From Ruin Theory to Solvency in Non-Life Insurance

Mario V. Wüthrich
RiskLab ETH Zurich & Swiss Finance Institute SFI

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Aim of this presentation

We start from Lundberg’s thesis (1903) on ruin theory and modify his model step by step until we arrive at today’s solvency considerations.
Consider the **surplus process** \((C_t)_{t \geq 0}\) given by

\[
C_t = c_0 + \pi t - \sum_{i=1}^{N_t} Y_i,
\]

where

- \(c_0 \geq 0\) initial capital,
- \(\pi > 0\) premium rate,
- \(L_t = \sum_{i=1}^{N_t} Y_i \geq 0\) homogeneous compound Poisson claims process,

satisfying the **net profit condition (NPC):** \(\pi > \mathbb{E}[L_1]\).
Ultimate ruin probability

The ultimate ruin probability for initial capital $c_0 \geq 0$ is given by

$$
\psi(c_0) = \mathbb{P} \left[ \inf_{t \in \mathbb{R}^+} C_t < 0 \mid C_0 = c_0 \right] = \mathbb{P}_{c_0} \left[ \inf_{t \in \mathbb{R}^+} C_t < 0 \right],
$$

i.e. this is the infinite time horizon ruin probability.

Under (NPC):
$$
\psi(c_0) < 1 \text{ for all } c_0 \geq 0.
$$
Assume (NPC) and that the **Lundberg coefficient** $\gamma > 0$ exists. Then, we have exponential bound

$$\psi(c_0) \leq \exp\{-\gamma c_0\},$$

for all $c_0 \geq 0$ (large deviation principle (LDP)).

This is the **light-tailed case**, i.e. for the existence of $\gamma > 0$ we need *exponentially decaying* survival probabilities of the claim sizes $Y_i$,

because we require $\mathbb{E}[\exp\{\gamma Y_i\}] < \infty$. 

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*Filip Lundberg*
Subexponential case

Von Bahr, Veraverbeke, Embrechts investigate the heavy-tailed case. In particular, for $Y_i \sim \text{i.i.d. Pareto}(\alpha > 1)$ and (NPC):

$$\psi(c_0) \sim \text{const } c_0^{-\alpha+1} \text{ as } c_0 \to \infty.$$ 

Heavy-tailed case provides a much slower decay.
Discrete time ruin considerations

Insurance companies cannot continuously control their surplus processes \((C_t)_{t \geq 0}\).

They close their books and check their surplus on a yearly time grid.

Consider the discrete time ruin probability

\[
P_{c_0} \left[ \inf_{n \in \mathbb{N}_0} C_n < 0 \right] \leq P_{c_0} \left[ \inf_{t \in \mathbb{R}_+} C_t < 0 \right] = \psi(c_0).
\]

This leads to the study of the random walk \((C_n - c_0)_{n \in \mathbb{N}_0}\) for (discrete time) accounting years \(n \in \mathbb{N}_0\).
One-period ruin problem

Insured buy *one-year* non-life insurance contracts: why bother about *ultimate* ruin probabilities?

Moreover, initial capital $c_0 \geq 0$ needs to be re-adjusted every accounting year.

Consider the (discrete time) *one-year ruin probability*

\[
\mathbb{P}_{c_0} [C_1 < 0] \leq \mathbb{P}_{c_0} \left[ \inf_{n \in \mathbb{N}_0} C_n < 0 \right] \leq \mathbb{P}_{c_0} \left[ \inf_{t \in \mathbb{R}_+} C_t < 0 \right] = \psi(c_0).
\]

This leads to the study of the surplus $C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i$ at time $1$. 
One-period problem and real world considerations

Why do we study so complex models when the real world problem is so simple?

- Total asset value at time 1: \( A_1 = c_0 + \pi \).

