Measure Theory
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Supplementary literature:

H. Bauer, *Mass- und Integrationstheorie*, de Gruyter, 1990.

Contents

Chapter 1

Measures and measurable mappings

Notation

Let Ω be a set, $A \subseteq \Omega$.

2^Ω power set of Ω, set of all subsets of Ω.

A^c complement of A.

$\overline{A}$ closure of A.

A° interior of A.

$\#A$ number of elements of A.

The terms function and mapping are used as synonyms. System, family and set all refer to sets.

1.1 σ-fields, rings and Dynkin systems

Definition 1.1. Let Ω be a set. A family $\mathfrak{F}$ of subsets of Ω is called *σ-field* (on Ω) if

(σ1) $\Omega \in \mathfrak{F}$,

(σ2) $A \in \mathfrak{F} \quad \Longrightarrow \quad A^c \in \mathfrak{F}$,

(σ3) $A_n \in \mathfrak{F}$ for all $n \in \mathbb{N} \quad \Longrightarrow \quad \bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}$.

Example 1.2.
1. The power set 2^Ω of Ω is a σ-field.
2. Let Ω' be a set and $\mathfrak{F}'$ a σ-field on Ω'. For a mapping $f\colon \Omega \to \Omega'$, the family

$$\sigma(f) := \{f^{-1}(A') \mid A' \in \mathfrak{F}'\}$$

   defines a σ-field.
3. Let $\Omega = \mathbb{N}$. Then $\mathfrak{F} = \{\mathbb{N}, \emptyset, \{5\}, \mathbb{N}\backslash\{5\}\}$ is a σ-field on Ω.

Remark 1.3. In Analysis, the following useful statements have been proved, and we will use them often throughout these lecture notes. Let Ω, Ω' be sets, $f : \Omega \to \Omega'$ a mapping and I an arbitrary index set.

1. Let $B_i \subseteq \Omega'$ for all $i \in I$. Then

$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i) \quad \text{und} \quad f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i).$$

2. Let $B \subseteq \Omega'$. Then $f^{-1}(\Omega' \backslash B) = \Omega \backslash f^{-1}(B)$.

Theorem 1.4. *Any intersection of (finitely or infinitely many) σ-fields on Ω is again a σ-field on Ω.*

Proof. Exercise. □

This theorem implies the following important corollary.

Corollary 1.5. *For any family $\mathcal{E}$ of subsets of Ω there exists the smallest σ-field $\sigma(\mathcal{E})$ containing $\mathcal{E}$. This means, if $\mathcal{G}$ is a σ-field with $\mathcal{E} \subseteq \mathcal{G}$, then $\sigma(\mathcal{E}) \subseteq \mathcal{G}$.*

Proof. Let Σ be the sets of all σ-fields containing $\mathcal{E}$. Σ is non-empty, because it contains 2^Ω, for example. We set

$$\sigma(\mathcal{E}) := \bigcap_{\mathfrak{F}\in\Sigma} \mathfrak{F}.$$

□

$\sigma(\mathcal{E})$ is also called the σ-field *generated by* $\mathcal{E}$, and the family of sets $\mathcal{E}$ is called the *generator of* $\sigma(\mathcal{E})$.

Example 1.6. 1. Let $\Omega = \mathbb{N}$ and $\mathcal{E} = \{\{5\}\}$. The σ-field generated by $\mathcal{E}$ equals

$$\sigma(\mathcal{E}) = \{\mathbb{N}, \emptyset, \{5\}, \mathbb{N} \setminus \{5\}\}.$$

2. Let $\Omega = \mathbb{R}$ and $\mathcal{E} = \{[0, \infty)\}$. The σ-field generated by $\mathcal{E}$ is

$$\sigma(\mathcal{E}) = \{\mathbb{R}, \emptyset, [0, \infty), (-\infty, 0)\}.$$

Remark 1.7. Condition (σ3) can be replaced by the condition

(σ3') $A_n \in \mathfrak{F}$ for all $n \in \mathbb{N} \implies \bigcap_{n=1}^\infty A_n \in \mathfrak{F}$.

Definition 1.8. Let Ω be a set. A family $\mathfrak{R}$ of subsets of Ω is called *ring* (on Ω), if

(R1) $\emptyset \in \mathfrak{R}$,

(R2) $A, B \in \mathfrak{R} \implies A \setminus B \in \mathfrak{R}$,

(R3) $A, B \in \mathfrak{R} \implies A \cup B \in \mathfrak{R}$.

If in addition $\Omega \in \mathfrak{R}$, then $\mathfrak{R}$ is called *algebra*.

Exercise 1.9. Prove the following statements.

1. Let $\mathfrak{R}$ be a ring. If $A, B \in \mathfrak{R}$, then $A \cap B \in \mathfrak{R}$.

2. Let $\mathfrak{R} \subseteq 2^\Omega$. $\mathfrak{R}$ is an algebra if and only if:

(A1) $\Omega \in \mathfrak{R}$,

(A2) $A \in \mathfrak{R} \implies A^c \in \mathfrak{R}$,

(A3) $A, B \in \mathfrak{R} \implies A \cup B \in \mathfrak{R}$.

Remark 1.10. Note that the difference between a σ-field and an algebra is that in (R3) only finite unions are allowed, while in (σ3) countable unions are possible. Here the "σ" means "countable".

Example 1.11. Let $\Omega = \mathbb{N}$. The system of sets $A \subseteq \mathbb{N}$ for which either A or A^c is finite is an algebra, but not a σ-Algebra.

Definition 1.12. Let Ω be a set. A family $\mathfrak{D}$ of subsets of Ω is called *Dynkin system* (on Ω) if

(D1) $\Omega \in \mathfrak{D}$,

(D2) $A, B \in \mathfrak{D}, A \subseteq B \implies B \setminus A \in \mathfrak{D}$,

(D3) $A_n \in \mathfrak{D}$ for all $n \in \mathbb{N}$ with $A_n \cap A_m = \emptyset$, $m \neq n \implies \bigcup_{n=1}^{\infty} A_n \in \mathfrak{D}$.

Example 1.13. Each σ-field is a Dynkin system.

A family $\mathcal{E}$ of sets is called $\cap$-*stable* if

$$A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$$

Theorem 1.14. *A Dynkin system is a σ-field if and only if it is $\cap$-stable.*

Proof. Since every σ-field is a Dynkin system, we only have to show that any $\cap$-stable Dynkin system $\mathfrak{D}$ is a is a σ-field

(σ1) follows by (D1) and (σ2) follows by (D2) with $B = \Omega$. Property (σ3) is obtained as follows. Let $A, B \in \mathfrak{D}$. Then $A \cap B \in \mathfrak{D}$ by assumption, and

$$A \cup B = A \cup (B \setminus (A \cap B))$$

and

$$A \cap (B \setminus (A \cap B)) = \emptyset,$$

so (D3) implies that $A \cup B \in \mathfrak{D}$. Let now $D_n \in \mathfrak{D}$ for all $n \in \mathbb{N}$. We can write

$$\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=0}^{\infty} D'_{n+1} \setminus D'_n,$$

where $D'_0 := \emptyset$ and $D'_n := D_1 \cup \cdots \cup D_n \in \mathfrak{D}$. The sets $D'_{n+1} \setminus D'_n$ are pairwise disjoint, so (D3) implies that $\bigcup_{n=1}^{\infty} D_n \in \mathfrak{D}$, and we obtain ($\sigma$3). □

Theorem 1.4 also holds for Dynkin systems and rings. Therefore, for any family of sets $\mathcal{E}$ there exists a *smallest Dynkin system* $\mathfrak{D}(\mathcal{E})$ *that contains* $\mathcal{E}$. We also call $\mathfrak{D}(\mathcal{E})$ the *Dynkin system generated by* $\mathcal{E}$.

Theorem 1.15. *Let $\mathcal{E}$ be a $\cap$-stable family of sets. Then*

$$\mathfrak{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

Proof. Every σ-field is a Dynkin system, so $\sigma(\mathcal{E})$ is a Dynkin system which contains $\mathcal{E}$. This implies $\mathfrak{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. If we can show that $\mathfrak{D}(\mathcal{E})$ is a σ-field, then we can conclude that $\sigma(\mathcal{E}) \subseteq \mathfrak{D}(\mathcal{E})$, because $\sigma(\mathcal{E})$ is the smallest σ-field containing $\mathcal{E}$.

We now show that $\mathfrak{D}(\mathcal{E})$ is $\cap$-stable, and then our claim follows by Theorem 1.14. Let $D \in \mathfrak{D}(\mathcal{E})$. We define

$$\mathfrak{D}_D := \{A \subseteq \Omega \,|A \cap D \in \mathfrak{D}(\mathcal{E})\}.$$

Now we verify that $\mathfrak{D}_D$ is a Dynkin system. Since $\mathcal{E}$ is $\cap$-stable, we know that $\mathcal{E} \subseteq \mathfrak{D}_E$ for any $E \in \mathcal{E}$, and therefore $\mathfrak{D}(\mathcal{E}) \subseteq \mathfrak{D}_E$. For any $E \in \mathcal{E}$ and $D \in \mathfrak{D}(\mathcal{E})$, it holds $E \cap D \in \mathfrak{D}(\mathcal{E})$. This in turn means that $\mathcal{E} \subseteq \mathfrak{D}_D$, and therefore $\mathfrak{D}(\mathcal{E}) \subseteq \mathfrak{D}_D$ for all $D \in \mathfrak{D}(\mathcal{E})$. So $\mathfrak{D}(\mathcal{E})$ is $\cap$-stable. □

1.2 Additive and σ-additive contents

Definition 1.16. Let $\mathfrak{R}$ be a ring on Ω. A mapping

$$\mu\colon \mathfrak{R} \to [0,\infty]$$

is called *content* on $(\Omega, \mathfrak{R})$ if

(I1) $\mu(\emptyset) = 0$,

(I2) for $A \cap B = \emptyset$, it holds $\mu(A \cup B) = \mu(A) + \mu(B)$.

The content μ is called *σ-additive*, if for $A_n \in \mathfrak{R}$, $n \in \mathbb{N}$, with $A_n \cap A_m = \emptyset$, $m \neq n$ and $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{R}$,

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark. In Measure Theory and Probability Theory we use the following rules for computing in $\mathbb{R} \cup \{-\infty, +\infty\}$:

1. $a \pm \infty = \pm\infty + a = \pm\infty$ for $a \in \mathbb{R}$,
2. $+\infty - \infty$ is not defined,
3. $0 \cdot \infty = 0$,
4. $a \cdot \infty = \operatorname{sign}(a) \cdot \infty$ for $a \in \mathbb{R}\backslash\{0\}$.

Example 1.17. 1. Let $\mathfrak{R}$ be a ring on Ω and $\omega \in \Omega$. Then

$$\delta_\omega(A) = \begin{cases} 0, & \omega \notin A, \\ 1, & \omega \in A, \end{cases} \qquad A \in \mathfrak{R}$$

is a content.

2. We consider the ring from example 1.11. (Each algebra is a ring.) Then

$$A \mapsto \begin{cases} 0, & \text{if } A \text{ is finite}, \\ \infty, & \text{if } A^c \text{ is finite}, \end{cases}$$

is a content. It is not σ-additive.

Proposition 1.18. *Let μ be a content on $\mathfrak{R}$. Then for $A, B \in \mathfrak{R}$, the following statements are true.*

1. *Let$A, B \in \mathfrak{R}$. Then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.*
2. *Let $A, B \in \mathfrak{R}$ with $A \subseteq B$. Then $\mu(A) \le \mu(B)$.*
3. *Let $A, B \in \mathfrak{R}$ with $A \subseteq B$ and $\mu(A) < \infty$. Then $\mu(B \setminus A) = \mu(B) - \mu(A)$.*
4. *Let $A_1, \dots, A_n \in \mathfrak{R}$. Then $\mu\big(\bigcup_{i=1}^n A_i\big) \le \sum_{i=1}^n \mu(A_i)$.*

Proof. Exercise. □

Notation. For a sequence of sets $A, A_1, A_2, \dots$ we write $A_n \uparrow A$, if $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup_{n=1}^\infty A_n$. We write $A_n \downarrow A$, if $A_1 \supseteq A_2 \supseteq \dots$ and $A = \bigcap_{n=1}^\infty A_n$.

Theorem 1.19 (Continuity of contents, part 1)**.** *Let μ be a content on a ring $\mathfrak{R}$. Then μ is σ-additive if and only if for all $A_n, A \in \mathfrak{R}$ with $A_n \uparrow A$,*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Proof. Assume first that μ σ-additive and $A_n, A \in \mathfrak{R}$ with $A_n \uparrow A$. We set $A_0 := \emptyset$ and $B_n := A_n \setminus A_{n-1}$, $n = 1, 2, \dots$. The B_n are pairwise disjoint. It holds $A_n = B_1 \cup \dots \cup B_n$ and $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n = A$, so

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \lim_{N\to\infty} \mu(A_N).$$

To show the reverse direction, let $A_n \in \mathfrak{R}$ be a sequence of disjoint sets with $A = \bigcup_{n=1}^\infty A_n \in \mathfrak{R}$. We define $B_n := A_1 \cup \dots \cup A_n$. Then $B_n \uparrow A$, so $\mu(A) = \lim_{n\to\infty} \mu(B_n)$. Finite additivity of the content μ gives

$$\mu(B_n) = \mu(A_1) + \dots + \mu(A_n),$$

which proves the claim. □

Theorem 1.20 (Continuity of contents, part 2)**.** *Let μ be a content on a ring $\mathfrak{R}$. If μ is σ-additive, then*

1. *for any sequence $A_n \in \mathfrak{R}$ with $A_n \downarrow A$, $A \in \mathfrak{R}$ and $\mu(A_1) < \infty$, it holds*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

2. *for any sequence $A_n \in \mathfrak{R}$ with $A_n \downarrow \emptyset$ and $\mu(A_1) < \infty$,*

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

If $\mu(A) < \infty$ for $A \in \mathfrak{R}$, then the conditions ((1.) or (2.)) are equivalent to σ-additivity.

