An Academic View on the Illiquidity Premium and Market-Consistent Valuation in Insurance

Mario V. Wüthrich*

April 15, 2011

Abstract

The insurance industry currently discusses to which extent they can integrate an illiquidity premium into their best estimate considerations of insurance liabilities. The present position paper studies this question from an actuarial perspective that is based on market-consistent valuation. We conclude that mathematical theory does not allow for discounting insurance liabilities with an illiquidity spread.

1 An actuarial view on the illiquidity premium

1.1 Aim and organization of this position paper

The aim of this position paper is to analyze the application of an illiquidity premium to discount the liability side of the balance sheet of an insurance company. We start with an introduction to market-consistent actuarial valuation in Section 1.2. In Section 1.3 we discuss the illiquidity premium from a rather non-mathematical perspective and give arguments why it should not be used for the above purpose. In Section 2 we present a simple model exemplifying the conclusions from Section 1.3.

1.2 Market-consistent actuarial valuation and regulation

The main task of an actuary is to predict and value insurance liability cash flows. These predictions and valuations form the basis for premium calculations as well as for solvency considerations of an insurance company. In most situations, insurance cash flows are not traded on a market. Therefore, current accounting and solvency regulation requires that these insurance

*ETH Zurich, RiskLab, Department of Mathematics, 8092 Zurich, Switzerland

cash flows are valued in a market-consistent mark-to-model approach. Article 75 of the Solvency II Framework Directive (Directive 2009/138/EC) states "liabilities shall be valued at the amount for which they could be transferred, or settled, between two knowledgeable willing parties in an arm's length transaction".

There is a consensus that (expected) insurance liability cash flows should be replicated by appropriate financial instruments hedging financial risks by an optimal asset allocation. The residual risks (between the expected values and the random variables) ask for an additional risk margin for the risk bearing of the run-off of these residual risks. The sum of these two elements (expected values and risk margin) then corresponds to the technical provisions. This is described in Article 77 of the Solvency II Framework Directive as follows "The value of technical provisions shall be equal to the sum of a best estimate and a risk margin ... The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. The calculation of the best estimate shall be based upon up-to-date and credible information ... The risk margin shall be such as to ensure that the value of technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations."

The crucial question now is: *what are admissible financial instruments for the replication of (expected) insurance liability cash flows?*

The aim of market-consistent actuarial valuation is to map the (expected) insurance liability cash flows to financial instruments that have a *reliable market value* (see also Article 77.4 of the Solvency II Framework Directive). That is, these financial instruments should meet the following criteria (see Article TP.4.5 of the QIS5 Technical Specifications Document):

(a) a large number of assets can be transacted without significantly affecting the price of the financial instruments used in the replications (*deep*),

(b) assets can be easily bought and sold without causing a significant movement in the price (*liquid*),

(c) current trade and price information are normally readily available to the public, in particular to the undertakings (*transparent*).

If these criteria are not met then there are no reliable market values for these financial instruments. Moreover, the above specifications mean that liquid (corporate) bonds can be traded at any time and the bondholder does not need to pursue a hold-to-maturity strategy in order to get a reasonable return on his investment.

1.3 Illiquidity premium

It is exactly this hold-to-maturity argument that is debated by several (life) insurance companies. They argue that insurance liabilities are illiquid and therefore it should be allowed to integrate an illiquidity spread into the discount function for these liabilities. They state: often insurance liability cash flows are highly predictable and therefore they have all the features of illiquid long-term bonds. Thus, these liabilities could be replicated by illiquid bonds and, as a consequence, these liabilities are valued at a much lower value because illiquidity massively beats down the prices, see also Keller [2].

We outline, why this argument contradicts market-consistent actuarial valuation. In this section we give rational arguments which will be mathematically supported in Section 2.

We give the following counter-arguments:

- Valuation of insurance liability cash flows with illiquid bonds contradicts market-consistent actuarial valuation because it corresponds to a hold-to-maturity view, but market-consistent values are based on a transfer view (with the appropriate reward for risk bearing). One may debate on a whole the concept of market-consistent valuation in accounting, but once this framework is given we are not allowed to measure some instruments by transfer values and others in a hold-to-maturity view, because this will lead to a rather inconsistent full balance sheet valuation that may allow for accounting arbitrage.

