs. of

$$\hat{f} = \delta^{-1}(\nabla_0 \hat{f}) + \delta^{-1}(\omega \hat{f}).$$

can be dropped and (21) easily follows. □

Proof of Proposition 18. We first recall the definition of formal exponential coordinates: If the connection ∇_0 is holomorphic, exponential coordinates around a point $m \in M$ define a holomorphic map $T_m M \to M$. But if ∇_0 is not holomorphic, neither are the exponential coordinates. Nonetheless, note that the k-jet of the exponential map $T_m M \to M$ depends only on the k-jet of the connection (coefficients of) ∇_0 at m for $k = 1, 2, ...$ This means in particular that the ∞-jet of this map is defined given any ∞ jet of connection coefficients, regardless of whether the Taylor series it defines converges. Now holomorphic exponential coordinates are those defined by the holomorphic jet of the connection ∇_0 at m. In local coordinates where

$$\nabla_0 = d + A$$

the holomorphic jet of A is simply the formal sum

$$\sum_I \frac{\partial A}{\partial z^I} y^I.$$

In case A is real analytic, this sum converges and so does the jet we called the "holomorphic exponential map".

Now we want to prove the proposition. We will work in the holomorphic exponential coordinates z^j around $m \in M$. Of course, these are in general not actually coordinates, but formal Taylor series of all function involved can still be defined. We need to show that the flat lift of the function z^k at the point $z = 0$ (i.e., at m) is y^k. But by the preceeding lemma, the lift is given explicitly by (21), i.e.,

$$\hat{(z)}^k = \sum_{j \geq 0} (\delta^{-1} \nabla_0)^j z^k. \tag{22}$$

Let the connection coefficients of ∇_0 be A^i_{kj}, so that

$$\nabla_0 = d + dz^k \left[A^i_{kj} y^j \frac{\partial}{\partial y^i}, \cdot \right].$$

Since the z^j are exponential coordinates, we know that the symmetrization of the s-th derivatives, $s = 0, 1, ..$ of A at 0 vanishes

$$A^i_{kj, r_1 \cdots r_s}(0) y^j y^j y^{r_1} \cdots y^{r_s} = 0.$$

But it is clear that at least one such term occurs in every term on the right of (22) that involves $A's$, when evaluated at 0. Hence, at 0, the only contributions left are

$$\hat{(z)}^k(0) = \left[(1 + (\delta^{-1} d)) z^k \right](0) = z^k(0) + y^k = y^k$$

and the proposition is proven. □

In general we have one-to-one correspondencies between (i) pairs (∇_0, Φ), (ii) flat connections ∇ and (iii) sections ϕ of $Exp_\infty M$. We can explicitly compute the linear response of a change of one of these onto the other two.

Proposition 20. *Let the objects (∇_0, Φ), ∇, ϕ be mapped to each other under the above one-to-one correspondence. Let further $\Delta \in \Gamma(D^{1 \geq 2}_\infty)$. Let μ be the unique solution to*

$$\delta^{-1} \nabla \mu = \Delta$$

Then to linear order in ϵ the following objects correspond to each other

(i) *The pair $(\nabla_0 + \epsilon \delta \Delta^2, \Phi + \epsilon \Delta^{\geq 3})$. Here $\Delta = \Delta^2 + \Delta^{\geq 3}$ is the splitting into quadratic and higher components in y.*
(ii) *The flat connection $\nabla + \epsilon \nabla \mu$.*
(iii) *The section of the formal exponential maps $\phi \circ (id - \epsilon \mu)$. Here $(id - \epsilon \mu)$ at a point $m \in M$ is considered as a diffeomorphisms of $T_m M$.*

Proof. □

2.6. **Interpolating Flat Connections.** In the following, it will be a central topic to interpolate between k flat connections $\nabla^0, \nabla^1, .., \nabla^k$ of $G_\infty M$ given on a common domain, say an open set $U \subset M$. Of course, we cannot just take the affine combination

$$\lambda_0 \nabla^0 + \lambda_1 \nabla^1 + \cdots + \lambda_k \nabla^k$$

where $\lambda_0 + .. + \lambda_k = 1$, because this will not be flat, in general. However, note that the space of flat connections is isomorphic to the space of sections of $Exp_\infty M$ and to the space of pairs (∇_0, Φ), both of which are endowed with non-isomorphic affine structures. We can use any of these to interpolate the connection, i.e.,

$$\nabla^\lambda := A(\lambda_0 \nabla_0^0 + \cdots + \lambda_k \nabla_0^k, \lambda_0 \Phi^0 + \cdots + \lambda_k \Phi^k)$$

or

$$\nabla^\lambda := B(\lambda_0 \phi^0 + \cdots + \lambda_k \phi^k).$$

These two ways ways to interpolate are different, as can be seen by an explicit calculation in one dimension. In this paper, we will generally stick to the first way, because it leads to explicit, relatively simple formulas. In contrast, the second way involves inversion of functions which is generally quite difficult.

The interpolated connection ∇^λ gives rise to a connection on the pullback bundle

$$\pi^* G_\infty M \to \Delta_k \times U$$

where Δ_k is the k-simplex and

$$\pi : \Delta_k \times U \to U$$

is the projection onto the second factor. This connection has the form

$$d_\lambda + \nabla^\lambda$$

where d_λ is the differential on Δ_k and is *not* flat. However, we can uniquely extend it to a flat connection

$$\nabla = d_\lambda + \nabla^\lambda + \omega_\lambda$$

where $\omega_\lambda \in \Gamma(\pi^* D_\infty^{1;\geq 2} U) \otimes \Omega^1(\Delta_k)$ is a formal vector field valued form. The uniqueness can be seen as follows. Note that the Maurer-Cartan equation for ∇ reads

$$\nabla^\lambda \omega_\lambda = -\left\{d_\lambda, \nabla^\lambda\right\} - d_\lambda \omega_\lambda - \frac{1}{2}\left[\omega_\lambda, \omega_\lambda\right].$$

The component in $\pi^* D_\infty U \otimes \Omega^1(\Delta_k) \otimes \Omega^1(U)$ reads

$$\nabla^\lambda \omega_\lambda = -\left\{d_\lambda, \nabla^\lambda\right\} \tag{23}$$

and can be uniquely solved by invertibility of ∇^λ. The remaining components read

$$d_\lambda \omega_\lambda + \frac{1}{2}\left[\omega_\lambda, \omega_\lambda\right] = 0. \tag{24}$$

This is a nontrivial *relation* that has to be proved to be satisfied by the unique solution to (23).

