Graph complexes

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Homology

- Abundant problem in mathematics:

Classify (some type of objects) up to (equivalence).
Homology

Often classification problems can be recast as follows:

– Collection of vector spaces $V_j$ with linear maps

\[ \cdots \to V_{j+1} \xrightarrow{\delta_{j+1}} V_j \xrightarrow{\delta_j} V_{j-1} \xrightarrow{\delta_{j-1}} V_{j-2} \to \cdots \]

...such that $\delta_j \delta_{j+1} = 0$.

– Objects to classify = elements $x \in V_j$ such that $\delta_j x = 0$. (closed elements)

– Equivalence: $x \simeq x'$ if there is a $y \in V_{j+1}$ such that $x - x' = \delta_{j+1} y$ (← exact element)

– Can solve classification problem by computing homology

\[ H_j = \ker(\delta_j)/\text{im}(\delta_{j+1}) \]
Homology

- Compress notation:

\[ V = \bigoplus_j V_j \]

*graded vector space, \( V_j \) in degree \( j \)*

- Linear map of degree \(-1\)

\[ \delta : V \to V \]

such that \( \delta^2 = 0 \). \((V, \delta)\) *chain complex*

- Homology (graded vector space)

\[ H(V) = \ker(\delta)/\text{im}(\delta) \]
Kontsevich’s graph complexes

- Chain complex of $\mathbb{Q}$-linear combinations of (isomorphism classes of) graphs

- Differential $\delta$: edge contraction

$$\delta \Gamma = \sum_{e \text{ edge}} \pm \frac{\Gamma / e}{\text{contract } e}$$

- $\delta^2 = 0$, $\Rightarrow$ can compute graph homology $\ker \delta / \text{im } \delta$. 
Kontsevich’s graph complexes $\text{GC}_n$

For $n \in \mathbb{Z}$ define

$$\text{GC}_n = \text{span}_{\mathbb{Q}}^{gr} \{\text{isomorphism classes of admissible graphs}\}$$

with

- Homological degree of vertices: $n$, of edges: $1 - n$.
- \textit{Admissible}:
  - connected
  - all vertices $\geq 2$-valent
  - no odd symmetries
- Differential: edge contraction
Example

Example for $n = 2$:

Differential:

$$\delta = + 2$$
Graph homology

- Main (long standing) open problem: Compute the graph homology $H(GC_n) = \ker \delta / \text{im} \delta$
Zoo of other versions

- Ribbon graphs (R. Penner ’88):

- Directed acyclic graphs:

- ...and a couple of others
Origins and applications

Topology
- Aut($E_n$)
- $H(\text{Diff}(S^n))$
- $\pi(\text{Emb}(\mathbb{R}^m, \mathbb{R}^n))$

Physics
- AKSZ Field theories
- Knot theory
- Def. quant
- Quantum groups

Algebra
- $H(\mathcal{M}_{g,n})$
- $H(\text{Out}(F_n))$
- $H(\text{Emb}(\mathbb{R}^m, \mathbb{R}^n))$
- Graph complexes
- Lie dec.
- directed
- ribbon
- plain GC
Plan for today

1. Graph homology: What is known?
2. Example of a reduction to graph homology
Graph complexes - state of the art in 2015

- What is known about graph homology?

  - only low degrees (computers)
  - understand some series of classes
  - full understanding

- other types
  - Ordinary graphs *Now*
  - Ribbon graphs $H(M_{g,n})$
Cheap information

- Differential does not change loop order ⇒ can study pieces of fixed loop order separately
- Have classes in $\text{GC}_n$

$W_k = \ldots$ (k vertices and k edges)

Theorem (Kontsevich)

$$H(\text{GC}_n) = H(\text{GC}_{n}^{\geq 3-\text{valent}}) \oplus \bigoplus_{k \equiv 2n+1 \mod 4} W_k$$
Cheap information II

