Graph complexes

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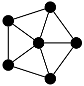
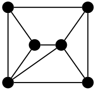
Outline

Kontsevich's graph complexes

- differential graded vector spaces of $\mathbb{Q}$-linear combinations of (isomorphism classes of) graphs

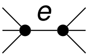


$+\frac{5}{2}$

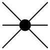


- Differential δ: edge contraction

$$\delta\Gamma = \sum_{e \text{ edge}} \pm \underbrace{\Gamma/e}_{\text{contract } e}$$



$\mapsto$



- $\delta^2 = 0$, $\Rightarrow$ can compute graph homology $ker\delta/im\delta$.

Kontsevich's graph complexes GC_n

For $n \in \mathbb{Z}$ define

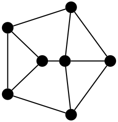
$$\mathrm{GC}_n = span^{gr}_{\mathbb{Q}} \{\text{isomorphism classes of admissible graphs}\}$$

with

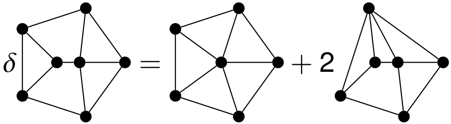
- Homological degree of vertices: n, of edges: $1 - n$.
- *Admissible*:
  - connected
  - all vertices $\geq$ 2-valent
  - no odd symmetries
- Differential: edge contraction

Example

Example for $n = 2$:



Differential:

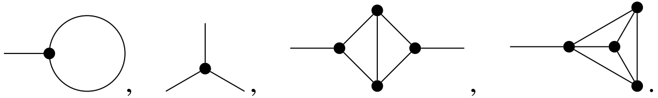


Graph homology

- Main (long standing) open problem: Compute the graph homology $H(GC_n) = ker\delta / im\delta$

Variant: Hairy graph complexes $\mathrm{HGC}_{m,n}$

$$\mathrm{HGC}_{m,n} = span^{gr}_{\mathbb{Q}}\{\text{isomorphism classes of admissible hairy graphs}\}$$



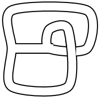
- Vertices have degree n, hairs m, edges $1 - n$.
- Differential again (non-hair-)edge contraction
- $H(\mathrm{HGC}_{m,n})$: hairy graph homology

Origins and applications (of GC_n, $HGC_{m,n}$)

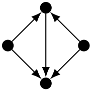
- Physics: Feynman diagrams of certain topological field theories (Chern-Simons, more generally AKSZ)
  - Hairy version contains Vassiliev invariants
- Algebra: Computes stable Lie algebra cohomology of $(\mathbb{Q}[x_1, \ldots, x_n, p_1, \ldots, p_n], \{-, -\})$
- Deformation Quantization
- Topology: Deformation theory of little cubes operads TODAY
- Rational homotopy of embedding spaces TODAY

Digression: Other versions

- Ribbon graphs (R. Penner '88): compute $H(\mathcal{M}_{g,n})$.



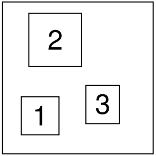
- "Lie decorated" graphs (Culler–Vogtman '86, Kontsevich '92): compute $H(Out(F_N))$.
- Directed acyclic graphs: Lie bialgebras, Quantum Groups



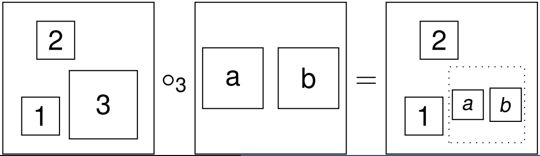
Little *n*-cubes operad

- Space of rectilinear embeddings of *n*-dimensional cubes

$$L_n(k) = \mathrm{Emb}_{rl}(\underbrace{[0,1]^n \sqcup \cdots \sqcup [0,1]^n}_{k\times}, [0,1]^n)$$



- Can glue configuration into another

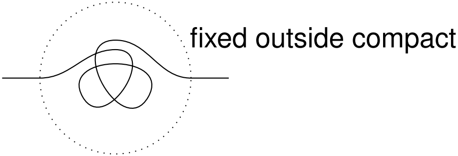


Little n-cubes operad

- Obvious relations:
  - Gluing into different slots commutes
  - Nested gluing associative
  - $\Rightarrow$ Operad structure
- L_n : Little n-cubes (balls/disks) operad, or (topological) E_n^{top} operad
- Very important and long studied in topology

Goodwillie-Weiss embedding calculus

- Goal: Space of long knots: $\overline{\mathrm{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$



Theorem (Goodwillie, Weiss, Dwyer–Hess, Arone–Lambrechts–Volic)

If $n - m \geq 3$ then

$$\overline{\mathrm{Emb}}(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1}\mathrm{Map}_{op}(L_m, L_n).$$

- Embedding calculus replaces hard topological problem by an equally hard algebraic problem, that we don't know how to solve either.