Lemma 21. *Eqn. (24) holds.*

Proof. The proof of Lemma 13 goes through almost without change: The constant order of (24) vanishes, hence it is sufficient to show that

$$\begin{aligned} 0 &= \nabla^\lambda \left(d_\lambda \omega_\lambda + \frac{1}{2}\left[\omega_\lambda, \omega_\lambda\right] \right) \\ &= \left\{\nabla^\lambda, d_\lambda\right\} \omega_\lambda - \nabla^\lambda \omega_\lambda + \left[\nabla^\lambda \omega_\lambda, \omega_\lambda\right] \\ &= -d_\lambda(\nabla^\lambda \omega_\lambda) = \left[d_\lambda, \left\{d_\lambda, \nabla^\lambda\right\}\right] = 0 \end{aligned}$$

□

3. The Construction of Engeli and Felder

3.1. **Hochschild Cohomology of the Weyl Algebra.** The key step in the construction of the $2n$-form μ in (1) is to write down an explicit formula for a $2n$-Hochschild-cocycle τ_{2n} of the "formal Weyl algebra" $\mathbb{C}[[y^1,..,y^n]][\frac{\partial}{\partial y^1},..,\frac{\partial}{\partial y^n}]$. This cocycle was constructed in [8], we merely state the formula here:

$$\tau_{2n}(a_0\otimes\cdots\otimes a_{2n}) = \mu_{2n}\int_{\Delta_{2n}}\prod_{0\le i<j\le 2n} e^{(2u_i-2u_j+1)\alpha_{ij}}\pi_{2n}(a_0\otimes\cdots\otimes a_{2n}). \tag{25}$$

The operator $\pi_2 n$ is defined as

$$\pi_{2n} = \det\begin{pmatrix} {}_1\partial_1 & {}_1\partial_2 & \cdots & {}_1\partial_{2n} & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_{2n}\partial_1 & {}_{2n}\partial_2 & \cdots & {}_{2n}\partial_{2n}. & \end{pmatrix}$$

where the operator ${}_i\partial_\alpha$ on a tensor product of differential operators $a_0\otimes\cdots\otimes a_{2n}$ acts as

$${}_i\partial_\alpha(a_0\otimes\cdots\otimes a_{2n}) = a_0\otimes\cdots\otimes a_{i-1}\otimes[Y_\alpha,a_i]\otimes a_{i+1}\otimes\cdots\otimes a_{2n}$$

where $Y_{2k-1}=\frac{\partial}{\partial y^k}$ and $Y_{2k}=y^k$.

Furthermore

$$\alpha_{ij} := \frac{1}{2}\sum_{k=1}^{n}({}_i\partial_{2k-1})({}_j\partial_{2k}) - ({}_i\partial_{2k})({}_j\partial_{2k-1})_i.$$

Finally μ_{2n} is symmetrization followed by evaluation at $y=0$ and multiplication

$$\mu_{2n}(a_0\otimes\cdots\otimes a_{2n}) = (Sa_0)(0)(Sa_1)(0)\cdots(Sa_{2n})(0)$$

Here, the symmetrization map S is defined as

$$S(Y_1\cdots Y_k) = \frac{1}{k!}\sum_\sigma \operatorname{sgn}(\sigma)Y_{\sigma_1}\cdots Y_{\sigma_k}.$$

Similarly, one can define a $2n$-cocycle τ^r_{2n} on the Hochschild cohomology of the algebra of matrices of differential operatos $\mathbb{C}[[y^1,..,y^n]][\frac{\partial}{\partial y^1},..,\frac{\partial}{\partial y^n}]\otimes\mathbb{C}^{r\times r}$ as follows.

$$\tau^r_{2n}((a_0\otimes M_0)\otimes\cdots\otimes(a_{2n}\otimes M_{2n})) = \tau_{2n}(a_0\otimes\cdots\otimes a_{2n})\operatorname{tr}(M_0\cdots M_{2n}) \tag{26}$$

Remark 22. These long definitions have been cited here for completeness. In the following, only a single feature will be important: The map π_{2n} is defined such that the commutator of one of the operators $a_1,..,a_{2n}$ with y^1 occurs in every term. This means that if all the a_k, $k=1,..,2n$ are independent of y^1, then the cocycle vanishes on them. This fact will imply lateron that the Lefschetz number vanishes if there exists a nonzero holomorphic vector field on M.

3.2. **Construction of μ.** The differential form $\mu(D)$ of equation (1) can now be defined as

$$\mu(D) = \tau^r_{2n}(\hat{D}\otimes\Omega^{\wedge n})$$

where we used the shorthand $\Omega = \delta+\omega+\bar{\omega}+E+\bar{E}$ for the part of the connection coefficients that is not linear in y. Here τ^r_{2n} is a Hochschild $2n$-cocycle on $\mathcal{D}_{n,r}$. This cocycle is given by an explicit integral formula in [8] and is invariant wrt. linear coordinate transformations.

3.3. **The Formula in Sheaf Language.** We now reformulate the formula (1) in a sheavy language. We start with the sheaf of algebras $\mathcal{D}_E$ and want to compute a Hochschild 0-cocycle of the algebra of global sections $H^0(M; \mathcal{D}_E)$. There is a natural map from this algebra into the zeroth hypercohomology of the complex of sheaves

$$\mathcal{D}_{E,\bullet} : \cdots \to \mathcal{D}_{E,-2n} \to \mathcal{D}_{E,-2n+1} \to \cdots \to \mathcal{D}_{E,0}.$$

where $\mathcal{D}_{E,-p}$ is the sheaf associated to the presheaf of Hochschild p-chains of holomorphic differential operators on M.

Lemma 23 (Brylinski, Engeli and Felder)**.** *The hypercohomology $H^\bullet(M; \mathcal{D}_{E,\bullet})$ of $\mathcal{D}_{E,\bullet}$ satisfies*

$$H^p(M; \mathcal{D}_{E,\bullet}) \cong H^{2n+p}(M; \mathbb{C}).$$

In particular, the zeroth hypercohomology is one-dimesional and the natural composition

$$H^0(H^0(M; \mathcal{D}_E)) \to H^0(M; \mathcal{D}_{E,\bullet}) \to H^{2n}(M; \mathbb{C}) \to \mathbb{C}$$

coincides, up to normalization, with the supertrace (1).

Proof. Let any finite good open cover $\underline{U}$ of M be given. Then Engeli and Felder [6] have essentially proved that the cohomology of the corresponding Čech -Hochschild double complex satisfies the above assertions. We only need to show that the statements remain true after passing to the direct limit over open covers. I.e., we need to show that the natural map from the Čech hypercohomology wrt. the given cover $\underline{U}$ into the sheaf hypercohomology is bijective. Since any open cover has a finite good refinement (???), it is sufficient to show that for any finite good refinement $\underline{V}$ of $\underline{U}$ the natural map

$$H^\bullet(\underline{U}; \mathcal{D}_{E,\bullet}) \to H^\bullet(\underline{V}; \mathcal{D}_{E,\bullet})$$

is an isomorphism. Since both spaces are isomorphic to $H^{2n+p}(M; \mathbb{C})$, it is sufficient to show that

$$\begin{array}{ccc} H^\bullet(\underline{U}; \mathcal{D}_{E,\bullet}) & \longrightarrow & H^\bullet(\underline{V}; \mathcal{D}_{E,\bullet}) \\ & \searrow^{\cong} \quad \swarrow^{\cong} & \\ & H^{2n+\bullet}(M; \mathbb{C}) & \end{array}$$

commutes. But, using Leray's Theorem, this is equivalent to showing that

$$\begin{array}{ccc} H^\bullet(\underline{U}; \mathcal{D}_{E,\bullet}) & \longrightarrow & H^\bullet(\underline{V}; \mathcal{D}_{E,\bullet}) \\ \cong \downarrow & & \downarrow \cong \\ H^{2n+\bullet}(\underline{U}; \mathbb{C}) & \xrightarrow{\cong} & H^{2n+\bullet}(\underline{V}; \mathbb{C}) \end{array}$$

commutes. But this is easily shown by tracing the definitions of the four maps involved. □

Although the map

$$H^\bullet(M; \mathcal{D}_{E,\bullet}) \cong H^{2n+\bullet}(M; \mathbb{C})$$

is given explicitly by the above construction, it is quite hard to compute in this language, and furthermore, is non-local. We would like to realize this map as the map on cohomology induced by a homomorphism of sheaves of complexes

$$\mathcal{D}_{E,\bullet} \to \Omega^\bullet$$

where $\Omega^\bullet$ is the sheaf of holomorphic forms on M with differential ∂. Its hypercohomology is the de Rham cohomology.

