

Virasoro constraints

history & moduli of sheaves

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I. History

Virasoro algebra

$$\text{Vir} := \text{span}_{\mathbb{C}} \left(\{L_n\}_{n \in \mathbb{Z}} \cup \{c\} \right)$$

central charge

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\delta_{n+m,0}(n^3-n)}{12} \cdot c$$

$$\text{Vir}_{\geq -1} := \text{span}_{\mathbb{C}} \{L_n\}_{n \geq -1} \subset \text{Vir}$$

Lie subalgebra

Witten conjecture

$$Z(\lambda, t_0, t_1, t_2, \dots)$$

$$:= \exp \left(\sum_{g \geq 0} \lambda^{2g-2} \sum_{\substack{n \geq 0 \\ k_1, \dots, k_n \geq 0}} \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \frac{t_{k_1} \dots t_{k_n}}{n!} \right)$$

$$\in \mathbb{Q}[\lambda^{\pm}, t_0, t_1, \dots] \quad \hookrightarrow \text{Vir}_{\geq -1} \text{-representation}$$

$$\text{e.g. } L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \quad \hookrightarrow \text{string equation}$$

$$\text{Thm (Kontsevich)} \quad \forall k \geq -1, \quad L_k(Z(\lambda, \vec{t})) = 0$$

Gromov-Witten theory

X smooth projective variety / \mathbb{C}

GW descendent invariants.




$$\int [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \prod_{i=1}^n \psi_i^{k_i} \underbrace{\text{ev}_i^*(\sigma_i)}_{\in H^*(X, \mathbb{C})}$$

Conj (Eguchi-Hori-Xiong) $\forall k \geq -1, L_k(Z^X) = 0$

↑ proven for $X = \begin{cases} \text{curves (Okounkov-Pandharipande)} \\ \text{toric varieties (Givental, Teleman)} \end{cases}$

Sheaf theory

- 1) $PT_n(X, \beta)$ via GW/PT correspondence
 - 2) $S^{[n]}$ via dimension reduction 
 - 3) $M_S^{\text{H-ss}}(r, c_1, c_2)$ [van Bree]
- [Morzina-Okounkov-O-P, M]

II. Moduli of sheaves

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Goal: "classify" coherent sheaves on X

e.g. vector bundles, $I_{Z/X}$, $i_* \mathcal{O}_Z$ ($i: Z \hookrightarrow X$)

$$\underline{\dim X = 1}$$

$$\text{Coh}_X(r, d) \supset \text{Vect}_C(r, d) \supset \text{Vect}_C^{\text{ss}}(r, d)$$

→
topological
classification.

Def. $F \in \text{Vect}_C(r, d)$ is semistable if

$$\forall \mathcal{G} \subsetneq F, \quad \frac{\deg(\mathcal{G})}{\text{rk}(\mathcal{G})} \leq \frac{d}{r}.$$

Thm. (Mumford) \exists projective variety $M_X^{\text{ss}}(r, d)$

which corepresents $\text{Vect}_X^{\text{ss}}(r, d)$.

Rmk.

$$\gcd(r, d) = 1 \Rightarrow \left(\begin{array}{l} M_X^{\text{ss}}(r, d) \text{ smooth of } \dim_{\mathbb{C}} = 1 - r^2(1-g) \\ \exists \text{ universal bundle } \mathcal{F} \rightarrow M_X^{\text{ss}}(r, d) \times X \end{array} \right)$$

General X

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* $v \in H^*(X, \mathbb{Q})$ Chern character

* H ample line bundle on X (a.k.a stability condition)

Thm. (Simpson, Huybrecht-Lehn)

highly singular

\exists projective coarse moduli scheme $M_X^{H-ss}(v)$

III. Enumerative invariants

Assumption on $M = M_X^{H-ss}(v)$

A1) no strictly H -semistable sheaves in M

A2) \exists universal sheaf $\mathbb{F} \rightarrow M \times X$

A3) $\forall F \in M, \text{Ext}_X^{\geq 3}(F, F) = 0$

not necessary

not unique

Rmk. 1) \exists numerical criterion for $\textcircled{A1}$, $\textcircled{A2}$.

2) $\textcircled{A3}$ \leftarrow automatic if $\dim X \leq 2$

often satisfied if X is Fano 3-fold.

