University of Zurich[UZH]

I·Math Institute of Mathematics

Graph complexes

Thomas Willwacher

Homology

- Abundant problem in mathematics:

  Classify (some type of objects) up to (equivalence).

Homology

Often classification problems can be recast as follows:

- Collection of vector spaces V_j with linear maps

$$\cdots \to V_{j+1} \stackrel{\delta_{j+1}}{\to} V_j \stackrel{\delta_j}{\to} V_{j-1} \stackrel{\delta_{j-1}}{\to} V_{j-2} \to \cdots$$

  ...such that $\delta_j \delta_{j+1} = 0$.
- Objects to classify = elements $x \in V_j$ such that $\delta_j x = 0$. (*closed* elements)
- Equivalence: $x \simeq x'$ if there is a $y \in V_{j+1}$ such that $x - x' = \delta_{j+1} y$ ($\leftarrow$ *exact* element)
- Can solve classification problem by computing *homology*

$$H_j = \ker(\delta_j)/\mathrm{im}(\delta_{j+1})$$

Homology

- Compress notation:

$$V = \bigoplus_j V_j$$

  graded vector space, V_j in degree j
- Linear map of degree -1

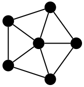
$$\delta\colon V \to V$$

  such that $\delta^2 = 0$. (V, δ) *chain complex*
- Homology (graded vector space)

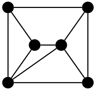
$$H(V) = \ker(\delta)/\mathrm{im}(\delta)$$

Kontsevich's graph complexes

– Chain complex of $\mathbb{Q}$-linear combinations of (isomorphism classes of) graphs

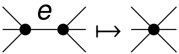


$+ \frac{5}{2}$



– Differential δ: edge contraction

$$\delta\Gamma = \sum_{e \text{ edge}} \pm \underbrace{\Gamma/e}_{\text{contract } e}$$



– $\delta^2 = 0$, $\Rightarrow$ can compute graph homology $\mathit{ker}\delta/\mathit{im}\delta$.

Kontsevich's graph complexes GC_n

For $n \in \mathbb{Z}$ define

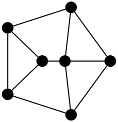
$$\mathrm{GC}_n = span^{gr}_{\mathbb{Q}} \{\text{isomorphism classes of admissible graphs}\}$$

with

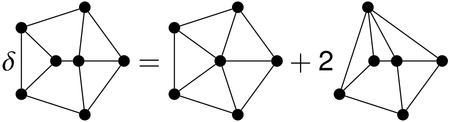
- Homological degree of vertices: n, of edges: $1 - n$.
- *Admissible*:
  - connected
  - all vertices $\geq$ 2-valent
  - no odd symmetries
- Differential: edge contraction

Example

Example for $n = 2$:



Differential:

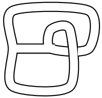


Graph homology

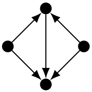
– Main (long standing) open problem: Compute the graph homology $H(GC_n) = ker\delta / im\delta$

Zoo of other versions

– Ribbon graphs (R. Penner ’88):

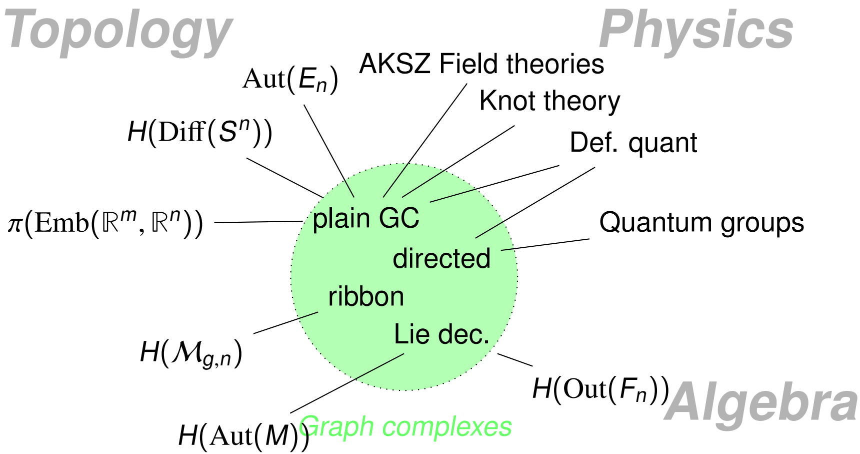


– Directed acyclic graphs:



– ...and a couple of others

Origins and applications

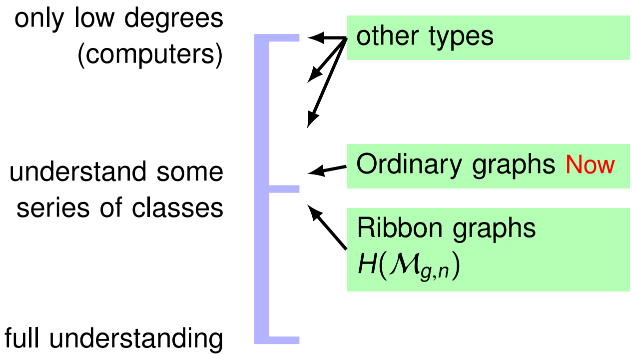


Plan for today

1. Graph homology: What is known?
2. Example of a reduction to graph homology

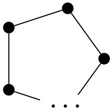
Graph complexes - state of the art in 2015

– What is known about graph homology?