However, such a map of complexes of sheaves does in general not exist, for it would in particular imply that $L(D) = 0$. Such a map can also be seen as a global section of the sheaf of homomorphims of sheaves of complexes

$$\mathcal{H}\wr\Updownarrow(\mathcal{D}_{E,\bullet}, \Omega^\bullet)$$

and the evaluation of such a map on a cocycle can be seen as the cup product coming from the pairing

$$\mathcal{H}\wr\Updownarrow(\mathcal{D}_{E,\bullet}, \Omega^\bullet) \otimes \mathcal{D}_{E,\bullet} \to \Omega^\bullet$$

coming from evaluation.

This viewpoint suggests that a natural way to explicitly construct the map

$$\Phi : \mathbb{H}^\bullet(M; \mathcal{D}_{E,\bullet}) \cong \mathbb{H}^{2n+\bullet}(M; \mathbb{C}) \tag{27}$$

is by taking the cup product with a cocycle ϕ in the hypercohomology

$$\mathbb{H}^{2n}(\mathcal{H}\wr\Updownarrow(\mathcal{D}_{E,\bullet}, \Omega^\bullet)).$$

So we will have

$$\Phi(\cdot) = [\phi] \cup (\cdot). \tag{28}$$

where a representative ϕ can be explicitly constructed. In fact, a Dolbeault representative is already provided by Engeli and Felder: It will be denoted by ϕ^{EF} and is defined as

$$\phi^{EF} = \sum_j \phi_j^{EF}$$

where

$$\phi_j^{EF} : \mathcal{D}_{E,j} \to \Omega^\bullet \tag{29}$$

is given by

$$\phi_j(\eta) = \tau_{2n}^r(\hat{\eta} *_{sh} 1 \otimes \omega \otimes \dots \omega) \tag{30}$$

where $*_{sh}$ is the shuffle product.

In this paper we will present a Čech representative ϕ. This also means, that we will deal exclusively with holomorphic objects.

4. Construction of ϕ

In this section, we will construct a Čech representative of the cocycle ϕ. For this, assume that we are given a good locally finite open cover U_α of M. The component of Φ of Čech degree k is defined to be

$$\phi[\alpha_0, .., \alpha_k](\eta) = \int_{\Delta_{2n-k}} \tau_{2n}^r(\hat{\eta} *_{sh} 1 \otimes \omega_\Delta \otimes \dots \omega_\Delta). \tag{31}$$

Here, the forms ω_Δ are "extended" Maurer-Cartan forms living on $\Delta_{2n-k} \times \bigcup_{j=0}^k U_{\alpha_k}$.

Note that this is similar to eqn. (30), but we integrate over the simplex and the integrand contains extended Maurer-Cartan forms ω_Δ living on $\Delta_{2n-k} \times \bigcup_{j=0}^k U_{\alpha_k}$. We will postpone the construction of these forms to section 4.1, but state the following lemma to be proved there.

Lemma 24. *Let $\mu(\eta)$ denote the integrand in eqn. (31). Then*

$$d\mu(\eta) = \mu_{b\eta}$$

where d is the de Rham differential on $\Delta_{2n-k} \times \bigcup_{j=0}^k U_{\alpha_k}$ and b is the Hochschild boundary operator.

Using this lemma, we can prove the following

Proposition 25. *ϕ as constructed above is a cocycle.*

Proof. Let η be a local section of $\mathcal{D}_{E,\bullet}$. We need to show that

$$(\delta + d)\phi(\eta) = \phi(b\eta).$$

But by Lemma 24 we know that

$$\begin{aligned} d\phi(\eta)[\underline{\alpha}] &= d\int_{\Delta_{2n-k}} \mu(\eta) \\ &= \int_{\partial\Delta_{2n-k}} \mu(\eta) + \int_{\Delta_{2n-k}} d\mu(\eta) \\ &= (\delta\phi)(\eta)[\underline{\alpha}] + \phi(b\eta)[\underline{\alpha}] \end{aligned}$$

(TODO:signs) □

4.1. **Construction of the $\omega_\Delta[\alpha]$.** In the following $\underline{\alpha} = (\alpha_0, .., \alpha_k)$. Let $\pi : \Delta_k \times U_{\underline{\alpha}} \to U_{\underline{\alpha}}$ be the projection onbto the second factor. The forms $\omega_\Delta[\underline{\alpha}]$ are holomorphic forms on $\Delta_k \times U_{\underline{\alpha}}$ with values in the first order formal differential operators on $\pi^* \, E|_{U_{\underline{\alpha}}}$.

4.2. $[\phi] \cup \cdot = [\phi^{EF}] \cup \cdot$.

Theorem 26. *The cocycle ϕ as constructed above induces, via (28), the isomorphism (27) on the hypercohomologies.*

Proof. Since any hypercohomology class in $\mathbb{H}^p(M; \mathcal{D}_{E,\bullet})$ can be represented by a cocycle

$$\eta \in \check{(C)}^{2n-p}(M; \mathcal{D}_{E,2n})$$

it is sufficient to check the Theorem on these cocycles. But there, the only nonvanishing contributions come from the components of ϕ in

$$\check{(C)}^0(\mathcal{H}\wr\Updownarrow(\mathcal{D}_{E,2n}, \Omega^0))$$

and these components are just the applications of τ^r_{2n} to lifts of the local Hochschild $2n$-cycles $\eta[\underline{\alpha}]$. These resulting numbers are, however, independent of local connections ∇ on $G_\infty(E)$ chosen and equal the maps in the definition (**??**) of Φ. □

5. Blowing ups

We can use the above construction of ϕ to prove Theorem 2. We will use the same notation as in the statement of the theorem. To define ϕ pick local coordinates z^j around m. Similarly to [1], construct an open cover of M as follows:

$$\begin{aligned} U_0 &= \{|z_j| < 1 \ \forall j\} \\ U_i &= \{|z_j| < 2 \ \forall j; z_i \neq 0\} \quad i = 1, .., n \end{aligned}$$

Complete the U_α to a cover with open sets V_β such that

$$V_\beta \cap U_0 = \emptyset.$$

Construct an open cover of $\tilde{M}$:

$$\begin{aligned} \tilde{V}_\beta &= \pi^{-1} V_\beta \\ \tilde{U}_j &= \left[\pi^{-1}\left(U_j \cup \{x\}\right)\right]^\circ \ j = 1, .., n \end{aligned}$$

To construct the $\omega_\Delta[\underline{\alpha}]$, we assign the following local holomorphic connections to these open sets.

- To the set U_0 the flat connection that vanishes in the coordinates z^j.

- To the set U_i, $i = 1, 2, ..$ we assign the flat connection vanishing in the coordinates w_i^j defined by

$$w_i^j = \begin{cases} z^i & i = j \\ \frac{z^j}{z^i} & i \neq j \end{cases}$$

  To the sets $\tilde{U}_j$ we assign the connection defined in the same way.
- To the sets V_β we arbitrarily assign some connections. To the set $\tilde{V}_\beta$ we assign the same connection as to the set V_β.