Useful because:
- Can obtain degree bounds:
  - Highest degree classes have many vertices \((v)\), few edges \((e)\)
  - Trivalence condition: \(e \geq \frac{3}{2}v\)
  - \(\Rightarrow\) upper bound on degree

\[
\text{(degree)} \leq \text{(\#loops)}(3 - n) - 3
\]
Not so cheap results (n=2)

Theorem (T.W., Invent. ’14)

\[ H_0(G\mathcal{C}_2) \cong \mathfrak{grt}_1 \]
\[ H_{-1}(G\mathcal{C}_2) \cong \mathbb{K} \]
\[ H_{<-1}(G\mathcal{C}_2) \cong 0 \]

\(\mathfrak{grt}_1: \text{Grothendieck-Teichmüller Lie algebra}\)

Theorem (F. Brown, Annals ’12)

\[ \text{FreeLie}(\sigma_3, \sigma_5, \sigma_7, \ldots) \hookrightarrow \mathfrak{grt}_1 \]

Deligne-Drinfeld conjecture: It is an isomorphism
Computer results

\( n = 2 \), degree (↑), loop order (→), values

\( \dim H_j(\text{GC}_2)_k \) loops

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Other degrees

Theorem (A. Khoroshkin, M. Živković, T.W., 2014)

*Graph cohomology classes come in pairs, that kill each other on some page of a spectral sequence.*
Cancellations in spectral sequence (even case)

\( n = 2 \), degree (↑), loop order (→)
In summary

- Have known series of classes in one degree + their "partners"
- Explains all classes in $H(GC_n)$ in computer accessible regime
- But: Computer cannot see very far
Origins and applications

- Graph complexes are linked to many problems in mathematics
- Today: Only discuss one specific case
- Goal: see interplay algebra - topology - physics
Topology: Little $n$-cubes operad

- Space of rectilinear embeddings of $n$-dimensional cubes

$$L_n(k) = \text{Emb}_{rl}([0, 1]^n \sqcup \cdots \sqcup [0, 1]^n, [0, 1]^n)$$

- Can glue configuration into another
Topology: Little $n$-cubes operad

- Obvious relations:
  - Gluing into different slots commutes
  - Nested gluing associative
  - $\Rightarrow$ Operad structure

- $L_n$: Little $n$-cubes (balls/disks) operad, or (topological) $E_n$ operad

- Very important and long studied in topology
Physics: (Topological) quantum field theories

Perturbative $n$-dimensional quantum field theory (simplified):

- Want: Expectation value of
  \[ O[\Psi] = \iiint f(x_1, \ldots, x_r) \psi(x_1)^{\alpha_1} \cdots \psi(x_r)^{\alpha_r} \]
- Perturbation theory

\[
\langle O \rangle = \sum_{\Gamma} c_\Gamma \int_{\text{Conf}_{\#\text{vert}(\Gamma)}(\mathbb{R}^n)} f(x_1, \ldots, x_r) \omega_\Gamma
\]

sum is over Feynman diagrams, e.g.,

the integrand is determined by Feynman rules.
Physics: (Topological) quantum field theories

- Our case: TFT of AKSZ type (kinetic part = de Rham differential)

\[
\omega_\Gamma = \bigwedge_{(i,j)\text{ edge}} \Omega_{S^{n-1}}(x_i - x_j)
\]
Link Physics - Topology

The connection is as follows:

- Shrinking cubes links \( L_n(r) \) to \( \text{Conf}_r(\mathbb{R}^n) \)

\[
\Rightarrow \text{can build an equivalent operad out of configuration spaces.}
\]

- Assemble linear combinations Feynman diagrams with \( r \) “external” vertices into space

\[
\text{Graphs}_n(r) = \text{span} \langle \text{Feynman diag. w/ } r \text{ ext. vert.} \rangle
\]
Link Physics - Topology

- Feynman rules give a map

\[ \omega : \text{Graphs}_n(r) \rightarrow \Omega(\text{Conf}_r(\mathbb{R}^n)) \]

**Theorem (Kontsevich)**

This map is compatible with the operad structure: The Feynman diagrams Graphs$_n$ can be made into a real Sullivan model for $L_n$. 
Link to graph complex

Theorem (T.W.)

