Coordinator
A. Gabrielli

Non-Life Insurance: Mathematics and Statistics

Exercise sheet 1

Exercise 1.1 Discrete Distribution
Suppose the random variable N follows a geometric distribution with parameter $p \in (0,1)$, i.e.

$$\mathbb{P}[N=k] = \begin{cases} (1-p)^{k-1}p & \text{if } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{else.} \end{cases}$$

(a) Show that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $n \in \mathbb{N} \setminus \{0\}$. Calculate $\mathbb{P}[N \geq n]$.

(c) Calculate $\mathbb{E}[N]$.

(d) Let $r < -\log(1-p)$. Calculate $M_N(r) = \mathbb{E}[\exp\{rN\}]$.
Remark: M_N is called the moment generating function of N.

(e) Calculate $\frac{d}{dr} M_N(r)|_{r=0}$. What do you observe?

Exercise 1.2 Absolutely Continuous Distribution
Suppose the random variable Y follows an exponential distribution with parameter $\lambda > 0$, i.e. the density f_Y of Y is given by

$$f_Y(x) = \begin{cases} \lambda \exp\{-\lambda x\} & \text{if } x \geq 0, \\ 0 & \text{else.} \end{cases}$$

(a) Show that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $0 < y_1 < y_2$. Calculate $\mathbb{P}[y_1 \leq Y \leq y_2]$.

(c) Calculate $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(d) Let $r < \lambda$. Calculate $\log M_Y(r) = \log \mathbb{E}[\exp\{rY\}]$.
Remark: $\log M_Y$ is called the cumulant generating function of Y.

(e) Calculate $\frac{d^2}{dr^2} \log M_Y(r)|_{r=0}$. What do you observe?

Exercise 1.3 Conditional Distribution
Suppose that an insurance company distinguishes between small and large claims, where a claim is called large if it exceeds a fixed threshold $\theta > 0$. We use the random variable I to indicate whether a claim is small or large, i.e. we have $I = 0$ for a small claim and $I = 1$ for a large claim. In particular, we get $\mathbb{P}[I=0] = 1-p$ and $\mathbb{P}[I=1] = p$ for some $p \in (0,1)$. Finally, we model the size of a claim that belongs to the large claims section by the random variable Y, where, given that we have a small claim, Y is equal to 0 almost surely and, given that we have a large claim, Y follows a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 0$, i.e. the density $f_{Y|I=1}$ of $Y \mid I = 1$ is given by

$$f_{Y|I=1}(x) = \begin{cases} \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} & \text{if } x \geq \theta, \\ 0 & \text{else.} \end{cases}$$

(a) Let $y > \theta$. Calculate $\mathbb{P}[Y \geq y]$.

(b) Calculate $\mathbb{E}[Y]$.

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Exercise sheet 2

Exercise 2.1 Gaussian Distribution

For a random variable X we write $X \sim \mathcal{N}(\mu, \sigma^2)$ if X follows a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. The moment generating function M_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$M_X(r) = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\} \qquad \text{for all } r \in \mathbb{R}.$$

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$. Show that

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2).$$

(b) Let $X_1, \ldots, X_n$ be independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i \in \{1, \ldots, n\}$. Show that

$$\sum_{i=1}^n X_i \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$

Exercise 2.2 Maximum Likelihood and Hypothesis Test

Let $Y_1, \ldots, Y_n$ be claim amounts in CHF that an insurance company has to pay. We assume that $Y_1, \ldots, Y_n$ are independent random variables that all follow a log-normal distribution with the same unknown parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Then, by definition, $\log Y_1, \ldots, \log Y_n$ are independent Gaussian random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Let $n = 8$ and suppose that we have the following observations for $\log Y_1, \ldots, \log Y_8$:

$$x_1 = 9, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 7, \quad x_5 = 3, \quad x_6 = 11, \quad x_7 = 6, \quad x_8 = 10.$$

(a) Write down the joint density $f_{\mu,\sigma^2}(x_1, \ldots, x_8)$ of $\log Y_1, \ldots, \log Y_8$.

(b) Calculate $\log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

(c) Calculate $(\hat{\mu}, \hat{\sigma}^2) = \arg \max_{(\mu,\sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}} \log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

Remark: Since the logarithm is a monotonically increasing function, $\hat{\mu}$ and $\hat{\sigma}^2$ are chosen such that the joint density of $\log Y_1, \ldots, Y_n$ at the given observations $x_1, \ldots, x_8$ is maximized. Hence $\hat{\mu}$ and $\hat{\sigma}^2$ are called maximum likelihood estimates.

(d) Now suppose that we are interested in the mean μ of the logarithm of the claim amounts and an expert claims that $\mu = 6$. Perform a statistical test to test the null hypothesis

$$H_0 : \mu = 6$$

against the (two-sided) alternative hypothesis

$$H_1 : \mu \neq 6.$$

Exercise 2.3 Variance Decomposition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$. Show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).$$

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Exercise sheet 3

Exercise 3.1 No-Claims Bonus
An insurance company decides to offer a no-claims bonus to good car drivers, namely

- a 10% discount after three years of no claim, and
- a 20% discount after six years of no claim.

If the base premium is defined as the expected claim amount, how does it need to be adjusted so that this no-claims bonus can be financed? For simplicity, we consider one car driver who has been insured for at least six years. Answer the question in the following two situations:

(a) The claim counts of the individual years of the considered car driver are independent, identically Poisson distributed random variables with frequency parameter $\lambda = 0.2$.