- The argument that insurance cash flows are predictable contradicts the very notion of insurability. However, we could measure the degree of predictability choosing an appropriate risk measure, but then this, of course, always depends on the choice of the risk measure. Since we have infinitely many acceptable risk measure choices we are again in an incomplete market setting which requires more information about preferences of financial agents.

- There is no clear concept how the illiquidity spread is measured and distinguished from the credit spread. Therefore, we can neither price this single component in a market-consistent way, nor can we replicate it in an appropriate way. Moreover, arguments that say that the illiquidity spread can be isolated and hedged contradict the "law-of-one-price" assumption which also allows for arbitrage.

- Often the degree of illiquidity is compared to a degree of predictability of insurance liability cash flows. We are not aware of a theory that would allow for a meaningful analysis.

- The general aim in market-consistent valuation is to replicate insurance liabilities with financial instruments that have reliable market prices. Therefore, we cannot choose illiquid instruments because their values are erratic and hence the market-consistent value of the insurance liabilities becomes arbitrary.

We believe that if we respect the *current accounting and solvency rules*, market-consistent actuarial valuation means that we need to replicate (expected) insurance cash flows with financial instruments that have reliable prices and, thus, are traded in deep, liquid and transparent financial markets (which excludes the illiquidity premium). This has the following consequences:

- Liabilities are replicated by liquid financial instruments that have reliable prices.

- On the asset side of the balance sheet we may hold liquid and illiquid financial instruments. If we hold illiquid instruments then they require a higher risk capacity because of possible (further) deterioration of their prices (especially for inter-temporal valuation according to the accounting time frame).

- This higher risk capacity has to be viewed in the trade-off of a higher expected return, see also (2.6) below. If the bearing of these illiquidity risks is rewarded, we will have a financial gain in the future, if these risks are not rewarded we will have a loss. This release of gains and losses will reveal over time according to the economic developments and essentially depends on the accounting rules, e.g. if the insurance companies have a yearly closing of their books, they give a "yearly solvency guarantee".

- If actuaries would value insurance liability cash flows with illiquid financial instruments they would *immediately release* these possible future illiquidity gains, see (2.7) below. That is, they would change the consumption stream of possible losses and gains (in a non-market-consistent way, as we will see in Section 2, below). This would contradict the actual solvency view because insurance companies need to hold best estimates and a risk (prudence) margin for possible adverse scenarios in a market-consistent one-year view. That is, the immediate release would give too low best estimates and would essentially weaken the financial strength of the insurance companies which is neither in the sense of the policyholder nor the aim of the regulator.

Conclusion.

The existence of the illiquidity spread can be observed on the market and can as well be incorporated in a theoretical model. However, under the current accounting and solvency rules (market-consistent one-year view) it can only be applied on the asset side of the balance sheet (with an appropriate risk margin). If regulation decides that also the liability side of the balance sheet can be discounted with the illiquidity spread then one *needs to change the accounting and solvency rules* such that they allow for a hold-to-maturity view (with all possible consequences), i.e. we believe that it is not the task of the actuary to introduce fancy arguments for the illiquidity discounting that circumvents the current regulation.

1.4 Current market situation and incentives in the future

A main trigger that has initiated the whole illiquidity premium discussion is the problematic financial state of several life insurance companies. There are several reasons for the financial distress situation of the life insurance market such as the financial crisis, low interest rates, high guarantees and a high market competition. In our opinion many of the sold insurance contracts were simply priced on a too low level. The insurance industry now looks for instantaneous additional income and gains through "smart" accounting in order to smooth this mis-pricing. Through the introduction of an illiquidity premium, in-transparent concepts are applied to the liability side of the balance sheet which are not really well understood. We believe that this should not be done! In the past, mistakes have been made selling life insurance contracts too cheaply. The solution to this problem should not be such as to give incentives for future contracts to be priced too low as well. Introducing illiquidity spread discounting on the liability side of the balance sheet could indeed lead to this undesirable effect. A more professional solution to the current problems would be, for instance, to temporarily lower the security level for the solvency considerations. For further discussions we also refer to Danielsson et al. [1].

