Chain-ladder method: dynamic run-off uncertainty analysis

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joint work with
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Outline

• Chain-ladder method

• Claims development result

• Examples
Chain-ladder algorithm

<table>
<thead>
<tr>
<th>accident year $i$</th>
<th>0</th>
<th>1</th>
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- $C_{i,j}$ = cumulative claim of accident year $i$ and development year $j$.

- $\mathcal{D}_t = \{C_{i,j}; \ i + j \leq t\}$ = observations at time $t$.

- Chain-ladder (CL) algorithm is based on the (regression) assumption

  $$C_{i,j+1} \approx f_j C_{i,j},$$

  for CL factors $f_j$ not depending on accident year $i$.  

$Ci,j$ to be predicted
Stochastic models underlying the CL algorithm

- CL algorithm is not based on a stochastic model (deterministic algorithm).

- We need a stochastic representation to quantify prediction uncertainty.

- Stochastic models introduced providing the CL reserves:
  - Mack’s distribution-free CL model (1993)
  - Poisson and over-dispersed Poisson (ODP) model of Renshaw-Verrall (1998)
  - Bayesian CL models by Gisler (2006), Bühlmann et al. (2009)
  - Gamma-gamma Bayesian CL model by Merz-Wüthrich (2008, 2014)
Bayesian chain-ladder (BCL) model

Model assumptions (gamma-gamma BCL model).
Assume there are fixed given variance parameters $\sigma^2_0, \ldots, \sigma^2_{J-1}$.

- Conditionally, given CL parameters $F = (F_0, \ldots, F_{J-1})$:
  - $\star (C_{i,j})_{j=0,\ldots,J}$ independent (in $i$) and Markovian (in $j$) with gamma innovations
  - with for all $1 \leq i \leq I$ and $0 \leq j \leq J - 1$

\[
\begin{align*}
\mathbb{E} [C_{i,j+1} | C_{i,j}, F] &= F_j C_{i,j}, \\
\text{Var} (C_{i,j+1} | C_{i,j}, F) &= \sigma_j^2 F_j^2 C_{i,j}.
\end{align*}
\]

- The components of $F^{-1}$ are independent and gamma distributed.

This model has the CL property: $C_{i,j+1} \approx F_j C_{i,j}$, for given CL factors $F_j$. 

$\square$
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▷ This model has the CL property: $C_{i,j+1} \approx F_j C_{i,j}$, for given CL factors $F_j$.  

\[\Box\]
BCL predictor

- Predictors can be calculated explicitly in the above model for observations $D_t$.

- BCL predictor at time $t \geq I > J$ for non-informative priors

$$\hat{C}_{i,J}^{(t)} = \mathbb{E}[C_{i,J}|D_t] = C_{i,t-i} \prod_{j=t-i}^{J-1} \hat{f}_j^{(t)} ,$$

with CL factor estimators

$$\hat{f}_j^{(t)} = \frac{\sum_{i=1}^{t-j-1} C_{i,j+1}}{\sum_{i=1}^{t-j-1} C_{i,j}} .$$
## CL claims prediction

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$\hat{f}_{ij}^{(t)}$ | 1.2343 | 1.2904 | 1.1918 | 1.1635 | 1.1457 | 1.1013 | 1.0702 | 1.0760 | 1.0444 |

**What about prediction uncertainty?**

**Consider the conditional mean square error of prediction (MSEP)**

$$
\text{msep}_{C_{i,j}|\mathcal{D}_t} \left( \hat{C}_{i,j}^{(t)} \right) = \mathbb{E} \left[ \left( C_{i,j} - \hat{C}_{i,j}^{(t)} \right)^2 \mid \mathcal{D}_t \right].
$$
Conditional MSEP formula

- Conditional MSEP can be calculated explicitly and exactly in the above model.

Conditional MSEP for non-informative priors for single accident years $i$:

$$
\text{msep}_{C_i,J|D_t} \left( \hat{C}_{i,J}^{(t)} \right) = \left( \hat{C}_{i,J}^{(t)} \right)^2 \left( \sum_{j=t-i}^{J-1} \left[ \frac{\sigma_j^2}{\hat{C}_{i,j}^{(t)}} + \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right] + o \left( \frac{\sigma_\ell^2}{C_{k,\ell}} \right) \right).
$$

- This is identical to the famous Mack formula (1993) up to:
  * a different variance parametrization, and
  * and a correction term of order $o \left( \sigma_\ell^2 / C_{k,\ell} \right)$.

- Aggregation over accident years $i$ is similar.
Outline

• Chain-ladder method
• Claims development result
• Examples
Claims development result (1/2)

- Conditional MSEP formula above considers the total prediction uncertainty over the entire run-off (static view).