(b) Suppose Θ follows a gamma distribution with shape parameter $\gamma = 1$ and scale parameter $c = 1$, i.e. the density f_Θ of Θ is given by

$$f_\Theta(x) = \begin{cases} \exp\{-x\} & \text{if } x \geq 0, \\ 0 & \text{else.} \end{cases}$$

Now, conditionally given Θ, the claim counts of the individual years of the considered car driver are independent, identically Poisson distributed random variables with frequency parameter $\Theta\lambda$, where $\lambda = 0.2$ as above.

Exercise 3.2 Central Limit Theorem
Let n be the number of claims and $Y_1, \ldots, Y_n$ the corresponding claim sizes, where we assume that $Y_1, \ldots, Y_n$ are independent, identically distributed random variables with expectation $\mathbb{E}[Y_1] = \mu$ and coefficient of variation $\mathrm{Vco}(Y_1) = 4$. Use the Central Limit Theorem to determine an approximate minimum number of claims such that with probability of at least 95% the deviation of the empirical mean $\frac{1}{n}\sum_{i=1}^n Y_i$ from μ is less than 1%.

Exercise 3.3 Compound Binomial Distribution
Assume $S \sim \mathrm{CompBinom}(v, p, G)$ for given $v \in \mathbb{N}$, $p \in (0,1)$ and individual claim size distribution G. Let $M > 0$ such that $G(M) \in (0,1)$. Define the compound distribution of claims Y_i exceeding threshold M by

$$S_{\mathrm{lc}} = \sum_{i=1}^{N} Y_i \, 1_{\{Y_i > M\}}.$$

Show that $S_{\mathrm{lc}} \sim \mathrm{CompBinom}(v, p[1 - G(M)], G_{\mathrm{lc}})$, where the large claims size distribution function G_{lc} satisfies $G_{\mathrm{lc}}(y) = \mathbb{P}(Y_1 \leq y | Y_1 > M)$.

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Exercise sheet 4

Exercise 4.1 Poisson Model and Negative-Binomial Model
Suppose that we are given the following claim count data of ten years:

t	1	2	3	4	5	6	7	8	9	10
N_t	1'000	997	985	989	1'056	1'070	994	986	1'093	1'054
v_t	10'000	10'000	10'000	10'000	10'000	10'000	10'000	10'000	10'000	10'000

Table 1: Observed claims counts N_t and corresponding volumes v_t.

(a) Estimate the claims frequency parameter $\lambda > 0$ of the Poisson model and calculate an estimated, roughly 70%-confidence interval for λ. What do you observe?

(b) Perform a χ^2-goodness-of-fit test at the significance level of 5% to test the null hypothesis of having Poisson distributions.

(c) Estimate the claims frequency parameter $\lambda > 0$ and the dispersion parameter $\gamma > 0$ of the negative-binomial model and calculate an estimated, roughly 70%-confidence interval for λ. What do you observe?

Exercise 4.2 Compound Poisson Distribution
For the total claim amount S of an insurance company we assume $S \sim \text{CompPoi}(\lambda v, G)$, where $\lambda = 0.06$, $v = 10$ and for a random variable Y with distribution function G we have

k	100	300	500	6'000	100'000	500'000	2'000'000	5'000'000	10'000'000
$\mathbb{P}[Y = k]$	3/20	4/20	3/20	2/15	2/15	1/15	1/12	1/24	1/24

Table 2: Distribution of $Y \sim G$.

Suppose that the insurance company wants to distinguish between

- small claims: claim size $\leq$ 1'000,
- medium claims: 1'000 $<$ claim size $\leq$ 1'000'000 and
- large claims: claim size $>$ 1'000'000.

Let S_{sc}, S_{mc} and S_{lc} be the total claims in the small claims layer, in the medium claims layer and in the large claims layer, respectively.

(a) Give definitions of S_{sc}, S_{mc} and S_{lc} in terms of mathematical formulas.

(b) Determine the distributions of S_{sc}, S_{mc} and S_{lc}.

(c) What is the dependence structure between S_{sc}, S_{mc} and S_{lc}?

(d) Calculate $\mathbb{E}[S_{\text{sc}}]$, $\mathbb{E}[S_{\text{mc}}]$ and $\mathbb{E}[S_{\text{lc}}]$ as well as $\text{Var}(S_{\text{sc}})$, $\text{Var}(S_{\text{mc}})$ and $\text{Var}(S_{\text{lc}})$. Use these values to calculate $\mathbb{E}[S]$ and $\text{Var}(S)$.

(e) Calculate the probability that the total claim in the large claims layer exceeds 5 millions.

Exercise 4.3 Method of Moments

We assume that the independent claim sizes $Y_1, \ldots, Y_8$ all follow a Gamma distribution with the same unknown shape parameter $\gamma > 0$ and the same unknown scale parameter $c > 0$ and that we have the following observations for $Y_1, \ldots, Y_8$:

$$x_1 = 7, \quad x_2 = 8, \quad x_3 = 10, \quad x_4 = 9, \quad x_5 = 5, \quad x_6 = 11, \quad x_7 = 6, \quad x_8 = 8.$$

Calculate the method of moments estimates of γ and c.

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Exercise sheet 5

Exercise 5.1 Kolmogorov-Smirnov Test
Suppose we are given the following claim size data (in increasing order) coming from independent realizations of an unknown claim size distribution:

$$x_1 = \left(-\log\frac{38}{40}\right)^2, x_2 = \left(-\log\frac{37}{40}\right)^2, x_3 = \left(-\log\frac{35}{40}\right)^2, x_4 = \left(-\log\frac{34}{40}\right)^2, x_5 = \left(-\log\frac{10}{40}\right)^2.$$

Perform a Kolmogorov-Smirnov test at significance level of 5% to test the null hypothesis of having a Weibull distribution with shape parameter $\tau = \frac{1}{2}$ and scale parameter $c = 1$ as claim size distribution. Moreover, explain why the Kolmogorov-Smirnov test is applicable in this example.

