EXPONENTIAL UTILITY WITH NON-NEGATIVE CONSUMPTION

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ABSTRACT. This paper investigates various aspects of the discrete-time exponential utility maximization problem, where feasible consumption policies are not permitted to be negative. By using the Kuhn-Tucker theorem, some ideas from convex analysis and the notion of aggregate state price density, we provide a solution to this problem, in the setting of both complete and incomplete markets (with random endowments). Then, we exploit this result and use certain fixed-point techniques to provide an explicit characterization of a heterogeneous equilibrium in a complete market setting. Moreover, we construct concrete examples of models admitting multiple (including infinitely many) equilibria. By using Cramer's large deviation theorem, we study the asymptotic behavior of endogenously determined in equilibrium prices of zero coupon bonds. Lastly, we show that for incomplete markets, un-insurable future income reduces the current consumption level, thus confirming the presence of the precautionary savings motive in our model.

1. INTRODUCTION

Utility maximization in the setting of decision making under uncertainty is a central topic both in mathematical finance and financial economics. The most prevalent class of preferences which has been extensively studied by numerous researchers over many decades is the so-called class of HARA (hyperbolic absolute risk-aversion) utility functions. HARA preferences are commonly split into two primary subclasses: CRRA (constant relative risk-aversion) preferences, which are also referred to as power utility functions; and CARA (constant absolute risk-aversion) preferences, which are known as exponential utilities. This paper concentrates on the latter type of preferences. Exponential preferences are widely used when dealing with indifference pricing (see e.g. Frei and Schweizer (2008), Henderson (2009), and Frei et. al. (2011)); incomplete markets with portfolio constraints (e.g. Svensson and Werner (1993)); and asset pricing in equilibrium (see Christensen et. al.

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We investigate various aspects of the discrete-time exponential utility maximization problem from consumption, where a significant and quite novel dimension introduced here (when comparing to other papers dealing with CARA preferences and consumption) is the constraint that feasible consumption policies are not permitted to be negative.

An essential drawback challenging the classical paradigm of exponential utility is the fact that negative consumption policies are included in the set of feasible controls (see Caballero (1990), Christensen et. al. (2011a, b), and Svensson and Werner (1993)). Economically, and intuitively, this is a vague assumption which cannot be reasonably justified, but is rather made to allow a simplified framework when solving analytically the corresponding problem. Namely, consumption of goods over a life-cycle cannot be apparently measured in negative units, since it models unrealistic situations when for instance one is capable of consuming a negative amount of bread. However, mathematically, exponential preferences often tend to extinguish some complicated affects (e.g., the indifference price is independent of the size of the wealth) caused by random shocks, and thus substantially ease the associated problems. Even though some seminal works rely on this negativity convention, many authors are aware of this problematic issue, for example, Caballero (1990) writes:

"This paper specializes to this type of preferences (exponential) in spite of some of its unpleasant features like the possibility of negative consumption. (Unfortunately, explicitly imposing non-negativity constraints impedes finding a tractable solution.)"

Nonetheless, it is demonstrated in the current paper that it makes a practical sense to impose these non-negativity constraints: not only that it genuinely refines the model, but it allows us finding the related solution in a closed form and employ it then for a variety of economic applications.

We explain now in more detail the contributions of this work and the links it has to some other related papers. First, by using some ideas from convex analysis, we solve the constrained exponential utility maximization problem in a complete market setting, with an arbitrary probability space. We show that the solution is equal to the non-negative part of the solution associated with the unconstrained problem (ass also Cox and Huang (1989), for a related problem in a continuous-time framework). Next, we use the Kuhn-Tucker theorem to solve the preceding maximization problem in a setting of incomplete markets, with a finite probability space, and express the solution in terms of the aggregate state price density introduced by Malamud and Trubowitz (2007). In particular, we show that the transparent relation between the constrained and unconstrained solution in complete markets
setting is violated for incomplete markets. Then, we turn to analyzing heterogeneous equilibria in the framework of complete markets. There is a vast body of literature studying equilibrium asset prices with heterogeneous investors (see e.g. Mas-Colell (1986), Karatzas et. al. (1990, 1991), Dana (1993a, 1993b), Constantinides and Duffie (1996), Malamud (2008)). Bühlmann (1980) was apparently the first one to write down the formulas in a one period risk-exchange economy with exponential preferences, and negative consumption. We express the equilibrium state price density as a ‘non-smooth’ sum over indicators depending on the agents’ characteristics and the total endowment of the economy. In effect, our finding is one of very few examples when one can characterize in a closed-form manner the associated state price density in heterogeneous economies. A further issue that we sort out here is the existence of equilibrium (this is a quite standard chronological order in equilibrium: first a characterization result, and then existence), by employing a fixed-point argument involving the excess-demand function in the spirit of Mas-Collel et. al. (1995). Some other authors have studied various equilibrium existence problems in a bit different settings: Dana’s papers (1993a, b) do not cover multi-period models and Karatzas et. al. (1990, 1991) work in continuous-time. Hence, we provide a self-contained proof that takes advantage of our underlying exponential preferences. Next, we turn treating the issue of non-uniqueness of equilibria. Non-uniqueness is usually anticipated in this type of models: it is resulted by the fact that the Inada condition is not fulfilled, which in turn leads to the associated excess-demand function not to satisfy the ‘gross-substitution’ property of Dana (1993a), that would guarantee uniqueness. However, we are not aware of any papers (apart from Malamud and Trubowitz (2006)) that construct an explicit example as ours of multiple equilibrium state price densities in a risk-exchange economy. We concentrate then on studying the important question of long-run behavior (see e.g. Wang (1996), Lengwiler (2005), and Malamud (2008)) of zero coupon bonds, whose price is determined endogenously in equilibrium. Despite of the non-uniqueness of the state price densities, modulo some multipliers, we show that asymptotically these multipliers carry no impact. We conduct our analysis in two settings: first we consider economies where agents differ with respect to their risk-aversion and then economies where agents differ only with respect to their impatience rate. The main tool we use for the preceding problems is Cramer’s large deviation theorem. Finally, we are concerned with the phenomenon of precautionary savings (see Kimball (1990) for a comprehensive introduction) in incomplete markets and exponential preferences. That is, we verify that un-insurable future income forces an investor to save more (or equivalently, consume less) in the present. We examine this phenomenon in rather enriched stochastic frameworks of incomplete markets (markets of type $C$, see Malamud and Trubowitz (2007)), whereas
most classical papers on this subject (see e.g. Dreze and Modigliani (1972) and Miller (1976)) consider mainly riskless bonds.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we establish a solution to the exponential utility maximization problem with a non-negativity constraint in both complete and incomplete markets. Section 4 analyzes the associated heterogeneous equilibrium for complete markets, and Section 5 deals with the non-uniqueness of this equilibrium. In Section 6 we study the long-run behavior of the equilibrium zero coupon bonds. Finally, in Section 7 we study the precautionary savings motive in incomplete markets.

2. Preliminaries

We consider a discrete-time market with a maturity date $T$. In Sections 2-5 and Section 7, we set $T \in \mathbb{N}$, and in Section 6, $T = \infty$. The uncertainty in our model is captured by a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_0 = \{\phi, \Omega\} \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$. Adaptedness and predictability of stochastic processes is always meant with respect to the filtration $(\mathcal{F}_k)_{k=0,...,T}$, unless otherwise stated. We use the notation $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. There is no-arbitrage in the market which consists of $n$ risky stocks: $(S^j_k)_{k=0,...,T}$, $j = 1, \ldots, n$ and one riskless bond paying an interest-rate $r_k$, at each period $k = 1, \ldots, T$. Each price process $(S^j_k)_{k=0,...,T}$ is non-negative and adapted, and the interest rate process $(r_k)_{k=1,...,T}$ is non-negative and predictable. The economy is inhabited by $N$ (types of) individuals, labeled by $i = 1, \ldots, N$. Each agent $i$ receives a random income $(\epsilon^i_k)_{k=0,...,T}$, which is assumed to be non-negative and adapted. The preferences of each agent $i$ are characterized by an impatience rate $\rho_i \geq 0$ and an exponential utility function $u_i(x) = - \exp(-\gamma_i x)$, defined on $\mathbb{R}_+$, for a given level of risk aversion $\gamma_i > 0$. The underlying utility maximization problem from consumption of each agent $i$ is formulated as

$$\sup_{(c_0, \ldots, c_T) \in B_i} \sum_{k=0}^T -e^{-\rho_i k} E[\exp(-\gamma_i c_k)],$$

where $(c_k)_{k=0,...,T}$ is a consumption stream lying in a certain set of budget constraints $B_i$. As described below, we will tackle problem (2.1) in both complete and incomplete markets, under different assumptions on the model and the budget set $B_i$. However, we will preserve the same notations as above for both settings without mentioning it explicitly.

2.1. Complete markets. When dealing with complete markets, the probability space is allowed to be infinite\(^1\). The budget set $B_i$ in this setting consists of all non-negative adapted processes $c_k$ that satisfy the equation

$$\sum_{k=0}^T E[\xi_k c_k] = \sum_{k=0}^T E[\xi_k c^i_k].$$

\(^1\)Therefore, the amount of securities completing the market might be infinite as well.
An equilibrium is a pair of processes \( (\xi_k)_{k=0,\ldots,T} \) and \( (\epsilon_k)_{k=0,\ldots,T} \) such that:

(a) The process \((\xi_k)_{k=0,\ldots,T}\) is a state price density (SPD) of the market, i.e.,

\[
S^{\epsilon}_{k+1} \xi_k = E \left[ S^{\epsilon}_{k+1} \xi_{k+1} | \mathcal{F}_k \right], \quad j = 1, \ldots, n,
\]

and

\[
\xi_k = E \left[ \xi_{k+1} (1 + r_{k+1}) | \mathcal{F}_k \right],
\]

for \( k = 0, \ldots, T - 1 \). Let us stress that the completeness of the market implies

that stock prices are not directly involved in the corresponding utility maximization problem \((2.1)\). We introduce now the standard notion of equilibrium in the framework of complete markets and risk exchange economies.

**Definition 2.1.** An equilibrium is a pair of processes \((\epsilon_k)_{k=0,\ldots,T}; i=1,\ldots,N\) and \((\xi_k)_{k=1,\ldots,T}\), such that:

(a) The process \((\xi_k)_{k=1,\ldots,T}\) is a SPD and \((\epsilon_k)_{k=0,\ldots,T}\) is the optimal consumption stream of each agent \( i \) (i.e., solving \((2.1)\)).

(b) The market clearing condition holds:

\[
\sum_{i=1}^{N} \epsilon^i_k = \epsilon_k := \sum_{i=1}^{N} \epsilon^i_k,
\]

for all \( k = 0, \ldots, T \).