Cheap information

- Differential does not change loop order $\Rightarrow$ can study pieces of fixed loop order separately
- Have classes in GC_n

$W_k =$  (k vertices and k edges)

Theorem (Kontsevich)

$$H(\mathrm{GC}_n) = H(\mathrm{GC}_n^{\geq 3-valent}) \oplus \bigoplus_{k \equiv 2n+1 \ mod\, 4} W_k$$

Cheap information II

Useful because:

- Can obtain degree bounds:
  - Highest degree classes have many vertices (v), few edges (e)
  - Trivalence condition: $e \geq \frac{3}{2}v$
  - $\Rightarrow$ upper bound on degree

$$(\text{degree}) \leq (\#\text{loops})(3-n) - 3$$

Not so cheap results (n=2)

Theorem (T.W., Invent. ’14)

$$H_0(\mathrm{GC}_2) \cong \mathfrak{grt}_1$$
$$H_{-1}(\mathrm{GC}_2) \cong \mathbb{K}$$
$$H_{<-1}(\mathrm{GC}_2) \cong 0$$

$\mathfrak{grt}_1$: *Grothendieck-Teichmüller Lie algebra*

Theorem (F. Brown, Annals ’12)

$$\mathit{FreeLie}(\sigma_3, \sigma_5, \sigma_7, \dots) \hookrightarrow \mathfrak{grt}_1$$

Deligne-Drinfeld conjecture: It is an isomorphism

Computer results

$n = 2$, degree (↑), loop order (→), values

$dim\, H_j(\mathrm{GC}_2)_{k \text{ loops}}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	1	0	0	0	0	0	0	0	0	1				
6	0	0	0	0	0	0	0	0	0	0				
5	0	0	0	0	0	0	0	0	0	0				
4	0	0	0	0	0	0	0	0	0	0				
3	1	0	0	0	0	1	0	1	1	2				
2	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	

Other degrees

Theorem (A. Khoroshkin, M. Živković, T.W., 2014)

Graph cohomology classes come in pairs, that kill each other on some page of a spectral sequence.

Cancellations in spectral sequence (even case)

$n = 2$, degree (↑), loop order (→)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	1								0	1				
6	0							0	0	0				
5	0						0	0	0	0				
4	0					0	0	0	0	0				
3	1	0	0	0	0	1	0	1	1	2				
2	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	

In summary

- Have known series of classes in one degree + their ”partners”
- Explains all classes in $H(\mathrm{GC}_n)$ in computer accessible regime
- But: Computer cannot see very far

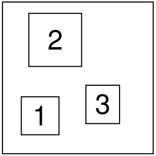
Origins and applications

- Graph complexes are linked to many problems in mathematics
- Today: Only discuss one specific case
- Goal: see interplay algebra - topology - physics

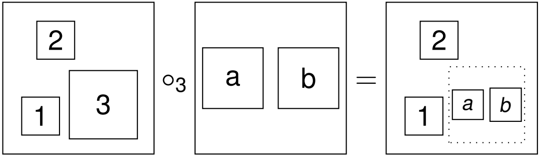
Topology: Little n-cubes operad

– Space of rectilinear embeddings of n-dimensional cubes

$$L_n(k) = \mathrm{Emb}_{rl}(\underbrace{[0,1]^n \sqcup \cdots \sqcup [0,1]^n}_{k\times}, [0,1]^n])$$



– Can glue configuration into another



Topology: Little n-cubes operad

- Obvious relations:
  - Gluing into different slots commutes
  - Nested gluing associative
  - $\Rightarrow$ Operad structure
- L_n : Little n-cubes (balls/disks) operad, or (topological) E_n operad
- Very important and long studied in topology

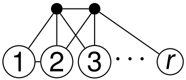
Physics: (Topological) quantum field theories

Perturbative n-dimensional quantum field theory (simplified):

- Want: Expectation value of
  $O[\Psi] = \iiint f(x_1, \ldots, x_r)\Psi(x_1)^{\alpha_1} \cdots \Psi(x_r)^{\alpha_r}$
- Perturbation theory

$$\langle O \rangle = \sum_{\Gamma} c_\Gamma \int_{\mathrm{Conf}_{\#\mathrm{vert}(\Gamma)}(\mathbb{R}^n)} f(x_1, \ldots, x_r)\omega_\Gamma$$

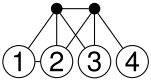
sum is over Feynman diagrams, e.g.,



the integrand is determined by Feynman rules.