Descendent invariants

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$$H^*(M, \mathbb{C}) \otimes H_*(M, \mathbb{C}) \xrightarrow{\int} \mathbb{C}$$

$$\sum_{\mathbb{F}} \uparrow \\ \mathbb{D}^x$$

(A2)

$$\psi \\ [M]^{vir}$$

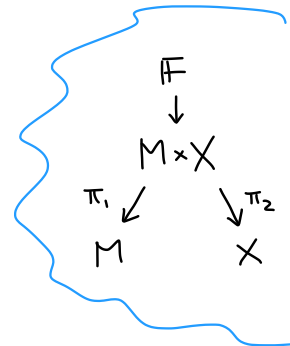
(A1), (A3)

homomorphism between
super-comm. ring

$$\sum_{\mathbb{F}} : \mathbb{D}^x \longrightarrow H^*(M, \mathbb{C})$$

$$ch_i(\sigma) \longmapsto \pi_{1,*} \left(ch_{\mathbb{F}} \cdot \pi_2^* \sigma \right)$$

$i \in \mathbb{Z}_{\geq 0}$ $\sigma \in H^{p,q}(X, \mathbb{C})$ $i + \dim_{\mathbb{C}} X - p$



Def. (Descendent invariants)

$$\int [M]^{vir} \sum_{\mathbb{F}}(\mathbb{D}) \in \mathbb{C} \text{ for each } \mathbb{D} \in \mathbb{D}^x$$

Question. What are the structures of descendent invariants?

e.g. dimension constraints, algebraicity constraints.

IV. Virasoro constraints in sheaf theory

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Representation of $\text{Vir}_{\geq -1}$

$$k \geq -1, \quad L_k = R_k + T_k \hookrightarrow \text{ID}^X$$



$$\text{derivation s.t. } R_k(ch_i(\gamma)) = i(i+1)\dots(i+k) ch_{i+k}(\gamma)$$

multiplication by $T_k = \sum_{\substack{i+j=k \\ i,j \geq 0}} i!j! \sum_s (-1)^{\dim X - p_s^L} ch_i(\gamma_s^L) ch_j(\gamma_s^R)$
 $(k \geq 0)$

$$\Delta_{*+d}(X) = \sum_s \gamma_s^L \otimes \gamma_s^R$$

$$\text{Indeed, } [L_n, L_m] = -(n-m)L_{n+m}$$

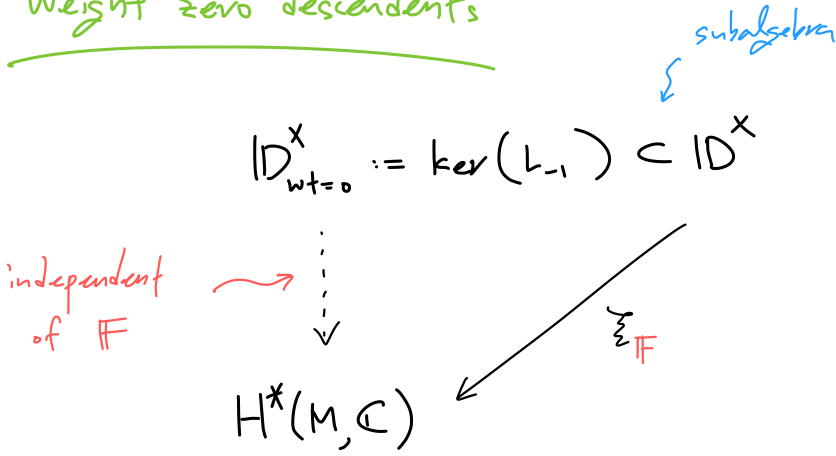
Unlike the Gromov-Witten case,

$$\int_{[M]^{vir}} \sum_{\mathbb{F}} (L_k(D)) \neq 0$$

in general. In fact, LHS depends on \mathbb{F} !

Weight zero descendants

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Main conjecture & result

Def. $L_{wt=0} := \sum_{k \geq -1} \frac{(-1)^k}{(k+1)!} L_k \circ (L_{-1})^{k+1} : ID^X \rightarrow ID_{wt=0}^X$

Conjecture (BLM) $\forall D \in ID^X, \int_{[M]^{vir}} L_{wt=0}(D) = 0$

van Breec's conjecture.

Thm (BLM) The conjecture holds for $M_c^{ss}(r, d), M_s^{H-ss}(r, c_1, c_2)$.

Remark. It is remarkable that the Virasoro constraints seem to exist in this great generality.

$H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$

e.g. all rational surfaces or Enriques surfaces.

odd degree line bundle.

e.g. $M = M_c^{ss}(2, \Delta)$: smooth of $\dim_{\mathbb{C}} = 3g-3$.

Newstead classes : $\alpha \in H^2(M)$, $\beta \in H^4(M)$, $\gamma \in H^6(M)$

\hookrightarrow comes from $c_2(E \oplus F)$

Virasoro constraints on M

$$\Leftrightarrow (g-k) \int_M \alpha^n \beta^m \gamma^k = -2n \int_M \alpha^{n-1} \beta^{m-1} \gamma^{k+1}$$

combinatorics
+
monodromy invariance.

" $\det(F^{norm}) \simeq \mathcal{O}_{M \times C}$ "

$\alpha = \sum_{F^{norm}} (-2 \text{ch}_1(1))$

$\beta = \sum_{F^{norm}} (4 \text{ch}_2(pt))$

$\gamma = \sum_{F^{norm}} \left(-2 \sum_{j=1}^g \text{ch}_2(f_j) \text{ch}_1(e_j) \right)$

$\langle e_i, f_j \rangle = \delta_{ij}$, $e_i \in H^0(C)$.