2 A simple insurance model for spread analysis

In this section we study a simple model that highlights the issues of the previous section from a more economic and mathematical point of view. The model stresses the more methodological features as detailed practicality. The main findings will be that the introduction of an illiquidity premium on the liability side of the balance sheet changes the underlying consumption stream in a non-market-consistent way, which gives wrong incentives in too low premiums and high risk appetite.

2.1 State price deflator modeling and default-free zero-coupon bonds

We define a financial market model that uses state price deflators for discounting, see Wüthrich et al. [6] for an extended introduction and discussion. We choose a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with finite time horizon $n \geq 2$ and discrete time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,n}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For the time grid we assume a yearly time scale (as for accounting). Assume that $\boldsymbol{\psi} = (\psi_t)_{t=0,\dots,n}$ and $\boldsymbol{\chi} = (\chi_t)_{t=0,\dots,n}$ are two strictly positive and $\mathbb{F}$-adapted stochastic processes. Moreover, we assume that ψ_0=1 and that χ_t is independent of $\sigma\{\mathcal{F}_{t-1}, \boldsymbol{\psi}\}$ with $\mathbb{E}[\chi_t] = 1$ for all t. Then, we define the state price deflator $\boldsymbol{\varphi} = (\varphi_t)_{t=0,\dots,n}$, for $t \geq 0$, by

$$\varphi_t \;=\; \psi_t \prod_{u=0}^{t} \chi_u.$$

This state price deflator $\boldsymbol{\varphi}$ has exactly the properties as defined in Chapter 2 of Wüthrich et al. [6] and it can be used as a stochastic discount function for (random) cash flows. The price of the *default-free zero-coupon bond* with maturity $m = 1, \dots, n$ at time $t \leq m$ is then given by (see (2.47) in Wüthrich et al. [6])

$$P(t,m) \;=\; \frac{1}{\varphi_t}\, \mathbb{E}\left[\varphi_m \middle| \mathcal{F}_t\right].$$

An explicit example is given by the discrete time one-factor Vasicek model in Exercises 2.3 and 2.5 of Wüthrich et al. [6], see also Vasicek [4]. The crucial property in the current model assumptions is the decoupling property of the state price deflator $\boldsymbol{\varphi}$ into two independent components $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$. We can interpret $\boldsymbol{\psi}$ as a "global market state price deflator" and $\boldsymbol{\chi}$ as idiosyncratic distortions, see also Malamud et al. [3]. Independence and $\mathbb{F}$-adaptedness imply

$$P(t,m) \;=\; \frac{1}{\varphi_t}\, \mathbb{E}\left[\varphi_m \middle| \mathcal{F}_t\right] \;=\; \frac{1}{\psi_t}\, \mathbb{E}\left[\psi_m \middle| \mathcal{F}_t\right]. \tag{2.1}$$

Proof of (2.1)**.** Using the $\mathbb{F}$-adaptedness of $\boldsymbol{\chi}$ implies

$$P(t,m) = \frac{1}{\varphi_t}\, \mathbb{E}\left[\varphi_m \middle| \mathcal{F}_t\right] = \frac{1}{\varphi_t}\, \mathbb{E}\left[\psi_m \prod_{u=0}^{m} \chi_u \middle| \mathcal{F}_t\right] = \frac{1}{\psi_t}\, \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m} \chi_u \middle| \mathcal{F}_t\right].$$

To the last term we now apply the tower property for conditional expectations, see Williams [5] p. 88,

$$\begin{aligned}\mathbb{E}\left[\psi_m \prod_{u=t+1}^{m} \chi_u \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\psi_m \prod_{u=t+1}^{m} \chi_u \middle| \mathcal{F}_{m-1}, \boldsymbol{\psi}\right]\middle| \mathcal{F}_t\right] = \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m-1} \chi_u \, \mathbb{E}\left[\chi_m \middle| \mathcal{F}_{m-1}, \boldsymbol{\psi}\right]\middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m-1} \chi_u \, \mathbb{E}\left[\chi_m\right]\middle| \mathcal{F}_t\right] = \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m-1} \chi_u \middle| \mathcal{F}_t\right],\end{aligned}$$

where in the second step we have used the adaptedness, in the third step the independence and in the last step the normalization. Iteration of this argument for $s = t, \dots, m-2$ completes the proof of (2.1).