Connection to graphs

- New goal: Study mapping spaces

$$\mathrm{Map}_{\mathsf{op}}(L_m, L_n)$$

- .... and automorphism groups

$$\mathrm{Aut}_{\mathsf{op}}(L_n)$$

$$(\mathrm{Aut}_{\mathsf{op}}(L_n) \to \mathrm{Map}_{\mathsf{op}}(L_m, L_n))$$

Operadic mapping spaces through graphs I

Theorem (T.W., Invent. Math 2014)

The graph complex GC_n acts on a combinatorial rational model for the operad L_n, such that

$$H(\mathrm{Der}(Chains(L_n, \mathbb{Q}))) \cong S^+\left(\mathbb{K}[n+1] \oplus H(\mathrm{GC}_n)[n+1]\right)[-1-n].$$

Theorem (B. Fresse, V. Turchin, T.W., in progress)

$$\pi_j(\mathrm{Aut}_{op}(L_n)) \otimes \mathbb{Q} \cong H_{-j}(\mathrm{GC}_n)$$
$$\pi_j(\mathrm{Map}_{op}(L_m, L_n)) \otimes \mathbb{Q} \cong H_{-j}(\mathrm{HGC}_{m,n})$$

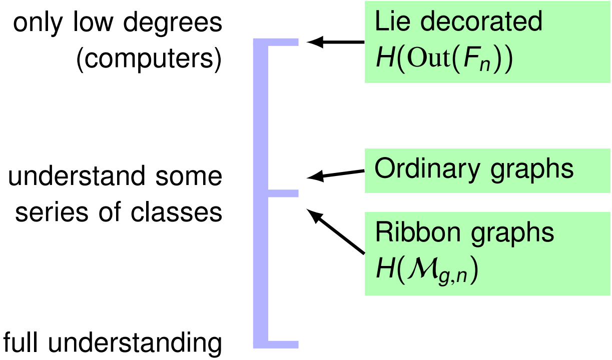
(for $n - m \geq 2$, $j \geq 1$)

Summary

- Have replaced hard topological problem by graph homology problem, that is still notoriously hard and nobody knows how to solve.
- ... well, not quite, ...

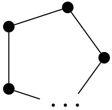
Graph complexes - state of the art in 2015

- What is known about graph homology?



Cheap information

- GC_n and $\mathrm{HGC}_{m,n}$ depend only on parity of m, n (up to degree shifts) $\Rightarrow$ periodicity.
- Grading by loop order
- Combinatorial degree bounds on graph complex $\Rightarrow$ Can understand low rational homotopy groups.
- Have classes in GC_n

$W_k =$  (k vertices and k edges)

Not so cheap results (n=2)

Study GC_2 and GC_3 $\Rightarrow$ understand all GC_n

Theorem (T.W., Invent. '14)

$$H_0(\mathrm{GC}_2) \cong \mathfrak{grt}_1$$
$$H_{-1}(\mathrm{GC}_2) \cong \mathbb{K}$$
$$H_{<-1}(\mathrm{GC}_2) \cong 0$$

$\mathfrak{grt}_1$: *Grothendieck-Teichmüller Lie algebra*

Theorem (F. Brown, Annals '12)

$$\mathit{FreeLie}(\sigma_3, \sigma_5, \sigma_7, \dots) \hookrightarrow \mathfrak{grt}_1$$

Deligne-Drinfeld conjecture: It is an isomorphism

Computer results

$n = 2$, degree ($\uparrow$), loop order ($\rightarrow$), values $dim\, H_j(\mathrm{GC}_2)_{k \text{ loops}}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	1	0	0	0	0	0	0	0	0	1				
6	0	0	0	0	0	0	0	0	0	0				
5	0	0	0	0	0	0	0	0	0	0				
4	0	0	0	0	0	0	0	0	0	0				
3	1	0	0	0	0	1	0	1	1	2				
2	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	

Not so cheap results (n=3)

- Have many nontrivial classes in $H_{-3}(\mathrm{GC}_3)$ from Chern-Simons theory.
- (Vogel, Kneissler) Have a map (conjecturally iso)

$$\mathbb{K}[t, \omega_0, ...\omega_p, ...]/\langle \omega_p\omega_q - \omega_0\omega_{p+q}, P\rangle \to H_{-3}(\mathrm{GC}_3)$$

Computer results

$n = 3$, degree ($\uparrow$), loop order ($\rightarrow$)

	1	2	3	4	5	6	7	8	9	10	11	12
8	1											
4	1											
0	1											
-2	0	0	0	0	0	0	0	0	0	0	0	0
-3	0	1	1	1	2	2	3	4	5	6	8	9
-4	0	0	0	0	0	0	0	0				
-5	0	0	0	0	0	0	0	0				
-6	0	0	0	0	0	1	1	2				
-7	0	0	0	0	0	0	0	0				
-8	0	0	0	0	0	0	0	0	0			

Other degrees

How to get information on other degrees?