### 2.2. Incomplete markets.

We assume that the probability space is finite. Thus incompleteness of markets is modeled standardly by allowing infinitely many SPDs. For each \( k = 0, \ldots, T \), we denote by \( L^2(\mathcal{F}_k) \) the Hilbert space of all \( \mathcal{F}_k \)-measurable random variables, endowed with the inner product \( \langle X, Y \rangle = E[XY], X, Y \in L^2(\mathcal{F}_k) \). Each agent \( i \) selects a portfolio strategy \((\pi^j_k)_{k=0,\ldots,T-1}, j = 1, \ldots, n, \) and \((\phi_k)_{k=0,\ldots,T-1}\). Here, \( \pi^j_k \) and \( \phi_k \) are \( \mathcal{F}_k \)-measurable and denote the shares invested in asset \( j \) and the riskless bond at period \( k \), respectively. We set further \( \pi^j_{T-1} = 0, j = 1, \ldots, n, \phi_{T-1} = 0, \pi^j_T = 0, j = 1, \ldots, n, \) and \( \phi_T = 0 \). The last two assumptions formalize the convention that no trading is executed in the last period \( T \). For each \( k = 1, \ldots, T \), we denote by

\[
\mathcal{L}_k = \left\{ \sum_{j=1}^{n} \pi^j_{k-1} S^j_k + \pi_{k-1}(1 + r_k) \mid \phi_{k-1}, \pi^j_{k-1} \in L^2(\mathcal{F}_{k-1}) \right\}, \quad j = 1, \ldots, n
\]

the wealth space at time \( k \), and remark that \( L^2(\mathcal{F}_{k-1}) \subseteq \mathcal{L}_k \subseteq L^2(\mathcal{F}_k) \). By Lemma 2.5 in Malamud and Trubowitz (2007) there exists a unique normalized SPD \((\mathcal{M}_k)_{k=0,\ldots,T}\) such that \( \mathcal{M}_k \in \mathcal{L}_k \), for all \( k = 1, \ldots, T \), called the aggregate SPD. For incomplete markets, the budget constraints are more sophisticated than the single constraint \((2.2)\) arising in a complete market setting. Namely, the budget set \( \mathcal{S}_i \) is composed of all non-negative adapted processes \((c_k)_{k=0,\ldots,T}\) of the form

\[
c_k = \epsilon^i_k + \sum_{j=1}^{n} \pi^j_{k-1} S^j_k + \pi_{k-1}(1 + r_k) - \sum_{j=1}^{n} \pi^j_{k} S^j_k - \pi_k.
\]
for all $k = 0, \ldots, T$, where $(\pi^j_k)_{k=0,\ldots,T-1}$, $j = 1, \ldots, n$, and $(\phi_k)_{k=0,\ldots,T-1}$ is a portfolio strategy. Notice that (2.4) can be rewritten as

\[(2.5) c_k = \epsilon_k + W_k - E \left[ \frac{M_{k+1} - W_{k+1}}{M_k} \right],\]

where $W_k \in L_k$, $k = 1, \ldots, T$, and $W_0 = W_{T+1} = 0$, see Section 2 in Malamud and Trubowitz (2007).

3. Optimal Consumption

The current section is devoted to the derivation of the optimal consumption policy of an individual solving the utility maximization problem (2.1) in both frameworks of complete and incomplete markets.

3.1. Complete Markets. We investigate the utility maximization problem (2.1) in the framework of complete markets, as specified in Subsection 2.1. We make use of convex conjugates and other related ideas from convex analysis to derive an explicit formula for the optimal consumption stream in this setup. As will be shown below, there is a link between the utility maximization problem (2.1) and the corresponding unconstrained (allowing negative consumption) version of this problem. Hence, we first solve the latter problem.

**Theorem 3.1.** Assume that $0 < \sum_{k=0}^T E[\xi_k \epsilon_k] < \infty$ and $-\infty < -\sum_{k=0}^T E[\xi_k \log \xi_k]$, for all $i = 1, \ldots, N$. Consider the utility maximization problem

\[(3.1) \sup_{(c_0, \ldots, c_T)} \sum_{k=0}^T -e^{-\rho_i k} E \left[ \exp (-\gamma_i c_k) \right],\]

where $(c_k)_{k=0,\ldots,T}$ is an adapted process that satisfies the equation (2.2). Then, there exists a unique solution given by

\[(3.2) \tilde{c}_k^i = \tilde{c}_k^i(\tilde{\lambda}) := \frac{1}{\gamma_i} \left( \log \left( \frac{\gamma_i}{\lambda} \right) - \rho_i k - \log (\xi_k) \right),\]

for all $k = 0, \ldots, T$, where $\tilde{\lambda}$ is a positive real number specified uniquely by the equation

\[(3.3) \frac{1}{\gamma_i} \sum_{k=0}^T E \left[ \xi_k \left( \log \left( \frac{\gamma_i}{\lambda} \right) - \rho_i k - \log (\xi_k) \right) \right] = \sum_{k=0}^T E \left[ \xi_k \epsilon_k^i \right].\]

**Proof of Theorem 3.1.** First, observe that the function $f_i : \mathbb{R}^+ \to \mathbb{R}$, $f_i(\lambda) = \sum_{k=0}^T E \left[ \xi_k \left( \log \left( \frac{\gamma_i}{\lambda} \right) - \rho_i k - \log (\xi_k) \right) \right]$ is strictly monotone decreasing with $\lim_{\lambda \to 0^+} f_i(\lambda) = \infty$ and $\lim_{\lambda \to \infty} f_i(\lambda) = -\infty$. Therefore, there exists a unique solution $\tilde{\lambda}$ for equation (3.3). Next, consider the Legendre transform $\tilde{v}_i(y) : \mathbb{R}^+ \to \mathbb{R}$ of the function $-e^{-\gamma_i x}$ with the domain of definition $\mathbb{R}$:

\[\tilde{v}_i(y) = \sup_{x \in \mathbb{R}} (-e^{-\gamma_i x} - xy).\]
Denote \( \tilde{I}_i(y) := \frac{1}{\gamma_i} \log \frac{n}{y} \), and note that \( \tilde{v}_i(y) = -e^{-\gamma_i \tilde{I}_i(y)} - \tilde{I}_i(y) y \). Therefore, we get

\[
\tilde{v}_i \left( \lambda e^{\rho_i k} \xi_k \right) \geq -e^{-\gamma_i c_k} - \lambda e^{\rho_i k} \xi_k c_k,
\]
or equivalently

\[
e^{-\rho_i k} e^{-\gamma_i \tilde{I}_i(\lambda e^{\rho_i k} \xi_k)} \geq -e^{-\rho_i k} e^{-\gamma_i c_k} + \lambda \xi_k \left( \tilde{I}_i \left( \lambda e^{\rho_i k} \xi_k \right) - c_k \right),
\]
for all \( F_k \)-measurable random variables \( c_k \). In particular, we have

\[
- \sum_{k=0}^{T} e^{-\rho_i k} E \left[ e^{-\gamma_i \tilde{I}_i(\lambda e^{\rho_i k} \xi_k)} \right]
\]

\[
\geq - \sum_{k=0}^{T} e^{-\rho_i k} E \left[ e^{-\gamma_i c_k} \right] + \lambda \sum_{k=0}^{T} E \left[ \xi_k \left( \tilde{I}_i \left( \lambda e^{\rho_i k} \xi_k \right) - c_k \right) \right],
\]
for all adapted processes \((c_k)_{k=0,...,T}\) such that \( \sum_{k=0}^{T} E \left[ \xi_k c_k \right] < \infty \). Thus, we conclude the existence of an optimal consumption stream of the form \( (3.2) \), because the last term disappears due to \((2.2)\). Uniqueness follows from the inequality \( \tilde{v}_i(y) > -e^{-\gamma_i x} - xy \), which holds for all \( x \neq \tilde{I}_i(y) \). \( \square \)

Next, we return to the treatment of the utility maximization problem with non-negative consumption policies.

**Theorem 3.2.** Assume that \( \sum_{k=0}^{T} E \left[ \xi_k c_k^i \right] > 0 \). Then, the constrained utility maximization problem \((2.1)\) in the setting of a complete market admits a unique solution given by

\[
c^*_k = (c^*_k(\lambda^*))^+ = \frac{1}{\gamma_i} \left( \log \left( \frac{\gamma_i}{\lambda^*} \right) - \rho_i k - \log (\xi_k) \right)^+,
\]
where the constant \( \lambda^* \) is determined as the unique positive solution of the equation

\[
\frac{1}{\gamma_i} \sum_{k=0}^{T} E \left[ \xi_k \left( \log \left( \frac{\gamma_i}{\lambda^*} \right) - \rho_i k - \log (\xi_k) \right)^+ \right] = \sum_{k=0}^{T} E \left[ \xi_k c_k^i \right].
\]

We first prove the following auxiliary lemma.

**Lemma 3.3.** Denote \( I_i(y) = \frac{1}{\gamma_i} \left( \log \left( \frac{\gamma_i}{y} \right) \right)^+ \), and consider the function \( \psi^i : \mathbb{R}_{++} \to \mathbb{R}_+ \) defined by

\[
\psi^i(\lambda) = \sum_{k=0}^{T} E \left[ \xi_k I_i(\lambda e^{\rho_i k} \xi_k) \right].
\]

Then, \( \psi^i(\lambda) \) is a decreasing continuous function of the following form: if \( \psi^i(b) > 0 \) for some \( b \in \mathbb{R}_{++} \), then \( \psi^i(a) > \psi^i(b) \) for all \( 0 < a < b \). Furthermore, \( \lim_{\lambda \to 0^+} \psi^i(\lambda) = \infty \) and \( \lim_{\lambda \to \infty} \psi^i(\lambda) = 0 \).
Proof of Lemma 3.3. First observe that \( E[\xi_k I_i(c\xi_k)] < \infty \) for all \( c > 0 \) since
\[
E[\xi_k I_i(c\xi_k)] = \frac{1}{\gamma_i} E \left[ \xi_k \log \left( \frac{\gamma_i}{c\xi_k} \right) 1_{\{1 \leq \frac{\gamma_i}{c\xi_k} \}} \right] < 1/c,
\]
which follows from the fact that \( \log t < t \), for all \( t \geq 1 \). This shows that \( \psi^{\gamma}(\lambda) \) is well defined for all \( \lambda \in \mathbb{R}_{++} \). Next, we prove that \( \psi^{\gamma} \) is continuous. Let \( \lambda_n \) be a sequence such that \( \lambda_n \uparrow \lambda \), hence, \( \lim_{n \to \infty} \xi_k I_i(\lambda_n e^{\rho_i k} \xi_k) = \xi_k I_i(\lambda e^{\rho_i k} \xi_k) \), \( P \)-a.s, and thus the dominated convergence theorem implies that \( \lim_{n \to \infty} \psi^{\gamma}(\lambda_n) = \psi^{\gamma}(\lambda) \). The same argument holds for a sequence \( \lambda_n \downarrow \lambda \). Now, we treat the limits. Consider an arbitrary increasing sequence \( \lambda_n \to \infty \), then, \( \xi_k I_i(\lambda_n e^{\rho_i k} \xi_k) \to 0 \) and \( \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) \geq \xi_k I_i(\lambda_n e^{\rho_i k} \xi_k) \), for all \( n \). Therefore, the dominated convergence theorem implies that \( \lim_{n \to \infty} \psi^{\gamma}(\lambda_n) = 0 \). Next, pick an arbitrary sequence \( \lambda_n \downarrow 0 \) and note that \( \lim_{n \to \infty} \xi_k I_i(\lambda_n e^{\rho_i k} \xi_k) = \infty \), \( P \)-a.s. Therefore, Fatou’s lemma implies that
\[
\lim_{n \to \infty} \inf E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) \right] \geq E \left[ \lim_{n \to \infty} \xi_k I_i(\lambda_n e^{\rho_i k} \xi_k) \right] = \infty.
\]
This shows that \( \lim_{s \to 0^+} \psi^{\gamma}(\lambda) = \infty \). Next, note that \( \lambda \mapsto E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) \right] \) is a decreasing function for each \( k = 1, ..., T \) since \( I_i \) is decreasing. Therefore, \( \psi^{\gamma}(\lambda) \) is decreasing. Finally, assume in a contrary that \( \psi^{\gamma}(b) > 0 \) and \( \psi^{\gamma}(a) = \psi^{\gamma}(b) \), for some \( a < b \). By the previous observations, it follows that \( E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) \right] = E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) \right] \), for each \( k = 0, ..., T \). By definition, \( I_i \) is a strictly decreasing function on the interval \( (0, \gamma_1] \), thus \( I_i(\gamma_1 e^{\rho_i k} \xi_k) < I_i(\gamma_1 e^{\rho_i k} \xi_k) \), on the set \( \{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \} \). Since \( \psi^{\gamma}(b) > 0 \), it follows that there exists some \( k \in \{0, ..., T \} \) such that \( P \{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \} > 0 \). This is a contradiction, since \( \{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \} \subseteq \{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \} \), and thus
\[
E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) 1_{\{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \}} \right] < E \left[ \xi_k I_i(\gamma_1 e^{\rho_i k} \xi_k) 1_{\{\gamma_1 e^{\rho_i k} \xi_k < \gamma_1 \}} \right],
\]
finishing the proof. \( \square \)