□

Therefore, the price of the default-free zero-coupon bond is solely determined by the stochastic process $\boldsymbol{\psi}$. The stochastic process $\boldsymbol{\chi}$ will be used for price distortions in defaultable zero-coupon bonds (leading to a possible illiquidity spread).

The continuously-compounded risk-free yield curve at time t for maturity $m > t$ is given by

$$R(t,m) = -\frac{1}{m-t} \, \log P(t,m).$$

This is the risk-free interest rate term structure for the calculation of the best estimate, see also Section 1.2. An explicit example on the basis of the discrete time one-factor Vasicek model is provided in Figure 1.

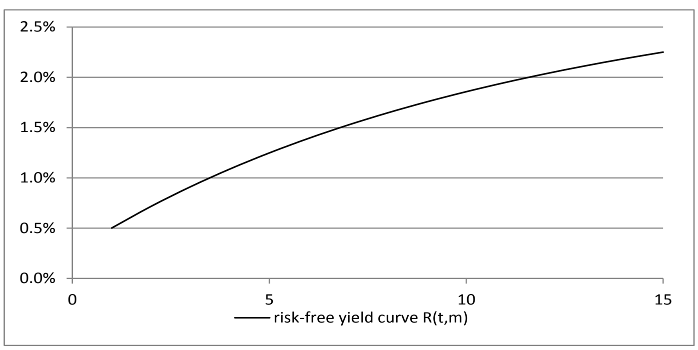

Figure 1: Continuously-compounded risk-free yield curve $R(t,\cdot)$ in the discrete time one-factor Vasicek model, see Exercises 2.3 and 2.5 of Wüthrich et al. [6], for parameters $b = 0.005$, $\beta = 0.8$, $\rho = 0.008$ and $\lambda = 8$ at time $t = 0$ with initial spot rate $r_0 = 0.5\%$. For the explicit parametrization of the discrete time one-factor Vasicek model we refer to Exercise 2.3 in Wüthrich et al. [6].

2.2 Defaultable zero-coupon bonds and the corresponding spreads

We assume that defaultable zero-coupon bonds cannot recover after default and there is also no recovery rate after default. We describe the default process $\boldsymbol{\Gamma} = (\Gamma_t)_{t=0,\dots,n}$ as follows: $\boldsymbol{\Gamma}$ is a non-increasing process with $\Gamma_0 = 1$ and for $t > 0$

$$\Gamma_t = \begin{cases} 0 & \text{if the bond has defaulted in } [0,t], \\ 1 & \text{if the bond has not defaulted in } [0,t]. \end{cases}$$

Assumption. We assume that the process $\mathbf{\Gamma}$ is $\mathbb{F}$-adapted, independent of $\boldsymbol{\psi}$ and for fixed $p \in (0,1)$ and $s \in [0,1)$ we have for all $t > 0$

$$\mathbb{E}\left[\Gamma_t \middle| \mathcal{F}_{t-1}, \boldsymbol{\psi}\right] = (1-p)\ \Gamma_{t-1}, \tag{2.2}$$

$$\mathbb{E}\left[\chi_t\ \Gamma_t \middle| \mathcal{F}_{t-1}, \boldsymbol{\psi}\right] = (1-p)\ (1-s)\ \Gamma_{t-1}. \tag{2.3}$$

Interpretation. From (2.2) we see that we have an annual default probability $p \in (0,1)$. The second property (2.3) adds an illiquidity spread to the pricing of defaultable zero-coupon bonds. Basically, for $s > 0$ it means that χ_t and Γ_t are negatively correlated, conditional on not having defaulted in $[0, t-1]$, which leads to a lower price compared to $s = 0$. We refer $s = 0$ to a liquid bond and $s > 0$ to an illiquid bond, see also Figure 2.

From a real-world probability measure $\mathbb{P}$ perspective, the default profile of $\mathbf{\Gamma}$ does not depend on $s \in [0,1)$, see (2.2).

