Virasoro constraints

history & moduli of sheaves

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I. History

Virasoro algebra

central charge

$$\mathrm{Vir} := \mathrm{span}_{\mathbb{C}} \left(\{ L_n \}_{n \in \mathbb{Z}} \cup \{ c \} \right)$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{\delta_{n+m,0} (n^3 - n)}{12} \cdot c$$

$$\mathrm{Vir}_{\geq -1} := \mathrm{span}_{\mathbb{C}} \{ L_n \}_{n \geq -1} \subset \mathrm{Vir}$$

Lie subalgebra.

Witten conjecture

$$Z(\lambda, t_0, t_1, t_2, \dots)$$

$$:= \exp \left(\sum_{g \geq 0} \lambda^{2g-2} \sum_{\substack{n \geq 0 \\ k_1, \dots, k_n \geq 0}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \frac{t_{k_1} \cdots t_{k_n}}{n!} \right)$$

$$\in \mathbb{Q}[\![\lambda^{\pm}, t_0, t_1, \dots]\!] \circlearrowleft \mathrm{Vir}_{\geq -1}\text{-representation}$$

e.g. $L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}$ $\leftrightarrow$ string equation

<u>Thm</u> (Kontsevich) $\forall k \geq -1, \; L_k \left(Z(\lambda, \vec{t}) \right) = 0$

Gromov-Witten theory

X smooth projective variety $/\mathbb{C}$

GW descendent invariants.

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \, ev_i^*(\underbrace{\gamma_i}_{\in H^*(X,\mathbb{C})})$$

<u>Conj</u> (Eguchi-Hori-Xiong) $\forall_{k \geq -1}$, $L_k(Z^X) = 0$.

proven for $X = \begin{cases} \text{curves} & \text{(Okounkov-Pandharipande)} \\ \text{toric varieties} & \text{(Givental, Teleman)} \end{cases}$

Sheaf theory

1) $PT_n(X,\beta)$ via GW/PT correspondence

2) $S^{[n]}$ via dimension reduction

[Moreira-Oblomkov-O-P, M]

3) $M_S^{H\text{-}ss}(r, c_1, c_2)$ [van Bree]

II. Moduli of sheaves

Goal: "classify" coherent sheaves on X

e.g. vector bundles, $I_{Z/X}$, $i_* \mathcal{O}_Z$ $\quad (i : Z \hookrightarrow X)$

dim $X = 1$

$$\underbrace{\mathrm{Coh}_X(r,d)}_{\text{topological classification}} \supset \mathrm{Vect}_c(r,d) \supset \mathrm{Vect}_c^{ss}(r,d)$$

Def. $F \in \mathrm{Vect}_c(r,d)$ is semistable if

$$\forall\, 0 \subsetneq G \subsetneq F, \quad \frac{\deg(G)}{\mathrm{rk}(G)} \leq \frac{d}{r}.$$

Thm (Mumford) $\exists$ projective variety $M_X^{ss}(r,d)$

which corepresents $\mathrm{Vect}_X^{ss}(r,d)$.

Rmk.

$$\gcd(r,d) = 1 \Rightarrow \left(\begin{array}{l} M_X^{ss}(r,d) \text{ smooth of } \dim_{\mathbb{C}} = 1 - r^2(1-g) \\ \\ \exists \text{ universal bundle } \mathbb{F} \to M_X^{ss}(r,d) \times X \end{array} \right.$$

General X

* $v \in H^*(X, \mathbb{Q})$ chern character
* H ample line bundle on X (a.k.a <u>stability</u> condition)

<u>Thm</u>. (Simpson, Huybrecht-Lehn)

highly singular

$\exists$ projective coarse moduli scheme $M_X^{H\text{-ss}}(v)$.

III. Enumerative invariants

<u>Assumption</u> on $M = M_X^{H\text{-ss}}(v)$

not necessary.

A1) no strictly H-semistable sheaves in M

A2) $\exists$ universal sheaf $\mathbb{F} \to M \times X$

not unique

A3) $\forall F \in M$, $\mathrm{Ext}_X^{\geq 3}(F, F) = 0$

<u>Rmk</u>. 1) $\exists$ numerical criterion for A1, A2.

2) A3 $\leftarrow$ automatic if $\dim X \leq 2$

often satisfied if X is Fano 3-fold.