Proof. It is obvious that (1.) implies (2.). Let $A_n \in \mathfrak{R}$ be a sequence with $A_n \downarrow A \in \mathfrak{R}$ and $\mu(A_1) < \infty$. Then also $\mu(A_n) < \infty$ for all n and $\mu(A) < \infty$, and $\mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$ due to Proposition 1.18 (3.) and $(A_1 \setminus A_n) \uparrow (A_1 \setminus A)$. Theorem 1.19 implies

$$\mu(A_1) - \mu(A) = \mu(A_1 \setminus A) = \lim_{n\to\infty} \mu(A_1 \setminus A_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n),$$

which is (1.).

We now assume that $\mu(B) < \infty$ for all $B \in \mathfrak{R}$ and we show that (2.) implies the σ-additivity of μ. For this we use Theorem 1.19. Let $A_n \in \mathfrak{R}$ with $A_n \uparrow A \in \mathfrak{R}$. Then $A \setminus A_n \downarrow \emptyset$, and because $\mu(A) < \infty$, $\mu(A_n) < \infty$, we obtain

$$0 = \lim_{n\to\infty} \mu(A \setminus A_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n).$$

□

We now consider a special ring on $\Omega = \mathbb{R}^d$. For $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{b} = (b_1, \ldots, b_d)$ we define the *half open rectangles*

$$(\mathbf{a}, \mathbf{b}] := \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_i < x_i \le b_i, i = 1, \ldots, d\}.$$

We consider the set $\mathfrak{R}^d$ of finite unions of rectangles, that is,

$$\mathfrak{R}^d := \Big\{ \bigcup_{k=1}^{n} (\mathbf{a}_k, \mathbf{b}_k] \mid n \in \mathbb{N}, \mathbf{a}_k, \mathbf{b}_k \in \mathbb{R}^d, k = 1, \ldots, n \Big\}.$$

Theorem 1.21. *$\mathfrak{R}^d$ is a ring in $\mathbb{R}^d$.*

Proof. Exercise. □

Remark 1.22. Each set in $A \in \mathfrak{R}^d$ admits a representation $A = \bigcup_{k=1}^{n} (\mathbf{a}_k, \mathbf{b}_k]$ with pairwise disjoint rectangles.

Definition 1.23. A function $H : \mathbb{R}^d \to [0, \infty)$ is called *rectangular monotone*, if for all $\mathbf{a}^1, \mathbf{a}^2 \in \mathbb{R}^d$ with $a_i^1 \le a_i^2$, $i = 1, \ldots, d$, it holds

$$\Delta_{\mathbf{a}^1}^{\mathbf{a}^2} H := \sum_{i_1,\ldots,i_d \in \{1,2\}} (-1)^{i_1+\cdots+i_d} H(a_1^{i_1}, \ldots, a_d^{i_d}) \ge 0.$$

Theorem 1.24. *Let $H : \mathbb{R}^d \to [0, \infty)$ be a rectangular monotone function. Then there exist a unique content μ on $\mathfrak{R}^d$ such that for each rectangle $(\mathbf{a}^1, \mathbf{a}^2] \ne \emptyset$,*

$$\mu((\mathbf{a}^1, \mathbf{a}^2]) = \Delta_{\mathbf{a}^1}^{\mathbf{a}^2} H.$$

For all $A \in \mathfrak{R}^d$, it holds $\mu(A) < \infty$.

Example 1.25. For a rectangle $(\mathbf{a}^1, \mathbf{a}^2] \ne \emptyset$ we define its *volume* by

$$\mathrm{Vol}((\mathbf{a}^1, \mathbf{a}^2]) = \prod_{i=1}^{d} (a_i^2 - a_i^1).$$

For $H(\mathbf{x}) = \prod_{i=1}^{d} x_i$, the volume satisfies $\mathrm{Vol}((\mathbf{a}^1, \mathbf{a}^2]) = \Delta_{a^1}^{a^2} H$. The unique content on $\mathfrak{R}^d$ corresponding to H is called the *Lebesgue content*, and we also denote it by Vol.

Exercise 1.26. Prove the claim in Example 1.25.

Proof of Theorem 1.24. **Uniqueness:** Let μ be such a content. Every set $A \in \mathfrak{R}^d$ can be represented as $A = \bigcup_{k=1}^n (\mathbf{a}^k, \mathbf{b}^k]$ with pairwise disjoint rectangles. Since μ is a content, we know

$$\mu(A) = \sum_{k=1}^n \mu((\mathbf{a}^k, \mathbf{b}^k]) = \sum_{k=1}^n \Delta_{\mathbf{a}^k}^{\mathbf{b}^k} H,$$

so μ is defined uniquely by its values on rectangles.
Existence: We only consider the case $d = 1$. For $A = \bigcup_{k=1}^n (a_k, b_k] \in \mathfrak{R}^1$ with pairwise disjoint intervals, we set

$$\mu(A) = \sum_{k=1}^n \Delta_{a_k}^{b_k} H = \sum_{k=1}^n (H(b_k) - H(a_k)).$$

We have to show that this definition does not depend on the representation of A.
For $c \in (a, b]$,

$$\begin{aligned}\mu((a,b]) &= \Delta_a^b H = H(b) - H(a) \\ &= H(b) - H(c) + H(c) - H(a) = \Delta_a^c H + \Delta_c^b H = \mu((a,c]) + \mu((c,b]).\end{aligned} \tag{1.1}$$

Let now $A = \bigcup_{l=1}^m (c_l, d_l]$ be another representation of A with pairwise disjoint rectangles. For all k, $(a_k, b_k] = \bigcup_{l=1}^m (a_k, b_k] \cap (c_l, d_l]$ is a partition of $(a_k, b_k]$ into disjoint intervals. We can now apply (1.1) inductively and obtain

$$\mu((a_k, b_k]) = \sum_{l=1}^m \mu((a_k, b_k] \cap (c_l, d_l]), \quad k = 1, \dots, n.$$

But also

$$\mu((c_l, d_l]) = \sum_{k=1}^n \mu((a_k, b_k] \cap (c_l, d_l]), \quad l = 1, \dots, m,$$

which proves the claim.
Properties of a content: For $A, B \in \mathfrak{R}^d$ with $A \cap B = \emptyset$, it holds by construction that $\mu(A \cup B) = \mu(A) + \mu(B)$, so μ is a content on $\mathfrak{R}^d$. □

Exercise 1.27. Prove the existence of the content μ in Theorem 1.24 for $d = 2$ or more generally for all d.

Theorem 1.28. *Let $H : \mathbb{R}^d \to [0, \infty)$ be a rectangular monotone function which is continuous from the right in each argument, i.e. for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$, it holds*

$$\lim_{s \downarrow x_i} H(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d) = H(x).$$

Then the content μ on $\mathfrak{R}^d$ from Theorem 1.24 is σ-additive.

Corollary 1.29. *The Lebesgue content* Vol *on $\mathfrak{R}^d$ is σ-additive.*

Proof of Theorem 1.28. For all $A \in \mathfrak{R}^d$, we know that $\mu(A) < \infty$. Due to Theorem 1.20 it is therefore sufficient to show that for any sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathfrak{R}^d$ with $A_n \downarrow \emptyset$, we have $\mu(A_n) \to 0$, $n \to \infty$.

Continuity from the right of H implies that for each rectangle $(\mathbf{a}, \mathbf{b}]$,

$$\mu((\mathbf{a}, \mathbf{b}]) = \sup\{\mu((\mathbf{c}, \mathbf{b}]) \mid c_i > a_i, i = 1, \dots, d\}. \tag{1.2}$$

Let now $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{R}^d$ be a sequence with $A_n \downarrow \emptyset$, and let $\varepsilon > 0$. Due to (1.2) we can choose for each $n \in \mathbb{N}$ a $C_n \in \mathfrak{R}^d$ such that

$$\overline{C}_n \subseteq A_n, \quad \mu(C_n) \geq \mu(A_n) - \varepsilon/2^n,$$

so that for $K_n = C_1 \cap \cdots \cap C_n$,

$$\mu(A_n \setminus K_n) = \mu\Big(\bigcup_{i=1}^{n} (A_n \setminus C_i)\Big) \leq \sum_{i=1}^{n} \mu(A_i \setminus C_i) = \sum_{i=1}^{n} \Big(\mu(A_i) - \mu(C_i)\Big) \leq \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \leq \varepsilon.$$

Furthermore $\bigcap_{n=1}^{\infty} \overline{K}_n \subseteq \bigcap_{n=1}^{\infty} \overline{C}_n \subseteq \bigcap_{n=1}^{\infty} A_n = \emptyset$, so that (due to compactness of the $\overline{K}_n$ and $K_{n+1} \subseteq K_n$; nested intervals) there exists a n_0 with $\overline{K}_n = \emptyset$ for all $n \geq n_0$, so $\mu(A_n) < \varepsilon$ for all $n \geq n_0$. □

Remark. We here use the following variant of the nested intervals principle: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets in a metric space with $K_{n+1} \subseteq K_n$ and $K_n \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.
This can be proved as follows: For all $n \in \mathbb{N}$ choose $x_n \in K_n \subset K_1$. Because K_1 is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $x_{n_k} \to x_0 \in K_1$. Assume that $x_0 \notin K_{n_0}$ for some n_0. Then also $x_0 \notin K_n$ for all $n \geq n_0$. Because $K_{n_0}^c$ is open, this implies $d(x_0, x_n) \geq d(x_0, K_n) \geq d(x_0, K_{n_0}) > 0$ for all $n \geq n_0$, which contradicts convergence of the subsequence. Hence $x_0 \in \bigcap_{n\in\mathbb{N}} K_n$.

1.3 General measures and the Lebesgue measure

Definition 1.30. Let $\mathfrak{F}$ be a σ-field on Ω. A mapping $\mu\colon \mathfrak{F} \to [0, \infty]$ is called *measure* if

(M1) $\mu(\emptyset) = 0$,

(M2) $A_n \in \mathfrak{F}$, $A_n \cap A_m = \emptyset$, $m \neq n$:

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triplet $(\Omega, \mathfrak{F}, \mu)$ is called *measure space.*

Remark. A measure is a σ-additive content, which is defined on a σ-field.

Example 1.31.

1. Let Ω be a set and $\mathfrak{F}$ a σ-field. For any $\omega \in \Omega$, δ_ω defines a measure on $\mathfrak{F}$; see also Example 1.17 (1.). This measure is called *Dirac measure* in ω.

2. Let Ω be a set and $\mathfrak{F}$ a σ-field. For any $A \in \mathfrak{F}$, let $\#A$ be the number of its elements. Then

$$A \mapsto \#A$$

defines a measure on $\mathfrak{F}$. It is called the *counting measure.*

3. The Lebesgue content Vol on $\mathfrak{R}^d$ is not a measure, because $\mathfrak{R}^d$ is not a σ-field.

To derive the *Lebesgue measure* from the Lebesgue content, we need the following extension theorem by Carathéodory.

Theorem 1.32 (Extension theorem)**.** *Let μ be a σ-additive content on a ring $\mathfrak{R}$ on Ω. Then there exists a measure $\tilde{\mu}$ (at least one such measure) on $\sigma(\mathfrak{R})$ which extends μ.*

The proof of this theorem relies on the notion of a *outer measure.*

Definition 1.33. A mapping $\mu^*\colon 2^\Omega \to [0,\infty]$ is called *outer measure*, if

(M*1) $\mu^*(\emptyset) = 0$,

(M*2) $A \subseteq B \subseteq \Omega \quad \Rightarrow \quad \mu^*(A) \le \mu^*(B)$,

(M*3) $A_n \subseteq \Omega, n = 1, 2, \dots \quad \Rightarrow \quad \mu^*\Big(\bigcup_{n=1}^\infty A_n\Big) \le \sum_{n=1}^\infty \mu^*(A_n)$.