The price of the defaultable zero-coupon bond with maturity m at time $t \leq m$ is given by, see also proof of (2.1),

$$\begin{aligned} B(t,m) &= \frac{1}{\varphi_t}\ \mathbb{E}\left[\varphi_m\ \Gamma_m \middle| \mathcal{F}_t\right] = \frac{1}{\psi_t}\ \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m} \chi_u\ \Gamma_m \middle| \mathcal{F}_t\right] \\ &= \frac{1}{\psi_t}\ \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m-1} \chi_u\ \mathbb{E}\left[\chi_m\ \Gamma_m \middle| \mathcal{F}_{m-1}, \boldsymbol{\psi}\right] \middle| \mathcal{F}_t\right] \\ &= (1-p)\ (1-s)\ \frac{1}{\psi_t}\ \mathbb{E}\left[\psi_m \prod_{u=t+1}^{m-1} \chi_u\ \Gamma_{m-1} \middle| \mathcal{F}_t\right] \\ &= \ldots = P(t,m)\ (1-p)^{m-t}\ (1-s)^{m-t}\ \Gamma_t. \end{aligned} \tag{2.4}$$

Pricing formula (2.4) is the basic relation we will need in Section 2.3. We will also need an analogous pricing formula for the case that the bond has survived up to time k, i.e. conditional on the event $\{\Gamma_k = 1\}$. Then we have for $k \leq t \leq m$ the price formula

$$B^{(k)}(t,m) = P(t,m)\ (1-p)^{m-t}\ (1-s)^{m-t}\ \Gamma_t^{(k)}, \tag{2.5}$$

where the probability law of $\Gamma_t^{(k)}$ corresponds to the conditional probability $\mathbb{P}(\Gamma_t \in \ \cdot\ | \Gamma_k = 1)$.

On the non-default set $\{\Gamma_t = 1\}$, for $m > t$, the continuously-compounded yield curve of the defaultable zero coupon is then given by, see (2.4),

$$Y(t,m) = -\frac{1}{m-t} \log B(t,m) = R(t,m) - \log(1-p) - \log(1-s) \; > \; R(t,m).$$

Figure 2 provides an example within the discrete time one-factor Vasicek deflator model.

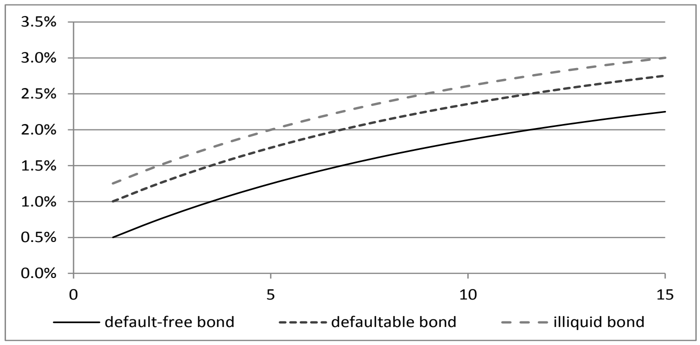

Figure 2: Continuously-compounded yield curves within the discrete time one-factor Vasicek model, see also Figure 1. The three curves correspond to the default-free yield curve $R(0,\cdot)$ ($p = s = 0$), defaultable bond yield curve $Y(0,\cdot)$ (only having credit spread $p > 0$ and $s = 0$) and an illiquid bond yield curve $Y(0,\cdot)$ (having credit and illiquidity spread $p, s > 0$).

2.3 Market-consistent solvency analysis

We now study a full balance sheet for an insurance company. We define a liability and an asset portfolio and then value these in a market-consistent way.

2.3.1 Insurance liability cash flows

For the insurance liability cash flows we choose a very simple model. Assume that the insurance liabilities are given by a default-free cash flow of size $M = 1$ at time $m = 2$. This means that, for simplicity, the insurance technical risk is deterministic (strictly speaking, this is not an insurance contract). Thus, we consider an example with completely predictable insurance liability cash flows, and thus, this example will also show that the predictability of insurance liability cash flows is not really the issue in an illiquidity premium discounting analysis!

The time series of the market-consistent value of the liability side of the balance sheet is given by

$$L_0 = P(0,2), \qquad L_1 = P(1,2), \qquad L_2 = 1 \qquad \text{and} \qquad L_t = 0 \;\; \text{for } t \geq 3.$$

A (single pure risk) premium of $\pi = L_0 = P(0,2)$ is charged to the policyholder at time $t = 0$.