Physics: (Topological) quantum field theories

– Our case: TFT of AKSZ type (kinetic part = de Rham differential)



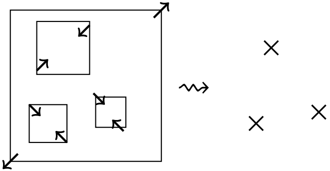
– Feynman rules assign to Γ a differential form on $\mathrm{Conf}_{k+r}(\mathbb{R}^n)$:

$$\omega_\Gamma = \bigwedge_{(i,j)\text{ edge}} \Omega_{S^{n-1}}(x_i - x_j)$$

Link Physics - Topology

The connection is as follows:

- Shrinking cubes links $L_n(r)$ to $\mathrm{Conf}_{\mathrm{r}}(\mathbb{R}^n)$



$\Rightarrow$ can build an equivalent operad out of configuration spaces.

- Assemble linear combinations Feynman diagrams with r "external" vertices into space

$$\mathsf{Graphs}_n(r) = \mathrm{span}\,\langle \text{Feynman diag. w/ } r \text{ ext. vert.} \rangle$$

Link Physics - Topology

– Feynman rules give a map

$$\omega : \mathrm{Graphs}_n(r) \to \Omega(\mathrm{Conf_r}(\mathbb{R}^n))$$

Theorem (Kontsevich)

This map is compatible with the operad structure: The Feynman diagrams Graphs_n *can be made into a real Suillvan model for* L_n.

Link to graph complex

Theorem (T.W.)

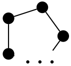
GC_n^* *is a Lie algebra and acts on* Graphs_n*, compatibly with the operad structure. This action exhausts all rational automorphisms of* L_n *up to homotopy.*

- Physically this action is analogous to a renormalization group action.

An application

- Of particular interest: $H_1(\mathrm{GC}_n)^*$ and $H_0(\mathrm{GC}_n)^*$, controlling obstructions and choices of weak equivalences
- Recall that

$$H(\mathrm{GC}_n) \cong \underbrace{\bigoplus_{k\equiv 2n+1 \bmod 4}}_{\leq 1 \text{ class can contribute}}$$



$$\oplus \underbrace{H(\mathrm{GC}_n^{\geq 3-\text{valent}})}_{\text{degree bounded} \Rightarrow \text{no contr.}}$$

Theorem (B. Fresse, T.W.)

The little n-cubes operads are rationally rigid and intrinsically formal for $n \geq 3$.

The End

Thanks for listening!

Peek into high loop orders

How to access high loop orders?

- Computer - no way.
- But: Can count graphs and compute Euler characteristic.

Theorem (T.W., M. Živković, Adv. in Math. '15)

Define generating functions for numbers of graphs:

$$P^{odd}(s,t) := \sum_{v,e} \dim\left(GC^{odd}_{v,e}\right) s^v t^e \quad P^{even}(s,t) := \sum_{v,e} \dim\left(GC^{even}_{v,e}\right) s^v t^e .$$

There exists an explicit formula.

$$P^{odd}(s,t) := \frac{1}{\left(-s,(st)^2\right)_\infty \left((st)^2,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{(-s)^{\alpha j_\alpha}}{j_\alpha!(-\alpha)^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \left(\frac{\left(t^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left((-s)^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left(t^\alpha,(st)^{2\alpha}\right)_\infty}{\left((-s)^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \frac{1}{\left(t^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}},$$

$$P^{even}(s,t) := \frac{\left(s,(st)^2\right)_\infty}{\left(-st,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \left(\frac{\left((-t)^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left(s^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left((-t)^\alpha,(st)^{2\alpha}\right)_\infty}{\left(s^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \left((-t)^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}$$

where $(a,q)_\infty = \prod_{k\geq 0}\left(1-aq^k\right)$ *is the q-Pochhammer symbol.*

	Even	Odd		Even	Odd
loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$	loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$
1	0	1	16	-3	6
2	1	1	17	-1	4
3	0	1	18	8	-5
4	1	2	19	12	-14
5	-1	1	20	27	-21
6	1	2	21	14	-11
7	0	2	22	-25	21
8	0	2	23	-39	44
9	-2	1	24	-496	504
10	1	3	25	-2979	2969
11	0	1	26	-412	413
12	0	3	27	38725	-38717
13	-2	4	28	10583	-10578
14	0	2	29	-667610	667596
15	-4	2	30	28305	-28290