- Idea: Deform differential on graph complex.

$$\delta \to \delta + D$$

such that $H(\mathrm{GC}_n, \delta + D)$ computable. $\Rightarrow$ Information from spectral sequence.

Theorem (A. Khoroshkin, M. Živković, T.W., 2014)

There is a spectral sequence E such that $E^1 \cong H(\mathrm{GC}_n)$ and

$$E \Rightarrow \begin{cases} \mathbb{K}[n-1] & n \text{ even} \\ \mathbb{K}[n] & n \text{ odd} \end{cases}$$

Cancellations in spectral sequence (even case)

$n = 2$, degree ($\uparrow$), loop order ($\rightarrow$)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	1								0	1				
6	0							0	0	0				
5	0						0	0	0	0				
4	0					0	0	0	0	0				
3	1	0	0	0	0	1	0	1	1	2				
2	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	

Cancellations in spectral sequence (odd case)

$n = 3$, degree (↑), genus (→)

	1	2	3	4	5	6	7	8	9	10	11	12
8	1											
4	1											
0	1											
-2	0	0	0	0	0	0	0	0	0	0	0	0
-3	0	1	1	1	2	2	3	4	5	6	8	9
-4	0	0	0	0	0	0	0	0				
-5	0	0	0	0	0	0	0	0				
-6	0	0	0	0	0	1	1	2				
-7	0	0	0	0	0	0	0	0				
-8	0	0	0	0	0	0	0	0	0			

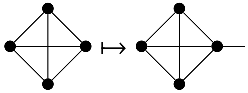
In summary

- Have known series of classes in one degree + their ”partners”
- Explains all classes in $H(\mathrm{GC}_n)$ in computer accessible regime
- But: Computer cannot see very far

Hairy case

- Have a map by adding one hair:

$$H(\mathrm{GC}_n^{\geq 3}) \to H(\mathrm{GC}_{m,n})[m-2n+1]$$



Theorem (V. Turchin, T.W.)

The above map is an injection in homology for all m, n.

Hairy case: Spectral sequences

- For m even, can deform the differential in two ways, $\delta + D_1$, $\delta + D_2$.

Theorem (A. K., V. T., M.Ž.,T. W., in preparation)

For m even we have two spectral sequences E, F, such that $E^1 = H(\mathrm{HGC}_{m,n}) = F^1$ *and*

$$E \Rightarrow 0$$

$$F \Rightarrow H(\mathrm{GC}_n^{\geq 3})$$

Computer data, $n = m = 2$

$\dim H(\mathrm{HGC}_{2,2})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree −3.

	1	2	3	4	5	6	7	8
9	1_{16}	1_{16}						
8		1_{15}						
7	1_{12}	1_{12}						
6		1_{11}						
5	1_8			1_{11}				
4				1_8				
3	1_4			1_7				
2								
1			1_1		1_1	1_4	1_1	$1_1, 1_4$

Computer data, $n = 3$, $m = 2$

$\dim H(\mathrm{HGC}_{3,2})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9		1_6						
8								
7	1_5	1_5	2_5					
6			1_2					
5		1_2	2_2	3_2	5_2			
4			1_1	1_1	1_{-1}			
3	1_1	1_1	2_1	2_1	3_1			
2					1_{-2}	1_{-2}		
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Computer data, $n = 3$, $m = 2$

$\dim H(\mathrm{HGC}_{2,3})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9		1_6						
8								
7	1_5	1_5	2_5					
6			1_2					
5		1_2	2_2	3_2	5_2			
4			1_1	1_1	1_{-1}			
3	1_1	1_1	2_1	2_1	3_1			
2					1_{-2}	1_{-2}		
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Summary

- Problems in several areas of mathematics reducible to graph homology computation.
- Computation of graph homology hard and long standing problem.
- But: Partial results and many classes known due to recent work.

Applications and Outlook

Three ”business branches” around graph complexes

1. Reducing other problems in math to graph homology computations
2. Relating various graph homology theories
3. Obtaining information about graph homology

A problem in algebraic geometry

- M complex mfd. or smooth algebraic variety
- Sheaf of multi vectors $\wedge\mathcal{T}M$
  1. ... algebra under wedge product
  2. ... carries compatible Lie bracket $\Rightarrow$ Gerstenhaber algebra
  3. ... carries an action of sheaf of diff. forms $\Omega(M)$ by contraction
- Sheaf cohomology $H(\wedge\mathcal{T}M)$, is a Gerstenhaber algebra, and carries action by $H(\Omega M)$.
- Statement/Conjecture of Kontsevich:

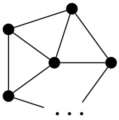
Theorem (Dolgushev,Rogers, T.W., Annals of Math. '15)

Action of odd Chern characters $c_{2n+1} \in H(\Omega M)$ compatible with Gerstenhaber structure on $H(\wedge\mathcal{T}M)$.