- Total liabilities at time 1: \( L_1 = \sum_{i=1}^{N_1} Y_i \).

\[
C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i = A_1 - L_1 \geq 0. \tag{1}
\]

There are many modeling issues hidden in (1)! We discuss them step by step.
Value-at-Risk (VaR) risk measure

\[ C_1 = A_1 - L_1 \geq 0. \]

- Value-at-Risk on confidence level \( p = 99.5\% \) (Solvency II): choose \( c_0 \) minimal such that

\[ \mathbb{P}_{c_0} [C_1 \geq 0] = \mathbb{P} [A_1 \geq L_1] = \mathbb{P} [L_1 - c_0 - \pi \leq 0] \geq p. \]

- Choose other (normalized) risk measures \( \varrho : \mathcal{M} \subset L^1(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R} \) and study

\[ \varrho(L_1 - A_1) = \varrho(L_1 - c_0 - \pi) \leq 0, \]

where “\( \leq \)” implies **Solvency** w.r.t. risk measure \( \varrho \).
Asset return and financial risk (1/2)

- Initial capital at time 0: \( c_0 \geq 0 \).

- Premium received at time 0 for accounting year 1: \( \pi > 0 \).

Total asset value at time 0: \( \ a_0 = c_0 + \pi > 0 \).

This asset value \( a_0 \) is invested in different assets \( k \in \{1, \ldots, K\} \) at time 0.

**asset classes**

- cash and cash equivalents
- debt securities (bonds, loans, mortgages)
- real estate & property
- equity, private equity
- derivatives & hedge funds
- insurance & reinsurance assets
- other assets
Asset return and financial risk (2/2)

Choose an asset portfolio \( \mathbf{x} = (x_1, \ldots, x_K)' \in \mathbb{R}^K \) at time 0 with initial value

\[
a_0 = \sum_{k=1}^{K} x_k S_0^{(k)},
\]

where \( S_t^{(k)} \) is the price of asset \( k \) at time \( t \). This provides value at time 1

\[
A_1 = \sum_{k=1}^{K} x_k S_1^{(k)} = a_0 (1 + \mathbf{w}' \mathbf{R}_1),
\]

for buy & hold asset strategy \( \mathbf{w} = \mathbf{w}(\mathbf{x}) \in \mathbb{R}^K \) and (random) return vector \( \mathbf{R}_1 \).

\[
\varrho (L_1 - A_1) = \varrho (L_1 - a_0 (1 + \mathbf{w}' \mathbf{R}_1)) \leq 0.
\]

where “\( \leq \)” implies solvency w.r.t. risk measure \( \varrho \) and business plan \((L_1, a_0, \mathbf{w})\).
Insurance claim (liability) modeling (1/2)

**MAIN ISSUE:** modeling of insurance claim \( L_1 = \sum_{i=1}^{N_1} Y_i \).

− Insurance claims are neither known nor can immediately be settled at occurrence!

Diagram:

− Insurance claims of accounting year 1 generate insurance liability cash flow \( X \):

\[
X = (X_1, X_2, \ldots) \text{ with } X_t \text{ being the payment in accounting year } t.
\]

**Question:** How is the cash flow \( X \) related to the insurance claim \( L_1 \)?
Main tasks:

- cash flow $X = (X_1, X_2, \ldots)$ modeling,

- cash flow $X = (X_1, X_2, \ldots)$ prediction,

- cash flow $X = (X_1, X_2, \ldots)$ valuation,

using all available relevant information:

- exactly here the one-period problem turns into a multi-period problem.
Best-estimate reserves

Choose a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) with filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}\) and assume cash flow \(X\) is \(\mathbb{F}\)-adapted.

**1st attempt** to define \(L_1\) (interpretation of Solvency II):

\[
L_1 = X_1 + \sum_{s \geq 2} P(1, s) \mathbb{E}[X_s | \mathcal{F}_1],
\]

where

- \(\mathbb{E}[X_s | \mathcal{F}_1]\) is the best-estimate reserve (prediction) of \(X_s\) at time 1;

- \(P(1, s)\) is the zero-coupon bond price at time 1 for maturity date \(s\).

Note that \(L_1\) is \(\mathcal{F}_1\)-measurable, i.e. observable w.r.t. \(\mathcal{F}_1\) (information at time 1).
1st attempt to define $L_1$

$$L_1 = X_1 + \sum_{s\geq 2} P(1, s) \mathbb{E} [X_s | \mathcal{F}_1].$$  \hspace{1cm} (2)

**Issue:** Solvency II asks for economic balance sheet, but $L_1$ is *not* an economic value.

(a) Risk margin is missing: any risk-averse risk bearer asks for such a (profit) margin.