Proof of Theorem 3.2. First, observe that Lemma 3.3 yields the existence of a unique solution to equation (3.5) denoted by \( \lambda^* > 0 \). Next, consider the Legendre transform of the function \(-e^{-\gamma_1 x}\) with the domain of definition \( \mathbb{R}_+ \):
\[
v_i(y) = \sup_{x \in \mathbb{R}_+} \left( -e^{-\gamma_1 x} - xy \right),
\]
for all \( y > 0 \). Note that \( v_i(0) = 0 \) and recall that \( I_i(y) = \frac{1}{\gamma_i} \left( \log \frac{2}{y} \right)^+ \). Observe that \( \frac{\partial}{\partial y} (u_i(x) - xy) = \gamma_i e^{-\gamma_1 x} - y \). Therefore, if \( y < \gamma_i \), then \( v_i(y) = u_i \left( \frac{1}{\gamma_i} \log \left( \frac{2}{y} \right) \right) - \frac{y}{\gamma_i} \log \left( \frac{2}{y} \right) \). If \( y \geq \gamma_i \), then \( v_i(y) = u_i(0) \). Thus, we can rewrite \( v_i(y) = u_i(I_i(y)) - I_i(y)y \). The rest of the proof is identical to the one of Theorem 3.1 and thus omitted. \( \square \)
For a sufficiently large level of endowments, and under some boundness assumptions on the SPD, the optimal consumption stream is strictly positive, and hence, in particular, coincides with the optimal consumption streams corresponding to the setting of unconstrained exponential utility.

**Corollary 3.4.** Assume that \( \xi_k < C \) and \( \epsilon^i_k > \frac{1}{\gamma_i} \left( \log \left( \frac{C}{\xi_k} \right) + \rho_i (T - k) \right) \), \( P \)-a.s., for all \( k = 0, ..., T \), and some constant \( C > 0 \). Then, the optimal consumption stream of the \( i \)-th agent (see (3.4)) is strictly positive and given by

\[
\epsilon^i_k = \frac{1}{\gamma_i} \left( \frac{\sum_{l=0}^{T} E \left[ \xi_l \left( \gamma_i \epsilon^i_l + \log \xi^l + \rho_i l \right) \right]}{\sum_{l=0}^{T} E \left[ \xi_l \right]} - \rho_i k - \log \xi_k \right) > 0,
\]

for all \( k = 0, ..., T \).

**Proof of Corollary 3.4.** Fix \( z = \frac{\gamma_i}{e^\rho_i} \) and observe that \( \frac{\gamma_i}{e^\rho_i \xi_k} > \frac{\gamma_i}{e^\rho_i} > 1 \), for all \( k = 0, ..., T \). Therefore, by applying the same argument as in the proof of Lemma 3.3, one concludes that \( \psi^i(\lambda) \) is a strictly decreasing function on the interval \((0, z)\). Moreover, note that

\[
\psi^i(z) = \frac{1}{\gamma_i} \sum_{k=0}^{T} E \left[ \xi_k \left( \log \left( \frac{C}{\xi_k} \right) + \rho_i (T - k) \right) \right] < \sum_{k=0}^{T} E \left[ \epsilon^i_k \xi_k \right],
\]

by assumption. Therefore, a solution \( \lambda^* \) to equation (3.5) is attained for some \( \lambda^* \in (0, z) \), and thus \( \frac{\gamma_i}{e^\rho_i \xi_k} > \frac{\gamma_i}{e^\rho_i} > 1 \). In view of (3.4), we obtain that the optimal consumption stream admits the form

\[
\epsilon^i_k = \frac{1}{\gamma_i} \log \left( \frac{\gamma_i}{\lambda^* e^\rho_i \xi_k} \right) > 0.
\]

By plugging this back into the budget constraints equation (3.5), one solves this equation explicitly and verifies the validity of (3.6).

### 3.2. Incomplete Markets

In the current section we deal with incomplete markets (see Subsection 2.2). In this setting, we provide an explicit construction of the optimal consumption for the utility maximization problem (2.1). The methods employed here rely on the Kuhn-Tucker theorem and the notion of aggregate SPD (defined in Subsection 2.2) introduced by Malamud and Trubowitz (2007).

**Theorem 3.5.** For an investor solving the utility maximization problem (2.1) in an incomplete market, the optimal consumption stream \((\tilde{c}_k)_{k=0,...,T}\) is uniquely determined through the following scheme:

\[
P^k_{\tilde{c}} \left[ \frac{e^{-\rho_i k \gamma_i} \exp(-\gamma_i \tilde{c}_k) + \lambda_k}{e^{-\rho_i (k-1) \gamma_i} \exp(-\gamma_i \tilde{c}_{k-1}) + \lambda_{k-1}} \right] = \frac{M_k}{M_{k-1}},
\]

for all \( k = 1, ..., T \), where \((\lambda_k)_{k=0,...,T}\) is a non-negative adapted process satisfying

\[
\lambda_k \tilde{c}_k = 0,
\]
for all $k = 0, ..., T$. Here, $(M_k)_{k=0,...,T}$ and $P^k_E$ stand for the aggregate SPD and the orthogonal projection on the space $L_k$, respectively (see Subsection 2.2).

**Proof of Theorem 3.5.** Consider the function

$$l^i(\pi, \phi, \lambda_0, ..., \lambda_T) =$$

$$\sum_{k=0}^T e^{-\rho_k} E \left[ -\exp \left( -\gamma_i \left( t^i_k + \sum_{j=1}^n \pi^j_{k-1} S^j_k + \phi_{k-1}(1 + r_k) - \sum_{j=1}^n \pi^j_k S^j_k - \phi_k \right) \right) \right] -$$

$$\sum_{k=0}^T E \left[ \lambda_k \left( t^i_k + \sum_{j=1}^n \pi^j_{k-1} S^j_k + \phi_{k-1}(1 + r_k) - \sum_{j=1}^n \pi^j_k S^j_k - \phi_k \right) \right],$$

where $\phi = (\phi_k)_{k=0,...,T-1}$, $\pi = (\pi^j_k)_{k=0,...,T-1}$, $j = 1, ..., n$, is a portfolio strategy and $\lambda_k \in L^2(F_k)$, $k = 0, ..., T$. Since our probability space is finite, we can employ the Kuhn-Tucker theorem. Namely, by differentiating the function $l^i$ with respect to the partial derivatives $\phi_k, \pi^j_k$, $k = 0, ..., T$, $j = 1, ..., n$, and equalizing the resulting expression to 0, we get that the optimal consumption stream $(\tilde{c}_k)_{k=0,...,T}$ is determined by requiring that the process

$$(e^{-\rho_k} \gamma_i \exp (-\gamma_i \tilde{c}_k) + \lambda_k)_{k=0,...,T}$$

is a SPD and $\lambda_k \tilde{c}_k = 0, k = 0, ..., T$, for some non-negative adapted process $(\lambda_k)_{k=0,...,T}$. Finally, Lemma 2.5 in Malamud and Trubowitz (2007) yields the validity of (3.7). Uniqueness follows from strict concavity. \(\square\)

**Remark 3.1.** The Kuhn-Tucker theorem could not be applied directly for complete markets, since we allowed for arbitrary probability spaces.

3.2.1. One-period incomplete markets. We fix $T = 1$ and consider a market of type C, introduced by Malamud and Trubowitz (2007). That is, we assume that $L_1 = L^2(H_1)$ and thus $P^k_E [\cdot] = E [\cdot | H_1]$, where $H_1$ is a sigma-algebra satisfying $F_0 \subset H_1 \subset F_1$. Let $\tilde{c}_0$ and $\tilde{c}_1$ denote the optimal consumption stream (we drop the index $i$). As above, $\lambda_0$ and $\lambda_1$ stand for the multipliers. Recall that by (2.5), we have $\tilde{c}_0 = \epsilon'_1 + \tilde{W}_1$, where $\tilde{W}_1 \in L_1$. Next, by Theorem 3.5 we get

$$\exp \left( -\gamma_i \tilde{W}_1 \right) = \frac{M_1 (\gamma_i \exp (-\gamma_i \tilde{c}_0) + \lambda_0) - E [\lambda_1 | H_1]}{e^{-\rho_i} \gamma_i E \left[ \exp \left(-\gamma_i \epsilon'_1 \right) \right] | H_1},$$

hence

$$(3.8) \quad \tilde{c}_1 = \epsilon'_1 + \tilde{W}_1 =$$

$$\frac{1}{\gamma_i} \log \left( \frac{e^{\gamma_i} \epsilon'_1 E \left[ e^{-\gamma_i \epsilon'_1} \right] | H_1]}{\left( \exp (\rho_i - \gamma_i \tilde{c}_0) + \lambda_0 \frac{e^{\gamma_i}}{\gamma_i} M_1 - \frac{e^{\gamma_i}}{\gamma_i} E [\lambda_1 | H_1] \right)} \right).$$
Now, denote \( \lambda = \exp(\mu - \gamma c_0) + \lambda_0 e^{\epsilon_1'}, \) or equivalently \( \tilde{c}_0 = \frac{1}{\gamma_1} \log \left( \frac{\gamma_1}{\gamma_1 e^{-\gamma_1'} - \lambda} \right). \) Recall that \( \lambda_0 \tilde{c}_0 = 0, \lambda_0 \geq 0 \) and \( \tilde{c}_0 \geq 0. \) Therefore, we get

\[ (3.9) \quad \tilde{c}_0 = \frac{1}{\gamma_1} \left( \log \left( \frac{1}{\lambda} \right) + \mu \right). \]

Now, we claim that

\[ (3.10) \quad \tilde{c}_1 = \frac{1}{\gamma_1} \log \left( \frac{e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right]}{\lambda M} \right) \bigg[ \text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] \bigg] + \frac{1}{\gamma_1} \log \left( \frac{e^{\gamma_1'} \lambda M}{\text{essinf} \left[ e^{-\gamma_1'} | H_1 \right]} \right) \bigg[ \text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] \bigg], \]

where \( \text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] \) is the essential infimum of the random variable \( e^{-\gamma_1'} \) conditioned on the sigma-algebra \( H_1. \) To this end, assume first that \( e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] > \lambda M_1. \) Then, identity (3.8) yields \( \tilde{c}_1 > 0 \) and thus \( \lambda_1 = 0, \) since \( \lambda_1 \tilde{c}_1 = 0. \) Thereby, we have

\[ \lambda_1 = \Lambda \left\{ e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1 \right\}, \]

for some \( F_1 \)-measurable non-negative random variable \( \Lambda. \) Now, the condition \( \lambda_1 \tilde{c}_1 = 0 \) can be rewritten as

\[ (3.11) \quad \Lambda \left\{ e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1 \right\} \times \log \left( \frac{e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right]}{\lambda M_1 - \frac{e^{\gamma_1'} E \left[ \Lambda \left\{ e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1 \right\} | H_1 \right]} \right) = 0. \]

Hence, if

\[ \text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] E \left[ e^{-\gamma_1'} | H_1 \right] > \lambda M_1, \]

then \( \tilde{c}_1 = \frac{1}{\gamma_1} \log \left( \frac{e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right]}{\lambda M_1} \right). \) On the other hand, we claim that if

\[ \text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1, \]

then

\[ \log \left( \frac{\text{essinf} \left[ e^{-\gamma_1'} | H_1 \right] E \left[ e^{-\gamma_1'} | H_1 \right]}{\lambda M_1 - \frac{e^{\gamma_1'} E \left[ \Lambda \left\{ e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1 \right\} | H_1 \right]} \right) = 0. \]