A problem in algebraic geometry II

Relation to graph complexes as follows:

- GC_2 acts on (resolution of) $\wedge \mathcal{T} M$
- $H^0(\mathrm{GC}_2) \cong \mathfrak{grt}_1$ acts on $H(\wedge \mathcal{T} M)$
- Recall that generators of $\mathfrak{grt}_1 = H^0(\mathrm{GC}_2)$ represented as

$\sigma_{2j+1} =$  $+ \cdots$

- Can check that leading term acts as contraction with odd Chern characters, other terms do not contribute (in cohomology).
- Using description as GC_2-action, check that it respects Gerstenhaber structure on $H(\wedge \mathcal{T} M)$.

Applications and Outlook

Three ”business branches” around graph complexes

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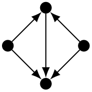
Relations between graph complexes

- Consider graph complex of directed acyclic graphs GC_n^{dag}.



Theorem (T.W., Comm. Math. Phys '15)

One has an isomorphism of Lie algebras $H(\mathrm{GC}_{n+1}^{dag}) \cong H(\mathrm{GC}_n)$.

- Has applications to Lie bialgebras and (infinite dimensional) deformation quantization

Applications and Outlook

Three ”business branches” around graph complexes

1. Reducing other problems in math to graph homology computations
2. Relating various graph homology theories
3. Obtaining information about graph homology

Peek into high loop orders

How to access high loop orders?

- Computer - no way.
- But: Can count graphs and compute Euler characteristic.

Theorem (T.W., M. Živković, Adv. in Math. '15)

Define generating functions for numbers of graphs:

$$P^{odd}(s,t) := \sum_{v,e} \dim\left(\mathrm{GC}^{odd}_{v,e}\right) s^v t^e \quad P^{even}(s,t) := \sum_{v,e} \dim\left(\mathrm{GC}^{even}_{v,e}\right) s^v t^e .$$

There exists an explicit formula.

$$P^{odd}(s,t) := \frac{1}{\left(-s,(st)^2\right)_\infty \left((st)^2,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{(-s)^{\alpha j_\alpha}}{j_\alpha!(-\alpha)^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \left(\frac{\left(t^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left((-s)^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left(t^\alpha,(st)^{2\alpha}\right)_\infty}{\left((-s)^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \frac{1}{\left(t^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}},$$

$$P^{even}(s,t) := \frac{\left(s,(st)^2\right)_\infty}{\left(-st,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \left(\frac{\left((-t)^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left(s^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left((-t)^\alpha,(st)^{2\alpha}\right)_\infty}{\left(s^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \left((-t)^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}$$

where $(a,q)_\infty = \prod_{k\geq 0}\left(1-aq^k\right)$ *is the q-Pochhammer symbol.*

	Even	Odd		Even	Odd
loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$	loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$
1	0	1	16	-3	6
2	1	1	17	-1	4
3	0	1	18	8	-5
4	1	2	19	12	-14
5	-1	1	20	27	-21
6	1	2	21	14	-11
7	0	2	22	-25	21
8	0	2	23	-39	44
9	-2	1	24	-496	504
10	1	3	25	-2979	2969
11	0	1	26	-412	413
12	0	3	27	38725	-38717
13	-2	4	28	10583	-10578
14	0	2	29	-667610	667596
15	-4	2	30	28305	-28290

The End

Thanks for listening!

Computer data, $n = 3$, $m = 3$

$\dim H(\mathrm{HGC}_{3,3})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9		2_{14}						
8	1_{13}	2_{13}						
7		2_{10}	5_{10}	1_8				
6	1_9	2_9	$3_9, 1_7$	4_7				
5		1_6	3_6	6_6	1_4			
4	1_5	1_5	2_5	$3_5, 1_3$	$4_5, 3_3$			
3		1_2	1_2	3_2	4_2			
2		1_1	1_1	2_1	2_1			
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Computer data, $n = 2$, $m = 1$

$\dim H(\mathrm{HGC}_{1,2})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9	1_8							
8		1_7	1_7					
7			1_8	1_6				
6		1_6	1_6	$1_6, 1_7$				
5	1_4		1_5	$1_5, 1_7$	2_5			
4		1_3		$1_5, 1_6$	1_6			
3			1_4		1_4			
2		1_2			1_5			
1			1_1		1_1	1_4	1_1	$1_1, 1_4$