Exercise 5.2 Large Claims
We define storm and flood events to be those events with total claim amount exceeding CHF 50 millions. Assume that for the yearly claim amount S of such storm and flood events we have $S \sim \text{CompPoi}(\lambda, G)$, where λ is unknown and G is the distribution function of a Pareto distribution with threshold $\theta = 50$ and unknown tail index $\alpha > 0$. Moreover, the number of claims and the claim sizes and thus also the total claim amounts are assumed to be independent across different years. During the years $1986 - 2005$ we observed the following 15 storm and flood events:

date	amount in millions	date	amount in millions
20.06.1986	52.8	18.05.1994	78.5
18.08.1986	135.2	18.02.1999	75.3
18.07.1987	55.9	12.05.1999	178.3
23.08.1987	138.6	26.12.1999	182.8
26.02.1990	122.9	04.07.2000	54.4
21.08.1992	55.8	13.10.2000	365.3
24.09.1993	368.2	20.08.2005	1'051.1
08.10.1993	83.8		

(a) Using n independent claim sizes $Y_1, \dots, Y_n$, show that the MLE $\hat{\alpha}_n^{\text{MLE}}$ of α is given by

$$\hat{\alpha}_n^{\text{MLE}} = \left(\frac{1}{n}\sum_{i=1}^{n} \log Y_i - \log\theta\right)^{-1}.$$

(b) According to Lemma 3.7 of the lecture notes,

$$\frac{n-1}{n}\hat{\alpha}_n^{\text{MLE}}$$

is an unbiased version of the MLE. Estimate α using the unbiased version of the MLE for the storm and flood data given above.

(c) Calculate the MLE of λ for the storm and flood data given above.

(d) Suppose that we introduce a maximal claims cover of $M = 2$ billions CHF per event, i.e. the individual claims are given by $\min\{Y_i, M\}$. Using the estimates of α and λ found above, calculate the estimated expected total yearly claim amount.

(e) Using the estimates of α and λ found above, calculate the estimated probability that we observe at least one storm and flood event next year which exceeds the level of $M = 2$ billions CHF.

Exercise 5.3 Pareto Distribution
Suppose the random variable Y follows a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 0$.

(a) Show that the survival function of Y is regularly varying at infinity with tail index α.

(b) Show that for $\theta \leq u_1 < u_2$ the expected value of Y within the layer $(u_1, u_2]$ is given by

$$\mathbb{E}\left[Y 1_{\{u_1 < Y \leq u_2\}}\right] = \begin{cases} \theta \frac{\alpha}{\alpha-1}\left[\left(\frac{u_1}{\theta}\right)^{-\alpha+1} - \left(\frac{u_2}{\theta}\right)^{-\alpha+1}\right] & \text{if } \alpha \neq 1, \\ \theta \log\left(\frac{u_2}{u_1}\right) & \text{if } \alpha = 1. \end{cases}$$

(c) Show that for $\alpha > 1$ and $y > \theta$ the loss size index function for level y is given by

$$\mathcal{I}[G(y)] = 1 - \left(\frac{y}{\theta}\right)^{-\alpha+1}.$$

(d) Show that for $\alpha > 1$ and $u > \theta$, the mean excess function of Y above u is given by

$$e(u) = \frac{1}{\alpha - 1} u.$$

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Exercise sheet 6

Exercise 6.1 Goodness-of-Fit Test
Suppose we are given the following claim size data (in increasing order) coming from independent realizations of an unknown claim size distribution:

210, 215, 228, 232, 303, 327, 344, 360, 365, 379, 402, 413, 437, 481, 521, 593, 611, 677, 910, 1623.

Use the intervals

$$I_1 = [200, 239), \quad I_2 = [239, 301), \quad I_3 = [301, 416), \quad I_4 = [416, 725), \quad I_5 = [725, +\infty)$$

to perform a χ^2-goodness-of-fit test at significance level of 5% to test the null hypothesis of having a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$ as claim size distribution.

Exercise 6.2 Log-Normal Distribution and Deductible
Assume that the total claim amount

$$S = \sum_{i=1}^{N} Y_i$$

in a given line of business has a compound distribution with $\mathbb{E}[N] = \lambda v$, where λ denotes the claims frequency, and with a log-normal distribution with mean parameter $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$ as claim size distribution.

(a) Show that

$$\mathbb{E}[Y_1] = \exp\left\{\mu + \frac{\sigma^2}{2}\right\},$$
$$\mathrm{Var}(Y_1) = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) \quad \text{and}$$
$$\mathrm{Vco}(Y_1) = \sqrt{\exp\left\{\sigma^2\right\} - 1}.$$

Hint: Use the moment generating function of a Gaussian distribution with mean μ and variance σ^2.

(b) Suppose that $\mathbb{E}[Y_1] =$ 3'000 and $\mathrm{Vco}(Y_1) = 4$. Up to now, the insurance company was not offering contracts with deductibles. Now it wants to offer a deductible of $d = 500$. Answer the following questions:

(i) How does the claims frequency λ change by the introduction of the deductible?

(ii) How does the expected claim size $\mathbb{E}[Y_1]$ change by the introduction of the deductible?

(iii) How does the expected total claim amount $\mathbb{E}[S]$ change by the introduction of the deductible?