2.3.2 Asset side of the balance sheet and consumption stream

We put ourselves in the situation of the insurer who chooses the asset side of the balance sheet with defaultable zero-coupon bonds that match the maturities of the insurance liabilities. These are chosen such that the time series on the asset side of the balance sheet always covers the market-consistent value of the insurance liabilities. Since we only have defaultable zero-coupon bonds for the replication (unavoidable market risk) we obtain a consumption stream $\mathbf{C} = (C_0, C_1, C_2)$ that balances losses and gains. One may debate the terminology consumption stream that we have borrowed from economic literature. However, we believe that it nicely illustrates the solvency picture.

Time $t = 0$. We choose the asset side of the balance sheet at time $t = 0$ to have value (matching maturities)

$$A_0 = (1-p)^{-2}\ (1-s)^{-2}\ B(0,2), \qquad \text{i.e. this provides}\ \ A_0 = L_0 = P(0,2).$$

Since the policyholder exactly pays the premium $\pi = P(0,2)$, we get an initial consumption $C_0 = 0$.

Time $t = 1$. The asset portfolio with initial value A_0 generates the value $A_1^- = (1-p)^{-2}\ (1-s)^{-2}\ B(1,2)$ at time $t = 1$. This value needs to be compared to the value of the liabilities $L_1 = P(1,2)$ at time $t = 1$. In order that these liabilities are exactly covered at time 1 the insurance company obtains a consumption

$$C_1 = A_1^- - L_1 = \left[(1-p)^{-1}\ (1-s)^{-1}\ \Gamma_1 - 1\right]\ P(1,2).$$

Note that this consumption can be positive or negative depending on the fact whether the bond defaults, $\Gamma_1 = 0$, or whether it does not default, $\Gamma_1 = 1$. In the case of default, the insurance company needs to inject capital L_1. We assume this is done through new (defaultable) bonds that have not defaulted yet. This provides, after consumption, the value of the asset portfolio at time $t = 1$

$$A_1 = (1-p)^{-1}\ (1-s)^{-1}\ B^{(1)}(1,2), \qquad \text{i.e. this provides}\ \ A_1 = L_1 = P(1,2),$$

where $B^{(1)}(1,2)$ is the price of a defaultable zero-coupon, conditional on the event that it has not defaulted in the first period, see (2.5).

Time $t = 2$. This asset portfolio chosen at time 1 generates value $A_2^- = (1-p)^{-1}\,(1-s)^{-1}\,B^{(1)}(2,2)$ at time $t=2$. This value needs to be compared to the liabilities $L_2 = P(2,2) = 1$ at time $t=2$. In order that these liabilities are covered at time 2 the insurance company obtains a final consumption

$$C_2 = A_2^- - L_2 = \left[(1-p)^{-1}\,(1-s)^{-1}\,\Gamma_2^{(1)} - 1\right].$$

Note that this consumption can again have both signs depending on the fact whether the bond defaults, $\Gamma_2^{(1)} = 0$, or whether it does not default, $\Gamma_2^{(1)} = 1$, in the second period.

2.3.3 Solvency under unlimited liability

We conclude that the insurance company faces the following consumption stream

$$\mathbf{C} = \Big(\,0\,,\;\left[(1-p)^{-1}(1-s)^{-1}\Gamma_1 - 1\right]P(1,2)\,,\;\left[(1-p)^{-1}(1-s)^{-1}\Gamma_2^{(1)} - 1\right]\,\Big).$$

If the insurance company completely meets this consumption stream $\mathbf{C}$ (i.e. cannot execute a limited liability option) then solvency is guaranteed with probability 1. In general, solvency is not guaranteed with probability 1, i.e. the insurance company may execute the limited liability option if a C_t is too negative for some $t = 0,1,2$ (for example measured by a value-at-risk risk measure). Note that traditional solvency definitions slightly differ from this view because here we interpret solvency capital as an unlimited liability option. For the purpose of this paper this difference is not important, but allows for a more clear description of the problem. In the present example we assume for simplicity that there is no limited liability option, i.e. the consumption stream $\mathbf{C}$ is completely met with probability 1. We can then easily calculate the price at time $t=0$ of this consumption stream $\mathbf{C}$ (see Theorem 2.5 in Wüthrich et al. [6]). As a consequence of no-arbitrage, the price $Q_0[\mathbf{C}]$ of the consumption stream $\mathbf{C}$ at time 0 is given by, see Section 2.3 in Wüthrich et al. [6],

$$Q_0[\mathbf{C}] \;=\; \sum_{u=0}^{2} \mathbb{E}\left[\varphi_u\, C_u\right] \;=\; 0.$$

The consequence of this no-arbitrage pricing is that the asset-liability pair (A_0, L_0) can be considered as a "fair" deal.