When constructing ϕ as above for the original manifold and the blowup, we see that the only difference is the term $\phi_n[0, 1, .., n]$ defined on

$$U_0 \cup U_1 \cup \cdots \cup U_n.$$

This term is present for the original manifold M, but missing for the blowup $\tilde{M}$. All other terms contributing to $\Phi(D)$ or $\Phi(\tilde{D})$ respectively are exactly the same. Hence the difference of the Lefschetz numbers is exactly the contribution of this term:

$$L(D) - L(\tilde{D}) = \int \phi_n[\{0, 1, .., n\}] \cup D$$

However, the map $\int$ on the right hand side reduces to the map Res_x, as shown in Lemma 45. Hence Theorem 2 is proven.

5.1. **Computation of the Maurer Cartan form and Lefschetz number.** Using the explicit connections given in the previous paragraph, we can calculate the Maurer Cartan form and the quantity $L(D)-L(\tilde{D})$ in lowest orders. Decompose ω_Δ into a form in $\Omega(M)$ and $\Omega(\Delta_n)$ respectively:

$$\omega_\Delta = \omega_M + \omega_\lambda$$

We have

$$\omega_M = \delta^{-1}\left(F + d\omega_M + [A, \omega_M] + \frac{1}{2}[\omega_M, \omega_M]\right) \tag{32}$$

$$\omega_\lambda = \delta^{-1}\left(d\omega_\lambda + [A, \omega_\lambda] + [\omega_M, \omega_\lambda] + d_\lambda A\right) \tag{33}$$

We will work throughout in the coordinate system on U_0. First, let us treat the classical RRH case where $D = \tilde{D} = 1$. The total number of derivatives in the formal variable involed in the computation of the integrand

$$\mu = \tau_{2n}(\hat{1}, \omega_\Delta, \ldots, \omega_\Delta)$$

is easily seen to be $2n$. Since the lowest order terms in ω_λ are already of order 2, and we need ω_λ n times, we see that we need to compute all quantities to lowest orders only. But then

$$\begin{aligned} \omega_M &= 0 \\ \omega_\lambda &= \delta^{-1} d_\lambda A = \frac{1}{2}\sum_{\alpha=1}^{n} A_{kl}^{(\alpha)m} y^k y^l \partial_m d\lambda_j \end{aligned}$$

Inserting this into

5.2. **Computation of Connection Coefficients.** Let us now compute the connection one-forms $A^{(\alpha)}$ of the connections $\nabla^{(\alpha)}$ on the open sets U_α. We will work in the above cordinates z^j. Then, of course, $\nabla^{(0)} = d$. Furthermore $\nabla^{(i)}$, $i = 1, .., n$ must annihilate the vectors

$$\begin{aligned} e_1^{(i)} &= z^i \frac{\partial}{\partial z^1} \\ e_2^{(i)} &= z^i \frac{\partial}{\partial z^2} \\ &\vdots \\ e_i^{(i)} &= \sum_{j=1}^n \frac{z^j}{z^i} \frac{\partial}{\partial z^j} \qquad \vdots \\ e_n^{(i)} &= z^i \frac{\partial}{\partial z^n} \end{aligned}$$

because these are the coordinate vector fields in the coordinates w_i^j described above, as is easily checked by applying them to these functions. It follows that for $k \neq i$

$$\nabla^{(i)} \frac{\partial}{\partial z^k} = -\frac{dz^i}{z^i} \frac{\partial}{\partial z^k}$$

and furthermore

$$\nabla^{(i)} \frac{\partial}{\partial z^i} = -\sum_{\substack{j=1 \\ j \neq i}}^n \left(d\frac{z^j}{z^i} - \frac{dz^i}{z^i} \right) \frac{\partial}{\partial z^j} = \frac{1}{(z^i)^2} \sum_{\substack{j=1 \\ j \neq i}}^n \left((z^j + z^i) dz^i - z^i dz^j \right) \frac{\partial}{\partial z^j}$$

From the l.h.s. of these equations we get the connection one-forms, which can be written in the form

$$A_k^{(i)j} = \frac{\delta_k^i}{(z^i)^2} \left((z^j + z^i) dz^i - z^i dz^j \right) - \frac{dz^i}{z^i} \delta_k^j$$

valid for all k.

The general formula is

$$A_{kl}^{(\alpha)m} = \frac{\partial^2 w_\alpha^i}{\partial z^k \partial z^l} \frac{\partial z^m}{\partial w_\alpha^i}.$$

6. A Bott Formula

6.1. **A Vanishing Theorem.** Now suppose that there exists a nowhere vanishing holomorphic vector field ξ on M, together with a holomorphic lift ∇_ξ to E. The following result and particularly its corollary is the main goal of this subsection.

Proposition 27. *If there exist ξ, ∇_ξ as above, we can find a flat connection $\boldsymbol{\nabla}$ on $G_\infty M$ and a compatible flat connection $\boldsymbol{\nabla}^E$ on $G_\infty E$ satisfying*

$$\hat{\nabla}_\xi = \delta(xi) \tag{34}$$

*Such a pair of connections will be called ξ-*constant.

Corollary 28. If there exists a nowhere vanishing holomorphic vector field on M together with a holomorphic lift to E, the Lefschetz number vanishes identically. I.e.,

$$L(D) = 0$$

for any holomorphic differential operator D on E.

of the Corollary. Pick a Maurer Cartan form ω as in Proposition 27. To evaluate τ^r_{2n} as in eqn. (26) at $m \in M$, pick a basis $\{\partial_1, .., \partial_n\}$ of $T_m M$, such that $\partial_1 = \xi_m$. As in Remark 22, note that he definition of τ_{2n} involves a ∂_1-derivative applied to ω. But this yields zero by (34). □

We now want to proof Proposition 27, i.e., construct a suitable Maurer Cartan form satisfying (34). Just as in sections 2.4 and 2.5 there are two ways to do this construction, a "Fedosov way" and a "formal exponential maps way". We will treat the Fedosov way first.

6.1.1. *Constructing ξ-constant Flat Connections I: Fedosov's Way.* This section will be almost a copy of section 2.4.

Definition 29. A pair of connections ∇_0 on TM and ∇_0^E on E is called ξ-*constant*, if for any vector field Y on M

$$\nabla_{0,\xi Y} = [\xi, Y]$$

and for any section s of E

$$\nabla^E_{0,\xi} s = \nabla_\xi s.$$

It is easily seen that the constant order of (34) says that a ξ-*constant* connection ∇^E on $G_\infty E$ comes from a pair of ξ-constant connections ∇^T on TM and ∇^E on E.

Similarly to Proposition 6 we have the following:

Proposition 30. *The set of ξ-constant flat connections ∇^E is* non-canonically *in one-to-one correspondence with the set of collections of the following data*

- *A ξ-constant torsion free affine connection ∇_0 and a ξ-constant connection ∇_0^E on E.*
- *A section $\Phi \in \Gamma(D^{1,\geq 3}_\infty M \oplus D^{0,\geq 2}_\infty M)$ such that*

$$[\delta(\xi), T] = 0.$$

The important difference to Proposition 6 is that the correspondence is not canonical and depends on some extra data, namely a section of the bundle of projections from TM to the subbundle spanned by ξ. Equivalently, we need a splitting of the exact sequence

$$0 \to Ann_\xi(T^*M) \to T^*M \to T^*M/Ann_\xi \to 0$$

of the cotangent space. This is, of course, undesirable, but at present we cannot give a canonical characterisation of the space of ξ-constant flat connections.