A set $A \subseteq \Omega$ is called *μ^*-measurable*, if

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \setminus A), \quad \text{for all } S \subseteq \Omega. \tag{1.3}$$

Theorem 1.34. *Let μ^* be an outer measure on Ω. Then the system $\mathfrak{F}^*$ of all μ^*-measurable sets is a σ-field and the restriction of μ^* to $\mathfrak{F}^*$ is a measure.*

Proof. We first show that $\mathfrak{F}^*$ is a σ-field. (σ1) and (σ2) follow by (1.3). Let now $A_n \in \mathfrak{F}^*$, $n = 1, 2, \dots$. We have to show that $A := \bigcup_{n=1}^\infty A_n \in \mathfrak{F}^*$. For this, let $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$. The sequence B_n is disjoint and $\bigcup_{n=1}^N B_n = \bigcup_{n=1}^N A_n$. Let now $S \subseteq \Omega$. Then,

$$\begin{aligned}
\mu^*(S) &= \mu^*(S \cap A_1) + \mu^*(S \cap A_1^c) \\
&= \mu^*(S \cap B_1) + \mu^*(S \cap A_1^c \cap A_2) + \mu^*(S \cap A_1^c \cap A_2^c) \\
&= \mu^*(S \cap B_1) + \mu^*(S \cap B_2) + \mu^*(S \cap A_1^c \cap A_2^c) \\
&= \dots \\
&= \sum_{n=1}^N \mu^*(S \cap B_n) + \mu^*\Big(S \cap \bigcap_{n=1}^N A_n^c\Big) \\
&\ge \sum_{n=1}^N \mu^*(S \cap B_n) + \mu^*(S \setminus A),
\end{aligned}$$

because $S \cap \bigcap_{n=1}^N A_n^c = S \cap \big(\bigcup_{n=1}^N A_n\big)^c \supseteq S \setminus A$. With $N \to \infty$ it follows

$$\begin{aligned}
\mu^*(S) &\ge \sum_{n=1}^\infty \mu^*(S \cap B_n) + \mu^*(S \setminus A) \\
&\ge \mu^*\Big(\bigcup_{n=1}^\infty (S \cap B_n)\Big) + \mu^*(S \setminus A) \\
&= \mu^*(S \cap A) + \mu^*(S \setminus A).
\end{aligned} \tag{1.4}$$

The inequality $\mu^*(S) \leq \mu^*(S \cap A) + \mu^*(S \setminus A)$ is true for all $A, S \subseteq \Omega$ due to (M*3), so $A \in \mathfrak{F}^*$ and we have shown that $\mathfrak{F}^*$ is a σ-field.

To show that μ^* defines a measure on $\mathfrak{F}^*$, we consider a sequence $A_n \in \mathfrak{F}^*$ of pairwise disjoint sets. Our construction above also applies to these A_n, and because they are disjoint, we have $B_n = A_n$. Now we set $S = A$ in (1.4) and obtain

$$\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A_n).$$

The other inequality holds due to (M*3). □

Proof of Theorem 1.32. For a set $Q \subseteq \Omega$ we define

$$\mu^*(Q) = \inf\Big\{ \sum_{n=1}^{\infty} \mu(A_n) \mid (A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{R} \text{ with } Q \subseteq \bigcup_{n=1}^{\infty} A_n \Big\}.$$

Note that here, we define the infimum of the empty set as $+\infty$. We now show that: (1.) μ^* is an outer measure, (2.) all sets in $\mathfrak{R}$ are μ^*-measurable, and (3.) μ^* and μ are equal on $\mathfrak{R}$.

First to (1.): (M*1) and (M*2) can be verified directly. Let now $(Q_n)_{n\in\mathbb{N}} \subseteq \Omega$ be a sequence of sets. We can assume that $\mu^*(Q_n) < \infty$ for all $n \in \mathbb{N}$. (Otherwise (M*3) is true.) Let $\varepsilon > 0$. For all n, there exists a sequence $(A_{nk})_{k\in\mathbb{N}} \subseteq \mathfrak{R}$ with

$$Q_n \subseteq \bigcup_{k=1}^{\infty} A_{nk}, \quad \text{und} \quad \sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(Q_n) + 2^{-n}\varepsilon.$$

Because of $\bigcup_{n=1}^{\infty} Q_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$, we know that

$$\mu^*\Big(\bigcup_{n=1}^{\infty} Q_n\Big) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(Q_n) + \varepsilon,$$

where we have used $\sum_{n=1}^{\infty} 2^{-n} = 1$. This shows (M*3).

Then (2.): Let $A \in \mathfrak{R}$, $Q \subseteq \Omega$ and $(B_n)_n \subseteq \mathfrak{R}$ with $Q \subseteq \bigcup_{n=1}^{\infty} B_n$. Then,

$$\sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(B_n \cap A) + \sum_{n=1}^{\infty} \mu(B_n \setminus A).$$

Because of $Q \cap A \subseteq \bigcup_{n=1}^{\infty}(B_n \cap A)$ and $Q \setminus A \subseteq \bigcup_{n=1}^{\infty}(B_n \setminus A)$, it follows

$$\sum_{n=1}^{\infty} \mu(B_n) \geq \mu^*(Q \cap A) + \mu^*(Q \backslash A)$$

and because the sequence (B_n) was arbitrary ($Q \subseteq \bigcup_{n=1}^{\infty} B_n$), also

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \backslash A).$$

And (3.): With the sequence $A, \emptyset, \emptyset, \ldots$ we obtain that $\mu^*(A) \leq \mu(A)$. Let $(A_n)_n \subseteq \mathfrak{R}$ with $A \subseteq \cup_{n=1}^{\infty} A_n$. We set $B_n = A \cap (A_n \backslash \bigcup_{i=1}^{n-1} A_i) \in \mathfrak{R}$. Then $A = \bigcup_{n=1}^{\infty} B_n$, and the sets B_n are disjoint. The σ-additivity of μ gives

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

because $B_n \subseteq A_n$. So $\mu(A) \leq \mu^*(A)$. □

In conclusion, we can extend the Lebesgue content on $\mathfrak{R}^d$ to a measure on $\sigma(\mathfrak{R}^d)$. However, we do not know if this extension is unique. In the proof of Theorem 1.32 only one such extension is constructed. The following theorem gives an answer to this question.

Theorem 1.35. *Let $\mathcal{E}$ be a $\cap$-stable generator of a σ-field $\mathfrak{F}$ on Ω. Let further $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ with $E_n \uparrow \Omega$. If μ_1, μ_2 are measures on $\mathfrak{F}$ with*

$$\mu_1(E) = \mu_2(E), \quad E \in \mathcal{E}$$

and

$$\mu_1(E_n) = \mu_2(E_n) < \infty, \quad n \in \mathbb{N},$$

then $\mu_1 = \mu_2$ on $\mathfrak{F}$.

Proof. Let $E \in \mathcal{E}$ with $\mu_1(E) = \mu_2(E) < \infty$. The set of sets

$$\mathfrak{D}_E := \{D \in \mathfrak{F} \mid \mu_1(E \cap D) = \mu_2(E \cap D)\}$$

is a Dynkin system (exercise). Because $\mathcal{E}$ is $\cap$-stable, it follows that $\mathcal{E} \subseteq \mathfrak{D}_E$, so also $\mathfrak{D}(\mathcal{E}) \subseteq \mathfrak{D}_E$. From Theorem 1.15, we deduce that $\mathfrak{D}(\mathcal{E}) = \sigma(\mathcal{E}) = \mathfrak{F}$, so also $\mathfrak{F} = \mathfrak{D}_E$, because $\mathfrak{D}_E \subseteq \mathfrak{F}$. This implies that

$$\mu_1(E \cap D) = \mu_2(E \cap D), \quad \text{für alle } D \in \mathfrak{F}. \tag{1.5}$$

Statement (1.5) in particular holds for $E = E_n$. It follows from $E_n \uparrow \Omega$ that $E_n \cap D \uparrow D$ and therefore, by Theorem 1.19,

$$\mu_1(D) = \lim_{n\to\infty} \mu_1(E_n \cap D) = \lim_{n\to\infty} \mu_2(E_n \cap D) = \mu_2(D), \quad \text{for all } D \in \mathfrak{F}.$$

□

We now apply Theorem 1.35 to show that the Lebesgue content Vol can be extended to a unique measure on $\sigma(\mathfrak{R}^d)$. Let μ_1, μ_2 be two extensions of Vol to $\sigma(\mathfrak{R}^d)$, which exist due to Theorem 1.32. Because μ_1, μ_2 are equal on $\mathfrak{R}^d$, the prerequisites of Theorem 1.35 are satisfied with $E_n = ((-n, \ldots, -n), (n, \ldots, n)]$. ($\mathfrak{R}$ is $\cap$-stable, because $\mathfrak{R}$ is a ring.) Thus $\mu_1 = \mu_2$.

Definition 1.36. The σ-field $\mathcal{B}(\mathbb{R}^d) := \sigma(\mathfrak{R}^d)$ is called *Borel σ-field* on $\mathbb{R}^d$. The measure which extends Vol to $\mathcal{B}(\mathbb{R}^d)$ is called *Lebesgue measure* on $\mathbb{R}^d$, and we denote it by $\mathcal{L}^d$.

The picture on the right shows the French mathematician Henri Léon Lebesgue, 1875–1941. He is the founder of modern measure theory and integration theory and introduced the Lebesgue measure in his dissertation in 1902.

Exercise 1.37. Let $\mathcal{O}$ be the system of open sets in $\mathbb{R}^d$, $\mathcal{A}$ the system of closed sets in $\mathbb{R}^d$ and $\mathcal{K}$ the system of compact sets in $\mathbb{R}^d$.

1. Show that $\sigma(\mathcal{O}) = \sigma(\mathcal{A}) = \sigma(\mathcal{K})$.

2. Show that $\mathcal{B}(\mathbb{R}^d) = \sigma(\mathcal{O})$.

1.4 Measurable mappings and image measures

Definition 1.38. Let $\mathfrak{F}_1$ be a σ-field on Ω_1 and $\mathfrak{F}_2$ a σ-field on Ω_2. A mapping $f\colon \Omega_1 \to \Omega_2$ is called $\mathfrak{F}_1$-$\mathfrak{F}_2$-*measurable*, if

$$f^{-1}(A) \in \mathfrak{F}_1, \quad \text{for all } A \in \mathfrak{F}_2.$$

Example 1.39. Let $c \in \Omega_2$. The constant mapping $f\colon \Omega_1 \to \Omega_2, \omega \mapsto c$ is $\mathfrak{F}_1$-$\mathfrak{F}_2$-measurable.

Exercise 1.40. Let $\mathfrak{F}$ be a σ-field on Ω and $A \in \mathfrak{F}$. We define the *indicator function* $\mathbb{1}_A$ *of* A by

$$\mathbb{1}_A : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Show that $\mathbb{1}_A$ is $(\mathfrak{F}, \mathcal{B}(\mathbb{R}))$-measurable.

Theorem 1.41. *Let $f\colon \Omega_1 \to \Omega_2$ be a $\mathfrak{F}_1$-$\mathfrak{F}_2$-measurable mapping and $g\colon \Omega_2 \to \Omega_3$ a $\mathfrak{F}_2$-$\mathfrak{F}_3$-measurable mapping. Then $g \circ f\colon \Omega_1 \to \Omega_3$ is a $\mathfrak{F}_1$-$\mathfrak{F}_3$-measurable mapping.*

Proof. For all $A \subseteq \Omega_3$ (in particular, for all $A \in \mathfrak{F}_3$),

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)).$$

This yields the claim. □

Theorem 1.42. *Let $\mathfrak{F}_2 = \sigma(\mathcal{A})$ for a system of sets $\mathcal{A} \subseteq 2^{\Omega_2}$. A mapping $f\colon \Omega_1 \to \Omega_2$ is $\mathfrak{F}_1$-$\mathfrak{F}_2$-measurable if and only if*

$$f^{-1}(A) \in \mathfrak{F}_1, \quad \textit{for all } A \in \mathcal{A}.$$

Proof. Let

$$\mathfrak{A} := \{A \subseteq \Omega_2 \mid f^{-1}(A) \in \mathfrak{F}_1\}.$$

$\mathfrak{A}$ is a σ-field; see Example 1.2 (2.). We have to show that $\mathfrak{F}_2 \subseteq \mathfrak{A}$. For this, it is sufficient to show that $\mathcal{A} \subseteq \mathfrak{A}$, but this holds by assumption. □

Exercise 1.43. Let $a \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + a$ is $\mathfrak{B}(\mathbb{R})$-$\mathfrak{B}(\mathbb{R})$-measurable.

Note that we only need sets Ω_1, Ω_2 and σ-fields to define measurability of mappings. The term MEASUrability might be misleading, because MEASURES are irrelevant in its definition. The next theorem shows that measurable mappings induce new measures, and in the next section we will see that measurability enables us to integrate a function with respect to some measure (i.e. to measure this function).

Theorem 1.44. *Let $f\colon \Omega_1 \to \Omega_2$ be a $\mathfrak{F}_1$-$\mathfrak{F}_2$-measurable mapping and μ a measure on $\mathfrak{F}_1$. Then*

$$\mu_f\colon \mathfrak{F}_2 \to [0, \infty], \quad B \mapsto \mu(f^{-1}(B))$$

is a measure on the σ-field $\mathfrak{F}_2$ on Ω_2. This measure is called the image measure *of μ under f.*

Proof. Due to $\mu_f(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$, statement (M1) is true. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{F}_2$ with $A_m \cap A_n = \emptyset$ for $n \neq m$. Due to remark 1.3,

$$f^{-1}\big(\bigcup_{n\in\mathbb{N}} A_n\big) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n).$$

$A_m \cap A_n = \emptyset$ implies that $f^{-1}(A_n) \cap f^{-1}(A_m) = \emptyset$, so

$$\mu_f\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mu\Big(f^{-1}\big(\bigcup_{n\in\mathbb{N}} A_n\big)\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} f^{-1}(A_n)\Big) = \sum_{n\in\mathbb{N}} \mu(f^{-1}(A_n)) = \sum_{n\in\mathbb{N}} \mu_f(A_n)$$

and we obtain (M2). □

Exercise 1.45. We consider the mapping $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. What is the image measure of the Dirac measure δ_5 under f? What is the image measure of the counting measure under f?