- Solvency considerations require a dynamic view: possible changes in predictions over the next accounting year(s).

\[ \text{end point of path (static view)} \iff \text{whole path behavior (dynamic view)} \]

- Define the claims development result of accounting year \( t + 1 > I \) by

\[
\text{CDR}_i(t + 1) = \hat{C}_{i,J}^{(t+1)} - \hat{C}_{i,J}^{(t)}.
\]
Claims development result (2/2)

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<th>accident year</th>
<th>development year</th>
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▷ Martingale property of \((\hat{C}_{i,J}^{(t)})_{t\geq I}\) implies

\[
\mathbb{E} \left[ \text{CDR}_{i}(t + 1) \mid \mathcal{D}_t \right] = \mathbb{E} \left[ \hat{C}_{i,J}^{(t+1)} - \hat{C}_{i,J}^{(t)} \bigg| \mathcal{D}_t \right] = 0.
\]

▷ Solvency: study the one-year uncertainty

\[
msep_{\text{CDR}_{i}(t+1)|\mathcal{D}_t}(0) = \mathbb{E} \left[ (\text{CDR}_{i}(t + 1) - 0)^2 \bigg| \mathcal{D}_t \right].
\]
Conditional MSEP can be calculated explicitly and exactly in the above model.

Conditional MSEP for non-informative priors for single accident years $i$:

$$\text{mse}_{\text{CDR}_i(t+1)|D_t(0)} = \left( \hat{C}_{i,J}(t) \right)^2$$

$$\times \left( \left[ \frac{\sigma_{t-i}^2}{C_{i,t-i}} + \frac{\sigma_{t-i}^2}{\sum_{\ell=1}^{i-1} C_{\ell,t-i}} + \sum_{j=t-i+1}^{J-1} \alpha_j^{(t)} \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right] + o \left( \frac{\sigma_{\ell}^2}{C_{k,\ell}} \right) \right),$$

with (credibility) weight

$$\alpha_j^{(t)} = \frac{C_{t-j,j}}{\sum_{\ell=1}^{t-j} C_{\ell,j}} \in (0, 1].$$

This is identical to Merz-Wüthrich formula (2008) up to the differences mentioned above.
Total uncertainty vs. one-year uncertainty

Total uncertainty:

$$\text{mse}_{pC_{i,J}|D_t} \left( \hat{C}_{i,J}^{(t)} \right) \approx \left( \hat{C}_{i,J}^{(t)} \right)^2 \sum_{j=t-i}^{J-1} \left[ \frac{\sigma_j^2}{\hat{C}_{i,j}^{(t)}} + \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right].$$

One-year uncertainty:

$$\text{mse}_{CDR_i(t+1)|D_t} (0) \approx \left( \hat{C}_{i,J}^{(t)} \right)^2 \times \left[ \frac{\sigma_{t-i}^2}{\hat{C}_{i,t-i}^{(t)}} + \frac{\sigma_{t-i}^2}{\sum_{\ell=1}^{i-1} C_{\ell,t-i}} + \sum_{j=t-i+1}^{J-1} \alpha_j^{(t)} \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right].$$

Process uncertainty, parameter estimation uncertainty and its reduction in time.
Residual uncertainty for remaining accounting years

This suggests for accounting year \( t + 2 \):

\[
\mathbb{E} \left[ \text{msep}_{\text{CDR}_i(t+2)} \big| \mathcal{D}_{t+1}(0) \big| \mathcal{D}_t \right] \\
\approx \left( \widehat{C}_{i,J}^{(t)} \right)^2 \left[ \frac{\sigma^2_{t-i+1}}{\widehat{C}_{i,t-i+1}^{(t)}} + \left(1 - \alpha_{t-i+1}^{(t)}\right) \frac{\sigma^2_{t-i+1}}{\sum_{\ell=1}^{i-2} C_{\ell,t-i+1}} \right] \\
+ \left( \widehat{C}_{i,J}^{(t)} \right)^2 \sum_{j=t-i+2}^{J-1} \left[ \alpha_{j-1}^{(t)} \left(1 - \alpha_{j}^{(t)}\right) \frac{\sigma^2_{j}}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right].
\]

This can be derived analytically and iterated!

It allocates the total MSEP formula across different accounting periods, i.e., this provides a run-off of risk pattern.

This was shown in Röhr (2013), Merz-Wüthrich (2014), Diers et al. (2016) and Gisler (2016).
Outline

- Chain-ladder method
- Claims development result
- Examples
Expected run-off of claims reserves is faster than the one of underlying risks.

Legal environment is important for run-off.
> Different lines of business behave differently (short- and long-tailed business).
Conclusions and implementation

- The one-year uncertainty formula was generalized to arbitrary accounting years.
- This allocates the total uncertainty formula across accounting years.
- This improves risk margin calculations under Solvency II.
- Standard approximation techniques typically under-estimate run-off risk.

- CRAN R package: ChainLadder