Exercise 6.3 Inflation and Deductible
This year's claims in a storm insurance portfolio have been modeled by a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 1$. The threshold θ (in CHF) can be understood as deductible, i.e. as the part that the insureds have to pay by themselves. Suppose that the inflation for next year is expected to be $100 \cdot r\,\%$ for some $r > 0$. By how much do we have to increase the deductible θ next year such that the average claim payment remains unchanged?

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Exercise sheet 7

Exercise 7.1 Hill Estimator (R Exercise)
Write an R-Code that samples 300 independent observations from a Pareto distribution with threshold $\theta = 10$ millions and tail index $\alpha = 2$. Use set.seed(100) for reproducibility reasons. Moreover, create a Hill plot and a log-log plot. What do you observe?

Exercise 7.2 Approximations for Compound Distributions
Assume that S has a compound Poisson distribution with expected number of claims $\lambda v = 1'000$ and claim sizes following a gamma distribution with shape parameter $\gamma = 100$ and scale parameter $c = \frac{1}{10}$. For all $\alpha \in (0, 1)$, let q_α denote the α-quantile of S.

(a) Use the normal approximation to estimate $q_{0.95}$ and $q_{0.99}$.

(b) Use the translated gamma approximation to estimate $q_{0.95}$ and $q_{0.99}$.

(c) Use the translated log-normal approximation to estimate $q_{0.95}$ and $q_{0.99}$.

(d) Comment on the results found above.

Exercise 7.3 Akaike Information Criterion and Bayesian Information Criterion
Assume that we have i.i.d. claim sizes $Y_1, \ldots, Y_n$, with $n = 1'000$, coming from a gamma distribution. Moreover, assume that if we fit a gamma distribution to this data, we obtain the following method of moments estimators and the MLEs:

$$\hat{\gamma}^{\mathrm{MM}} = 0.9794 \quad \text{and} \quad \hat{c}^{\mathrm{MM}} = 9.4249,$$
$$\hat{\gamma}^{\mathrm{MLE}} = 1.0013 \quad \text{and} \quad \hat{c}^{\mathrm{MLE}} = 9.6360.$$

The corresponding log-likelihoods are given by

$$\ell_{\mathbf{Y}}\left(\hat{\gamma}^{\mathrm{MM}}, \hat{c}^{\mathrm{MM}}\right) = 1264.013 \quad \text{and} \quad \ell_{\mathbf{Y}}\left(\hat{\gamma}^{\mathrm{MLE}}, \hat{c}^{\mathrm{MLE}}\right) = 1264.171.$$

(a) Why is $\ell_{\mathbf{Y}}\left(\hat{\gamma}^{\mathrm{MLE}}, \hat{c}^{\mathrm{MLE}}\right) > \ell_{\mathbf{Y}}\left(\hat{\gamma}^{\mathrm{MM}}, \hat{c}^{\mathrm{MM}}\right)$? Which fit should be preferred according to the Akaike Information Criterion (AIC)?

(b) The estimates of γ are very close to 1 and, thus, we could also use an exponential distribution as claim size distribution. For the exponential distribution we obtain the MLE $\hat{c}^{\mathrm{MLE}} = 9.6231$ and the corresponding log-likelihood $\ell_{\mathbf{Y}}\left(\hat{c}^{\mathrm{MLE}}\right) = 1264.169$. According to the AIC and the Bayesian Information Criterion (BIC), should we prefer the gamma model or the exponential model?

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Exercise sheet 8

Exercise 8.1 Panjer Algorithm
In this exercise we calculate monthly health insurance premiums for different franchises d using the Panjer algorithm. In particular, we assume that the yearly claim amount

$$S = \sum_{i=1}^{N} Y_i$$

of a given customer is compound Poisson distributed with $N \sim \text{Poi}(1)$ and $Y_1 \overset{(d)}{=} k + Z$, where $k = 100$ CHF and $Z \sim \text{LN}(\mu = 7.8, \sigma^2 = 1)$. In health insurance, the policyholder can choose between different franchises $d \in \{300, 500, 1'000, 1'500, 2'000, 2'500\}$. Moreover, he has to pay $\alpha = 10\%$ of the part of the total claim amount that exceeds the franchise d, but only up to a maximum of $M = 700$ CHF. Thus, the yearly amount paid by the customer is given by

$$S_{\text{ins}} = \min\{S, d\} + \min\{\alpha \cdot (S - d)_+, M\}.$$

If we define $\pi_0 = \mathbb{E}[S]$ and $\pi_{\text{ins}} = \mathbb{E}[S_{\text{ins}}]$, then the monthly pure risk premium π is given by

$$\pi = \frac{\pi_0 - \pi_{\text{ins}}}{12}.$$

Calculate π for the different franchises $d = 300, 500, 1'000, 1'500, 2'000$ and $2'500$ using the Panjer algorithm. In order to do that, discretize the translated log-normal distribution using a span of $s = 10$ and putting all the probability mass to the upper end of the intervals.

Exercise 8.2 Variance Loading Principle
We would like to insure the following car fleet:

i	v_i	λ_i	$\mathbb{E}[Y_1^{(i)}]$	$\text{Vco}(Y_1^{(i)})$
passenger car	40	25%	2'000	2.5
delivery van	30	23%	1'700	2.0
truck	10	19%	4'000	3.0

Assume that the total claim amounts for passenger cars, delivery vans and trucks can be modeled by independent compound Poisson distributions.

(a) Calculate the expected claim amount of the car fleet.