(b) Zero-coupon bond prices and claims cash flows $X_s, s \geq 2$, may be influenced by the same risk factors and, thus, *there is no decoupling* such as (2).
2nd attempt to define $L_1$

Choose an appropriate state-price deflator $\varphi = (\varphi_t)_{t \geq 1}$ and

$$L_1 = X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E}[\varphi_s X_s | \mathcal{F}_1].$$

- $\varphi = (\varphi_t)_{t \geq 1}$ is a strictly positive, a.s., and $\mathbb{F}$-adapted.

- $\varphi = (\varphi_t)_{t \geq 1}$ reflects price formation at financial markets, in particular,

$$P(1, s) = \frac{1}{\varphi_1} \mathbb{E}[\varphi_s | \mathcal{F}_1].$$

- If $\varphi_s$ and $X_s$ are positively correlated, given $\mathcal{F}_1$, then

$$L_1 \geq X_1 + \sum_{s \geq 2} P(1, s) \mathbb{E}[X_s | \mathcal{F}_1].$$
Solvency at time 0

▷ Choose a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) such that it carries the random vectors \(\varphi\) (state-price deflator), \(R_1\) (returns of assets) and \(X\) (insurance liability cash flows) in a reasonable way.

▷ The business plan \((X, a_0, w)\) is solvent w.r.t. the risk measure \(\varrho\) and state-price deflator \(\varphi\) if

\[
\varrho(L_1 - A_1) = \varrho \left( X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E}[\varphi_s X_s | \mathcal{F}_1] - a_0 (1 + w' R_1) \right) \leq 0.
\]

Thus, it is likely (measured by \(\varrho\) and \(\varphi\)) that the liabilities \(L_1\) are covered by assets \(A_1\) at time 1 in an economic balance sheet.
Acceptability arbitrage

• The choice of the state-price deflator $\varphi$ and the risk measure $\varrho$ cannot be done independently of each other:
  - $\varphi$ describes the risk reward;
  - $\varrho$ describes the risk punishment.

• Assume there exist acceptable zero-cost portfolios $Y$ with

$$\mathbb{E}[\varphi'Y] = 0$$

and

$$\varrho \left( Y_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E}[\varphi_s Y_s | \mathcal{F}_1] \right) < 0.$$

Then, unacceptable positions can be turned into acceptable ones just by loading on more risk $\implies$ acceptability arbitrage.

• Reasonable solvency models $(\varphi, \varrho)$ should exclude acceptability arbitrage, see Artzner, Delbaen, Eisele, Koch-Medina.
Asset & liability management (ALM)

The business plan \((X, a_0, w)\) is solvent w.r.t. risk measure \(\varrho\) and state-price deflator \(\varphi\) if

\[
\varrho(L_1 - A_1) = \varrho \left( X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E}[\varphi_s X_s | \mathcal{F}_1] - a_0 (1 + w' R_1) \right) \leq 0.
\]

ALM optimize this business plan \((X, a_0, w)\):
Which asset strategy \(w \in \mathbb{R}^K\) minimizes the capital \(a_0 = c_0 + \pi\) and we still remain solvent?

▷ This is a non-trivial optimization problem.

▷ Of course, we need to exclude acceptability arbitrage, which may also provide restrictions on the possible asset strategies \(w \implies\) eligible assets.
Summary of modeling tasks

- Provide reasonable stochastic models for $R_1$, $X$ and $\varphi$ (yield curve extrapolation).

- What is a reasonable profit margin for risk bearing expressed by $\varphi$?

- Which risk measure(s) $\varrho$ should be preferred? ($\Rightarrow$ No-acceptability arbitrage!)

- Modeling is often split into different risk modules:
  - (financial) market risk
  - insurance risk (underwriting and reserve risks)
  - credit risk
  - operational risk

  ▶ Issue: dependence modeling and aggregation of risk modules.

- Aggregation over different accounting years and lines of business?
Dynamic considerations

Are we happy with the above considerations?

▶ Not entirely!

Liability run-off is a multi-period problem:

We also want sensible dynamic behavior.

This leads to the consideration of multi-period problems and super-martingales.