Assume that it is not the case. It follows that

\[ \log \left( \frac{e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right]}{\lambda M_1 - \frac{e^{\gamma_1'} E \left[ \Lambda \left\{ e^{\gamma_1'} E \left[ e^{-\gamma_1'} | H_1 \right] \leq \lambda M_1 \right\} | H_1 \right]} \right) > 0, \]
and thus by (3.11) we get

$$\Lambda_1 \{ e^{\gamma_i \epsilon_i^1} E [ e^{-\gamma_i \epsilon_i^1} | H_1 ] \leq \lambda M_1 \} = 0.$$ 

By substituting it back, we get

$$\inf_{E [ e^{\gamma_i \epsilon_i^1} | H_1 ]} [ e^{\gamma_i \epsilon_i^1} E [ e^{-\gamma_i \epsilon_i^1} | H_1 ] ] > \lambda M_1,$$

and this is a contradiction, proving the identity (3.10). Finally, let us remark that \( \lambda \) is derived from the equation

$$b c_0 + E [ M_1 b c_1 ] = \epsilon_0 + E [ M_1 \epsilon_1 ].$$

Remark 3.2. For complete markets, the optimal consumption stream in the constrained case (non-negative consumption) is given in the form of a call-option on the unconstrained (possibly negative) optimal consumption stream with a strike being equal to zero (see (3.4)). In contrast to this, for incomplete markets this relation is no longer valid. Indeed, if we allow for negative consumption, one can show that the associated unconstrained optimal consumption stream is given by

$$c_1' = \frac{1}{\gamma_i} \log \left( \frac{e^{\gamma_i \epsilon_i^1} E [ e^{-\gamma_i \epsilon_i^1} | H_1 ]}{\lambda M_1} \right),$$

and

$$c_0' = \frac{1}{\gamma_i} \left( \rho_i + \log \frac{1}{\lambda'} \right),$$

where the constant \( \lambda' \) is determined by the equation

$$c_0' + E [ M_1 c_1' ] = \epsilon_0 + E [ M_1 \epsilon_1 ].$$

Thus by comparing it to (3.9) and (3.10), we conclude that the structure of the solution presented in the complete setting disappears here.

4. Equilibrium

We adapt the notion of a complete market equilibrium as introduced in Subsection 2.1 (see Definition 2.1). In the current section we show that there exists an equilibrium under the assumption that all endowments are positive. Moreover, we describe the set of all feasible equilibrium SPDs and the associated optimal consumption policies. For this purpose we introduce the following quantities. For each vector \((\lambda_1, ..., \lambda_N) \in \mathbb{R}_+^N\), we define

$$\beta_i(k) = \beta_i^{(\lambda_i)}(k) = \frac{\gamma_i}{\lambda_i e^{\rho_i k}},$$

for all \( i = 1, ..., N \) and all \( k = 0, ..., T \). For a fixed \( k = 0, ..., T \), let \( i_1(k), ..., i_N(k) \) denote the order statistics of \( \beta_1(k), ..., \beta_N(k) \), that is, \( \{ i_1(k), ..., i_N(k) \} = \{ 1, ..., N \} \) and \( \beta_{i_1(k)}(k) \leq ... \leq \beta_{i_N(k)}(k) \). We set \( \beta_{i_0(k)}(k) = 0 \), for all \( k = 0, ..., T \). With the preceding notations, we denote

$$\eta_j(k) = \eta_j^{(\lambda_1, ..., \lambda_N)}(k) = \sum_{l=j+1}^N \frac{\log (\beta_{i_l(k)}(k)) - \log (\beta_{i_j(k)}(k))}{\gamma_i(k)} \geq 0.$$
Note that $\eta_0(k) = +\infty$ and $\eta_N(k) = 0$, for all $k = 0, ..., T$. At last, we introduce a candidate for the equilibrium SPD

\begin{equation}
\xi_k (\lambda_1, ..., \lambda_N) = \sum_{j=1}^{N} \left( \prod_{t=j}^{j+1} \beta ji(k) \right)^{-1} \exp\left( -\frac{\epsilon_k}{1/\gamma_{1i}(k)} \right) 1_{\{\eta_{j}(k) \leq \epsilon_k < \eta_{j-1}(k)\}},
\end{equation}

for all $k = 0, ..., T$. Recall that $I_i(y) = \left( \log \frac{y}{\gamma_i} \right)^+$. 

**Theorem 4.1.** Assume that $\epsilon_k > 0$ for each period $k$ and each agent $i$, $P$–a.s. Then, every equilibrium SPD is given by $(\xi_k (\lambda_1, ..., \lambda_N))_{k=0, ..., T}$, where $\lambda_1^*, ..., \lambda_N^* \in \mathbb{R}_{++}$ are constants that solve the following system of equations

\begin{equation}
I_0 (\xi_0 (\lambda_1^*, ..., \lambda_N^*)) = \sum_{i=1}^{N} \frac{\gamma_i}{\gamma_i (\lambda_1^*, ..., \lambda_N^*)},
\end{equation}

for $i = 1, ..., N$. The optimal consumption stream of agent $i$ is given by

\begin{equation}
c_i^* = I_i (\lambda_i^*, e^{\rho_i k} \xi_k (\lambda_1^*, ..., \lambda_N^*)),
\end{equation}

for all $k = 0, ..., T$.

**Remark.** Theorem 4.1 constitutes a preliminary tool for proving the existence of an equilibrium (see Theorem 4.3) and yields a closed-form characterization of all feasible equilibrium state price densities.

**Proof of Theorem 4.1.** Let $(c_i^*_k)_{k=0, ..., T}$ denote the optimal consumption stream of agent $i$. Recall that, by (3.4), we have that $c_i^* = I_i (\lambda_i^*, e^{\rho_i k} \xi_k)$ for some $\lambda_i^* > 0$. Plugging this into the market clearing condition (2.3), we obtain that the following holds in equilibrium:

\begin{equation}
\sum_{i=1}^{N} I_i (\lambda_i^*, e^{\rho_i k} \xi_k) = \epsilon_k,
\end{equation}

for all $k = 0, ..., T$. Here, $\lambda_1^*, ..., \lambda_N^*$ are constants that will be derived from the budget constraints in the sequel. Using the explicit form of $I_i$, this is equivalent to

\begin{equation}
\sum_{i=1}^{N} \log \left[ \gamma_i 1_{\{\lambda_i^* e^{\rho_i k} \xi_k > \gamma_i\}} + \lambda_i^* e^{\rho_i k} \xi_k 1_{\{\lambda_i^* e^{\rho_i k} \xi_k \leq \gamma_i\}} \right]^{1/\gamma_i} = \epsilon_k,
\end{equation}

a further transformation yields,

\begin{equation}
\prod_{i=1}^{N} (\gamma_i 1_{\{\xi_k > \beta_i(k)\}} + \lambda_i^* e^{\rho_i k} \xi_k 1_{\{\xi_k \leq \beta_i(k)\}})^{1/\gamma_i} = \prod_{i=1}^{N} \gamma_i^{1/\gamma_i} \exp(-\epsilon_k),
\end{equation}
where \( \beta_i(k) = \beta_i^{(\lambda^*)}(k) \) was defined in (4.1). This is equivalent to

\[
\sum_{j=1}^{N} Y_j^{(k)} 1\{\beta_{j-1}(k) < \xi_k \leq \beta_{j}(k)\} + \prod_{i=1}^{N} \gamma_i^{1/\gamma_i} 1\{\xi_k > \beta_N(k)\} = \prod_{i=1}^{N} \gamma_i^{1/\gamma_i} \exp(-\epsilon_k),
\]

where

\[
Y_j^{(k)} = \prod_{l=1}^{j-1} \gamma_l^{1/\gamma_l} \prod_{l=j}^{N} (\lambda^*_l)^{1/\gamma_l} \exp\left( k \sum_{l=j}^{N} \frac{\beta_l(k)}{\gamma_l} \right). \]

The strict-positivity assumption on the endowments implies that \( \epsilon_k > 0 \), P-a.s., and thus \( \xi_k \leq \beta_{i_k}(k) \) holds P-a.s. Next, for each \( k = 0, \ldots, T \), we have that

\[
\sum_{j=1}^{N} Y_j^{(k)} 1\{\beta_{j-1}(k) < \xi_k \leq \beta_{j}(k)\} = \prod_{i=1}^{N} \gamma_i^{1/\gamma_i} \exp(-\epsilon_k) 1\{\beta_{j-1}(k) < \xi_k \leq \beta_{j}(k)\},
\]

which implies that the following holds on each set \( \{\beta_{j-1}(k) < \xi_k \leq \beta_{j}(k)\} \):

\[
\xi_k = \prod_{l=j}^{N} \gamma_l^{1/\gamma_l} (\sum_{m=j}^{N} \gamma_m) \exp\left( -\frac{\epsilon_k}{\sum_{l=j}^{N} \gamma_l} \right).
\]

In particular, one checks that \( \xi_k \leq \beta_{i_k}(k) \) is equivalent to \( \epsilon_k \geq \eta_j(k) \) and \( \beta_{j-1}(k) < \xi_k \) is equivalent to \( \epsilon_k < \eta_{j-1}(k) \), where, \( \eta_j(k) = \eta^*_j \cdots \eta^*_N(k) \) is given in (4.2). Now, one revises the above identity in terms of \( \lambda^*_1, \ldots, \lambda^*_N \) and concludes that every equilibrium SPD is of the form (4.3) for some \( \lambda^*_1, \ldots, \lambda^*_N \). At last, observe that due to the budget constraints (2.2), in equilibrium, the constants \( \lambda^*_1, \ldots, \lambda^*_N \) solve the system of equations (4.4). This concludes the proof. \( \square \)

The following result is essential for proving an existence of equilibrium.

**Lemma 4.2.** For each \( i = 1, \ldots, N \), consider the excess demand function \( g^i : \mathbb{R}^{+}_1 \rightarrow \mathbb{R} \) defined by

\[
g^i(\lambda_1, \ldots, \lambda_N) = \lambda_i^{-1} \sum_{k=0}^{T} \left( E \left[ \xi_k (\lambda^{-1}_1, \ldots, \lambda^{-1}_N) I_i (\lambda^{-1}_i e^{\rho k} \xi_k (\lambda^{-1}_1, \ldots, \lambda^{-1}_N)) \right] - E \left[ \xi_k (\lambda^{-1}_1, \ldots, \lambda^{-1}_N) \epsilon_k^i \right] \right),
\]

where \( \xi_k (\lambda_1, \ldots, \lambda_N) \) is defined in (4.3). Then, the following properties are satisfied:

1. Each function \( g^i(\lambda_1, \ldots, \lambda_N) \) is a homogeneous function of degree 0.
2. We have

\[
\sum_{i=1}^{N} \lambda_i g^i(\lambda_1, \ldots, \lambda_N) = 0,
\]
for all \((\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_{++}\).

(3) Each function \(g'((\lambda_1, \ldots, \lambda_N))\) is continuous.

(4) (i) The following limit holds

\[
\lim_{\lambda_i \to 0^+} g'((\lambda_1, \ldots, \lambda_N)) = -\infty.
\]

(ii) Each function \(g'((\lambda_1, \ldots, \lambda_N))\) is bounded from above.

**Proof of Lemma 4.2.** (1) The assertion follows from the identity

\[
\xi_k((c\lambda_1)^{-1}, \ldots, (c\lambda_N)^{-1}) = c\xi_k((1, \ldots, 1),
\]

for all \(c > 0\), which holds due to (4.3) and the observations that \(\eta_j((\lambda_1, \ldots, \lambda_N)) = \eta_j((c\lambda_1, \ldots, c\lambda_N))\).