(b) Calculate the premium for the car fleet using the variance loading principle with $\alpha = 3 \cdot 10^{-6}$.

Exercise 8.3 Panjer Distribution
Let N be a random variable that has a Panjer distribution with parameters $a, b \in \mathbb{R}$. Calculate $\mathbb{E}[N]$ and $\text{Var}(N)$. What can you say about the ratio of $\text{Var}(N)$ to $\mathbb{E}[N]$?

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Exercise sheet 9

Exercise 9.1 Utility Indifference Price
In this exercise we calculate the premium for the accident insurance of a given company COMP using the utility indifference price principle. We suppose that all employees of COMP have been divided into two groups, depending on their work, and that the total claim amounts S_1 and S_2 of the two groups are independent and compound Poisson distributed:

Group i	v_i	λ_i	$Y_1^{(i)}$
Group 1	2'000	50%	$\Gamma(\gamma = 20, c = 0.01)$
Group 2	10'000	10%	expo(0.005)

We write $S = S_1 + S_2$ for the total claim amounts of COMP. Let c_0 be the initial capital of the insurance company that sells accident insurance to COMP.

(a) Let u be a utility function. Show that if the utility indifference price $\pi = \pi(u, S, c_0)$ exists, then it is unique and satisfies $\pi > \mathbb{E}[S]$.

(b) Calculate $\mathbb{E}[S]$.

(c) Calculate π using the utility indifference price principle for the exponential utility function with parameter $\alpha = 1.5 \cdot 10^{-6}$.

(d) What happens to π if we replace the compound Poisson distributions by Gaussian distributions with the same corresponding first two moments?

Exercise 9.2 Value-at-Risk and Expected Shortfall
Suppose that for the yearly claim amount S of an insurance company in a given line of business we have $S \sim \mathrm{LN}(\mu, \sigma^2)$ with $\mu = 20$ and $\sigma^2 = 0.015$. Moreover, we set the cost-of-capital rate $r_{\mathrm{CoC}} = 6\%$. Then the premium π_{CoC} for the considered line of business using the cost-of-capital pricing principle is given by

$$\pi_{\mathrm{CoC}} = \mathbb{E}[S] + r_{\mathrm{CoC}}\, \rho(S - \mathbb{E}[S]),$$

where ρ is a risk measure.

(a) Calculate π_{CoC} using the VaR risk measure at security level $1 - q = 99.5\%$.

(b) Calculate π_{CoC} using the expected shortfall risk measure at security level $1 - q = 99\%$.

(c) Which security level is needed such that π_{CoC} using the VaR risk measure is equal to the price calculated in (b)?

(d) Let U and V be two independent copies of $\log S$. Show that on the one hand

$$\mathrm{VaR}_{0.45}(U + V) > \mathrm{VaR}_{0.45}(U) + \mathrm{VaR}_{0.45}(V),$$

but on the other hand

$$\mathrm{VaR}_{0.55}(U + V) < \mathrm{VaR}_{0.55}(U) + \mathrm{VaR}_{0.55}(V).$$

In particular, the VaR is not subadditive, and hence not coherent.

Exercise 9.3 Esscher Premium

Let S be a random variable with distribution function F and moment generating function M_S. Assume that there exists $r_0 > 0$ such that $M_S(r) < \infty$ for all $r \in (-r_0, r_0)$. For $\alpha \in (0, r_0)$, let π_α denote the Esscher premium of S.

(a) Show that if S is non-deterministic, then π_α is strictly increasing in α.

(b) Show that the Esscher premium for small values of α boils down to a variance premium principle.

(c) Suppose that $S \sim \text{CompPoi}(\lambda v, G)$, where $\lambda v > 0$ and G is the distribution function of a gamma distribution with shape parameter $\gamma > 0$ and scale parameter $c > 0$. For which values of α does π_α exist? Calculate π_α where it is defined.

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Exercise sheet 10

Exercise 10.1 Tariffication Methods (R Exercise)
Suppose that a car insurance portfolio of an insurance company has been divided according to two tariff criteria:

- vehicle type: {passenger car, delivery van, truck} = {1,2,3}
- driver age : {21-30 years, 31-40 years, 41-50 years, 51-60 years} = {1,2,3,4}

For simplicity, we set the number of policies $v_{i,j} = 1$ for all risk classes (i, j), $1 \leq i \leq 3$, $1 \leq j \leq 4$. Moreover, assume that we observed the following claim amounts

	21-30y	31-40y	41-50y	51-60y
passenger car	2'000	1'800	1'500	1'600
delivery van	2'200	1'600	1'400	1'400
truck	2'500	2'000	1'700	1'600

and that we work with a multiplicative tariff structure.

(a) Apply the method of Bailey & Simon in order to calculate the tariffs.
Hint: In order to get a unique solution, set $\mu = \chi_{1,1} = 1$.

(b) Apply the method of Bailey & Jung (i.e. the method of total marginal sums) in order to calculate the tariffs.
Hint: In order to get a unique solution, set $\mu = \chi_{1,1} = 1$.

(c) Determine the design matrix Z of the log-linear regression model and write an R-Code that applies the MLE method in the Gaussian approximation framework. Discuss the output. Is there statistical evidence that the classification into different types of vehicles could be omitted?
Hint: In order to get a unique solution, set $\beta_{1,1} = \beta_{2,1} = 0$.

(d) Compare the results found in parts (a) - (c).