If the σ-fields are clear from the context, we simply call a mapping $f\colon \Omega_1 \to \Omega_2$ measurable instead of $\mathfrak{F}_1$-$\mathfrak{F}_2$-measurable. On $\mathbb{R}^d$ one (almost) always takes the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ as σ-field, unless it is defined differently explicitly.

Let now Ω be a set and $\mathfrak{F}$ a σ-field on Ω. To us, functions $f\colon \Omega \to \bar{\mathbb{R}}$ are of particular interest, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$. Usually we define on $\bar{\mathbb{R}}$ the σ-field $\bar{\mathcal{B}}$, given by

$$\bar{\mathcal{B}} := \{A \subseteq \bar{\mathbb{R}} \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}.$$

Proposition 1.46. *A mapping $f\colon \Omega \to \bar{\mathbb{R}}$ is measurable (i.e. $\mathfrak{F}$-$\bar{\mathcal{B}}$-measurable) if and only if one of the following equivalent conditions holds:*

$$\begin{aligned}
\{\omega \mid f(\omega) \le t\} &\in \mathfrak{F}_1 \quad \textit{for all } t \in \mathbb{R},\\
\{\omega \mid f(\omega) < t\} &\in \mathfrak{F}_1 \quad \textit{for all } t \in \mathbb{R},\\
\{\omega \mid f(\omega) \ge t\} &\in \mathfrak{F}_1 \quad \textit{for all } t \in \mathbb{R},\\
\{\omega \mid f(\omega) > t\} &\in \mathfrak{F}_1 \quad \textit{for all } t \in \mathbb{R}.
\end{aligned}$$

Proof. We show that the system of sets $\bar{\mathcal{E}} = \{[-\infty, a] \mid a \in \mathbb{R}\}$ generates the σ-field $\bar{\mathcal{B}}$, so that 1.42 implies the first condition. With analogous proofs one can also show that the sets $\{[-\infty, a) \mid a \in \mathbb{R}\}$, $\{[a, \infty] \mid a \in \mathbb{R}\}$, $\{(a, \infty] \mid a \in \mathbb{R}\}$ generate $\bar{\mathcal{B}}$, which gives the other conditions.

Clearly, $\bar{\mathcal{E}} \subseteq \bar{\mathcal{B}}$, so also $\sigma(\bar{\mathcal{E}}) \subseteq \bar{\mathcal{B}}$. For $a, b \in \mathbb{R}$ with $a < b$, it holds $(a, b] = [-\infty, b]\setminus[-\infty, a] \in \sigma(\bar{\mathcal{E}})$, so $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\bar{\mathcal{E}})$. Also, $\{-\infty\} = \bigcap_{n\in\mathbb{N}}[-\infty, -n] \in \sigma(\bar{\mathcal{E}})$, $\{+\infty\} = \bigcap_{n\in\mathbb{N}}[n, \infty] \in \sigma(\bar{\mathcal{E}})$. Let now $A \in \bar{\mathcal{B}}$. Then $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \subseteq \sigma(\bar{\mathcal{E}})$ and

$$A = (A \cap \mathbb{R}) \cup (A \cap \{+\infty, -\infty\}) \in \sigma(\bar{\mathcal{E}}).$$

□

For sets of the form $\{\omega \in \Omega \mid f(\omega) \text{ has property } E(w)\}$, we use the abbreviation

$$\{f \text{ has property } E\} = \{\omega \in \Omega \mid f(\omega) \ \text{ has property } E(\omega)\},$$

for example $\{f > 0\} = \{\omega \in \Omega \mid f(\omega) > 0\}$. Sometimes, it is better to use the complete notation so as not to be confused by the abbreviations!

Theorem 1.47. *Let $f, g : \Omega \to \mathbb{R}$ be measurable mappings. Then the sets $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$, $\{f \neq g\}$ are all contained in $\mathfrak{F}$.*

Proof. Clearly,

$$\{f < g\} = \bigcup_{r \in \mathbb{Q}} \Big(\{f < r\} \cup \{r < g\} \Big).$$

Due to Theorem 1.46, all sets of the type $\{f < r\}$, $\{r < g\}$ for $r \in \mathbb{Q}$ in are contained in the σ-field $\mathfrak{F}$, so also the countable union over $\mathbb{Q}$. Moreover, $\{f \leq g\} = \{f > g\}^c$, $\{f = g\} = \{f \leq g\} \cap \{f \geq g\}$ and $\{f \neq g\} = \{f = g\}^c$. □

Theorem 1.48. *Let $f, g\colon \Omega \to \bar{\mathbb{R}}$ be measurable mappings, $c \in \mathbb{R}$. Then,*

1. *the function $cf\colon \Omega \to \bar{\mathbb{R}}, \omega \mapsto cf(\omega)$ is measurable.*
2. *the function $f + g\colon \Omega \to \bar{\mathbb{R}}, \omega \mapsto f(\omega) + g(\omega)$ is measurable.*

Proof. If $c = 0$, then the first claim is clearly true. If $c > 0$ and $a \in \mathbb{R}$, then for all $t \in \mathbb{R}$,

$$\{cf + a \leq t\} = \{f \leq (t - a)/c\} \in \mathfrak{F},$$

which implies the first claim and the measurability of $cf + a$. The case $c < 0$ is analogous. For all $t \in \mathbb{R}$,

$$\{f + g \leq t\} = \{f \leq -g + t\} \in \mathfrak{F},$$

because $-g + t$ is measurable due to the above considerations and Theorem 1.47. □

Theorem 1.49. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets, $f_n\colon \Omega \to \bar{\mathbb{R}}$. Then,*

1. $\inf_{n\geq 1} f_n$ *and* $\sup_{n\geq 1} f_n$ *are measurable,*
2. $\liminf_{n\to\infty} f_n$ *and* $\limsup_{n\to\infty} f_n$ *are measurable.*

Proof. 1. Follows by $\{\sup_{n\geq 1} f_n \leq t\} = \bigcap_{n\geq 1}\{f_n \leq t\}$ and similarly $\{\inf_{n\geq 1} f_n \geq t\} = \bigcap_{n\geq 1}\{f_n \geq t\}$.

2. It holds

$$\limsup_{n\to\infty} f_n = \lim_{n\to\infty} \sup_{m\geq n} f_m = \inf_n \sup_{m\geq n} f_m.$$

Use (1.). The claim for $\liminf$ follows by similar arguments.

□

Theorem 1.50. *Every continuous function $f\colon \mathbb{R}^d \to \mathbb{R}^k$ is measurable.*

Proof. Preimages of open sets under continuous mappings are open. □

The reverse statement of Theorem 1.50 is wrong. There are many more measurable mappings than there are continuous mappings; see the exercises.

Chapter 2

Integration and product measures

Bernhard Riemann
1826–1866

Henry Léon Lebesgue
1875–1941

2.1 Integration with respect to measures

Let Ω be a set, $\mathfrak{F}$ a σ-field on Ω and μ a measure on $(\Omega, \mathfrak{F})$.

2.1.1 Integration of non-negative stepfunctions

Definition 2.1. A mapping $f\colon \Omega \to [0, \infty]$ is called *non-negative stepfunction*, if it is $\mathfrak{F}$-measurable and only takes finitely many values.

Definition 2.2. For a set $A \subseteq \Omega$ we define the *indicator function*

$$\mathbb{1}_A\colon \Omega \to \{0,1\}, \quad \omega \mapsto \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

A *partition* of Ω is a set of sets $(C_j)_{j\in J}$ such that $C_j \cap C_i = \emptyset$ for $j \neq i$ and $\bigcup_{j\in J} C_j = \Omega$.

Proposition 2.3. *1. Let f be a non-negative stepfunction. Then there exists a partition $A_1, \dots, A_n \in \mathfrak{F}$ and numbers $c_1, \dots, c_n \in [0, \infty]$, such that*

$$f(\omega) = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}(\omega), \quad \omega \in \Omega.$$

2. Let $B_1, \dots, B_n \in \mathfrak{F}$ and $d_1, \dots, d_n \in [0, \infty]$. Then $f = \sum_{k=1}^{n} d_k \mathbb{1}_{B_k}$ is a non-negative stepfunction.

Proof. Exercise. □

Lemma 2.4. *Let f be a non-negative stepfunction. For two representations*

$$f = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k} = \sum_{l=1}^{m} d_l \mathbb{1}_{B_l},$$

it holds

$$\sum_{k=1}^{n} c_k \mu(A_k) = \sum_{l=1}^{m} d_l \mu(B_l).$$

Proof. We can assume without loss of generality that the families of sets $(A_k)_{k\in\{1,\dots,n\}}$ and $(B_l)_{l\in\{1,\dots,m\}}$ are partitions of Ω, because any finite family of sets can be separated into partitions.

By assumption,

$$\begin{aligned}\sum_{k=1}^{n} c_k\mu(A_k) &= \sum_{k=1}^{n} c_k\mu(A_k\cap\Omega) = \sum_{k=1}^{n} c_k\mu\Big(A_k\cap\bigcup_{l=1}^{m} B_l\Big)\\ &= \sum_{k=1}^{n} c_k\mu\Big(\bigcup_{l=1}^{m}(A_k\cap B_l)\Big) = \sum_{k=1}^{n}\sum_{l=1}^{m} c_k\mu(A_k\cap B_l).\end{aligned}$$

If $A_k\cap B_l\neq\emptyset$, then $f(\omega)=c_k=d_l$ for $\omega\in A_k\cap B_l$, so in any case $c_k\mu(A_k\cap B_l)=d_l\mu(A_k\cap B_l)$. Hence,

$$\sum_{k=1}^{n} c_k\mu(A_k) = \sum_{k=1}^{n}\sum_{l=1}^{m} c_k\mu(A_k\cap B_l) = \sum_{k=1}^{n}\sum_{l=1}^{m} d_l\mu(A_k\cap B_l) = \sum_{l=1}^{m} d_l\mu(B_l).$$

□

Lemma 2.4 justifies the following definition.

Definition 2.5. The μ-integral of a non-negative stepunction $f=\sum_{k=1}^{n} c_k\mathbb{1}_{A_k}$ is defined as

$$\int f\,\mathrm{d}\mu := \sum_{k=1}^{n} c_k\mu(A_k)\in[0,\infty].$$

Proposition 2.6. *Let f, g be two non-negative stepfunctions, $\alpha\in[0,\infty]$. Then,*

1. $\int(\alpha f)\,\mathrm{d}\mu = \alpha\int f\,\mathrm{d}\mu$,
2. $\int(f+g)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu$,
3. $f\le g \implies \int f\,\mathrm{d}\mu \le \int g\,\mathrm{d}\mu$.

Proof. Exercise. Hint: Use representations of f and g as in Proposition 2.3, (1.). □

Example 2.7. Let $(a_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ be sequences of real numbers with $a_n\ge 0$. We consider the measure $\mu=\sum_{n=1}^{\infty} a_n\delta_{x_n}$ on $\mathbb{R}$. Then for a non-negative stepfunction $f=\sum_{k=1}^{m} c_k\mathbb{1}_{A_k}$,

$$\int f\,\mathrm{d}\mu = \sum_{k=1}^{m} c_k\mu(A_k) = \sum_{k=1}^{m} c_k\sum_{n=1}^{\infty} a_n\delta_{x_n}(A_k) = \sum_{n=1}^{\infty} a_n\sum_{k=1}^{m} c_k\mathbb{1}_{A_k}(x_n) = \sum_{n=1}^{\infty} a_n f(x_n).$$

Exercise 2.8. Is the function

$$f:\mathbb{R}\to\mathbb{R},\quad x\mapsto\begin{cases}1, & \text{if } x\in\mathbb{Q},\\ 0, & \text{if } x\notin\mathbb{Q},\end{cases}$$

a non-negative stepfunction? If yes, what is its integral with respect to the Lebesgue measure $\mathcal{L}^1$? What is its integral with respect to the Dirac measure δ_1? What is its integral with respect to $\mu=\sqrt{2}\delta_1+\delta_{\sqrt{2}}$?

2.1.2 Integration of non-negative functions

Notation. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, $f_n\colon \Omega \to \bar{\mathbb{R}}$. We call $(f_n)_{n\in\mathbb{N}}$ a *monotone increasing* sequence, if $f_n(\omega) \le f_{n+1}(\omega)$, $\omega \in \Omega$, and we write $f_n \uparrow$. Because $\lim_{n\to\infty} f_n(\omega) = \sup_n f_n(\omega) =: f(\omega) \in \bar{\mathbb{R}}$, it is often written $f_n \uparrow f$, if there is a notation for the limit (in this case f).

The following theorem shows that non-negative stepfunctions are useful for the approximation of general functions.