Descendent invariants

$$
\begin{array}{ccc}
H^*(M,\mathbb{C}) \otimes H_*(M,\mathbb{C}) & \xrightarrow{\int} & \mathbb{C} \\
\xi_{\mathbb{F}} \uparrow \quad\quad \Psi & & \\
\mathbb{D}^X \quad\quad [M]^{vir} & &
\end{array}
$$

A2   A1, A3

homomorphism between super-comm. ring

$$\xi_{\mathbb{F}} : \mathbb{D}^X \longrightarrow H^*(M,\mathbb{C})$$

$$\mathrm{ch}_i(\gamma) \longmapsto \pi_{1*}\left(\mathrm{ch}_{i+\dim_{\mathbb{C}} X - p}\,\mathbb{F} \cdot \pi_2^*\gamma\right)$$

$i \in \mathbb{Z}_{\geq 0}$ $\gamma \in H^{p,*}(X,\mathbb{C})$ $i+\dim_{\mathbb{C}} X - p$

$$
\begin{tikzcd}
 & \mathbb{F} \arrow[d] & \\
 & M \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
M & & X
\end{tikzcd}
$$

<u>Def</u>. (Descendent invariants)

$$\int_{[M]^{vir}} \xi_{\mathbb{F}}(D) \in \mathbb{C} \quad \text{for each } D \in \mathbb{D}^X$$

<u>Question</u>. What are the structures of descendent invariants?

e.g. dimension constraints, algebraicity constraints.

IV. Virasoro constraints in sheaf theory

Representation of $\mathrm{Vir}_{\geq -1}$

$$k \geq -1, \quad L_k = R_k + T_k \curvearrowright \mathbb{D}^X$$

derivation s.t. $R_k\big(\mathrm{ch}_i(\gamma)\big) = i(i+1)\cdots(i+k)\,\mathrm{ch}_{i+k}(\gamma)$.

multiplication by $T_k = \sum_{\substack{i+j=k \\ i,j\geq 0}} i!\,j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i(\gamma_s^L)\,\mathrm{ch}_j(\gamma_s^R)$
$(k \geq 0)$

$$\Delta_* \mathrm{td}(X) = \sum_s \gamma_s^L \otimes \gamma_s^R$$

Indeed, $[L_n, L_m] = -(n-m)L_{n+m}$.

Unlike the Gromov-Witten case,

$$\int_{[M]^{vir}} \xi_{\mathbb{F}}\big(L_k(D)\big) \neq 0$$

in general. In fact, LHS depends on $\mathbb{F}$!

Weight zero descendents

subalgebra

$$\mathbb{D}^X_{wt=0} := \ker(L_{-1}) \subset \mathbb{D}^X$$

$$\begin{tikzcd} \mathbb{D}^X_{wt=0} \arrow[d, dashed] \arrow[r, phantom, "\subset"] & \mathbb{D}^X \arrow[dl, "\xi_{\mathbb{F}}"] \\ H^*(M, \mathbb{C}) & \end{tikzcd}$$

independent of $\mathbb{F}$

Main conjecture & result

<u>Def</u>. $$L_{wt=0} := \sum_{k \geq -1} \frac{(-1)^k}{(k+1)!} L_k \circ (L_{-1})^{k+1} : \mathbb{D}^X \longrightarrow \mathbb{D}^X_{wt=0}.$$

<u>Conjecture</u>. (BLM) $\forall D \in \mathbb{D}^X$, $$\int_{[M]^{vir}} L_{wt=0}(D) = 0.$$

<u>Thm</u>. (BLM) The conjecture holds for $M^{ss}_C(r,d)$, $\underbrace{M^{H\text{-}ss}_S(r,c_1,c_2)}_{\text{van Bree's conjecture}}$.

$H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$.

e.g. all rational surfaces or Enrique surfaces.

<u>Rmk</u>. It is remarkable that the Virasoro constraints seem to exist in this great generality.

odd degree line bundle.

e.g. $M = M_C^{ss}(2, \Delta)$: smooth of $\dim_{\mathbb{C}} = 3g-3$.

Newstead classes : $\alpha \in H^2(M)$, $\beta \in H^4(M)$, $\gamma \in H^6(M)$

$\hookrightarrow$ comes from $c_2(\mathrm{End}\,\mathbb{F})$.

Virasoro constraints on M

$$\Longleftrightarrow \quad (g-k)\int_M \alpha^n \beta^m \gamma^k = -2n \int_M \alpha^{n-1}\beta^{m-1}\gamma^{k+1}$$

combinatorics

+

monodromy invariance.

"$\det(\mathbb{F}^{norm}) \simeq \mathcal{O}_{M\times C}$"

$$\alpha = \xi_{\mathbb{F}^{norm}}\left(-2\,\mathrm{ch}_1(1)\right)$$

$$\beta = \xi_{\mathbb{F}^{norm}}\left(4\,\mathrm{ch}_2(\mathrm{pt})\right)$$

$$\gamma = \xi_{\mathbb{F}^{norm}}\left(-2\sum_{j=1}^{g}\mathrm{ch}_2(f_j)\,\mathrm{ch}_1(e_j)\right)$$

$\langle e_i, f_j\rangle = \delta_{i,j}$, $e_i \in H^{0,1}(C)$.