Exercise 10.2 Claims Frequency Modelling with GLM (R Exercise)
Suppose that a motorbike insurance portfolio of an insurance company has been divided according to three tariff criteria:

- vehicle class: {weight over 60 kg and more than two gears, other} = {1,2}
- vehicle age: {at most one year, more than one year} = {1,2}
- geographic zone: {large cities, middle-sized towns, smaller towns and countryside} = {1,2,3}

Assume that we observed the following claim frequencies:

class	age	zone	volumes	number of claims	claim frequencies
1	1	1	100	25	0.250
1	1	2	200	15	0.075
1	1	3	500	15	0.030
1	2	1	400	60	0.150
1	2	2	900	90	0.100
1	2	3	7'000	210	0.030
2	1	1	200	45	0.225
2	1	2	300	45	0.150
2	1	3	600	30	0.050
2	2	1	800	80	0.100
2	2	2	1'500	120	0.080
2	2	3	5'000	90	0.018

(a) Write an R-Code that performs a GLM analysis for the claim frequencies.

(b) Plot the observed and the fitted claim frequencies against the vehicle class, the vehicle age and the geographic zone.

(c) Create a Tukey-Anscombe plot of the fitted claim frequencies versus the deviance residuals and a QQ plot of the deviance residuals versus the theoretical (estimated) quantiles.

(d) Is there statistical evidence that the classification into the geographic zones could be omitted?

Exercise 10.3 Tweedie's Compound Poisson Model

Let $S \sim \text{CompPoi}(\lambda v, G)$, where $\lambda > 0$ is the unknown claims frequency parameter, $v > 0$ the known volume and G the distribution function of a gamma distribution with known shape parameter $\gamma > 0$ and unknown scale parameter $c > 0$. Then S has a mixture distribution with a point mass of $\mathbb{P}[S = 0]$ in 0 and a density f_S on $(0, \infty)$.

(a) Calculate $\mathbb{P}[S = 0]$ and f_S.

(b) Show that S belongs to the exponential dispersion family with

$$
\begin{aligned}
w &= v, \\
\phi &= \frac{\gamma+1}{\lambda\gamma}\left(\frac{\lambda v \gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}, \\
\theta &= -(\gamma+1)\left(\frac{\lambda v \gamma}{c}\right)^{-\frac{1}{\gamma+1}}, \\
\boldsymbol{\Theta} &= (-\infty, 0), \\
b(\theta) &= \frac{\gamma+1}{\gamma}\left(\frac{-\theta}{\gamma+1}\right)^{-\gamma}, \\
c(x, \phi, w) &= \log\left(\sum_{n=1}^{\infty}\left[\frac{(\gamma+1)^{\gamma+1}}{\gamma}\left(\frac{\phi}{w}\right)^{-\gamma-1}\right]^n \frac{1}{\Gamma(n\gamma)n!}\, x^{n\gamma-1}\right), \quad \text{if } x > 0, \text{ and} \\
c(0, \phi, w) &= 0.
\end{aligned}
$$

Non-Life Insurance: Mathematics and Statistics

Exercise sheet 11

Exercise 11.1 (Inhomogeneous) Credibility Estimators for Claim Counts
Suppose that we are given the following current year's claim counts data for 5 different regions, where $v_{i,1}$ = number of policies in region i and $N_{i,1}$ = number of claims in region i, for all $i \in \{1, \dots, 5\}$:

region i	$v_{i,1}$	$N_{i,1}$
1	50'061	3'880
2	10'135	794
3	121'310	8'941
4	35'045	3'448
5	4'192	314
total	220'743	17'377

We assume that we are in the Bühlmann-Straub model framework with $I = 5$, $T = 1$ and, for all $i \in \{1, \dots, 5\}$,

$$N_{i,1} \,|\, \Theta_i \sim \text{Poi}(\mu(\Theta_i) v_{i,1}),$$

where $\mu(\Theta_i) = \Theta_i \lambda_0$ and $\lambda_0 = 0.088$. Moreover, we assume that the pairs $(\Theta_1, N_{1,1}), \dots, (\Theta_5, N_{5,1})$ are independent and $\Theta_1, \dots, \Theta_5$ are i.i.d. with $\Theta_1 > 0$ a.s. and

$$\mu_0 = \mathbb{E}[\mu(\Theta_1)] = \lambda_0 \qquad \text{and} \qquad \tau^2 = \text{Var}(\mu(\Theta_1)) = 0.00024.$$

(a) Calculate the inhomogeneous credibility estimator $\widehat{\widehat{\mu(\Theta_i)}}$ for each region i.

(b) We denote next year's numbers of policies by $v_{1,2}, \dots, v_{5,2}$ and next year's numbers of claims by $N_{1,2}, \dots, N_{5,2}$. Suppose that $N_{i,1}$ and $N_{i,2}$ are independent, conditionally given Θ_i, for all $i \in \{1, \dots, 5\}$, and that the number of policies grows 5% in each region. Calculate the mean square error of prediction

$$\mathbb{E}\left[\left(\frac{N_{i,2}}{v_{i,2}} - \widehat{\widehat{\mu(\Theta_i)}}\right)^2\right],$$

with $N_{i,2} \,|\, \Theta_i \sim \text{Poi}(\mu(\Theta_i) v_{i,2})$, for all $i \in \{1, \dots, 5\}$.