(2) By the construction of the SPDs in the proof of Theorem 4.1 (see (4.6)), it follows that \(c_k = I_i(1, \ldots, e^{\rho_i}k \xi_k((1, \ldots, 1)))\) satisfies the market clearing condition (2.3), for all \((\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_{++}\). By changing the order of summation, the claim becomes equivalent to

\[
\sum_{k=0}^T E \left[ \xi_k((\lambda_1^{-1}, \ldots, \lambda_N^{-1}) \left( \sum_{i=1}^N I_i(\lambda_i^{-1}, e^{\rho_i}k \xi_k((1, \ldots, 1))) - \sum_{i=1}^N c_k \right) \right] = 0,
\]

which follows from the latter observation concerning the market clearing condition.

(3) Consider some \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N_{++}\) and a sequence \(x_m = (x^{m}_1, \ldots, x^{m}_N) \in \mathbb{R}^N_{++}\) such that \(x_m \to x\). One checks that the random function \(\xi_k((\lambda_1, \ldots, \lambda_N)) : \mathbb{R}^{++} \to \mathbb{R}_{++}\) is \(P\)-a.s continuous, and hence \(\lim_{m \to \infty} \xi_k(x_m) = \xi_k(x)\), and consequently \(\lim_{m \to \infty} \xi_k(x_m) I_i(x^m e^{\rho_i}k \xi_k(x_m)) = \xi_k(x) I_i(x e^{\rho_i}k \xi_k(x))\). Therefore, it suffices to show uniform integrability in order to obtain \(L^1\)–convergence. Thus, it is enough to check that

\[
\sup_{m \geq 1} E \left[ \left( \xi_k(x_m) I_i(x^m e^{\rho_i}k \xi_k(x_m)) \right)^p \right] < \infty, \quad \sup_{m \geq 1} E \left[ (\xi_k(x_m) e_k)^p \right] < \infty,
\]

for some \(p > 1\). Since \(x_i > 0\), for all \(i = 1, \ldots, N\), we can assume that there exists \(\varepsilon > 0\) such that \(x^m_i > \varepsilon\), for all \(i = 1, \ldots, N\) and all \(m\). Therefore, we can estimate \(\xi_k(x_m) \leq Ke^{-M\varepsilon}\), for all \(m\) and some constants \(K, M > 0\). Hence, \(\sup_{m \geq 1} E \left[ (\xi_k(x_m) e_k)^p \right] \leq K^p E \left[ e^{-M\varepsilon} \right] < \left( \frac{K}{Me} \right)^p\). Next, one checks that

\[
\sup_{m \geq 1} E \left[ \left( \xi_k(x_m) I_i(x^m e^{\rho_i}k \xi_k(x_m)) \right)^p \right] \\
\leq \frac{1}{\gamma_i} \sup_{m \geq 1} E \left[ \left( \frac{\gamma_i}{x^{m}_i e^{\rho_i}k} - \xi_k(x_m) \right)^p 1_{x^m_i e^{\rho_i}k - \xi_k(x_m) \geq 0} \right] \\
\leq \frac{1}{\gamma_i} \sup_{m \geq 1} E \left[ \left( \frac{\gamma_i}{x^{m}_i e^{\rho_i}k} + Ke^{-M\varepsilon} \right)^p \right] < \frac{1}{\gamma_i} \left( \frac{\gamma_i}{e^{\rho_i}k} + K \right)^p,
\]

where the first inequality follows from the fact that \(1 + \log t \leq t\), for all \(t \geq 1\). This shows that \(g((\lambda_1, \ldots, \lambda_N))\) is continuous.
We show that \( \lim_{\lambda_i \to \infty} \xi_k(\lambda_1, \ldots, \lambda_N) = \lambda_i \xi_k(\lambda_1, \ldots, \lambda_N) \) \( I_i \left( \lambda_i e^{\rho_i^k} \xi_k(\lambda_1, \ldots, \lambda_N) - \epsilon_i^k \right) \).

The claim is equivalent to showing that \( \lim_{\lambda_i \to \infty} \) \( f^i(k, \lambda_1, \ldots, \lambda_N) = -\infty \). First, we show that \( \lim_{\lambda_i \to \infty} \xi_k(\lambda_1, \ldots, \lambda_N) \) \( I_i \left( \lambda_i e^{\rho_i^k} \xi_k(\lambda_1, \ldots, \lambda_N) - \epsilon_i^k \right) < 0 \), \( P\)-a.s. Assume that \( \lambda_i \) is sufficiently large so that \( i = i_1(k) \) and \( \lambda_i = \lambda_{i_1(k)} > \max\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_N\} \). One checks that \( \lim_{\lambda_i \to \infty} \eta_1(k) = \infty \). This implies that

\[
\lim_{\lambda_i \to \infty} \xi_k(\lambda_1, \ldots, \lambda_N) = \prod_{i=1}^{N} \left[ \sum_{j=1}^{N} \frac{\gamma_j(\lambda_j / \gamma_j(\lambda_i))}{\gamma_i(\lambda_i)} \right]^{-1} \exp \left( -\frac{\epsilon_i}{\sum_{j=1}^{N} \frac{1}{\gamma_j(\lambda_i)}} \right) 1_{\{\eta_2(k) \leq \epsilon_i < +\infty\}} + \frac{1}{\gamma_i} \log \left( \frac{\gamma_i}{\lambda_i e^{\rho_i^k} \xi_k(\lambda_1, \ldots, \lambda_N)} \right) \cdot \frac{1}{\left\{ e^{\rho_i^k} \xi_k(\lambda_1, \ldots, \lambda_N) \geq \lambda_i \right\}} = 0
\]

is satisfied identically \( P\)-a.s. for sufficiently large \( \lambda_i \). This yields that

\[
\lim_{\lambda_i \to \infty} \xi_k(\lambda_1, \ldots, \lambda_N) \left( I_i \left( \lambda_i e^{\rho_i^k} \xi_k(\lambda_1, \ldots, \lambda_N) - \epsilon_i^k \right) \right) < 0.
\]

Now, consider an arbitrary sequence \( (a_n)_{n=1}^{\infty} \) such that \( a_n \to \infty \), and denote \( b_n = (\lambda_1, \ldots, \lambda_{i-1}, a_n, \lambda_{i+1}, \ldots, \lambda_N) \in \mathbb{R}_+^N \). The same arguments as in (3) can be applied to show that the sequence \( \{\xi_k(b_n) \left( I_i \left( a_n e^{\rho_i^k} \xi_k(b_n) - \epsilon_i^k \right) \right) \}_{n=1}^{\infty} \) is uniformly integrable. Therefore, \( \lim_{\lambda_i \to \infty} \) \( f^i(k, \lambda_1, \ldots, \lambda_N) \) is bounded from above, for any fixed \( a > 0 \).

This accomplishes the proof of part (i).

(ii) This follows by the fact that the function \( h : \mathbb{R}_+ \to \mathbb{R} \) given by \( h(x) = x \left( (\log 1/x)^+ - a \right) \) is bounded from above, for any fixed \( a > 0 \). □

The next statement establishes the existence of an equilibrium.
**Theorem 4.3.** Assume that $\epsilon_i^k > 0$ for each period $k$ and each agent $i$, $P$-a.s. Then, there exists an equilibrium. Furthermore, a process $(\xi_k^i)_{k=0,...,T}$ is an equilibrium SPD if and only if $\xi_k^i = \xi_k(\lambda_1^i, ..., \lambda_N^i)$ for some $(\lambda_1^i, ..., \lambda_N^i) \in \mathbb{R}_+^N$ that solves the system of equations (4.4). The optimal consumption stream is given by (4.5).

**Proof of Theorem 4.3.** By Theorem 4.1 it is sufficient to prove that $\xi_k(\lambda_1^i, ..., \lambda_N^i)$ and $I_i(\lambda_1^i e^{\rho k} \xi_k(\lambda_1^i, ..., \lambda_N^i))$ is an equilibrium. Therefore, it is left to check that the system of equations (4.4) has a solution, or equivalently, the system of equations $g^i(\lambda_1, ..., \lambda_N) = 0$, for $i = 1, ..., N$ (see (4.8)) has a solution. The existence of a solution follows by properties (1)-(4) in Lemma 4.2 and fixed point arguments similar to those appearing in Theorem 17.C.1 in Mas-Colell et al. (1995).

As will be shown in the next section, the equilibrium SPD is, in general, not unique. Nevertheless, when the economy is homogeneous, the utility maximization problem is similar to the one in the non-constrained setting for consumption, and in particular assures uniqueness.

**Example: Homogeneous Economy.** In an economy populated only by agents of type $i$ that hold strictly positive endowment streams $(\epsilon_i^k)_{k=0,...,T}$, there exists a unique (normalized) equilibrium and the corresponding homogeneous SPD process $\{\xi_k^i\}_{k=0,...,T}$ is given by
\[
\xi_k^i = e^{\gamma_i (\epsilon_i^0 - \epsilon_i^k) - \rho_i k},
\]
for all $k = 1, ..., T$. The optimal consumptions obviously coincide with the endowments: $c_k^i = \epsilon_k^i$, for all $k = 0, ..., T$.

5. **Non-uniqueness of equilibrium**

In the present section we show that the equilibrium established in Section 4 is generally non-unique. First, we construct a specific example when endowments are positive, and then explore the case when endowments may vanish with a positive probability and provide some examples.

5.1. **Non-uniqueness with positive endowments.** It is evident that the system of equations (4.4) is related to the uniqueness of the equilibrium. The system of equations (4.4) is somewhat cumbersome in certain aspects: the functions $\xi_k(\lambda_1, ..., \lambda_N)$ are not differentiable with respect to each variable $\lambda_i$; it might happen that (4.4) has infinitely many different solutions but the equilibrium is still unique; the property of “gross substitution” (see Definition 3.1 in Dana (1993b)), which would be a sufficient condition for the uniqueness of the equilibrium, is not necessarily satisfied. The next example demonstrates the existence of multiple equilibria.
Example: Non-Uniqueness of Equilibrium. We assume a one period market with \( \mathcal{F}_0 = \mathcal{F}_1 = \{ \Omega, \emptyset \} \). Consider two agents \( i = 1, 2 \) represented by \( u_1(x) = u_2(x) = -e^{-x} \) and \( \rho_1 = \rho_2 = 0 \). The agents hold different endowments \( \epsilon_0^i, \epsilon_1^i \) and \( \epsilon_0^j, \epsilon_1^j \) respectively. Let \( \epsilon_0 \) and \( \epsilon_1 \) denote the aggregate endowments. By Theorem 4.1, every equilibrium state price density is of the form

\[
\xi_1(x,y) = \frac{1}{\min\{x,y\}} \, 1_{\{\epsilon_1 < \log \frac{\max(x,y)}{\min(x,y)}\}} e^{-\epsilon_1} + \frac{1}{\sqrt{xy}} \, 1_{\{\log \frac{\max(x,y)}{\min(x,y)} \leq \epsilon_1\}} e^{-\epsilon_1/2},
\]

where \( x \) and \( y \) are to be determined by the budget constraints. One can check that it is possible to rewrite it as

\[
\xi_1(x,y) = \begin{cases} 
\frac{1}{e^{\epsilon_1^2} x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1^2}}, \\
\frac{1}{\sqrt{y e^{\epsilon_1^1} x^2}} \frac{1}{\sqrt{y}} & \text{if } ye^{\epsilon_1^1} \leq x \leq ye^{\epsilon_1^1}, \\
\frac{1}{y e^{\epsilon_1^1}} & \text{if } x > ye^{\epsilon_1^1}.
\end{cases}
\]

The positive arguments \( x, y \) solve equations (4.4) which take the form:

1. \( \log(1/y)1_{\{y\leq1\}} + \xi_1(x,y) \log \left( \frac{1}{y \xi_1(x,y)} \right) 1_{\{x \xi_1(x,y) \leq 1\}} = \epsilon_0 + \epsilon_1^1 \xi_1(x,y), \)

2. \( \log(1/x)1_{\{x\leq1\}} + \xi_1(x,y) \log \left( \frac{1}{x \xi_1(x,y)} \right) 1_{\{x \ xi_1(x,y) \leq 1\}} = \epsilon_0 + \epsilon_1^2 \xi_1(x,y). \)