Theorem 2.9. *For any measurable function $f \ge 0$ there exist non-negative stepfunctions f_n such that $f_n \uparrow f$, that is, $f_n(\omega) \le f_{n+1}(\omega)$ and*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega) \quad \text{for all } \omega \in \Omega.$$

Proof. We define

$$f_n = \sum_{k=0}^{n2^n-1} k2^{-n}\mathbb{1}_{A_k^{(n)}} + n\mathbb{1}_{B^{(n)}},$$

with

$$\begin{aligned} A_k^{(n)} &= \{\omega \in \Omega \mid k2^{-n} \le f(\omega) < (k+1)2^{-n}\} \\ B^{(n)} &= \{\omega \in \Omega \mid f(\omega) \ge n\}. \end{aligned}$$

□

Lemma 2.10. *Let g and f_n, $n \ge 1$ be non-negative stepfunctions and $f_n \uparrow f \ge g$. Then,*

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \ge \int g \, \mathrm{d}\mu. \tag{2.1}$$

Corollary 2.11. *Let $f\colon \Omega \to [0,\infty]$ be measurable and $(f_n)_{n\in\mathbb{N}}$, $(\bar{f}_m)_{m\in\mathbb{N}}$ two sequences of non-negative stepfunctions $f_n \uparrow f$ and $\bar{f}_m \uparrow f$. Then,*

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \lim_{m\to\infty} \int \bar{f}_m \, \mathrm{d}\mu.$$

Proof. Due to Lemma 2.10 and $f_n \uparrow f \ge \bar{f}_m$ for all $m \in \mathbb{N}$, we know that

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \ge \int \bar{f}_m \, \mathrm{d}\mu,$$

so also $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \ge \lim_{m\to\infty} \int \bar{f}_m \, \mathrm{d}\mu$. Interchanging the roles of f_n and $\bar{f}_m$ proves the claim. □

Corollary 2.11 shows that the following definition is sensible.

Definition 2.12. The *μ-integral* of a non-negative measurable function f is defined as

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \in [0,\infty],$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence of non-negative stepfunctions with $f_n \uparrow f$. Such a sequence exists by Theorem 2.9.

Corollary 2.13. *The integral of a measurable non-negative function f equals the supremum of $\int g \, \mathrm{d}\mu$ over all non-negative stepfunctions $g \le f$.*

Proof of lemma 2.10. First notice that the limit above exists due to monotonicity of the sequence $(\int f_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$. It may be equal to $+\infty$. Let $g = \sum_{l=1}^m d_l \mathbb{1}_{B_l}$.

Case 1: If $\int g \, \mathrm{d}\mu < \infty$, then $d_l \mu(B_l) < \infty$ for all $l \in \{1, \dots, m\}$. We can assume without loss of generality that $d_l \in (0, \infty)$ and $\mu(B_l) \in (0, \infty)$ for all $l \in \{1, \dots, m\}$. We set $\tilde{B} := \bigcup_{l=1}^m B_l$. Then, $\mu(\tilde{B}) \le \sum_{l=1}^m \mu(B_l) < \infty$.
For $\varepsilon > 0$ we define

$$A_n := \{\omega \in \tilde{B} \mid f_n(\omega) \ge g(\omega) - \varepsilon\}.$$

Then $A_n \subseteq A_{n+1}$ and $\bigcup_n A_n = \tilde{B}$ (short: $A_n \uparrow \tilde{B}$) and

$$f_n = f_n \mathbb{1}_{A_n} + f_n \mathbb{1}_{A_n^c} \ge f_n \mathbb{1}_{A_n} \ge (g - \varepsilon) \mathbb{1}_{A_n}.$$

The integral of non-negative stepfunctions is monotone and linear, so

$$\begin{aligned} \int \big(f_n + \varepsilon \mathbb{1}_{A_n}\big) \, \mathrm{d}\mu &= \int f_n \, \mathrm{d}\mu + \varepsilon \mu(A_n) \\ &\ge \int g \mathbb{1}_{A_n} \, \mathrm{d}\mu = \sum_{l=1}^m d_l \mu(B_l \cap A_n). \end{aligned}$$

For $n \to \infty$ we therefore obtain

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu + \varepsilon \mu(\tilde{B}) \ge \sum_{l=1}^m d_l \mu(B_l) = \int g \, \mathrm{d}\mu.$$

Since $\mu(\tilde{B}) < \infty$, the claim follows by $\varepsilon \to 0$.

Case 2: Let now $\int g \, \mathrm{d}\mu = \infty$. Then there exists l_0 such that $d_{l_0} \mu(B_{l_0}) = \infty$ and $0 < d_l \le \infty$, $0 < \mu(B_{l_0}) \le \infty$. We choose x, y with $0 < x < d_{l_0}$ and $0 < y < \mu(B_{l_0})$. Define $A_n := \{\omega \in B_{l_0} \mid f_n(\omega) > x\}$. Because of $f_n \uparrow f \ge g$, it follows that $A_n \uparrow B_{l_0}$, so $\mu(A_n) > y$ for all n large enough. For these n it now follows that

$$\int f_n \, \mathrm{d}\mu \ge x \mu(A_n) > xy,$$

so also $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \ge xy$. If $d_l = \infty$, then let $x \to \infty$; if $\mu(B_{l_0}) = \infty$, then let $y \to \infty$. □

Proposition 2.14. *Let $f, g \colon \Omega \to [0, \infty]$ be measurable functions. Then,*

1. $\int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu$ *for all* $c \ge 0$,
2. $f \le g \implies \int f \, \mathrm{d}\mu \le \int g \, \mathrm{d}\mu$,
3. $\int (f + g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$.

Proof. Exercise. □

Theorem 2.15 (Monotone convergence). *Let f, f_n, $n \in \mathbb{N}$ be measurable non-negative functions with $f_n \uparrow f$. Then,*

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Proof. For each n let $(f_n^{(k)})_{k\in\mathbb{N}}$ be a sequence of non-negative stepfunctions with $f_n^{(k)} \uparrow f_n$, $k\to\infty$. We define

$$g^{(k)} := \max_{m\in\{1,\dots,k\}} f_m^{(k)}.$$

The functions $g^{(k)}$ are also non-negative stepfunctions. It holds $f_m^{(k-1)} \le f_m^{(k)} \le f_m$, so

$$g^{(k-1)} = \max_{m\in\{1,\dots,k-1\}} f_m^{(k-1)} \le \max_{m\in\{1,\dots,k\}} f_m^{(k)} = g^{(k)} \le \max_{m\in\{1,\dots,k\}} f_m = f_k.$$

Let $g = \lim_{n\to\infty} g^{(n)}$ and $m\in\mathbb{N}$. For $k\ge m$, $f_m^{(k)} \le g^{(k)}$, and so with $k\to\infty$, it follows that

$$f_m \le g \le f$$

for all $m\in\mathbb{N}$, so $f=g$. With Proposition 2.14 (2.) and $g^{(n)} \le f_n$ we get

$$\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu = \lim_{n\to\infty}\int g^{(n)}\,\mathrm{d}\mu \le \lim_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

Because of $f_n \le f$, it holds $\int f_n\,\mathrm{d}\mu \le \int f\,\mathrm{d}\mu$, which proves the claim. □

Example 2.16 (Continuation of example 2.7)**.** Let again $\mu = \sum_{k=1}^{\infty} a_k\delta_{x_k}$. Let f be a measurable non-negative function and $f_n \uparrow f$ be non-negative stepfunctions. Then due to Theorem 2.15,

$$\int f\,\mathrm{d}\mu = \lim_{n\to\infty}\int f_n\,\mathrm{d}\mu = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_k f_n(x_k) = \sum_{k=1}^{\infty} a_k f(x_k).^{1}$$

We see that sums can be interpreted as integrals.

Theorem 2.17 (Fatou's Lemma)**.** *Let f_n, $n\in\mathbb{N}$ be measurable non-negative functions. Then,*

$$\int \liminf_{n\to\infty} f_n\;\mathrm{d}\mu \le \liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

Proof. Due to the theorem on measurability of limits (Theorem 1.49), $f := \liminf_{n\to\infty} f_n \ge 0$ is a measurable function. Define $g_n := \inf_{m\ge n} f_m$. Then $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n = f$, and because $g_n \le g_{n+1}$, we can apply the monotone convergence theorem (Theorem2.15), which gives

$$\int f\,\mathrm{d}\mu = \int \lim_{n\to\infty} g_n\,\mathrm{d}\mu = \lim_{n\to\infty}\int g_n\,\mathrm{d}\mu.$$

Since $g_n \le f_m$ for $n\le m$, we obtain

$$\int g_n\,\mathrm{d}\mu \le \inf_{m\ge n}\int f_m\,\mathrm{d}\mu \quad\Longrightarrow\quad \lim_{n\to\infty}\int g_n\,\mathrm{d}\mu \le \lim_{n\to\infty}\inf_{m\ge n}\int f_m\,\mathrm{d}\mu = \liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

□

Exercise 2.18. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{F}$. Show that $\liminf_{n\to\infty}\mathbb{1}_{A_n}$ is the indicator function of the set

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty}\bigcap_{m\ge n} A_m.$$

Conclude that

$$\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty}\mu(A_n).$$

[1] In the last step the limits can be interchanged because $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_k f_n(x_k) = \lim_{n\to\infty}\lim_{N\to\infty}\sum_{k=1}^{N} a_k f_n(x_k) = \sup_{n\in\mathbb{N}}\sup_{N\in\mathbb{N}}\sum_{k=1}^{N} a_k f_n(x_k) = \sup_{N\in\mathbb{N}}\sup_{n\in\mathbb{N}}\sum_{k=1}^{N} a_k f_n(x_k) = \sup_{N\in\mathbb{N}}\sum_{k=1}^{N} a_k f(x_k) = \sum_{k=1}^{\infty} a_k f_n(x_k)$.

2.1.3 Integrals of measurable functions

Let now $f\colon \Omega \to \bar{\mathbb{R}}$ be a measurable function. Then also $f_+ := \max\{f, 0\}$ and $f_- := -\min\{f, 0\}$ are measurable, and they are non-negative. Moreover,

$$f = f_+ - f_- \quad \text{und} \quad |f| = f_+ + f_-.$$

Definition 2.19. Let $f\colon \Omega \to \bar{\mathbb{R}}$ be a measurable function. We say that *the integral of f exists*, if at least one of the values $\int f_+ \,\mathrm{d}\mu$, $\int f_- \,\mathrm{d}\mu$ is finite. In this case, we define

$$\int f \,\mathrm{d}\mu := \int f_+ \,\mathrm{d}\mu - \int f_- \,\mathrm{d}\mu \quad \in [-\infty, \infty].$$

We call f *μ-integrable* if $\int f_+ \,\mathrm{d}\mu < \infty$ and $\int f_- \,\mathrm{d}\mu < \infty$. Because $|f| = f_+ + f_-$, the function f is integrable if and only if $\int |f| \,\mathrm{d}\mu < \infty$.

Proposition 2.20. *Let $f, g\colon \Omega \to \bar{\mathbb{R}}$ be integrable functions. Then,*

1. $\int cf \,\mathrm{d}\mu = c \int f \,\mathrm{d}\mu$ *for all* $c \in \mathbb{R}$*;*
2. $f \leq g \quad \Rightarrow \quad \int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$*;*
3. $f + g$ *is integrable too, and* $\int (f + g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$*;*
4. $\left| \int f \,\mathrm{d}\mu \right| \leq \int |f| \,\mathrm{d}\mu$.

Proof. For (1.)-(3.), apply the definition of the integral and Proposition 2.14. For (4.), use that $-|f| \leq f \leq |f|$ and apply (2.). □

Definition 2.21 (Null sets)**.** A set $N \in \mathfrak{F}$ is called μ *null set*, if $\mu(N) = 0$.

Definition 2.22 (Properties 'almost everywhere')**.** Let $E(\omega)$ be a property which may be true or not for each single $\omega \in \Omega$. We say that is true E μ *almost everywhere* or μ *almost surely*, if there exists a μ null set $N \in \mathfrak{F}$ such that E is true for all $\omega \in \Omega \setminus N$.

Example 2.23. Let f, g be measurable functions.

1. ‚$f = g$ μ almost everywhere means:
   $\mu(\{f \neq g\}) = \mu(\{\omega \mid f(\omega) \neq g(\omega)\}) = 0$, so there exists $N := \{\omega \mid f(\omega) \neq g(\omega)\} \in \mathfrak{F}$ with $\mu(N) = 0$ and $f(\omega) = g(\omega)$ for all $\omega \in \Omega \setminus N$.
2. $\lim_{n\to\infty} f_n$ exists μ almost everywhere means:
   The set $\{\omega \mid \lim_{n\to\infty} f_n(\omega) \text{ does not exist}\}$ is a μ null set.