Choosing such a projection is equivalent to choosing a (smooth) section $\eta \in \Gamma(T^{*1,0})$ such that

$$\iota_\xi \eta = 1.$$

Similarly to eqns. (7), (8) we define the following operators

$$\delta_\xi = \delta - \epsilon_\eta [\delta(\xi), \cdot] \tag{35}$$

$$\delta^*_\xi = \delta^* - y_\eta \iota_\xi \tag{36}$$

(37)

suppressing the dependence on η. In these formulas, we denoted by ϵ_η exterior multiplication by η and by y_η the image of the canonical inclusion

$$T^{*1,0} \hookrightarrow G_\infty M.$$

Similarly to Lemma 10 we have

Lemma 31. *The operators δ_ξ, δ_ξ^* satisfy*

$$\left\{\delta_\xi, \delta_\xi^*\right\} = G_\xi \tag{38}$$

where G_ξ is the gradation operator counting (form degree+degree in y) on $Ann_\xi T^*M$. *For example*

$$G_\xi(y^{i_1}\cdots y^{i_p} y^i_\eta dz^{k_1}\cdots dz^{k_r}\wedge \eta d\bar{z}^{l_1}\cdots d\bar{z}^{l_s}) \\ = (p+r) y^{i_1}\cdots y^{i_p} y^i_\eta dz^{k_1}\cdots dz^{k_r}\wedge \eta d\bar{z}^{l_1}\cdots d\bar{z}^{l_s}.$$

where now $\left[\delta(\xi), y^{i_a}\right] = \iota_\xi dz^{k_b} = 0$ for $a=1,..,p$ and $b=1,..,r$.

Again, as in (8), we can define

$$\delta_\xi^{-1} = G_\xi^{-1}\delta_\xi^*$$

which satisfies

$$\left\{\delta(\xi), \delta_\xi^{-1}\right\} = G^{-1}G = 1 - \Pi_0 \tag{39}$$

where Π_0 is the projector onto the kernel of G_ξ.

Applying the operator δ_ξ^{-1} to both sides of the Maurer-Cartan equation and using that $\Pi_0\omega = 0$ we obtain the recursive definition

$$\omega = \delta_\xi^{-1}\left(R + \nabla_0\omega + \frac{1}{2}\left[\omega,\omega\right]\right) + \delta\Phi. \tag{40}$$

where we set $\delta^{-1}\omega = \Phi$.

Lemma 32. *The ω thus defined satisfies (34).*

Proof. In view of (39) it suffices to show that

$$[\delta(\xi), F] = 0$$

and

$$[\delta(\xi),\cdot]\,\nabla_0 = \nabla_0\,[\delta(\xi),\cdot]$$

The first equation easily follows from the second since

$$[R,\cdot] = \nabla_0^2.$$

Furthermore, since ∇ is a derivation

$$\begin{aligned} \nabla_0\,[\delta(\xi),\cdot] &= [\nabla_0\delta(\xi),\cdot] + [\delta(\xi),\nabla\cdot] \\ &= [\delta(\nabla_0\xi),\cdot] + [\delta(\xi),\cdot]\,\nabla_0 \\ &= 0 + [\delta(\xi),\cdot]\,\nabla_0 \end{aligned}$$

showing the second equation and thus the lemma. □

Just as section 2.4 we can compute $\bar{\omega}$, E and $\bar{E}$. Explicit formulas can be obtained by merely replacing δ by δ_ξ.

6.1.2. *Constructing ξ-constant Flat Connections II: Formal Exponential Maps.* By section 2.5 we can interpret any flat connection ∇ on $G_\infty M$ as a section of $Exp_\infty M$. Under this correspondence, the ξ-constant flat connections are mapped to those local coordinates in which ξ is a constant vector field, as is evident already by the definition (34). Furthermore, the affine linear combination of two sets of coordinate functions, in which ξ is constant, still has the property that ξ is constant in the linearily combined coordinates. In other words, ξ-constancy is compatible with the affine structure on $\Gamma(Exp_\infty M)$. This also implies that the subbundle of $Exp_\infty M$ formed by the ξ-constant section has contractible fibers and hence a global section. Of course, this result would also follow from the construction in the last section and the correspondence of section 2.5.

These considerations suggest that the "formal exponential maps-framework" is more natural to use than the "Fedosov-framework". However, practical calculations also quickly become quite involved.

Remark 33. We saw that, given an affine connection ∇_0, there was a canonical choice of flat connection ∇, namely $\Phi = 0$, which was equivalent to taking holomorphic exponential coordinates. Now however, in exponential coordinates ξ need not be constant, even if $\nabla_0 \xi = 0$. So there is not any more a canonical choice.

6.1.3. *Interpolating ξ-constant Flat Connections.* To adapt Theorem 26 to the ξ-constant context, we need to be able to interpolate between ξ-constant flat connections. In the formal exponential framework this is easily done since the affine structure on $\Gamma(Exp_\infty M)$ is compatible with ξ-constancy as mentioned in the last subsection.

In the Fedosov-framework, we can still interpolate the connections ∇_0 and the functions $\Phi = \delta_\xi^{-1}\omega$, but we also need to interpolate the auxiliary one-forms η. Hence the interpolation is non-canonical and depends on the special choice of η. This is an undesired fact.

For the proof of Theorem 1 we will also need to interpolate ξ-constant and non-ξ-constant flat connections. In the formal exponential maps framework this is not a problem, since we use the same interpolation method for ξ-constant and non-ξ-constant connections anyways. In the Fedosov framework, we first interpolate the ξ-constant connections using the method described above and then interpolate the resulting connection with the remaining ones using the standard interpolation method.

6.2. **A Bott Formula.** Given the vanishing theorem, Corollary 28, we can now prove Theorem 1 in the spirit of Baum and Bott [1]. Concretely, assume that there is a homolomorphic vector field ξ on M with holomorphic lift ∇_ξ to E, and that ξ has isolated zeros at $p_1, .., p_l$. For notational convenience, we will discuss the case of a single isolated zero at p. The generalization to $l > 0$ will be obvious.

Similarly to [1] and section 5, construct an open covering of M as follows. Pick coordinates z_j on a neighbourhood of p, such that $z_j(p) = 0$ for all $j = 1, .., n$. The vector field ξ will in these coordinates take on the form

$$\xi = \sum_{j=1}^{n} a^j \frac{\partial}{\partial z^j}.$$

Then define, for a sufficiently small constant ϵ,

$$\begin{aligned} U_0 &= \{|z_j| < \epsilon \forall j\} \\ U_i &= \{|z_j| < 2\epsilon \forall j; a^i \neq 0\} \; i = 1, .., n \end{aligned}$$

By rescaling the coordinates, we can, of course, assume that $\epsilon = 1$. Complete the U_α to a cover of M with open sets V_β such that

$$V_\beta \cap U_0 = \emptyset.$$

On $U_1, .., U_n$ and V_β, we can construct ξ-constant Maurer Cartan forms by one of the two methods described above. We then apply Theorem 26 and see that all contributions from intersections not involving U_0 vanish by Theorem 1: The intersection of U_0 with any set V_β is empty by construction. Furthermore, since the Maurer-Cartan forms on the U_α are all holomorphic, any intersection of less than $n+1$ of them does not contribute. Hence the only contribution present is that on

$$U_0 \cap U_1 \cap \cdots \cap U_n$$

Finally, by lemma 45 we get the residue formula, just as we got in the blow-up case, and the proof is complete.