Let us note that we work with a normalized state price density, i.e., \( \xi_0 = 1 \). We denote

\[
h(x,y) = \xi_1(x,y) \left( \log \left( \frac{1}{y \xi_1(x,y)} \right) 1_{\{x \ xi_1(x,y) \leq 1\}} - \epsilon_1^1 \right),
\]

and

\[
g(x,y) = \xi_1(x,y) \left( \log \left( \frac{1}{x \ xi_1(x,y)} \right) 1_{\{x \ xi_1(x,y) \leq 1\}} - \epsilon_1^2 \right).
\]

Observe that

\[
h(x,y) = \begin{cases} 
-\epsilon_1^1 \frac{1}{x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1^2}}, \\
\frac{1}{\sqrt{y e^{\epsilon_1^1} x^2}} \frac{1}{\sqrt{y}} \left( \log \left( \frac{\sqrt{ye^{\epsilon_1^1}}}{\sqrt{x}} \right) - \epsilon_1^1 \right) & \text{if } ye^{\epsilon_1^1} \leq x \leq ye^{\epsilon_1^1}, \\
\frac{1}{y e^{\epsilon_1^1}} (\epsilon_1 - \epsilon_1^1) & \text{if } x > ye^{\epsilon_1^1}.
\end{cases}
\]

and that

\[
g(x,y) = \begin{cases} 
-\epsilon_1^2 \frac{1}{x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1^2}}, \\
\frac{1}{\sqrt{y e^{\epsilon_1^1} x^2}} \frac{1}{\sqrt{x}} \left( \log \left( \frac{\sqrt{ye^{\epsilon_1^1}}}{\sqrt{x}} \right) - \epsilon_1^2 \right) & \text{if } ye^{\epsilon_1^1} \leq x \leq ye^{\epsilon_1^1}, \\
\frac{1}{y e^{\epsilon_1^1}} (\epsilon_1 - \epsilon_1^2) & \text{if } x > ye^{\epsilon_1^1}.
\end{cases}
\]

Hence, equations (1) and (2) can be rewritten as

\[
(1') \quad \log(1/y)1_{\{y\leq1\}} + h(x,y) = \epsilon_0^1,

(2') \quad \log(1/x)1_{\{x\leq1\}} + g(x,y) = \epsilon_0^2.
\]

We are going to define the endowments in a particular way that will yield two distinct solutions to (1') and (2'). More precisely, we are going to construct two
In Theorems 4.1 and 4.3 assume that many equilibria all of the same canonical form. It turns out that once this assumption is relaxed, there exist necessarily infinitely many equilibria. We start by treating equation (2'). Consider the function \( h(x) = \log(1/x) - x^{-\frac{1}{e_1}} \) on the interval \([0, ye^{-\epsilon_1}]\), for arbitrary \( y > 0 \).

Note that \( h'(x) = \frac{1}{x^2 \, e_1} - \frac{1}{x^2} \), and hence \( h'(x) > 0 \) if \( x < \frac{e_1}{e_1} \), and \( h'(x) < 0 \) if \( x > \frac{e_1}{e_1} \), which implies that \( x_{\max} = \frac{e_1}{\epsilon_1} \) is a maximum of \( h \). Furthermore, \( y_l \) (to be determined explicitly in the sequel) will satisfy \( \epsilon_1 < y_l \), and this will guarantee that the maximum is indeed in the domain of definition of \( h \), i.e., \( \frac{e_1}{\epsilon_1} \in [0, ye^{-\epsilon_1}] \).

Next, note that \( h(\frac{e_1}{\epsilon_1}) = \log \left( \frac{e_1}{\epsilon_1} \right) - 1 > 0 \) due to the assumption \( e^{2\epsilon_1-1} > \epsilon_1 \).

Now let \( \delta > 0 \) be some small quantity to be determined below. One can pick \( \epsilon_0 \) such that the equation \( h(x) = \epsilon_0 \) has exactly two solutions \( x_1 \) and \( x_2 \) (see Figure 1) in the interval \( [\frac{e_1}{\epsilon_1} - \delta, \frac{e_1}{\epsilon_1} + \delta] \). Now, equation (1') has two solutions denoted by \( y_1 \) and \( y_2 \) (depending on \( \epsilon_0 \)) corresponding to \( x_1 \) and \( x_2 \) that are given by

\[
y_l = \exp \left( -\epsilon_0 - \frac{\epsilon_1}{e^{\epsilon_1} \, x_l} \right),
\]

for \( l = 1, 2 \). Obviously, \( y_1, y_2 < 1 \). It is left to check that \( \max\{x_l, \frac{e_1}{\epsilon_1} \} < ye^{-\epsilon_1} \), for \( l = 1, 2, \) that is,

\[
\max\left\{x_l, \frac{\epsilon_1}{e^{\epsilon_1}} \right\} < \exp \left( -\epsilon_1 - \epsilon_0 - \frac{\epsilon_1}{e^{\epsilon_1}} \, \frac{1}{x_l} \right).
\]

Since \( x_l \in [\frac{\epsilon_1}{e^{\epsilon_1}} - \delta, \frac{\epsilon_1}{e^{\epsilon_1}} + \delta] \), it suffices to verify that

\[
e^{\epsilon_1} \left( \frac{\epsilon_1}{e^{\epsilon_1}} + \delta \right) < \exp \left( -\epsilon_1 - \epsilon_0 - \frac{\epsilon_1}{e^{\epsilon_1}} \, \frac{1}{\epsilon_1 - \delta e^{\epsilon_1}} \right),
\]

for an appropriate choice of \( \delta > 0 \) and \( \epsilon_0 \). This inequality is equivalent to

\[
\epsilon_1 + \delta e^{\epsilon_1} < \exp \left( -\epsilon_0 - \frac{\epsilon_1}{\epsilon_1 - \delta e^{\epsilon_1}} \right).
\]

By continuity, it suffices to prove this inequality for \( \delta = 0 \) and \( \epsilon_0 = 0 \), which becomes \( \epsilon_1 e < 1 \), and follows from the assumptions imposed on \( \epsilon_1 \). \( \square \)

5.2. Non-uniqueness with vanishing endowments. In Theorems 4.1 and 4.3 we have assumed that \( P(\epsilon_k > 0) = 1 \), for all \( k = 1, \ldots, T \) and all \( i = 1, \ldots, N \). This assumption was crucial for proving that every equilibrium SPD is of the form (4.3). It turns out that once this assumption is relaxed, there exist necessarily infinitely many equilibria all of the same canonical form.

Theorem 5.1. Assume that \( P(\epsilon_k = 0) > 0 \), for all \( k = 0, \ldots, T \) and \( P(\cup_{k=0}^T \{\epsilon_k > 0\}) > 0 \), for all \( i = 1, \ldots, N \). Then, in the setting of Section 4, there exist infinitely many equilibria. Every equilibrium SPD \( \tilde{\zeta}_k \) for \( 0 \leq k \leq T \) is of the form

\[
\tilde{\zeta}_k(\lambda_1, \ldots, \lambda_N) = \xi_k(\lambda_1, \ldots, \lambda_N)1_{\{\epsilon_k \neq 0\}} + X_k 1_{\{\epsilon_k = 0\}},
\]
for all $k = 1, \ldots, T$, where $\xi_k(\lambda_1, \ldots, \lambda_N)$ is given by (4.3) and $X_k$ is some non-negative $F_k$-measurable random variable. The constants $\lambda_1, \ldots, \lambda_N$ are determined by the budget constraints

$$
\sum_{k=0}^{T} (E\left[\xi_k(\lambda_1, \ldots, \lambda_N)I_i\left(\lambda_i e^{\rho k} \xi_k(\lambda_1, \ldots, \lambda_N)\right)\right] - E\left[\xi_k(\lambda_1, \ldots, \lambda_N)\epsilon_k^i\right]) = 0
$$

for $i = 1, \ldots, N$.

**Proof of Theorem 5.1.** The proof is identical to the proofs of Theorem 4.1 and Theorem 4.3 apart from a slight modification as follows. Consider equation (4.7) and note that in the current context this equation admits the form $1_{\{\epsilon_k > \beta_{\omega(k)}(k)\}} = 1$ on the set $\{\epsilon_k = 0\}$, which implies that $\xi_k$ is of the form (5.1). The rest follows by similar arguments to those found in Section 4. \qed

We illustrate the above phenomenon in the following elementary example.

**Example: Infinitely Many Equilibria.** Let $(\Omega, \mathcal{F}_1, P)$ be a probability space where $\Omega = \{\omega_1, \omega_2\}$, $P(\{\omega_1\}), P(\{\omega_2\}) > 0$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_1 = 2^\Omega$. Consider a one period homogeneous economy with an individual represented by the utility function $u(x) = -e^{-x}$ and $\rho = 0$. The endowments of the agent are denoted by $\epsilon_0$ and $\epsilon_1$. For the sake of transparency, we analyze the following two simple cases directly by using the definition of equilibrium rather than by using Theorem 5.1.

(i) Let $\epsilon_0 = 0$ and $\epsilon_1$ be an arbitrary $\mathcal{F}_1$-measurable positive random variable.

To illustrate the above phenomenon in the following elementary example.

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(i) Let $\epsilon_0 = 0$ and $\epsilon_1$ be an arbitrary $\mathcal{F}_1$-measurable positive random variable. Theorem 3.2 implies that the optimal consumption policies $c_0$ and $c_1$ are given by
\[ c_0 = -\log \left( \mathbf{1}_{\{\lambda > 1\}} + \lambda \mathbf{1}_{\{\lambda \leq 1\}} \right) \quad \text{and} \quad c_1 = -\log \left( \mathbf{1}_{\{\lambda \xi_1 > 1\}} + \lambda \xi_1 \mathbf{1}_{\{\lambda \xi_1 \leq 1\}} \right). \]

The market clearing condition \( c_0 = 0 \) and \( c_1 = \epsilon_1 \) implies that \( \xi_1 = e^{-\epsilon_1} \) is an equilibrium SPD, for all \( \lambda \geq 1 \). Note that the budget constraints of the type (2.2) are automatically satisfied due to the fact that the market clears and due to the homogeneity of the economy. We stress out that for the corresponding unconstrained problem

\[ \sup_{(c_0, c_1)} -e^{-c_0} - E[e^{-c_1}], \]

where \( c_0 \in \mathbb{R} \) and \( c_1 \in L^2(\mathcal{F}_1) \) are such that

\[ c_0 + E[\xi_1 \epsilon_1] = \epsilon_0 + E[\xi_1 \epsilon_1], \]

there exists a unique equilibrium corresponding to \( \lambda = 1 \), that is, \( \xi_1 = e^{-\epsilon_1} \).

(ii) Let \( \epsilon_0 > 0 \) be arbitrary, \( \epsilon_1(\omega_1) > 0 \) and \( \epsilon_1(\omega_2) = 0 \). Then, by Theorem 3.2 we obtain that \( c_0 = \log(1/\lambda) = \epsilon_0 \) and \( c_1 = \max\{\log(1/\lambda), 0\} = \epsilon_1 \). It follows that \( \lambda = e^{-\epsilon_0} \), and that there are infinitely many equilibrium state price densities of the form \( \xi_1(y) = e^{\epsilon_0 - \epsilon_1} 1_{\{\epsilon_1 \neq 0\}} + y 1_{\{\epsilon_1 = 0\}} \), for every \( y > e^{\epsilon_0} \). As in (i), the market clearing condition is redundant due to the homogeneity of the economy.