Exercise 11.2 (Homogeneous) Credibility Estimators for Claim Sizes
Suppose that we are given the following claim size data for two different years and four different risk classes, where $v_{i,t}$ = number of claims in risk class i and year t and $Y_{i,t}$ = total claim size in risk class i and year t, for all $i \in \{1, 2, 3, 4\}, t \in \{1, 2\}$:

risk class i	$v_{i,1}$	$Y_{i,1}$	$v_{i,2}$	$Y_{i,2}$
1	1'058	8'885'738	1'111	13'872'665
2	3'146	7'902'445	3'303	4'397'183
3	238	2'959'517	250	6'007'351
4	434	10'355'286	456	15'629'998
total	4'876	30'102'986	5'120	39'907'197

We assume that we are in the Bühlmann-Straub model framework with $I = 4$, $T = 2$ and, for all $i \in \{1, 2, 3, 4\}, t \in \{1, 2\}$,

$$Y_{i,t} \,|\, \Theta_i \sim \Gamma(\mu(\Theta_i) c v_{i,t}, c),$$

where $\mu(\Theta_i) = \Theta_i$ and $c > 0$. Moreover, we assume that $(\Theta_1, Y_{1,1}, Y_{1,2}), \dots, (\Theta_4, Y_{4,1}, Y_{4,2})$ are independent as well as $Y_{i,1}$ and $Y_{i,2}$, conditionally given Θ_i, for all $i \in \{1,2,3,4\}$, and that $\Theta_1, \dots, \Theta_4$ are i.i.d. with $\Theta_1 > 0$ a.s. and $\mathbb{E}[\Theta_1^2] < \infty$.

(a) Calculate the homogeneous credibility estimator $\widehat{\widehat{\mu(\Theta_i)}}^{\text{hom}}$ for each risk class i.

(b) We denote next year's numbers of claims by $v_{1,3}, \dots, v_{4,3}$ and next year's total claim sizes by $Y_{1,3}, \dots, Y_{4,3}$. Suppose that $Y_{i,1}, Y_{i,2}$ and $Y_{i,3}$ are independent, conditionally given Θ_i, for all $i \in \{1,2,3,4\}$, and that the number of claims grows 5% in each region. Estimate the mean square error of prediction

$$\mathbb{E}\left[\left(\frac{Y_{i,3}}{v_{i,3}} - \widehat{\widehat{\mu(\Theta_i)}}^{\text{hom}}\right)^2\right],$$

with $Y_{i,3} \,|\, \Theta_i \sim \Gamma(\mu(\Theta_i) c v_{i,3}, c)$, for all $i \in \{1,2,3,4\}$.

Exercise 11.3 Pareto-Gamma Model
Suppose that $\Lambda \sim \Gamma(\gamma, c)$ with prior shape parameter $\gamma > 0$ and prior scale parameter $c > 0$ and that, conditionally given Λ, the components of $\mathbf{Y} = (Y_1, \dots, Y_T)$ are independent with $Y_t \sim \text{Pareto}(\theta, \Lambda)$ for some threshold $\theta > 0$, for all $t \in \{1, \dots, T\}$.

(a) Show that the posterior distribution of Λ, conditional on $\mathbf{Y}$, is given by

$$\Lambda \,|\, \mathbf{Y} \sim \Gamma\left(\gamma + T, c + \sum_{t=1}^{T} \log \frac{Y_t}{\theta}\right).$$

(b) For the estimation of the unknown parameter Λ we define the prior estimator λ_0 and the posterior estimator $\widehat{\lambda}_T^{\text{post}}$ as

$$\lambda_0 = \mathbb{E}[\Lambda] = \frac{\gamma}{c} \qquad \text{and} \qquad \widehat{\lambda}_T^{\text{post}} = \mathbb{E}[\Lambda \,|\, \mathbf{Y}] = \frac{\gamma + T}{c + \sum_{t=1}^{T} \log \frac{Y_t}{\theta}}.$$

Show that the posterior estimator $\widehat{\lambda}_T^{\text{post}}$ has the following credibility form

$$\widehat{\lambda}_T^{\text{post}} = \alpha_T \, \widehat{\lambda}_T + (1 - \alpha_T)\, \lambda_0$$

with credibility weight α_T and observation based estimator $\widehat{\lambda}_T$ given by

$$\alpha_T = \frac{\sum_{t=1}^{T} \log \frac{Y_t}{\theta}}{c + \sum_{t=1}^{T} \log \frac{Y_t}{\theta}} \qquad \text{and} \qquad \widehat{\lambda}_T = \frac{T}{\sum_{t=1}^{T} \log \frac{Y_t}{\theta}}.$$

(c) Show that the (mean square error) uncertainty of the posterior estimator $\widehat{\lambda}_T^{\text{post}}$ is given by

$$\mathbb{E}\left[\left(\Lambda - \widehat{\lambda}_T^{\text{post}}\right)^2 \middle|\, \mathbf{Y}\right] = (1 - \alpha_T)\, \frac{1}{c}\, \widehat{\lambda}_T^{\text{post}}.$$

(d) Let $\widehat{\lambda}_{T-1}^{\text{post}}$ denote the posterior estimator in the sub-model where we only have observed $(Y_1, \dots, Y_{T-1})$. Show that the posterior estimator $\widehat{\lambda}_T^{\text{post}}$ has the following recursive update structure

$$\widehat{\lambda}_T^{\text{post}} = \beta_T \, \frac{1}{\log \frac{Y_T}{\theta}} + (1 - \beta_T)\, \widehat{\lambda}_{T-1}^{\text{post}}$$

with credibility weight

$$\beta_T = \frac{\log \frac{Y_T}{\theta}}{c + \sum_{t=1}^{T} \log \frac{Y_t}{\theta}}.$$