7. Example: 1-dim case

Unfortunately, explicit calculations in the above framework quickly become quite complicated due to the nonlinearity of the Maurer-Cartan equations. The very simplest example where this does not happen is the case However, on a curve the nonlinear terms vanish and we can compute everything explicitly. Wlog. we can treat the case of a neighbourhood of the origin in $\mathbb{C}^1$ with the origin being an isolated zero of the vector field

$$\xi = a(z)\partial = a(z)\frac{\partial}{\partial z}.$$

On $U_0 = \{|z| < 1; z \neq 0\}$ we pick the connection $\nabla_{00} = d$. On $U_1 = \{|z| < 2; z \neq 0\}$ we pick the connection defined by $\nabla_{01}\partial = -\frac{a'(z)}{a(z)}dz\partial$ which is obviously ξ-constant. We obviously obtain that $\omega = 0$ by dimensional reasons. Furthermore, the recursive definition for ω_λ is

$$\omega_\lambda = \delta^{-1}\left(\frac{a'(z)}{a(z)}y\partial_y d\lambda dz + d\omega_\lambda + \frac{a'(z)}{a(z)}dz\,[y\partial_y, \omega_\lambda]\right).$$

We can set

$$\omega_\lambda = \sum_{j\geq 2}\omega_j\frac{y^j}{j!}\partial_y d\lambda$$

and obtain the recursion relation

$$\begin{aligned}\omega_{j+1} &= (\partial + u(j-1))\omega_j \\ \omega_2 &= -u\end{aligned}$$

for $u = \frac{a'}{a}$ the logarithmic derivative of a. The solution is obviously

$$\omega_j = -(\partial + u(j-2))\cdots(\partial + u)u.$$

For a linear vector field $u = \frac{1}{z}$ and hence $\omega_j = 0$ for $j \geq 3$.

Appendix A. The Simplex Space of M

A.1. **Basic Definitions.** Let $\bigcup_{\alpha\in A} U_\alpha = M$ be an open cover of M. The disjoint union of the U_α and all their finite intersections

$$\mathbb{U} = \coprod_{\substack{B\subset A\\|B|<\infty}}\bigcap_{\beta\in B} U_\beta$$

is naturally a simplicial space. Its geometric realization ([11], p. 403f) will be denoted ΣM and called the "simplex space of M".[2]

$$\Sigma M = \mathbb{U} \times_\Delta \mathbb{Delta}$$

There is a natural projection

$$\pi : \Sigma M \to M$$

coming from the projection

$$\mathbb{U} \to M.$$

We will here consider only *locally finite* open covers U_α and give less formal down-to-earth construction of ΣM:

(i) Begin with the manifold M.

[2]Although this notation does not mention the cover U_α for simplicity, it should be clearly kept in mind that ΣM essentially depends on it.

(ii) Construct the "bundle with varying fiber" ΣM over M which has as fiber over a point $m \in M$ the simplex Δ_k where

$$k = |\{\alpha \in A; m \in U_\alpha\}| - 1 \ \in \{0, 1, ..\}.$$

Label each corner of the simplex by one element of $\{\alpha \in A; m \in U_\alpha\}$.

(iii) Endow the set ΣM with the obvious topology such that corners with the same label are glued together. See Figure **??** for examples.

The central point is now, that we can endow the topological space ΣM with a kind of "smooth structure". Note first that for any $B \subset B' \subset A$ we have the inclusion

$$\Sigma M M \supset U_B \times \Delta_{|B|-1} \supset U_{B'} \times \Delta_{|B|-1} \overset{\iota_{BB'}}{\hookrightarrow} \supset U_{B'} \times \Delta_{|B'|-1} \subset \Sigma M.$$

The map $\iota_{BB'}$ is a smooth injective map between manifolds with boundary.

Definition 34. The space of *smooth p-forms on* ΣM, denoted $\Omega^p(\Sigma M)$, $p = 0, 1, \ldots$, consists of collections of p-forms η_B on the manifold with boundary $U_B \times \Delta_{|B|-1}$, such that for all $B \subset B' \subset A$ we have

$$\omega_B|_{U_{B'} \times \Delta_{|B|-1}} = \iota^*_{BB'} \omega_{B'}. \tag{41}$$

Lemma 35. *On the space $\Omega^\bullet(\Sigma M)$ the operations of exterior product "$\wedge$" and differentiation "d" are well defined.*

Proof. Both operations commute with pullbacks. □

Example 36. For each $\alpha \in A$, there is canonically defined smooth function $\lambda_\alpha \in \Omega^0(\Sigma M)$. It is zero on the sets

$$U_B \times \Delta_{|B|-1}$$

if $\alpha \notin B$ and equals the barycentric coordinate on $\Delta_{|B|-1}$ associated to the corner labeled by α if $\alpha \in B$. It is easily seen that this definition satisfies (41). These functions satisfy $\sum_{\alpha \in A} \lambda_\alpha = 1$

Let $(\rho_\alpha)_{\alpha \in A}$ be a partition of unity subordinate to the cover U_α, i.e., $\sum_{\alpha \in A} \rho_\alpha = 1$, $0 \leq \rho_\alpha \leq 1$ and $\overline{\operatorname{supp} \rho_\alpha} \subset U_\alpha$. It obviously defines a "smooth section"

$$\rho : M \to \Sigma M$$

of ΣM.

Definition 37. A *smooth section of* ΣM is a continuous map

$$\rho : M \to \Sigma M$$

that comes from a partition of unity of M subordinate to U_α.

Let now

$$\rho : M \to \Sigma M$$

be a smooth section and let ω be a smooth p-form on ΣM. Let $m \in M$ and $V \ni m$ be a small enough open neighbourhood, such that $V \subset U_{B_x}$ where

$$B_x = \{\alpha \in A; m \in U_\alpha\}$$

and such that

$$V \cap \overline{\operatorname{supp}} \rho_\beta = \emptyset$$

for all $\beta \notin B_x$. We have a smooth section of a bundle (in the standard sense)

$$\rho_{B_x,V} : V \to V \times \Delta_{|B_x|-1}$$

where $\rho_{B_x,V}$ is constructed in the obvious manner from $\rho_\beta|_V$, $\beta \in B_x$.

Hence can define the pullback

$$\rho^*_{B_x,V}(\omega|_{V \times \Delta_{|B_x|-1}}) \in \Gamma(\Omega^p(V)).$$

Lemma 38. *The above local pullbacks glue together to a smooth p-form*

$$\rho^*\omega \in \Gamma(\Omega^p(M))$$

which we call the pullback of ω by ρ. *Furthermore, pullback commutes with the exterior differential*

$$\rho^*(d_{\Sigma M}\omega) = d_M \rho^*\omega.$$

Proof. The proof is more a notational challenge. For the first part it suffices to show that for m, V as above and $m' \in V$ and a small open neighbourhood $x' \in V' \subset V$ we have

$$\rho^*_{B_{x'},V'}(\omega|_{V'\times\Delta_{|B_{x'}|-1}}) = \left[\rho^*_{B_x,V}(\omega|_{V\times\Delta_{|B_x|-1}})\right]_{V'} = \rho^*_{B_x,V'}(\omega|_{V'\times\Delta_{|B_x|-1}})$$

But, by construction $B_{x'} \supset B_x$ and

$$\rho_{B_{x'},V'} = \iota_{B_x B_{x'}} \circ \rho_{B_x,V'}.$$

Inserting this into the previous equation and using (41) the first part of the lemma is proven.