6. Long-run yields of zero coupon bonds

In the current section we work with an infinite time horizon \( T = \infty \). We emphasize that all our results in the context of equilibrium hold in this setting due to a specific choice of the aggregate endowment process (see (6.3)). Denote by \( \xi_k = \xi_k(\lambda_1, \ldots, \lambda_N) \), \( k \in \mathbb{N} \) the equilibrium SPD (see (4.3)). The equilibrium price at time 0 of a zero coupon bond maturing at a period \( t \in \mathbb{N} \) is defined by

\[ B^t = E[\xi_t]. \]

Based mainly on Cramer’s large deviation theorem, we study the asymptotic behavior (as \( t \to \infty \)) of the yield of this bond, defined by

\[ \frac{\log B^t}{t}. \]

Let us stress that we omit the weights \( \lambda_1, \ldots, \lambda_N \) (in most cases) since they will not be crucial when dealing with the corresponding limits. As will be shown in the sequel, the long-run behavior of this bond will depend on the agents’ characteristics. Throughout this section we assume that the total endowment process in the economy is a random walk with a drift, i.e.,

\[ \epsilon_k = \sum_{j=1}^{k} X_j, \]

for all \( k \in \mathbb{N} \), where \( X_0 = 0 \) and \( X_1, X_2, \ldots \) are non-negative independent identically distributed random variables with a finite mean \( E[X_1] > 0 \). This specification of endowments is quite natural in this setting, due to the exponential preferences of
the investors. We consider two categories of economies: agents with heterogeneous risk-aversion and agents with heterogeneous impatience rates.

6.1. Heterogeneous risk-aversion. Consider an economy where agents differ only with respect to the risk-aversion, that is, \( \gamma_1 < \ldots < \gamma_N, \) \( \epsilon_i = \epsilon_k / N, \) \( i = 1, \ldots, N, \) \( k \in \mathbb{N}; \) \( \rho_1 = \ldots = \rho_N = \rho. \) Recall (4.1) and note that in the current setting, equations (4.4) can be rewritten as

\[
\sum_{k=0}^{\infty} E \left[ \xi_k \left( \log \left( \beta_i(k) \right) - \log \xi_k \right) \right] = \frac{\gamma_i}{N} \sum_{k=0}^{\infty} E \left[ \xi_k \xi_k \right].
\]

Now, let \( i, j \in \{1, \ldots, N\} \) be arbitrary. Notice that the homogeneity of the impatience rate among agents implies that either \( \beta_i(k) \leq \beta_j(k), \) or \( \beta_i(k) \geq \beta_j(k), \) for all \( k \in \mathbb{N}. \) Therefore, by (6.4), we conclude that \( \beta_i(k) < \ldots < \beta_N(k), \) for all \( k \in \mathbb{N} \) and thus \( i_j(k) = l, \) for all \( l = 1, \ldots, N. \) Hence, we get (see (4.2))

\[
\eta_j = \log \left( \frac{\lambda_j}{\gamma_j} \right)^{1/\gamma_{j+1}+\ldots+1/\gamma_N} \left( \frac{\gamma_{j+1}}{\lambda_{j+1}} \right)^{1/\gamma_{j+1}} \ldots \left( \frac{\gamma_N}{\lambda_N} \right)^{1/\gamma_N},
\]

for all \( j = 1, \ldots, N, \) and (see (4.3))

\[
\xi_k = \sum_{j=1}^{N} \prod_{i=j}^{N} \left( \frac{\gamma_i}{\lambda_i e^{\rho k}} \right) \left( \sum_{m=j}^{N} \frac{\gamma_m}{\lambda_m e^{\rho k}} \right)^{-1} \exp \left( -\frac{\epsilon_k}{\sum_{l=j}^{N} 1/\gamma_l} \right) 1_{\{\eta_j \leq \epsilon_k < \eta_{j-1}\}},
\]

for all \( k \in \mathbb{N}. \)

**Theorem 6.1.** In heterogeneous risk-aversion economies, we have

\[
\lim_{t \to \infty} -\frac{\log B^t}{t} = \rho - \log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma_l} \right) \right].
\]

**Proof of Theorem 6.1.** First, we have evidently

\[
-\frac{\log B^t}{t} \leq -\frac{\log E \left[ \prod_{i=1}^{N} \left( \frac{\gamma_i}{\lambda_i e^{\rho k}} \right) \left( \sum_{m=i}^{N} \frac{\gamma_m}{\lambda_m e^{\rho k}} \right)^{-1} \exp \left( -\frac{\epsilon_k}{\sum_{l=i}^{N} 1/\gamma_l} \right) 1_{\{\eta_i \leq \epsilon_t < \eta_0\}} \right]}{t}.
\]

Next, note that the law of large numbers implies that

\[
limit_{t \to \infty} P(a \leq \epsilon_t \leq b) = 0,
\]

for all \( 0 \leq a \leq b, \) and

\[
limit_{t \to \infty} P(c \leq \epsilon_t) = 1, \quad \text{for any } 0 \leq c.
\]

Therefore, since \( \eta_0 = \infty, \) we get

\[
-\frac{\log E \left[ N \prod_{i=1}^{N} \left( \frac{\gamma_i}{\lambda_i e^{\rho k}} \right) \left( \sum_{m=i}^{N} \frac{\gamma_m}{\lambda_m e^{\rho k}} \right)^{-1} \exp \left( -\frac{\epsilon_k}{\sum_{l=i}^{N} 1/\gamma_l} \right) 1_{\{\eta_i \leq \epsilon_t < \eta_0\}} \right]}{t} \leq -\frac{\log B^t}{t},
\]

for sufficiently large \( t. \) It is left to prove that

\[
\lim_{t \to \infty} -\frac{\log E \left[ \exp \left( -\frac{\epsilon_k}{\sum_{l=i}^{N} 1/\gamma_l} \right) \right] 1_{\{\eta_i \leq \epsilon_t < \eta_0\}}}{t} = -\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=i}^{N} 1/\gamma_l} \right) \right].
\]
First, observe that
\[
\lim_{t \to \infty} -\log E \left[ \exp \left( -\frac{\epsilon t}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right] \geq 0
\]
\[
= -\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma} \right) \right] .
\]
On the other hand, we have
\[
-\log E \left[ \exp \left( -\frac{\epsilon t}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right] \leq \frac{\sum_{l=1}^{N} 1/\gamma}{\sum_{l=1}^{N} 1/\gamma} \right] .
\]
\[
-\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right] = -\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right] .
\]
lastly, the dominated convergence theorem yields
\[
\lim_{t \to \infty} -\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right] = -\log E \left[ \exp \left( -\frac{X_1}{\sum_{l=1}^{N} 1/\gamma} \right) 1_{\{\eta_1 \leq \epsilon t < \eta_0\}} \right],
\]
accomplishing the proof. □

6.2. Heterogeneous impatience rates. Consider an economy which is composed of agents who differ only with respect to the impatience rates, that is, \( \gamma_1 = \ldots = \gamma_N = \gamma; \epsilon_k = \epsilon_k/N, i = 1, \ldots, N, k = 1, 2, \ldots; \rho_n < \ldots < \rho_1. \) Let \( \lambda_1, \ldots, \lambda_N \) be the weights corresponding to the equilibrium SPD \( \xi_k = \xi_k(\lambda_1, \ldots, \lambda_N), \) \( k \in \mathbb{N}. \) Note that there exists \( t' \in \mathbb{N} \) such that \( \lambda_1 e^{\rho_1 t} > \ldots > \lambda_N e^{\rho_N t}, \) for all \( t > t'. \) Therefore, by recalling (4.1), we get \( i_l(t) = l, \) for all \( t > t', \) and consequently (see (4.2) and (4.3)), we have
\[
(6.5) \quad \eta_j(t) = \frac{1}{\gamma} \sum_{l=j+1}^{N} \log(\beta_l/\beta_j) = \frac{1}{\gamma} \sum_{l=j+1}^{N} \log(\lambda_j/\lambda_l) + \frac{1}{\gamma} \sum_{l=j+1}^{N} (\rho_j - \rho_l)t,
\]
for all \( j = 1, \ldots, N, \) and
\[
\xi_t = \gamma \sum_{j=1}^{N} (\lambda_j \ldots \lambda_N)^{-1} (N-j+1)^{-1} \exp \left( -\frac{\rho_j + \ldots + \rho_N}{N-j+1} t - \frac{\gamma}{N-j+1} \epsilon_j \right) \times 1_{\{\eta_j(t) \leq \epsilon_t < \eta_{j-1}(t)\}},
\]
for all \( t > t'. \) Consider the logarithmic moment generating function of \( X_1 \)
\[
\Lambda(x) = \log E \left[ e^{x X_1} \right],
\]
and denote by
\[
\Lambda^*(y) = \sup_{x \in \mathbb{R}} (xy - \Lambda(y)),
\]
the corresponding Legendre transform of $\Lambda$. We set
\[ a_j = \rho_j + \inf_{x \in \left[ \frac{1}{N} \sum_{i=j+1}^N (\rho_i - \rho_j), \frac{1}{N} \sum_{i=j}^N (\rho_i - \rho_j) \right]} \Lambda^*(x), \]
\[ b_j = \rho_{j-1} + \inf_{x \in \left( \frac{1}{N} \sum_{i=j+1}^N (\rho_i - \rho_j), \frac{1}{N} \sum_{i=j}^N (\rho_i - \rho_j) \right)} \Lambda^*(x), \]
for $j = 2, \ldots, N$, and
\[ a_1 = \rho_1 + \inf_{x \in \left[ \frac{1}{N} \sum_{i=2}^N (\rho_i - \rho_1), \infty \right)} \Lambda^*(x). \]

We are ready to state now the main result of this subsection.

**Theorem 6.2.** In economies with heterogeneous impatience rates, we have
\[ \lim_{t \to \infty} \sup \frac{-\log B_t}{t} \leq \min \{ b_2, \ldots, b_N \}, \]
and
\[ \lim_{t \to \infty} \inf \frac{-\log B_t}{t} \geq \min \{ a_1, \ldots, a_N \}. \]

**Proof of Theorem 6.2.** Observe that the following inequality is satisfied
\[ \xi_t \leq \gamma \sum_{j=1}^N (\lambda_j \cdots \lambda_N)^{-1} (N-j+1)^{-1} \exp \left( -\frac{1}{N-j+1} ((\rho_j + \ldots + \rho_N) t + \gamma \eta_j(t)) \right) \]
\[ \times 1_{\{ \eta_j(t) \leq \epsilon_t < \eta_{j-1}(t) \}} \leq \gamma \sum_{j=1}^N (\lambda_j)^{-1} \exp (-\rho_j t) 1_{\{ \eta_j(t) \leq \epsilon_t < \eta_{j-1}(t) \}}, \]
for all $t > t'$. Denote $a_j(t) := (\lambda_j)^{-1} \exp (-\rho_j t) 1_{\{ \eta_j(t) \leq \epsilon_t < \eta_{j-1}(t) \}}$. We have
\[ \lim_{t \to \infty} \inf \frac{-\log B_t}{t} \geq \lim_{t \to \infty} \inf \frac{-\log E[\sum_{j=1}^N a_j(t)]}{t} \]
\[ \geq \lim_{t \to \infty} \inf \log \left( \frac{1}{(N \max \{ E[a_1(t)], \ldots, E[a_N(t)] \})^{1/t}} \right) \]
\[ = \lim_{t \to \infty} \inf \log \left( \frac{1}{\max \{ E[a_1(t)], \ldots, E[a_N(t)] \}} \right) \]
\[ = \lim_{t \to \infty} \min \left\{ \log \left( \frac{1}{(E[a_1(t)])^{1/t}} \right), \ldots, \log \left( \frac{1}{(E[a_N(t)])^{1/t}} \right) \right\} \]
\[ = \min \left\{ \lim_{t \to \infty} \inf \log \left( \frac{1}{(E[a_1(t)])^{1/t}} \right), \ldots, \lim_{t \to \infty} \inf \log \left( \frac{1}{(E[a_N(t)])^{1/t}} \right) \right\}. \]