Proposition 2.24. *Let $f, g\colon \Omega \to \bar{\mathbb{R}}$ be integrable functions. Then,*

1. *If $f = 0$ μ almost everywhere, then* $\int f \,\mathrm{d}\mu = 0$.
2. *If $f = g$ μ almost everywhere, then* $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$.
3. *If $f \geq 0$ μ almost everywhere and $\int f \,\mathrm{d}\mu = 0$, then $f = 0$ μ almost everywhere.*

Proof. 1. Let first $f \geq 0$. Measurability of f implies that $N := \{f \neq 0\} \in \mathfrak{F}$, and by assumption $\mu(N) = 0$. Set $g = \infty \mathbb{1}_N$. Then $f \leq g$, so

$$0 \leq \int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu = \infty \cdot 0 = 0.$$

If f is an arbitrary integrable function with $f = 0$ μ almost everywhere, then f_+, f_- are non-negative measurable functions with $f_+ = 0$ μ almost everywhere and $f_- = 0$ μ almost everywhere.

2. Apply (1.) to $f - g$.

3. If $f \geq 0$ μ almost everywhere, then $f_- = 0$ μ almost everywhere, so it only remains to show that $f_+ = 0$ μ almost everywhere. We define $N := \{f_+ > 0\}$ and $A_n := \{f_+ \geq 1/n\}$, $n \in \mathbb{N}$. Then $N, A_n \in \mathfrak{F}$ due to measurability of f_+, and also $N = \bigcup_{n\in\mathbb{N}} A_n$. We will show that $\mu(A_n) = 0$, which concludes the proof.
Due to (1.) we obtain that $\int f_- \, \mathrm{d}\mu = 0$, and $f_+ \geq (1/n)\mathbb{1}_{A_n}$, so

$$0 = \int f \, \mathrm{d}\mu = \int f_+ \, \mathrm{d}\mu \geq \int \frac{1}{n} \mathbb{1}_{A_n} \, \mathrm{d}\mu = \frac{1}{n}\mu(A_n).$$

□

Exercise 2.25. Let $f : \Omega \to \bar{\mathbb{R}}$ be a μ-integrable function. Show that f is finite μ almost everywhere.

2.1.4 Riemann and Lebesgue integrals

Let now $\Omega = \mathbb{R}^d$ with the Borel σ-field $\mathfrak{F} = \mathcal{B}(\mathbb{R}^d)$ and the Lebesgue measure $\mu = \mathcal{L}^d$; see Section 1.3. Let $f \colon \mathbb{R}^d \to \bar{\mathbb{R}}$ be a $\mathcal{L}^d$ integrable function. The integral

$$\int f(x) \, \mathrm{d}x := \int f \, \mathrm{d}\mathcal{L}^d = \int f(x) \, \mathrm{d}\mathcal{L}^d(x) = \int f(x_1, \dots, x_d) \, \mathrm{d}\mathcal{L}^d(x_1, \dots, x_d)$$

is called the *Lebesgue integral* of f. For Borel sets $B \in \mathcal{B}(\mathbb{R}^d)$ we write

$$\int_B f(x) \, \mathrm{d}x := \int \mathbb{1}_B(x) f(x) \, \mathrm{d}x.$$

To compute Lebesgue integrals, you will use your knowledge about Riemann integrals, which is justified by the following theorem.

Theorem 2.26. *Let $f \colon [a, b] \to \mathbb{R}$ be a measurable function, $-\infty < a < b < \infty$. If f is Riemann integrable, then f is Lebesgue integrable and the integrals are equal.*

Proof. See Bauer, Mass- und Integrationstheorie, Satz 16.4. □

The reverse statement of Theorem 2.26 is not true, as the following example shows.

Example 2.27. Es sei

$$f \colon [0, 1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

f is measurable and equals 0 almost everywhere. So f is Lebesgue integrable with $\int_{[0,1]} f(x) \, \mathrm{d}x = 0$. However, f is not Riemann integrable.

2.2 Important theorems on integrals

Let Ω be a set, $\mathfrak{F}$ a σ-field on Ω and μ a measure on $(\Omega, \mathfrak{F})$.

Theorem 2.28 (Hölder's inequality)**.** *Let $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g : \Omega \to \bar{\mathbb{R}}$ measurable functions. Then,*

$$\int |fg| \, \mathrm{d}\mu \leq \Big(\int |f|^p \, \mathrm{d}\mu \Big)^{1/p} \Big(\int |g|^q \, \mathrm{d}\mu \Big)^{1/q}. \tag{2.2}$$

Proof. Let $a > 0$. Consider the function

$$h \colon (0, \infty) \to \mathbb{R}, \quad b \mapsto \frac{a^p}{p} + \frac{b^q}{q} - ab.$$

One can show that h attains a global minimum at $b_0 = a^{\frac{1}{q-1}}$ and $h(b_0) = 0$, so

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab, \quad \text{for all } a, b \geq 0. \tag{2.3}$$

If $\int |f|^p \, \mathrm{d}\mu$ or $\int |g|^q \, \mathrm{d}\mu$ equal $+\infty$, then (2.2) is certainly true. If $\int |f|^p \, \mathrm{d}\mu = 0$ (or $\int |g|^q \, \mathrm{d}\mu = 0$), then $|f| = 0$ almost surely, so also $|fg| = |f||g| = 0$ almost surely and (2.2) is true. We can therefore assume that

$$0 < \int |f|^p \, \mathrm{d}\mu < \infty, 0 < \int |g|^q \, \mathrm{d}\mu < \infty.$$

We set

$$a = \frac{|f|}{(\int |f|^p \, \mathrm{d}\mu)^{1/p}}, \quad b = \frac{|g|}{(\int |g|^q \, \mathrm{d}\mu)^{1/q}}$$

and apply (2.3). This gives

$$|fg| \leq \Big(\frac{1}{p} \frac{|f|^p}{\int |f|^p \, \mathrm{d}\mu} + \frac{1}{q} \frac{|g|^q}{\int |g|^q \, \mathrm{d}\mu} \Big) \Big(\int |f|^p \, \mathrm{d}\mu \Big)^{1/p} \Big(\int |g|^q \, \mathrm{d}\mu \Big)^{1/q}.$$

Computing the integral on both sides yields Hölder's inequality. □

Corollary 2.29 (Cauchy-Schwarz inequality)**.** *For measurable functions $f, g : \Omega \to \bar{\mathbb{R}}$,*

$$\int |fg| \, \mathrm{d}\mu \leq \sqrt{\int f^2 \, \mathrm{d}\mu \int g^2 \, \mathrm{d}\mu}.$$

Theorem 2.30 (Lebesgue convergence theorem, dominated convergence theorem)**.** *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of integrable functions such that $\lim_{n \to \infty} f_n =: f$ μ exists μ almost everywhere. Let g be an integrable function with $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then,*

$$\int f \, \mathrm{d}\mu = \lim_{n \to \infty} \int f_n \, \mathrm{d}\mu$$

so in particular, f is integrable. The function g is called dominating function.

Proof. Let $A = \{\omega \mid \lim_{n\to\infty} f_n(\omega) = f(\omega)\}$. Then $A \in \mathfrak{F}$ and $\mu(A^c) = 0$. On A, we know that $|f| \le g$ almost surely. Because $\mu(A^c) = 0$ it follows that $|f| \le g$ μ almost surely, and integrability of g implies integrability of f.
Set $\tilde{f} := f\mathbb{1}_A$ and $\tilde{f}_n := f_n\mathbb{1}_A$. Then $\tilde{f}_n \to \tilde{f}$ for all ω and $\int f \,\mathrm{d}\mu = \int \tilde{f} \,\mathrm{d}\mu$ as well as $\int f_n \,\mathrm{d}\mu = \int \tilde{f}_n \,\mathrm{d}\mu$. So we can assume that f_n converges to f for all ω, and drop the ˜ for notational convenience.

Because $|f_n - f| \le g + |f|$, the Fatou Lemma (Theorem 2.17) implies that

$$\begin{aligned}\int g + |f| \,\mathrm{d}\mu &= \int \liminf_{n\to\infty} (g + |f| - |f_n - f|) \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int (g + |f| - |f_n - f|) \,\mathrm{d}\mu \\ &= \int g + |f| \,\mathrm{d}\mu - \limsup_{n\to\infty} \int |f_n - f| \,\mathrm{d}\mu.\end{aligned}$$

This gives

$$0 \ge \limsup_{n\to\infty} \int |f_n - f| \,\mathrm{d}\mu \ge \limsup_{n\to\infty} |\int f_n \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu| \ge 0.$$

□

Theorem 2.31 (Integration with respect to image measures). *Let $g\colon \Omega \to \mathbb{R}$ be a measurable function and μ_g its image measure. For any measurable function $f\colon \mathbb{R} \to [0,\infty]$,*

$$\int_{\mathbb{R}} f \,\mathrm{d}\mu_g = \int_{\Omega} f \circ g \,\mathrm{d}\mu. \tag{2.4}$$

Proof. First verify this formula for non-negative stepfunctions. Notice that if $f\colon \mathbb{R} \to [0,\infty]$ is a non-negative stepfunction, then $f \circ g\colon \Omega \to [0,\infty]$ is also a non-negative stepfunction, but on Ω. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative stepfunctions with $f_n \uparrow f$. Then also $f_n \circ g \uparrow f \circ g$, so the formula follows by the definition of the integral. □

Corollary 2.32. *Formula* (2.4) *also holds for integrable $f\colon \mathbb{R} \to \bar{\mathbb{R}}$.*

Proof. Decompose into positive and negative part. □

The following proposition shows that one can use non-negative measurable functions to define new measures.

Proposition 2.33. *Let μ be a measure on $(\Omega, \mathfrak{F})$, and let $f\colon \Omega \to [0,\infty]$ be a measurable function. Then*

$$\lambda(A) := \int \mathbb{1}_A f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu$$

defines a measure on $(\Omega, \mathfrak{F})$.

Proof. Exercise. □

Definition 2.34. Let μ, λ be measures on $(\Omega, \mathfrak{F})$. If the measure λ is given by

$$\lambda(A) = \int_A f \,\mathrm{d}\mu$$

for a measurable function $f\colon \Omega \to [0,\infty]$, then f is called a *density of λ with respect to μ*.

Theorem 2.35 (Integration with respect to measures with densities)**.** *Assume that the measure λ admits a density f with respect to μ. Then for each non-negative measurable (or integrable) function g,*

$$\int g \,\mathrm{d}\lambda = \int gf \,\mathrm{d}\mu.$$

Proof. Let first $g = \sum_{k=1}^n c_k \mathbb{1}_{A_k}$ be a non-negative stepfunction. Then,

$$\int g \,\mathrm{d}\lambda = \sum_{k=1}^n c_k \lambda(A_k) = \sum_{k=1}^n c_k \int \mathbb{1}_{A_k} f \,\mathrm{d}\mu = \int \sum_{k=1}^n c_k \mathbb{1}_{A_k} f \,\mathrm{d}\mu = \int gf \,\mathrm{d}\mu.$$

Let g be a non-negative measurable function and $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-negative stepfunctions $g_n \uparrow g$. Then also $g_n f \uparrow gf$, so the claim follows by the monotone convergence theorem (Theorem 2.15). The case for integrable g follows by decomposition into positive and negative part. □

2.3 Product of two measures

Consider two measure spaces $(\Omega_1, \mathfrak{F}_1, \mu_1)$ and $(\Omega_2, \mathfrak{F}_2, \mu_2)$. We are interested in the product set

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

Definition 2.36. The *product σ-field* $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ is given by

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 := \sigma(\{A_1 \times A_2 \mid A_1 \in \mathfrak{F}_1, A_2 \in \mathfrak{F}_2\}).$$

Remark 2.37. Let $\tilde{\Omega}$ be a set and $\mathfrak{F}$ a σ-field on $\tilde{\Omega}$. For two mappings $f : \tilde{\Omega} \to \Omega_1$, $g : \tilde{\Omega} \to \Omega_2$ we define the σ-field generated by f and g by

$$\sigma(f, g) = \sigma\big(\sigma(f) \cup \sigma(g)\big),$$

where $\sigma(f) = \{f^{-1}(B) \mid B \in \mathfrak{F}_1\}$, $\sigma(g) = \{g^{-1}(B) \mid B \in \mathfrak{F}_2\}$ are defined as in Example 1.2, part 2. The σ-field $\sigma(f, g)$ is the smallest σ-field $\mathcal{A}$ such that f is $\mathcal{A}$-$\mathfrak{F}_1$-measurable and g is $\mathcal{A}$-$\mathfrak{F}_2$-measurable.