The second part is a trivial consequence of the commutativity of pullback with d. □

A.2. **De Rham Cohomology.** We denote the cohomology of the complex $(\Omega^\bullet, d)$ by $H^\bullet(\Sigma M)$ and call it the de Rham cohomology of ΣM.

In this section, let furthermore the Čech -de Rham hypercohomology wrt. the cover U_α be denoted by $\check{H}^\bullet(M, \Omega^\bullet(M))$. There is a natural chain map between the underlying complexes, which we call the *fiber integral*:

$$\int_{fiber} : \Omega^\bullet(\Sigma M) \to \check{C}^\bullet(M, \Omega^\bullet(M)).$$

On a form $\omega \in \Omega^\bullet(\Sigma M)$ it is defined as

$$\left(\int_{fiber}\omega\right)[\alpha_0, .., \alpha_k] = \int_{\Delta_k} \omega|_{U_{\underline{\alpha}}\times\Delta_k}.$$

In the integral, the orientation of Δ_k is that of the standard simplex when the vertex labeled by α_0 is identified with $(1, 0, .., 0)$, that labeled by α_1 is identified with $(0, 1, 0, .., 0)$ etc.

Lemma 39. *The fiber integral $\int_{fiber}$ is a chain map.*

Proof.

$$\begin{aligned} \left[(d \circ \int_{fiber}\omega)\right][\underline{\alpha}] &= d\int_{\Delta_k} \omega|_{U_{\underline{\alpha}}\times\Delta_k} \\ &= \int_{\partial\Delta_k} \omega|_{U_{\underline{\alpha}}\times\Delta_k} + \int_{\Delta_k} d\omega|_{U_{\underline{\alpha}}\times\Delta_k} \\ &= \left[(\delta \circ \int_{fiber})\omega\right][\underline{\alpha}] + \left[(\int_{fiber} \circ d)\omega\right][\underline{\alpha}] \end{aligned}$$

□

Proposition 40. *Assume that the covering U_α of M is good. Then the following diagram commutes for any smooth section ρ of ΣM and the homomorphism involved* *are all isomorphisms.*

$$\begin{array}{ccc} H^\bullet(\Sigma M) & \xrightarrow{\int_{fiber}} & \check{H}^\bullet(M, \Omega^\bullet(M)) \\ & {\scriptstyle \rho^*}\searrow \quad \swarrow{\scriptstyle \hat{\rho}} & \\ & H^\bullet_{dR}(M) & \end{array}$$

Note that the right downwards map does not depend on the ρ chosen and hence the left downwards map does neither.

Proof. The map ρ^* is an isomorphism. The proof of this statement is word-by-word the same as that for the analogous statement for fiber bundles with contractible fibers.

Next we construct a right inverse I. For an ordered subset

$$\underline{\alpha} = \{\alpha_0, \dots, \alpha_k\} \subset A$$

we first define the k-form

$$\mu_{\underline{\alpha}} := \lambda_{\alpha_0} d\lambda_{\alpha_1} \cdots d\lambda_{\alpha_k} \in \Omega^k(\Sigma M)$$

where the functions λ_α are those of Example 36. Then the operator I is given by

$$I(\eta) = \sum_{\underline{\alpha}} \mu_{\underline{\alpha}} \wedge \pi^* \eta[\underline{\alpha}].$$

This function is a chain map since

$$\begin{aligned} d(I(\eta)) = \sum_{\underline{\alpha}} (d\mu_{\underline{\alpha}}) \wedge \pi^* \eta[\underline{\alpha}] \\ + \pi^* d\eta[\underline{\alpha}] \\ = \sum_{\alpha} \sum_{\underline{\alpha}} \lambda_\alpha (d\mu_{\underline{\alpha}}) \wedge \pi^* \eta[\underline{\alpha}] \\ + \pi^* d\eta[\underline{\alpha}] \\ = \sum_{\underline{\alpha}} \mu_{\underline{\alpha}} \wedge \sum_{\alpha} \pm \pi^* \eta[\underline{\alpha} - \{\alpha\}] \\ + \pi^* d\eta[\underline{\alpha}] \\ = I((d+\delta)\eta). \end{aligned}$$

It is a right inverse since for all $\underline{\beta} = \{\beta_0, .., \beta_k\} \subset A$

$$\begin{aligned} \left[\int_{fiber} \circ \pi^*(\eta)\right] [\underline{\beta}] = \left[\int_{fiber} \sum_{\underline{\alpha}} \mu_{\underline{\alpha}} \wedge \pi^* \eta[\underline{\alpha}]\right] [\underline{\beta}] \\ = \int_{\Delta_k} \sum_{\sigma} \mu_{\sigma \underline{\beta}} \wedge \eta[\sigma \underline{\beta}] \\ = \eta[\underline{\beta}] \int_{\Delta_k} \sum_{\sigma} \operatorname{sgn}(\sigma) \mu_{\sigma \underline{\beta}} \\ = \eta[\underline{\beta}]. \end{aligned}$$

In the second equality we used that $\mu_{\underline{\alpha}}$ vanishes on the domain of integration except $\underline{\beta} \supset \underline{\alpha}$. But by dimensional reasons, the cardinalities of these sets must be equal and hence $\alpha = \sigma\beta$ for some permutation σ. The computation of the integral in the last step is elementary: W.l.o.g. we can assume $\underline{\beta} = \{0, .., k\}$. Then

$$\int_{\Delta_k} \sum_{\sigma} \mathrm{sgn}(\sigma)\mu_{\sigma\underline{\beta}} = k! \int_{\Delta_k} \sum_{i=0}^{k} (-1)^i \lambda_i d\lambda_0 \cdots d\hat{\lambda}_i \cdots d\lambda_k = k! \int_{\Delta_k} (1 - \sum_{j=1}^{k} \lambda_j) d\lambda_1 \cdots d\lambda_k + \sum_{i=1}^{k}\sum_{j=1}^{k} d\lambda_j \cdots d\hat{\lambda}_i \cdots$$

$$= k! \int_{\Delta_k} d\lambda_1 \cdots d\lambda_k = n! Vol(\Delta_k) = 1$$

It is easy to see that $\rho^* \circ I = \hat{\rho}$ and hence the commutativity of the diagram has been proven.

For the last statement note that if the cover is good, the map $\hat{\rho}$ and ρ^* are both isomorphisms and hence $\int_{fiber}$ is.

□

A.3. **The Holomorphic Case.** Next we treat the case of actual interest in our present context. If the manifold M is holomorphic and $U \subset M$ is open, we can define the holomorphic p forms on the product

$$U \times \Delta_k$$

for some $k = 0, 1, ..$ to be the sections of the pullback bundle

$$\pi^* \Omega^{p,0}(U)$$

which can be extended to a holomorphic section on a neighbourhood of $U \times \Delta_k$ in $U \times \mathbb{C}^k$. This means that the dependence on the U-argument is holomorphic and the dependence on the Δ_k-argument is real analytic.