\( \text{GC}_n^* \) is a Lie algebra and acts on \( \text{Graphs}_n \), compatibly with the operad structure. This action exhausts all rational automorphisms of \( L_n \) up to homotopy.

- Physically this action is analogous to a renormalization group action.
An application

- Of particular interest: \( H_1(GC_n)^* \) and \( H_0(GC_n)^* \), controlling obstructions and choices of weak equivalences
- Recall that

\[
H(GC_n) \cong \bigoplus_{k \equiv 2n+1 \mod 4} \cdots \bigoplus \underbrace{H(GC_n^{\geq 3\text{-valent}})}_{\text{degree bounded} \Rightarrow \text{no contr.}}
\]

\( \leq 1 \text{ class can contribute} \)

Theorem (B. Fresse, T.W.)

*The little n-cubes operads are rationally rigid and intrinsically formal for \( n \geq 3 \).*
The End

Thanks for listening!
Peek into high loop orders

How to access high loop orders?

– Computer - no way.

– But: Can count graphs and compute Euler characteristic.
Theorem (T.W., M. Živković, Adv. in Math. ’15)

Define generating functions for numbers of graphs:

\[ P^{\text{odd}}(s, t) := \sum_{v, e} \dim(GC_{v, e}^{\text{odd}}) s^v t^e \quad P^{\text{even}}(s, t) := \sum_{v, e} \dim(GC_{v, e}^{\text{even}}) s^v t^e. \]

There exists an explicit formula.

\[
P^{\text{odd}}(s, t) := \frac{1}{(s, (st)^2)_\infty ((st)^2, (st)_\infty)_{j_1 j_2 \ldots} \prod_{\alpha} \frac{(-s)^{\alpha j_\alpha}}{j_\alpha !(-\alpha)^{j_\alpha}} \frac{1}{((-st)^\alpha, (-st)_\infty)_{j_\alpha}} \left( \frac{(t^{2\alpha - 1}, (st)^{4\alpha - 2})_{\infty}}{((-s)^{2\alpha - 1} t^{4\alpha - 2}, (st)^{4\alpha - 2})_{\infty}} \right)^{j_2 \alpha - 1/2} \]

\[
\left( \frac{(t^{\alpha}, (st)^{2\alpha})_{\infty}}{((-s)^{\alpha} t^{2\alpha}, (st)^{2\alpha})_{\infty}} \right)^{j_1 \alpha} \prod_{\alpha, \beta} \frac{1}{(t^{\lcm(\alpha, \beta)}, (-st)^{\lcm(\alpha, \beta)})_{\infty}}^{\gcd(\alpha, \beta)} \frac{1}{j_\alpha j_\beta / 2},
\]

\[
P^{\text{even}}(s, t) := \frac{(s, (st)^2)_\infty}{(-st, (st)^2)_\infty} \sum_{j_1 j_2 \ldots} \prod_{\alpha} \frac{s^{\alpha j_\alpha}}{j_\alpha !\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha, (-st)_\infty)_{j_\alpha}} \left( \frac{((-t)^{2\alpha - 1}, (st)^{4\alpha - 2})_{\infty}}{(s^{2\alpha - 1} t^{4\alpha - 2}, (st)^{4\alpha - 2})_{\infty}} \right)^{j_2 \alpha - 1/2} \]

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\left( \frac{((-t)^{\alpha}, (st)^{2\alpha})_{\infty}}{(st^{\alpha} t^{2\alpha}, (st)^{2\alpha})_{\infty}} \right)^{j_1 \alpha} \prod_{\alpha, \beta} \frac{1}{((-t)^{\lcm(\alpha, \beta)}, (-st)^{\lcm(\alpha, \beta)})_{\infty}}^{\gcd(\alpha, \beta)} \frac{1}{j_\alpha j_\beta / 2}
\]

where \((a, q)_\infty = \prod_{k\geq 0} (1 - aq^k)\) is the q-Pochhammer symbol.
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