We can also study the expected gains generated by the consumption stream $\mathbf{C}$, they are

$$\mathbb{E}[C_0] = C_0 = 0, \qquad \mathbb{E}[C_1] = \frac{s}{1-s}\,\mathbb{E}[P(1,2)] \;\geq\; 0 \qquad \text{and} \qquad \mathbb{E}[C_2] = \frac{s}{1-s} \;\geq\; 0. \tag{2.6}$$

This shows that investing in illiquid bonds is, on average, rewarded by a positive consumption (but of course there is also a downside risk which results in the no-arbitrage price $Q_0[\mathbf{C}] = 0$).

2.4 Illiquidity premium

In this subsection we demonstrate what happens if we also allow for discounting with an illiquidity spread on the liability side of the balance sheet.

2.4.1 Liability side of the balance sheet

Choose $s \in (0,1)$, i.e. $s > 0$. The time series of the spread discounted liabilities is given by

$$\widetilde{L}_0 = (1-s)^2 \; P(0,2), \quad \widetilde{L}_1 = (1-s) \; P(1,2), \quad \widetilde{L}_2 = L_2 = 1 \quad \text{and} \quad \widetilde{L}_t = 0 \;\; \text{for } t \geq 3.$$

As in the previous example, we assume that we still charge the (single pure risk) premium $\pi = L_0 = P(0,2) > \widetilde{L}_0$ to the policyholder at time $t = 0$.

2.4.2 Asset side of the balance sheet and consumption stream

Next, we consider the consumption stream that balances losses and gains in this new situation.

Time $t = 0$. We choose the asset side of the balance sheet at time $t = 0$ to be

$$\widetilde{A}_0 = (1-p)^{-2} \; B(0,2), \qquad \text{i.e. this provides } \; \widetilde{A}_0 = \widetilde{L}_0 = (1-s)^2 \; P(0,2).$$

Moreover, since the policyholder pays premium $\pi = P(0,2)$, we get a first (positive) consumption

$$\widetilde{C}_0 = \pi - \widetilde{A}_0 = (2s - s^2) \; P(0,2) \; > \; 0.$$

Time $t = 1$. This asset portfolio with initial value $\widetilde{A}_0$ generates value $\widetilde{A}_1^- = (1-p)^{-2} \; B(1,2)$ at time $t = 1$ that needs to be compared to the value of the liabilities $\widetilde{L}_1 = (1-s) \; P(1,2)$ at time $t = 1$. The second consumption is then given by

$$\widetilde{C}_1 = \widetilde{A}_1^- - \widetilde{L}_1 = \left[(1-p)^{-1} \; \Gamma_1 - 1\right] \; (1-s) \; P(1,2).$$

After this consumption, the asset portfolio at time $t = 1$ is given by

$$\widetilde{A}_1 = (1-p)^{-1} \; B^{(1)}(1,2), \qquad \text{i.e. this provides } \; \widetilde{A}_1 = \widetilde{L}_1 = (1-s) \; P(1,2),$$

under the side constraint that we have a bond that has not defaulted at time 1, i.e. conditional on $\{\Gamma_1 = 1\}$.