Lemma 2.38. *Let $p_i : \Omega_1 \times \Omega_2 \to \Omega_i$, $(\omega_1, \omega_2) \mapsto \omega_i$, $i = 1, 2$ be the i-th projection mapping. The product σ-field $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ is the smallest σ-field $\mathcal{A}$ such that p_i is $\mathcal{A}$-$\mathfrak{F}_i$-measurable for $i = 1, 2$.*

Proof. Let $A_1 \in \mathfrak{F}_1$. Then,

$$p_1^{-1}(A_1) = A_1 \times \Omega_2 \in \mathfrak{F}_1 \otimes \mathfrak{F}_2,$$

so $\sigma(p_1) \subseteq \mathfrak{F}_1 \otimes \mathfrak{F}_2$. Analogously $\sigma(p_2) \subseteq \mathfrak{F}_1 \otimes \mathfrak{F}_2$, and therefore $\sigma(p_1, p_2) \subseteq \mathfrak{F}_1 \otimes \mathfrak{F}_2$. On the other hand, for $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$,

$$A_1 \times A_2 = p_1^{-1}(A_1) \cap p_2^{-1}(A_2) \in \sigma(p_1, p_2),$$

so $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \sigma(p_1, p_2)$. The claim follows by Remark 2.37. □

Proposition 2.39. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be generators of $\mathfrak{F}_1$, $\mathfrak{F}_2$, that is, $\mathfrak{F}_1 = \sigma(\mathcal{A}_1)$, $\mathfrak{F}_2 = \sigma(\mathcal{A}_2)$. Assume that $\mathcal{A}_1$ contains a sequence $E_{1,n} \uparrow \Omega_1$ and $\mathcal{A}_2$ contains a sequence $E_{2,n} \uparrow \Omega_2$. Then,*

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \sigma(\{A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}).$$

Proof. We define $\mathcal{E} := \{A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$. For $i = 1, 2$,

$$p_i \text{ is } \mathfrak{F}_1 \otimes \mathfrak{F}_2\text{-}\mathfrak{F}_i\text{-measurable.} \quad \Longleftrightarrow \quad p_i^{-1}(A_i) \in \mathfrak{F}_1 \otimes \mathfrak{F}_2,\ \forall A_i \in \mathcal{A}_i.$$

For $i = 1$ and $A_1 \in \mathcal{A}_1$, $A_1 \times E_{2,n} \in \mathcal{E}$ is true for all $n \in \mathbb{N}$, so $A_1 \times E_{2,n} \uparrow A_1 \times \Omega_2 \in \sigma(\mathcal{E})$, and so $p_1^{-1}(A_1) = A_1 \times \Omega_2 \in \sigma(\mathcal{E})$, hence p_1 is $\sigma(\mathcal{E})$-$\mathcal{A}_1$-measurable, and p_2 is $\sigma(\mathcal{E})$-$\mathcal{A}_2$-measurable by analogous arguments. Due to Lemma 2.38, we get $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \sigma(\mathcal{E})$. □

Example 2.40. The Borel σ-field $\mathcal{B}(\mathbb{R}^2)$ on $\mathbb{R}^2$ is generated by the set of rectangles $(\mathbf{a}, \mathbf{b}]$, $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, and

$$(\mathbf{a}, \mathbf{b}] = (a_1, b_1] \times (a_2, b_2],$$

so by Proposition 2.39, we obtain that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

We now want to define a measure π on $(\Omega, \mathfrak{F}_1 \otimes \mathfrak{F}_2)$ in such a way that

$$\pi(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \quad A_1 \in \mathfrak{F}_1, A_2 \in \mathfrak{F}_2. \tag{2.5}$$

This is possible, but requires some preliminary results.

Definition 2.41. A measure μ on $\mathfrak{F}$ is called *σ-finite*, if there exits a sequence $A_n \uparrow \Omega$ with $\mu(A_n) < \infty$.

Remark. All important measures in this lecture are σ-finite, in particular the Lebesgue measure, measures with densities with respect to the Lebesgue measure, and discrete measures with finite values.

Theorem 2.42 (Uniqueness of product measures)**.** *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be $\cap$-stable generators of $\mathfrak{F}_1$, $\mathfrak{F}_2$, so $\mathfrak{F}_1 = \sigma(\mathcal{A}_1)$, $\mathfrak{F}_2 = \sigma(\mathcal{A}_2)$. Assume further that $\mathcal{A}_1$ contains a sequence $A_{1,n} \uparrow \Omega_1$, $\mu_1(A_{1,n}) < \infty$, and $\mathcal{A}_2$ contains a sequence $A_{2,n} \uparrow \Omega_2$, $\mu_2(A_{2,n}) < \infty$. Then there exists at most one measure π on $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ with property* (2.5)*.*

Proof. The system of sets $\{A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ is also $\cap$-stable. It generates $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ due to Proposition 2.39. It holds $A_{1,n} \times A_{2,n} \uparrow \Omega = \Omega_1 \times \Omega_2$, and by (2.5) we obtain

$$\pi(A_{1,n} \times A_{2,n}) = \mu_1(A_{1,n})\mu_2(A_{2,n}) < \infty$$

so the claim follows by Theorem 1.35. □

Definition 2.43. Let $A \subset \Omega_1 \times \Omega_2$. For each $\omega_1 \in \Omega_1$ we define the *ω_1-cut* of A by

$$A_{\omega_1} := \{\omega_2 \in \Omega_2 \mid (\omega_1, \omega_2) \in A\} \subseteq \Omega_2$$

and analogously for $\omega_2 \in \Omega_2$, ${}_{\omega_2}A := \{\omega_1 \in \Omega_1 \mid (\omega_1, \omega_2) \in A\} \subseteq \Omega_1$.

Lemma 2.44. *For $A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$, it holds $A_{\omega_1} \in \mathfrak{F}_2$ and ${}_{\omega_2}A \in \mathfrak{F}_1$ for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$.*

Proof. Let $\omega_1 \in \Omega_1$. We define

$$\mathfrak{A} := \{A \subseteq \Omega \mid A_{\omega_1} \in \mathfrak{F}_2\}.$$

$\mathfrak{A}$ is a σ-field (Exercise). For all $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$, it holds $A_1 \times A_2 \in \mathfrak{A}$, so

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{A}.$$

□

Lemma 2.45. *Assume that* μ_1*,* μ_2 *are* σ*-finite. Then for all* $A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$*, the function*

$$s_A\colon \Omega_1 \to [0,\infty], \quad \omega_1 \mapsto \mu_2(A_{\omega_1})$$

is $\mathfrak{F}_1$*-measurable, and analogously, the function*

$$t_A\colon \Omega_2 \to [0,\infty], \quad \omega_2 \mapsto \mu_1({}_{\omega_2}A)$$

is $\mathfrak{F}_2$*-measurable.*

Proof. We first show that s_A is $\mathfrak{F}_1$-measurable if $\mu_2(\Omega_2) < \infty$. Define

$$\mathfrak{D} := \{A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2 \mid s_A \text{ is } \mathfrak{F}_1\text{-measurable.}\}$$

and show that $\mathfrak{D}$ is a Dynkin system containing all sets of the form $A_1 \times A_2$, $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$. Since $\{A_1 \times A_2 \mid A_1 \in \mathfrak{F}_1, A_2 \in \mathfrak{F}_2\}$ is $\cap$-stable, $\mathfrak{D} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$ follows by Theorem 1.15. Let μ_2 be σ-finite and $B_n \to \Omega_2$ with $\mu_2(B_n) < \infty$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, $\mu_{2,n} : \mathfrak{F}_2 \to [0,\infty), A \mapsto \mu_2(A \cap B_n)$ is a finite measure on $\mathfrak{F}_2$. So for all $A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$, the mapping $\Omega_1 \to [0,\infty], \omega_1 \mapsto \mu_{2,n}(A_{\omega_1})$ is $\mathfrak{F}_1$-measurable, and

$$\sup_{n\in\mathbb{N}} \mu_{2,n}(A_{\omega_1}) = \sup_{n\in\mathbb{N}} \mu_2(A_{\omega_1} \cap B_n) = \mu_2(A_{\omega_1}).$$

So also $\Omega_1 \to [0,\infty], \omega_1 \mapsto \mu_2(A_{\omega_1})$ is $\mathfrak{F}_1$-measurable. □

Theorem 2.46 (Existence of product measures)**.** *Assume that* μ_1*,* μ_2 *are* σ*-finite. There is a unique measure* π *on* $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ *with property* (2.5)*. For each* $A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$*,*

$$\pi(A) = \int \mu_2(A_{\omega_1})\,\mathrm{d}\mu_1(\omega_1) = \int \mu_1({}_{\omega_2}A)\,\mathrm{d}\mu_2(\omega_2).$$

We denote the product measure π *by* $\mu_1 \otimes \mu_2$*.*

Proof. Let $A \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$. Due to Lemma 2.45, s_A is a non-negative $\mathfrak{F}_1$-measurable function on Ω_1. We define

$$\pi(A) := \int s_A\,\mathrm{d}\mu_1.$$

Verify that π is a measure on $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ (Exercise). We now verify property (2.5). Let $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$. Then,

$$\pi(A_1 \times A_2) = \int s_{A_1 \times A_2}\,\mathrm{d}\mu_1 = \int \mathbb{1}_{A_1}\mu_2(A_2)\,\mathrm{d}\mu_1 = \mu_1(A_2)\mu_2(A_2).$$

Another measure on $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ is obtained by

$$\pi'(A) = \int t_A\,\mathrm{d}\mu_2 = \int \mu_1({}_{\omega_2}A)\,\mathrm{d}\mu_2(\omega_2).$$

It always satisfies (2.5), so $\pi = \pi'$ by the theorem on uniqueness of product measures (Theorem 2.42). □

Example 2.47. For the Lebesgue measure $\mathcal{L}^2$ on $\mathbb{R}^2$, we obtain that with $(\mathbf{a},\mathbf{b}] = (a_1,b_1] \times (a_2,b_2]$,

$$\mathcal{L}^2((\mathbf{a},\mathbf{b}]) = (b_1 - a_1)(b_2 - a_2) = \mathcal{L}^1((a_1,b_1])\mathcal{L}^1((a_2,b_2]),$$

so $\mathcal{L}^2 = \mathcal{L}^1 \otimes \mathcal{L}^1$.

Theorem 2.48 (Fubini's Theorem)**.** *Let μ_1, μ_2 be σ-finite measures. Let $f\colon \Omega = \Omega_1 \times \Omega_2 \to [0,\infty]$ be a $\mathfrak{F}_1 \otimes \mathfrak{F}_2$-measurable function. Then the function $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1,\omega_2)\,\mathrm{d}\mu_2(\omega_2)$ is $\mathfrak{F}_1$-measurable, and analogously for the other component, and*

$$\int f\,\mathrm{d}(\mu_1 \otimes \mu_2) = \int\int f(\omega_1,\omega_2)\,\mathrm{d}\mu_1(\omega_1)\,\mathrm{d}\mu_2(\omega_2) = \int\int f(\omega_1,\omega_2)\,\mathrm{d}\mu_2(\omega_2)\,\mathrm{d}\mu_1(\omega_1).$$

An analogous statement holds for integrable functions.

Proof. See *Bauer, Mass- und Integrationstheorie*, Satz 23.3. □

Remark 2.49. One can show that the product operation on σ-fields is associative, i.e. $\mathfrak{F}_1 \otimes (\mathfrak{F}_2 \otimes \mathfrak{F}_3) = (\mathfrak{F}_1 \otimes \mathfrak{F}_2) \otimes \mathfrak{F}_3$. This justifies writing $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \otimes \mathfrak{F}_3$. For n σ-finite measures $\mu_1,\dots,\mu_n$ on $\mathfrak{F}_1,\dots,\mathfrak{F}_n$ one can show by induction that there exists a unique product π on $\mathfrak{F}_1 \otimes \cdots \otimes \mathfrak{F}_n$ with

$$\pi(A_1 \times \cdots \times A_n) = \mu_1(A_1) \times \cdots \times \mu_n(A_n)$$

for all $A_i \in \mathfrak{F}_i$, $i = 1,\dots,n$.

Chapter 3

Radon-Nikodym Theorem

3.1 Uniqueness of densities

Let Ω be a set and $\mathfrak{F}$ a σ-field on Ω. In Section 1 we have shown the following theorem.

Theorem (Proposition 2.33)**.** *Let μ be a measure on $(\Omega, \mathfrak{F})$, and $f : \Omega \to [0, \infty]$ a measurable function. Then*

$$\lambda(A) := \int_A f d\mu$$

defines a measure on $(\Omega, \mathfrak{F})$.

This theorem motivated the following definition.

Definition (Definition 2.34)**.** Let μ, λ be measures on $(\Omega, \mathfrak{F})$. If the measure λ is given by

$$\lambda(A) = \int_A f d\mu$$

for a measurable function $f : \Omega \to [0, \infty]$, then f is called *density of λ with respect to μ.*

The uniqueness of densities is described by the following theorem.

Theorem 3.1. *Let μ be a σ-finite measure and assume that the measure λ admits density $f : \Omega \to [0, \infty]$ with respect to μ. Then f is μ almost surely unique.*

The proof of this theorem uses the following lemma.

Lemma 3.2. *A measure μ is σ-finite if and only if there exists a μ-integrable function $h : \Omega \to \mathbb{R}$ such that $0 < h(\omega) < \infty$ for all $\omega \in \Omega$.*

Proof. Let μ be σ-finite, i.e. there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{F}$ with $A_n \uparrow \Omega$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. We define $\varepsilon_n := 2^{-n} \min\{1, 1/\mu(A_n)\}$. Then the function $h := \sum_{n=1}^{\infty} \varepsilon_n \mathbb{1}_{A_n}$ satisfies the claimed properties (Exercise). On the other hand, if $h > 0$ is μ-integrable, we set $A_n := \{h \geq 1/n\} \in \mathfrak{F}$. Since $h > 0$, it follows $A_n \uparrow \Omega$, and because $\mathbb{1}_{A_n} \leq nh$, we obtain

$$\mu(A_n) = \int \mathbb{1}_{A_n} \, \mathrm{d}\mu \leq n \int h \, \mathrm{d}\mu < \infty.$$

$\square$

Proof of Theorem 3.1. **Step 1:** We assume that $\lambda(\Omega) < \infty$. Then $\int f d\mu = \lambda(\Omega) < \infty$. Let $g : \Omega \to [0, \infty]$ be another density of λ with respect to μ. Because of $\int g d\mu = \lambda(\Omega) < \infty$, the function g is μ-integrable. Consider

$$A = \{f > g\} \in \mathfrak{F}.$$

Then,

$$\lambda(A) = \int_A g d\mu \leq \int_A f d\mu = \lambda(A),$$

so

$$\int_A (f - g) d\mu = 0,$$

which implies that $\mathbb{1}_A(f - g) = 0$ μ almost surely, so $\mu(A) = 0$ because $f - g > 0$ on A. Analogous arguments can be used for $B = \{f < g\}$.
Step 2: Let now $\mu(\Omega) < \infty$, but $\lambda(\Omega) = \infty$. We use the following Lemma.