Non-Life Insurance: Mathematics and Statistics

Exercise sheet 12

Exercise 12.1 Chain-Ladder and Bornhuetter-Ferguson
Set $I = J + 1 = 10$. Assume that for the cumulative payments $C_{i,j}$ with $1 \leq i \leq 10$ and $0 \leq j \leq 9$ we have observations

$$\mathcal{D}_{10} = \{C_{i,j} \,|\, i + j \leq 10,\, 1 \leq i \leq 10,\, 0 \leq j \leq 9\}$$

given by the following claims development triangle:

accident	development year j									
year i	0	1	2	3	4	5	6	7	8	9
1	5'946'975	9'668'212	10'563'929	10'771'690	10'978'394	11'040'518	11'106'331	11'121'181	11'132'310	11'148'124
2	6'346'756	9'593'162	10'316'383	10'468'180	10'536'004	10'572'608	10'625'360	10'636'546	10'648'192	
3	6'269'090	9'245'313	10'092'366	10'355'134	10'507'837	10'573'282	10'626'827	10'635'751		
4	5'863'015	8'546'239	9'268'771	9'459'424	9'592'399	9'680'740	9'724'068			
5	5'778'885	8'524'114	9'178'009	9'451'404	9'681'692	9'786'916				
6	6'1847'93	9'013'132	9'585'897	9'830'796	9'935'753					
7	5'600'184	8'493'391	9'056'505	9'282'022						
8	5'288'066	7'728'169	8'256'211							
9	5'290'793	7'648'729								
10	5'675'568									

Note that you can download this data set from

https://people.math.ethz.ch/~wueth/exercises2.html

by clicking on "Data to the Examples".

(a) Predict the lower triangle

$$\mathcal{D}_{10}^c = \{C_{i,j} \,|\, i + j > 10,\, 1 \leq i \leq 10,\, 0 \leq j \leq 9\}$$

using the chain-ladder (CL) method. Calculate the CL reserves $\widehat{\mathcal{R}}_i^{CL}$ for all accident years $i \in \{1, \ldots, 10\}$.

(b) Assume we have prior informations $\widehat{\mu}_1, \ldots, \widehat{\mu}_{10}$ for the total expected ultimate claims $\mathbb{E}[C_{1,J}], \ldots, \mathbb{E}[C_{10,J}]$ given by

accident year	1	2	3	4	5
prior information	11'653'101	11'367'306	10'962'965	10'616'762	11'044'881
accident year	**6**	**7**	**8**	**9**	**10**
prior information	11'480'700	11'413'572	11'126'527	10'986'548	11'618'437

Calculate the claim reserves $\widehat{\mathcal{R}}_i^{BF}$ for all accident years $i \in \{1, \ldots, 10\}$ using the Bornhuetter-Ferguson (BF) method.

(c) Explain why in this example we have

$$\widehat{\mathcal{R}}_i^{CL} < \widehat{\mathcal{R}}_i^{BF},$$

for all accident years $i \in \{2, \ldots, 10\}$.

Exercise 12.2 Mack's Formula and Merz-Wüthrich (MW) Formula (R Exercise)
Consider the identical setup as in Exercise 12.1. Write an R-Code using the R-package `ChainLadder` in order to calculate the conditional mean square error of prediction

$$\mathrm{msep}^{\mathrm{Mack}}_{C_{i,j}|\mathcal{D}_I}\left(\widehat{C}^{CL}_{i,J}\right),$$

for all $i \in \{1, \ldots, 10\}$, given in formula (9.21) of the lecture notes, as well as

$$\mathrm{msep}^{\mathrm{Mack}}_{\sum_{i=1}^{I} C_{i,j}|\mathcal{D}_I}\left(\sum_{i=1}^{I}\widehat{C}^{CL}_{i,J}\right),$$

for aggregated accident years, given in formula (9.22) of the lecture notes. Interpret the square-rooted conditional mean square errors of prediction relative to the claims reserves calculated in Exercise 12.1, (a). Moreover, determine

$$\mathrm{msep}^{\mathrm{MW}}_{\mathrm{CDR}_{i,I+1}|\mathcal{D}_I}(0),$$

for all $i \in \{1, \ldots, 10\}$, given in formula (9.34) of the lecture notes as well as

$$\mathrm{msep}^{\mathrm{MW}}_{\sum_{i=1}^{I}\mathrm{CDR}_{i,I+1}|\mathcal{D}_I}(0),$$

for aggregated accident years, given in formula (9.35) of the lecture notes.

Exercise 12.3 Conditional MSEP and Claims Development Result
In the first part of this exercise we show that the conditional mean square error of prediction can be decoupled into the process uncertainty and the parameter estimation error. In the second part we show that the claims development results are uncorrelated.

(a) Let $\mathcal{D}$ be a σ-algebra, $\widehat{X}$ a $\mathcal{D}$-measurable predictor for the random variable X and both $\widehat{X}$ and X square-integrable. Show that

$$\mathrm{msep}_{X|\mathcal{D}}\left(\widehat{X}\right) = \mathbb{E}\left[\left(X - \widehat{X}\right)^2 \,\middle|\, \mathcal{D}\right] = \mathrm{Var}(X \mid \mathcal{D}) + \left(\mathbb{E}[X \mid \mathcal{D}] - \widehat{X}\right)^2 \quad \text{a.s.}$$

(b) Suppose that we are in the framework of Chapter 9.4.1 of the lecture notes. Let $t_1 < t_2 \in \mathbb{N}$ such that $t_1 \geq I$ and $i > t_2 - J$ and assume that $C_{i,J}$ has finite second moment. Show that

$$\mathrm{Cov}\left(\mathrm{CDR}_{i,t_1+1}, \mathrm{CDR}_{i,t_2+1}\right) = 0.$$