As in section A.1 the pullbacks $\iota^*_{BB'}$ map holomorphic $p-forms$ to holomorphic p-forms. Hence, as before, we can define the space of holomorphic p-forms $\Omega^p_h(\Sigma M)$ on ΣM to be the collections of holomorphic p-forms $(\phi_B)_{B \subset A}$, where ϕ_B is a holomorphic p-form on

$$U \times \Delta_{|B|-1},$$

that satisfy

$$\phi_B|_{U_{B'} \times \Delta_{|B'|-1}} = \iota^*_{BB'} \phi_{B'}$$

for all $B \subset B' \subset A$. Again, there can be defined an exterior differential $d = \partial$ on $\Omega^p_h(\Sigma M)$, and we can compute the cohomology $H^\bullet_h(\Sigma M)$ of $\Omega^\bullet_h(\Sigma M)$. In analogy to Proposition 40 we have the following

Proposition 41. *Assume that the covering U_α of M is* acyclic. *Then the following diagram commutes for any smooth section ρ of ΣM and the homomorphism involved are all isomorphisms.*

$$\begin{array}{ccc} H^\bullet_h(\Sigma M) & \xrightarrow{\int_{fiber}} & \check{H}^\bullet(M, \Omega^\bullet_h(M)) \\ & {\scriptstyle\rho^*}\searrow \quad \swarrow{\scriptstyle\hat{\rho}} & \\ & H^\bullet_{dR}(M) & \end{array}$$

A.4. **A First Application.** Let C_α be a covering of M by closed sets such that

- $C_\alpha \subset U_\alpha$.
- Each C_α is the continuous image of a polyhedron in $\mathbb{R}^{2n}$.
- The intersections of two distinct polyhedra is either empty or a common face.

Such a covering will be called a *cell decomposition subordinate to U_α*. Fix such a cell decomposition.

Definition 42. The *cell integral*

$$\int_{cell} : \check{C}^\bullet(M, \Omega^\bullet(M)) \to \mathbb{C}$$

is defined as the map

$$\eta[\bar{\alpha}] \mapsto \sum_{\bar{\alpha}} \int_{C_{\bar{\alpha}}} \eta[\bar{\alpha}].$$

Here

$$C_{\bar{\alpha}} := C_{\alpha_0} \cap ... \cap C_{\alpha_k}.$$

Lemma 43. *$\int_{cell}$ is a cochain map, i.e., vanishes on exact elements. Hence it descend to a map on cohomology.*

Proof. This follows from the proof of the next proposition. □

Proposition 44. *The following diagram commutes*

$$\begin{array}{ccc} H^\bullet(M) & \xrightarrow{\check{\rho}} & \check{H}^\bullet(M, \Omega^\bullet(M)) \\ & {\scriptstyle \int_M} \searrow \quad \swarrow {\scriptstyle \int_{cell}} & \\ & \mathbb{C} & \end{array}$$

Proof. (TODO:noch gepfuscht) Consider the chain map

$$\breve{\omega} : \check{C}^\bullet(M, \Omega^\bullet(M)) \to \Omega^\bullet(\Sigma M)$$

defined by

$$\eta[\bar{\alpha}] \mapsto \sum_{\bar{\alpha}} \pi^* \eta[\bar{\alpha}] \wedge \omega_{\bar{\alpha}}.$$

Here $\omega_{\bar{\alpha}}$ is the volume form on the fiber over $U_{\bar{\alpha}}$. The map $\breve{\omega}$ is defined such that

$$\check{\rho} = \rho^* \circ \breve{\omega}.$$

For closed $\eta[\bar{\alpha}]$, the form $\breve{\omega}(\eta[\bar{\alpha}])$ is closed and hence the integral

$$\int_M \check{\rho}(\eta[\bar{\alpha}]) = \int_M \rho^* \breve{\omega}(\eta[\bar{\alpha}]) = \int_{\rho(M) \subset M} \breve{\omega}(\eta[\bar{\alpha}])$$

is independent of the embedding[3] ρ of M into ΣM chosen. Hence we can deform the embedding to one containing horizontal patches in the interior of the C_α and vertical patches along their boundaries as indicated in figure **??**. However, the sum of the integrals over all those patches is exactly the sum of integrals occuring in the definition of $\int_{cell}$. □

Appendix B. Residues and relative cohomology

Let $\{U_\alpha, V_\beta\}$ be a covering of M as in sections 6 and 5. Let an element in $\check{H}^n(M, \Omega^{n,0}(M))$ be given that is represented by

$$\eta[U_0, .., U_n] = \omega$$

with all other components vanishing and ω a holomorphic n-form on $U_0 \cap .. \cap U_n$.

Let Res_M be the composition

$$\check{H}^\bullet(M, \Omega(M)) \cong H^\bullet(M) \xrightarrow{\int} \mathbb{C}.$$

Lemma 45.

$$Res_M(\eta) = Res_x(\omega)$$

where Res_x is the residue at the special point x (...which was blown up or a zero of the vector field respectively).

[3] Note that any partition of unity is an embedding of M into ΣM.

First Proof. We will apply Lemma 44 to the following cell decomposition:

$$\begin{aligned} C_0 &= \{|z_1| \leq 1, .., |z_n| \leq 1\} \\ C_1 &= \{|z_1| \geq 1\} \\ &\dots \\ C_i &= \{|z_1| \leq 1, .., |z_{i-1}| \leq 1, |z_i| \geq 1\} \\ &\dots \end{aligned}$$

The common boundary $\bigcap_{i=0}^n C_i$ is exactly the torus

$$\{|z_1| = |z_2| = \cdots = |z_n| = 1\}.$$

Hece Lemma 44 yields the desired result. □

Second Proof. I repeat here the argument from [1]. η defines an element of the relative cohomology

$$\check{H}^\bullet(M, W; \Omega(M))$$

where $W = M - \{x\}$. By excision

$$\begin{aligned} &\check{H}^\bullet(M, W; \Omega^{n,0}(M)) \\ &\qquad \cong \check{H}^\bullet(M - (M - D), W - (M - D); \Omega^{n,0}(M)) \\ &\qquad\qquad \cong \check{H}^\bullet(D, D - \{x\}; \Omega^{n,0}(M)) \end{aligned}$$

where D is a small disk around x. By what is said in [1] the map Res_M descends to the map Res_p on $\check{H}^\bullet(D, D - \{x\}; \Omega(M))$. □

Third Partial Proof (?TODO: still not so good). In the local coordinates $z_1, ..z_n$ we have by Laurent expansion

$$\omega = dz_1 \cdots dz_n \sum_{j_1,..,j_n \in \mathbb{Z}} c_{j_1..j_n} z_1^{j_1} \cdots z_n^{j_n}.$$

Any term in the sum with $(j_1, .., j_n) \neq (-1, .., -1)$ is de Rham-exact. E.g., for $j_1 \neq -1$

$$c_{j_1..j_n} z_1^{j_1} \cdots z_n^{j_n} dz_1 \cdots dz_n = d\left(\frac{c_{j_1..j_n}}{j_1 + 1} z_1^{j_1+1} z_2^{j_2} \cdots z_n^{j_n} dz_2 \cdots dz_n\right).$$

Hence this defines a Čech -de Rham exact element since the intersection of $U_0 \cap .. \cap U_n$ with any other open set from the cover vanishes. But Res_M vanishes on Čech -de Rham exact elements and hence is proportional to Res_x. □

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