Lemma. *There is a sequence of disjoint sets $(\Omega_n)_{n\in\mathbb{N}} \subseteq \mathfrak{F}$ with $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and the following properties:*

1. *For all $A \in \mathfrak{F}$,*

$$\mu(A \cap \Omega_1) = \lambda(A \cap \Omega_1) = 0 \quad \textit{or} \quad \Big(\mu(A \cap \Omega_1) > 0 \textit{ and } \lambda(A \cap \Omega_1) = \infty.\Big) \tag{3.1}$$

2. *For $n \geq 2$, $\lambda(\Omega_n) < \infty$.*

Let g be another density of λ with respect to μ. We define $A := \{f > g\}$. For $n \geq 2$, show as in Step 1 (with $A \cap \Omega_n$ instead of A) that $\mu(A \cap \Omega_n) = 0$. Now we will show that $\mathbb{1}_{\Omega_1} f = \infty$ μ almost surely. For $k \in \mathbb{N}$,

$$\lambda(\{f \leq k\} \cap \Omega_1) = \int_{\{f \leq k\}} \mathbb{1}_{\Omega_1} f \, \mathrm{d}\mu \leq k\mu(\{f \leq k\} \cap \Omega_1).$$

By (3.1) it follows that $\mu(\{f \leq k\} \cap \Omega_1) = 0$.

Proof of the Lemma. We define

$$\mathcal{Q} := \{B \in \mathfrak{F} \mid \lambda(B) < \infty\}, \quad \alpha := \sup_{B \in \mathcal{Q}} \mu(B) < \infty.$$

Let $(B'_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}$ with $\lim_{n\to\infty} \mu(B'_n) = \alpha$. Since $\mathcal{Q}$ is closed under finite unions, we have that $B_n := B'_1 \cup \cdots \cup B'_n \in \mathcal{Q}$. Define $B_0 := \bigcup_{n\in\mathbb{N}} B_n \in \mathfrak{F}$. Then $B_n \uparrow B_0$ and $\mu(B_0) = \alpha$. We define for $n \geq 2$

$$\Omega_1 := \Omega \backslash B_0 \quad \Omega_2 := B_1, \quad \Omega_{n+1} = B_n \backslash B_{n-1}.$$

Now we show that (3.1) is satisfied. Let $A \in \mathfrak{F}$ with $\lambda(A \cap \Omega_1) < \infty$. For all $n \in \mathbb{N}$, it holds $B_n \cup (A \cap \Omega_1) \in \mathcal{Q}$, so $\mu(B_n \cup (A \cap \Omega_1)) \leq \alpha$. Therefore,

$$\alpha + \mu(A \cap \Omega_1) = \mu(B_0) + \mu(A \cap \Omega_1) = \mu(B_0 \cup (A \cap \Omega_1)) = \lim_{n\to\infty} \mu(B_n \cup (A \cap \Omega_1)) \leq \alpha$$

which implies that $\mu(A \cap \Omega_1) = 0$ and thus (3.1). □

Step 3: Let now μ be σ-finite and h as in Lemma 3.2. The measure

$$\mathfrak{F} \to [0, \infty], \quad A \mapsto \int_A h\, f \,\mathrm{d}\mu$$

admits density f with respect to the finite measure

$$\nu : \mathfrak{F} \to [0, \infty], \quad A \mapsto \int_A h \,\mathrm{d}\mu,$$

so f is unique ν almost everywhere. Because $h > 0$, the inequality $\mu(A) > 0$ is true if and only if $\nu(A) > 0$, so f is also μ almost surely unique. □

3.2 Existence of densities

We now turn to the question under what conditions a measure λ admits a density with respect to μ. A necessary condition is the following. Assume that λ admits the density f with respect to μ. Let $A \in \mathfrak{F}$ be a set with $\mu(A) = 0$, so $\mathbb{1}_A = 0$ μ almost everywhere and therefore $\mathbb{1}_A f = 0$ μ almost everywhere. So,

$$\lambda(A) = \int \mathbb{1}_A f d\mu = 0.$$

This motivates the following definition.

Definition 3.3. Let λ, μ be measures on a measurable space $(\Omega, \mathfrak{F})$. The measure λ is called *absolutely continuous* with respect to μ, if for all $A \in \mathfrak{F}$,

$$\mu(A) = 0 \quad \Longrightarrow \quad \lambda(A) = 0,$$

i.e. every μ null set is also a λ null set.

Exercise 3.4. Which of the measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are absolutely continuous with respect to each other?

$$\mu_1 = \delta_1, \quad \mu_2 = \sum_{n=1}^{\infty} \delta_n, \quad \mu_3 = \mathcal{L}^1, \quad \mu_4 = \mathcal{L}^1(\cdot \cap [0, \infty)).$$

The following important theorem in probability theory shows that in many cases absolute continuity is not only a necessary but also a sufficient condition for the existence of densities.

Theorem 3.5 (Radon-Nikodym Theorem)**.** *Let λ be a measure and μ a σ-finite meausre on a measurable space $(\Omega, \mathfrak{F})$. Then the following statements are equivalent.*

1. *λ admits a density with respect to μ.*
2. *λ is absolutely continuous with respect to μ.*

The proof requires the following lemma.

Lemma 3.6. *Let η and ν be finite measures on $(\Omega, \mathfrak{F})$ with $\nu(\Omega) < \eta(\Omega)$. Then there exists a set $\Omega' \in \mathfrak{F}$ with*

$$\nu(\Omega') < \eta(\Omega')$$

and

$$\nu(\Omega' \cap A) \leq \eta(\Omega' \cap A), \quad \textit{for all } A \in \mathfrak{F}.$$

Proof. The mapping

$$\delta : \mathfrak{F} \to \mathbb{R},\ A \mapsto \delta(A) := \eta(A) - \nu(A)$$

is bounded because $-\nu(\Omega) \leq \delta(A) \leq \eta(\Omega)$. If $\inf_{A\in\mathfrak{F}} \delta(A) \geq 0$, we can choose $\Omega' = \Omega$ and the proof is complete. Otherwise, define inductively two sequences $(A_n)_{n\in\mathbb{N}}$ and $(\Omega_n)_{n\in\mathbb{N}}$ of sets, as explained as follows. We set $A_0 := \emptyset$ and $\Omega_0 := \Omega \setminus A_0 = \Omega$. If now A_n, Ω_n are defined, we set

$$\alpha_n := \inf_{A\in\mathfrak{F}} \delta(A \cap \Omega_n).$$

If $\alpha_n \geq 0$, choose $A_{n+1} = \emptyset$ and $\Omega_{n+1} = \Omega_n = \Omega_n \setminus A_{n+1}$. If $\alpha_n < 0$, then there exists $A \in \mathfrak{F}$ with $\delta(A \cap \Omega_n) \leq \alpha_n/2$. We define $A_{n+1} = A \cap \Omega_n$ and $\Omega_{n+1} = \Omega_n \setminus A_{n+1}$.

The sets in the sequence $(A_n)_{n\in\mathbb{N}}$ are pairwise disjoint. Therefore the series $\sum_{n=0}^{\infty} \delta(A_n)$ converges (to $\eta(\cup A_n) - \nu(\cup A_n)$), which implies $\lim_{n\to\infty} \delta(A_n) = 0$. This gives $\lim_{n\to\infty} \alpha_n = 0$. We set $\Omega' = \bigcap_{n=0}^{\infty} \Omega_n$. The sequence $(\Omega_n)_{n\in\mathbb{N}}$ is monotone decreasing, so (by continuity of the measures η and ν)

$$\lim_{n\to\infty} \delta(\Omega_n) = \delta(\Omega').$$

We now show that Ω' satisfies the two claimed properties. By construction of the sequence we know that $\delta(A_n) \leq 0$, so

$$\delta(\Omega_{n+1}) = \delta(\Omega_n) - \delta(A_{n+1}) \geq \delta(\Omega_n) \geq \cdots \geq \delta(\Omega_0) = \delta(\Omega) > 0.$$

This gives $\delta(\Omega') = \lim_{n\to\infty} \delta(\Omega_n) > 0$. Let now $A \in \mathfrak{F}$. By $A \cap \Omega' = A \cap \Omega' \cap \Omega_n$ for all n, we conclude $\delta(A \cap \Omega') \geq \alpha_n$ for all n, so because $\lim_{n\to\infty} \alpha_n = 0$ we obtain that $\delta(A \cap \Omega') \geq 0$. □

Proof of Theorem 3.5. The direction "'1. $\Longrightarrow$ 2."' was proved before Definition 3.3. We now show the reverse direction "'2. $\Longrightarrow$ 1."'. The proof is separated into three parts.

Step 1: Assume that the measures μ and λ are finite. Let $\mathcal{G}$ be the set of all measurable functions $g : \Omega \to [0, \infty]$ with

$$\int_A g d\mu \leq \lambda(A), \quad \text{for all } A \in \mathfrak{F}.$$

The set $\mathcal{G}$ is non-empty, because it contains the function which is constant zero. For two functions $g, h \in \mathcal{G}$, it also holds $\max\{g, h\} \in \mathcal{G}$, because

$$\int_A \max\{g, h\} d\mu = \int_{A\cap\{g\geq h\}} g d\mu + \int_{A\cap\{g<h\}} h d\mu \leq \lambda(A \cap \{g \geq h\}) + \lambda(A \cap \{g < h\}) = \lambda(A). \tag{3.2}$$

For all $g \in \mathcal{G}$, $\int g d\mu \leq \lambda(\Omega) < \infty$, so also

$$\gamma := \sup_{g\in\mathcal{G}} \int g d\mu < \infty$$

and there exists a sequence $(g'_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}$ with $\lim_{n\to\infty} \int g'_n d\mu = \gamma$. Because of (3.2),

$$g_n := \max\{g'_1, \ldots, g'_n\} \in \mathcal{G}$$

and $g'_n \leq g_n$ implies $\int g'_n d\mu \leq \int g_n d\mu$, so also $\lim_{n\to\infty} \int g_n d\mu = \gamma$. The monotone convergence theorem now yields $f := \sup_n g_n \in \mathcal{G}$ as well as $\int f d\mu = \gamma$. We now show that λ admits the density f with respect to μ. We define

$$\tau(A) = \lambda(A) - \int_A f d\mu, \quad A \in \mathfrak{F}.$$

By construction of f, we know that $\tau(A) \geq 0$, and τ is a finite measure which is absolutely continuous with respect to μ by assumption. We have to show that $\tau = 0$. Assume for a contradiction that $\tau(\Omega) > 0$. Then $\mu(\Omega) > 0$ by absolute continuity and so

$$\beta := \frac{\tau(\Omega)}{2\mu(\Omega)} > 0,$$

hence $\tau(\Omega) = 2\beta\mu(\Omega) > \beta\mu(\Omega)$. Lemma 3.6 applied to $\eta = \tau$ and $\nu = \beta\mu$ yields a set $\Omega' \in \mathfrak{F}$ with

$$\beta\mu(\Omega') < \tau(\Omega')$$

and

$$\beta\mu(\Omega' \cap A) \leq \tau(\Omega' \cap A), \quad \text{for all } A \in \mathfrak{F}.$$

We define $f_0 = f + \beta \mathbb{1}_{\Omega'}$. Then $f_0 \in \mathcal{G}$, because

$$\int_A f_0 d\mu = \int_A f d\mu + \beta\mu(\Omega' \cap A) \leq \int_A f d\mu + \tau(A) = \lambda(A), \quad A \in \mathfrak{F}.$$

It holds $\mu(\Omega') > 0$, because $\mu(\Omega') = 0$ would imply $\tau(\Omega') = 0$, in contradiction to $\tau(\Omega') > \beta\mu(\Omega')$. On the other hand, it now follows that

$$\int_\Omega f_0 d\mu = \int_\Omega f d\mu + \beta\mu(\Omega') = \gamma + \beta\mu(\Omega') > \gamma$$

and this is a contradiction to the construction of γ. So we have shon that $\tau = 0$.
Step 2: Assume that $\mu(\Omega) < \lambda(\Omega) \leq \infty$. We skip the details.
Step 3: We now allow that $\mu(\Omega) = \infty$, but we assume that μ is σ-finite. Also here we skip the details.
The missing parts can be found in Bauer, Mass- und Integrationstheorie, Satz 17.10. □

Exercise 3.7. Find the densities of the measures in Exercise 3.4 which are absolutely continuous with respect to each other.