Time $t = 2$. This asset portfolio $\widetilde{A}_1$ generates value $\widetilde{A}_2^- = (1-p)^{-1}\ B^{(1)}(2,2)$ and provides a final consumption at time $t = 2$ given by

$$\widetilde{C}_2 = \widetilde{A}_2^- - \widetilde{L}_2 = \left[(1-p)^{-1}\ \Gamma_2^{(1)} - 1\right].$$

2.4.3 Analysis of the new consumption stream

In this situation we have a new consumption stream

$$\widetilde{\mathbf{C}} = \Big(\ (2s - s^2)P(0,2)\ ,\ \left[(1-p)^{-1}\Gamma_1 - 1\right](1-s)P(1,2)\ ,\ \left[(1-p)^{-1}\Gamma_2^{(1)} - 1\right]\ \Big).$$

Under the unlimited liability assumption we again calculate its no-arbitrage price at time $t = 0$:

$$Q_0[\widetilde{\mathbf{C}}] \ = \ \sum_{u=0}^{2} \mathbb{E}\left[\varphi_u\ \widetilde{C}_u\right] \ = \ 0.$$

The expected consumptions under the real-world probability measure $\mathbb{P}$ are given by, $s > 0$,

$$\mathbb{E}[\widetilde{C}_0] = \widetilde{C}_0 = (2s - s^2)P(0,2) \ > \ 0 \qquad \text{and} \qquad \mathbb{E}[\widetilde{C}_1] = \mathbb{E}[\widetilde{C}_2] = 0. \tag{2.7}$$

This shows that if we allow for an illiquidity premium discounting on the liability side of the balance sheet all the expected (illiquidity) gains from periods $t = 1, 2$ (see (2.6)) are shifted to the first period $t = 0$, see (2.7). This implies that if we consume these expected gains already in the first period, there is no reward for doing the risk bearing of the run-off of the liabilities. Indeed, what happens in this situation is that the insurance company consumes $\widetilde{C}_0 = (2s - s^2)P(0,2) > 0$ at the beginning and after this consumption it remains with the asset-liability pair $(\widetilde{A}_0, \widetilde{L}_0)$. This asset-liability pair generates a consumption stream $(0, \widetilde{C}_1, \widetilde{C}_2)$ with market-consistent value at time 0 given by, $s > 0$,

$$Q_0[(0, \widetilde{C}_1, \widetilde{C}_2)] \ = \ -(2s - s^2)P(0,2) \ < \ 0. \tag{2.8}$$

As a consequence of the over-consumption of capital in the first period, the insurance company lacks capital to allow for a market-consistent transfer of its business at a later period; hence *violating the core idea of market-consistent valuation and solvency*!

Note that a similar picture is obtained if it charges a too low premium $\pi < P(0,2)$. Mis-pricing cannot be corrected by an aggressive asset strategy and smart accounting.

2.5 Conclusions

The insurance companies should NOT integrate an illiquidity premium on the *liability side* of the balance sheet. An illiquidity premium is in-consistent with market-consistent actuarial valuation. It shifts possible future gains to the starting point. Therefore, these possible gains are not consumed when they are due but rather in the beginning, and the company is left with a negative market value, see (2.8). In particular, this means that there is no risk margin left for the orderly run-off of the insurance liabilities, which contradicts any regulatory effort. Moreover, it gives wrong incentives for an over-aggressive consumption, a high risk appetite and too low insurance premiums. Putting arguments together, it undermines adequate insurance prices and a competitive well-functioning insurance market with a long-term perspective.

Acknowledgment. I would like the express my thankfulness to Prof. H. Bühlmann, Prof. P. Embrechts, Dr. H. Furrer, Prof. A. Gisler, Dr. P. Keller and Prof. M. Koller for fruitful discussions on this subject and various comments on earlier versions of this paper.

References

[1] Danielsson, J., de Jong, F., Laeven, R., Laux, C., Perotti, E., Wüthrich, M.V. (2011). A prudential regulatory issue at the heart of Solvency II. VoxEU, 31 March 2011. Available under `http://www.voxeu.org/index.php?q=node/6305`

[2] Keller, P. (2010). Some comments on the illiquidity premium. Swiss Association of Actuaries. Available under `www.actuaries.ch`

[3] Malamud, S., Trubowitz, E., Wüthrich, M.V. (2008). Market consistent pricing of insurance products. ASTIN Bulletin 38/2, 483-526.

[4] Vasicek, O. (1977). An equilibrium characterization of the term structure. J. Financial Economics 5, 177-188.

[5] Williams, D. (1991). Probability with Martingales. Cambridge Mathematical Textbooks.

[6] Wüthrich, M.V., Bühlmann, H., Furrer, H. (2010). Market-Consistent Actuarial Valuation. 2nd edition. Springer.