Fix an arbitrary $\varepsilon > 0$. Next, recall (6.5) and observe that the following inequality holds true for each $j \in \{ 2, \ldots, N \}$,
\[ \lim_{t \to \infty} \inf \frac{-\log E[a_j(t)]}{t} = \rho_j + \lim_{t \to \infty} \inf \frac{-\log P(\eta_j(t) \leq \epsilon_t < \eta_{j-1}(t))}{t} \]
\[ \geq \rho_j + \liminf_{t \to \infty} \frac{\log P \left( \frac{1}{t} \sum_{i=j}^{N} (\rho_{i} - \rho) - \varepsilon \leq \epsilon_t \leq \frac{1}{t} \sum_{i=j}^{N} (\rho_{i-1} - \rho) + \varepsilon \right)}{t} \]

\[ \geq \rho_j + \inf_{x \in \left[ \frac{1}{N} \sum_{i=j+1}^{N} (\rho_i - \rho), \frac{1}{N} \sum_{i=j}^{N} (\rho_{i-1} - \rho) + \varepsilon \right]} \Lambda^*(x), \]

where in the last inequality we have employed Cramer’s large deviation theorem (see e.g. Theorem 2.2.3 in Dembo and Zeitouni (1998)). Since \( \varepsilon > 0 \) was arbitrary, we conclude that

\[ \liminf_{t \to \infty} - \frac{\log a_j(t)}{t} \geq \rho_j + \inf_{x \in \left[ \frac{1}{N} \sum_{i=j}^{N} (\rho_i - \rho), \frac{1}{N} \sum_{i=j}^{N} (\rho_{i-1} - \rho) + \varepsilon \right]} \Lambda^*(x). \]

Similarly, we have

\[ \liminf_{t \to \infty} - \frac{\log E[a_1(t)]}{t} \geq \rho_1 + \inf_{x \in \left[ \frac{1}{N} \sum_{i=2}^{N} (\rho_i - \rho), \infty \right]} \Lambda^*(x). \]

The preceding inequalities combined with (6.8) prove the validity of (6.7). On the other hand, we have

\[ \xi_t(\lambda_1, ..., \lambda_N) \geq \gamma \sum_{j=2}^{N} (\lambda_j - \lambda_N)^{-(N-j+1)} \exp \left( - \frac{1}{N-j+1} \left( (\rho_j + ... + \rho_N) t + \gamma \eta_{j-1}(t) \right) \right) \]

\[ \times 1_{\{\eta_j(t) \leq \epsilon_t < \eta_{j-1}(t)\}} \geq \gamma \sum_{j=2}^{N} (\lambda_{j-1})^{-1} \exp (-\rho_{j-1} t) 1_{\{\eta_j(t) \leq \epsilon_t < \eta_{j-1}(t)\}}. \]

Finally, by exploiting the preceding inequality, one can use the same arguments as in the lower-limit case to obtain inequality (6.6). \( \square \)

### 7. Precautionary savings

The current section is concerned with the phenomenon of *precautionary savings*, namely, savings resulting from future uncertainty. When markets are complete, one would not anticipate this phenomenon to occur, since in principle, all risks can be hedged. Thus we concentrate on incomplete markets. Here, as commonly referred to in the literature (see e.g. Carroll and Kimball (2008)), *savings* should be understood literally as less consumption. We consider a one-period incomplete market of type \( \mathcal{C} \), as in Subsection 3.2.1, and stick to the same notation. The only distinction is the particular specification of the random endowment. Let \( X \in L^2(\mathcal{F}_1) \) be arbitrary non-negative random variable. Denote by

\[ (7.1) \quad \epsilon_1 = \epsilon_1(\varepsilon) := \frac{e^{\varepsilon X}}{E[e^{\varepsilon X} | \mathcal{H}_1]}. \]

the endowment at time \( T = 1 \) of the agent, for some \( \varepsilon \in [0, 1] \). Note that \( E[\epsilon_1(\varepsilon) | \mathcal{H}_1] = 1 \), and \( \epsilon_1 = 1 \) in case that \( X \in L^2(\mathcal{H}_1) \) or in case that the market is complete (i.e., if \( \mathcal{F}_1 = \mathcal{H}_1 \)). As shown in the next statement, the variance of \( \epsilon_1(\varepsilon) \) is an increasing function of \( \varepsilon \).
Lemma 7.1. The conditional variance on $H_1$ of the random variable $\epsilon_1(\epsilon)$ is an increasing function of $\epsilon$, namely, the function

$$\text{Var}(\epsilon_1(\epsilon)) = E \left[ (\epsilon_1(\epsilon) - E[\epsilon_1(\epsilon)|H_1])^2 |H_1 \right],$$

satisfies the inequality

$$\text{Var}(\epsilon_1(\epsilon_1)) \leq \text{Var}(\epsilon_1(\epsilon_2)),$$

$P-a.s.$, for all $\epsilon_1 \leq \epsilon_2$.

Proof of Lemma 7.1. First note that

$$\text{Var}(\epsilon_1(\epsilon)) = \frac{E[e^{2\epsilon X}|H_1]}{(E[e^{\epsilon X}|H_1])^2} - 1.$$  

We can view $\text{Var}(\epsilon_1(\epsilon)) : [0,1] \rightarrow L^2(H_1)$ as a curve. Since the probability space $\Omega$ is finite, the function $\text{Var}(\epsilon_1(\epsilon))$ is differentiable. Furthermore, to complete the proof, it suffices to show that the corresponding differential is non-negative. One checks that the differentiable is non-negative if and only if

$$E \left[ Xe^{2\epsilon X}|H_1 \right] E \left[ e^{\epsilon X}|H_1 \right] - E \left[ e^{2\epsilon X}|H_1 \right] E \left[ Xe^{\epsilon X}|H_1 \right] \geq 0.$$  

To see this, consider the measure $Q$ defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = \frac{e^{\epsilon X}}{E[e^{2\epsilon X}|H_1]},$$

and note that the above inequality is equivalent to

$$E^Q [X|H_1] E^Q [e^{\epsilon X}|H_1] \geq E^Q [X e^{\epsilon X}|H_1].$$

The latter inequality follows from the Fortuin-Kasteley-Ginibre (see Fortuin et. al. (1971); FGK, for short) inequality, since it can be rephrased as

$$E^Q [f(X)|H_1] E^Q [g(X)|H_1] \geq E^Q [f(X)g(X)|H_1],$$

where $f(x) = x$ is increasing and $g(x) = e^{-\epsilon x}$ is decreasing. □

According to Lemma 7.1, the parameter $\epsilon \in [0,1]$ quantifies the degree of income uncertainty. More precisely, an increase in $\epsilon$ yields an increase in the variance of the income but preserves the mean. Therefore, we learn that the un-insurability of the income is an increasing function of the parameter $\epsilon$. Now, recall the explicit formulas (see (3.9) and (3.10)) for the optimal consumption stream in the current setting. One can show (similarly to Corollary 3.4) that by choosing a sufficiently large $\epsilon_0$ (the initial endowment), the corresponding optimal consumption stream coincides with unconstrained one and is given by

$$\tilde{c}_0(\epsilon) = \frac{1}{\gamma} \left( \log \left( \frac{1}{\lambda(\epsilon)} \right) + \rho \right),$$

where $\lambda(\epsilon)$ is the solution to the equation $\text{Var}(\epsilon_1(\epsilon)) = \text{Var}(\epsilon_1(\epsilon_2))$. 

(7.2)
and

\[ \tilde{c}_1(\epsilon) = \frac{1}{\gamma} \log \left( \frac{e^{\gamma \epsilon} E \left[ e^{-\gamma \epsilon} | H_1 \right]}{\lambda(\epsilon) M_1} \right) \]

\[ = \frac{1}{\gamma} \log \left( \frac{e^{\gamma \epsilon} E \left[ e^{-\gamma \epsilon} | H_1 \right]}{M_1} \right) + \tilde{c}_0(\epsilon) - \frac{1}{\gamma} \rho \]

where \( \lambda(\epsilon) \) (or equivalently, \( \tilde{c}_0(\epsilon) \)) solves uniquely the equation

\[ \tilde{c}_0(\epsilon) + E \left[ M_1 \tilde{c}_1(\epsilon) \right] = \epsilon_0 + E \left[ M_1 \epsilon_1(\epsilon) \right]. \]

Now, we are ready to state the main result of the section.

**Theorem 7.2.** In the setting of the current section, the precautionary savings motive holds true. Namely, \( \tilde{c}_0(\epsilon) \) is a decreasing function of \( \epsilon \). Hence, in particular, investors consume less in the present, as the variance of future-income increases.

**Proof of Theorem 7.2.** First, it is possible to rewrite the constraint (7.4) as

\[ \tilde{c}_0(\epsilon) (1 + E [M_1]) + \frac{1}{\gamma} E \left[ M_1 \log E \left[ e^{-\gamma \epsilon_1(\epsilon)} | H_1 \right] \right] = K, \]

where \( K := \frac{\rho}{\gamma} E [M_1] + \epsilon_0 + \frac{1}{\gamma} E [M_1 \log M_1] \) is a constant not depending on \( \epsilon \). Obviously, \( \tilde{c}_0(\epsilon) \) is a differentiable function of \( \epsilon \). By differentiating the above identity with respect to \( \epsilon \), we get

\[ \frac{\partial \tilde{c}_0}{\partial \epsilon} = \frac{1}{1 + E [M_1]} E \left[ M_1 \log E \left[ e^{-\gamma \epsilon_1(\epsilon)} | H_1 \right] \right], \]

where the differential of \( \epsilon_1(\epsilon) \) is given by (see (7.1))

\[ \frac{\partial \epsilon_1}{\partial \epsilon} = \frac{e^{\epsilon X} E \left[ X e^{\epsilon X} | H_1 \right]}{e^{\epsilon X} E \left[ e^{\epsilon X} | H_1 \right]} - \frac{e^{\epsilon X} E \left[ X e^{\epsilon X} | H_1 \right]}{(E [e^{\epsilon X} | H_1])^2}. \]

Therefore, to accomplish the proof, it suffices to check that

\[ E \left[ e^{-\gamma \epsilon_1(\epsilon)} \frac{\partial \epsilon_1}{\partial \epsilon} | H_1 \right] \leq 0. \]

This boils down to proving that

\[ E \left[ X e^{\epsilon X} e^{-\gamma \epsilon_1(\epsilon)} | H_1 \right] E \left[ e^{\epsilon X} | H_1 \right] \leq E \left[ e^{-\gamma \epsilon_1(\epsilon)} e^{\epsilon X} | H_1 \right] E \left[ X e^{\epsilon X} | H_1 \right]. \]

Set a new measure \( Q \) given by the Radon-Nykodym derivative

\[ \frac{dQ}{dP} := \frac{e^{\epsilon X}}{E [e^{\epsilon X} | H_1]}, \]

and note that by dividing the preceding inequality by \( (E [e^{\epsilon X} | H_1])^2 \), we arrive at

\[ E^Q \left[ X e^{-\gamma \epsilon_1(\epsilon)} | H_1 \right] \leq E^Q \left[ e^{-\gamma \epsilon_1(\epsilon)} | H_1 \right] E^Q \left[ X | H_1 \right]. \]
Assume first that $\varepsilon > 0$. By (7.1), we have

$$X = (\log \epsilon_1(\varepsilon) + \log E \left[ e^{\varepsilon X} \big| H_1 \right]) \frac{1}{\varepsilon}.$$ 

Therefore, the required inequality admits the form

$$E_Q \left[ e^{-\gamma \epsilon_1(\varepsilon) \log \epsilon_1(\varepsilon)} \big| H_1 \right] \leq E_Q \left[ e^{-\gamma \epsilon_1(\varepsilon)} \big| H_1 \right] E_Q \left[ \log \epsilon_1(\varepsilon) \big| H_1 \right],$$

which follows by the FGK inequality, as in Lemma 7.1. Finally, if $\varepsilon = 0$, the proof follows from the fact that $\epsilon_1(0) = 1$. □

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